Geom Dedicata
DOI 10.1007/s10711-006-9103-y

ORIGINAL PAPER

The true prosoluble completion of a group: Examples and open problems

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Received: 5 January 2006 / Accepted: 5 October 2006

Abstract The *true prosoluble completion* $\mathcal{PS}(\Gamma)$ of a group Γ is the inverse limit of the projective system of soluble quotients of Γ. Our purpose is to describe examples and to point out some natural open problems. We discuss a question of Grothendieck for profinite completions and its analogue for true prosoluble and true pronilpotent completions.

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Goulnara Arzhantseva and Zoran Šunić were the authors of the Appendix.

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Keywords Soluble group · Residual properties · True prosoluble completion · Profinite completion · Open problems · Grigorchuk group

Mathematics Subject Classifications (2000) 20E18 · 20F14 · 20F22

1 Introduction

A group Γ has a *profinite topology*, for which the set $\mathcal{F}$ of normal subgroups of finite index is a basis of neighbourhoods of the identity, and the resulting *profinite completion*, hereafter denoted by $P\mathcal{F}(\Gamma)$. The canonical homomorphism

$$\varphi_{\mathcal{F}} : \Gamma \longrightarrow P\mathcal{F}(\Gamma)$$

is injective if the group Γ is *residually finite* (by definition). The notion of profinite completion is relevant in various domains (outside pure group theory), including Galois theory of infinite fields extensions and fundamental groups in algebraic topology [25]. For the theory of profinite groups, there are many papers and several books available [18,47,54]; see Sect. 1.1 in [50] for a quick introduction and [27] for an early paper.

Besides $\mathcal{F}$, there are other natural families of normal subgroups of Γ which give rise to other "procompletions". The purpose of this report is to consider some variants, with special emphasis on the *true prosoluble completion* $P\mathcal{S}(\Gamma)$ associated to the family $\mathcal{S}$ of all normal subgroups of Γ with soluble quotients. The corresponding homomorphism

$$\varphi_{\mathcal{S}} : \Gamma \longrightarrow P\mathcal{S}(\Gamma)$$

is injective if the group Γ is *residually soluble* (by definition).

On the one hand, we discuss examples including free groups, free soluble groups, wreath products, $SL_d(\mathbf{Z})$ and its congruence subgroups, the Grigorchuk group, and parafree groups. On the other hand, we discuss some open problems, of which we would like to point out from the start the following ones.

(i) Let Γ, Δ be M two residually finite groups and let $\psi : \Gamma \longrightarrow \Delta$ be a homomorphism such that, at the profinite level, the corresponding homomorphism $P\mathcal{F}(\psi)$: $P\mathcal{F}(\Gamma) \longrightarrow P\mathcal{F}(\Delta)$ is an isomorphism. How far from an isomorphism can ψ be? The problem goes back to Grothendieck [25], and is motivated by the need to compare two notions of a fundamental group for algebraic varieties. There are examples with ψ not an isomorphism and $P\mathcal{F}(\psi)$ an isomorphism, with Γ, Δ finitely generated [44] and finitely presented [13]. If $P\mathcal{F}(\psi)$ is an isomorphism, there are known sufficient conditions [1] on Γ and Δ for ψ to be an isomorphism.
Our main interest in this paper is the *prosoluble analogue of Grothendieck's problem.* More precisely, let again Γ, Δ be two groups and ψ: $\Gamma \longrightarrow \Delta$ a homomorphism, but assume now that the groups are residually soluble. Assume that, at the true prosoluble level, the corresponding homomorphism $P\mathcal{S}(\psi)$: $P\mathcal{S}(\Gamma) \longrightarrow P\mathcal{S}(\Delta)$ is an isomorphism. How far from an isomorphism can ψ be? Additional requirements can be added on Γ and Δ (such as finite generation, finite presentation, ...).

[1] Proposition 2 of [45]: if Γ, Δ are finitely generated residually finite groups, if Δ is a soluble subgroup of $GL_n(\mathbf{C})$ for some $n \geq 1$, and if $P\mathcal{F}(\psi)$ is an isomorphism, then so is ψ.

(ii) True prosoluble completions provide a natural setting to turn qualitative statements of the kind "some group Γ is not residually soluble" in more precise statements concerning the *true prosoluble kernel* $\mathrm{Ker}(\varphi_{\mathcal{S}} : \Gamma \longrightarrow P\mathcal{S}(\Gamma))$. One example is worked out in (6.F).

(iii) Can one find interesting characterizations of those groups which are true prosoluble completions of residually soluble group? More precisely, let G be a complete Hausdorff topological group such that, for any $g \in G$, $g \neq 1$, there exists an open normal subgroup N of G not containing g such that G/N is soluble; how can it be decided whether G is isomorphic to $P\mathcal{S}(\Gamma)$ for some finitely generated group Γ? for some finitely presented group Γ? The corresponding questions for profinite groups are standard, and mostly open [34].

Other open problems occur in (4.H), (6.F), (6.G), and (7.C).

We are grateful to Gilbert Baumslag, Slava Grigorchuk, Said Sidki, and John Wilson for valuable remarks. We also thank Dan Segal and an anonymous referee for pointing out a mistake in a previous version of this paper.

2 Completion with respect to a directed set of normal subgroups

In this section, we review some classical constructions and facts. See in particular [53], with Sect. 5 on projective limits, [9], with Chapter 3, Sect. 3, No. 4 on completions and Sect. 7 on projective limits, and [35], with Problem Q of Chapter 6 on completions. Defining a topology on a group using a family of subgroups goes back at least to Garrett Birkhoff [8], see pp. 52–54, and André Weil.

Let Γ be a group. Let $\mathcal{N}$ be a family of normal subgroups of Γ which is directed, namely which is such that the intersection of two groups in $\mathcal{N}$ contains always a group in $\mathcal{N}$.

(2.A) Denote by $\mathcal{CN}(\Gamma)$ the intersection of all elements in $\mathcal{N}$ (the letter $\mathcal{C}$ stands for "core"), by $\underline{\Gamma}$ the quotient group $\Gamma/\mathcal{CN}(\Gamma)$, and by $\underline{\mathcal{N}}$ the family of normal subgroups of $\underline{\Gamma}$ which are images of groups in $\mathcal{N}$. Then $\underline{\mathcal{N}}$ is a basis of neighbourhoods of the identity for a Hausdorff topology on $\underline{\Gamma}$. The corresponding left and right uniformities have the same Cauchy nets; indeed, for $x, y \in \underline{\Gamma}$ and $\underline{N} \in \underline{\mathcal{N}}$, we have $x^{-1}y \in \underline{N}$ if and only if $xy^{-1} \in \underline{N}$. It follows that $\underline{\Gamma}$ can be completed, say with respect to the left uniformity, to a Hausdorff complete [2] group which is called here the *pro-$\mathcal{N}$-completion of* Γ and which is denoted by $P\mathcal{N}(\Gamma)$. The canonical homomorphism

$$\varphi_{\mathcal{N}} : \Gamma \longrightarrow P\mathcal{N}(\Gamma)$$

has kernel $\mathcal{CN}(\Gamma)$ and image dense in $P\mathcal{N}(\Gamma)$. For $N \in \mathcal{N}$, the projection $\Gamma/\mathcal{CN}(\Gamma) \longrightarrow \Gamma/N$ extends uniquely to a continuous homomorphism

$$p_N : P\mathcal{N}(\Gamma) \longrightarrow \Gamma/N$$

which is onto.

[2] Recall that a topological group G is *complete* if both its left and right uniform structures are complete uniform structures, or equivalently if *one* of these structures is a complete uniform structure; see [[9], Chapter 3, §3]. Let Γ be a topological group and let G denote its completion, as a topological space, with respect to the left uniform structure; a sufficient condition for G to be a completion of Γ as a topological group is that the left and right uniform structures on Γ have the same Cauchy nets; see Theorem 1 of No. 4 in the same book.

(2.B) Observe that the following properties are equivalent

(i) $\{1\} \in \mathcal{N}$,
(ii) the topology defined by $\mathcal{N}$ on Γ is discrete,
(iii) $P\mathcal{N}(\Gamma) = \Gamma$ and $\varphi_{\mathcal{N}} : \Gamma \longrightarrow P\mathcal{N}(\Gamma)$ is the identity.

More generally, $C\mathcal{N}(\Gamma) \in \mathcal{N}$ if and only if the topology defined by $\underline{\mathcal{N}}$ on $\Gamma/C\mathcal{N}(\Gamma)$ is discrete, if and only if the natural homomorphism $\Gamma/C\mathcal{N}(\Gamma) \longrightarrow P\mathcal{N}(\Gamma)$ is the identity.

(2.C) Assume, moreover, that the family $\mathcal{N}$ is countable. We can assume without loss of generality that the elements of $\mathcal{N}$ constitute a nested sequence $N_1 \supset N_2 \supset \cdots$ (otherwise, if $\mathcal{N} = \{\tilde{N}_j\}_{j\geq 1}$, set $N_1 = \tilde{N}_1$ and choose inductively N_{j+1} as a group in $\mathcal{N}$ contained in $N_j \cap \tilde{N}_{j+1}$). Assume also that $\cap_{i\geq 1} N_i = \{1\}$.

The topology defined by $\mathcal{N}$ on Γ is metrisable. Indeed, define first $v_i : \Gamma \longrightarrow \{0,1\}$ by $v_i(\gamma) = 0$ if $\gamma \in N_i$ and $v_i(\gamma) = 1$ if not. Define next $w : \Gamma \longrightarrow [0,1]$ by $w(\gamma) = \sum_{i\geq 1} 2^{-i} v_i(\gamma)$. Then the mapping $d : \Gamma \times \Gamma \longrightarrow [0,1]$ defined by $d(\gamma_1, \gamma_2) = w(\gamma_1^{-1}\gamma_2)$ is a left-invariant ultrametric on Γ which defines the same topology as $\mathcal{N}$.

(2.D) The quotients

$$\Gamma/N \quad \text{where } N \in \mathcal{N},$$

and the canonical projections

$$p_{M,N} : \Gamma/N \longrightarrow \Gamma/M \quad \text{where } M, N \in \mathcal{N} \text{ are such that } N \subset M,$$

constitute an inverse system of groups of which the inverse limit (also called the projective limit) $\varprojlim \Gamma/N$ can be identified with $P\mathcal{N}(\Gamma)$. The following properties are standard: $P\mathcal{N}(\Gamma)$ is totally disconnected and complete.

For a group Γ, denote by $Z(\Gamma)$ its centre and by $D(\Gamma)$ the subgroup generated by the commutators; for a topological group G, denote by $\overline{D}(G)$ the closure of $D(G)$. We have:

$$Z(P\mathcal{N}(\Gamma)) = \varprojlim Z(\Gamma/N) \qquad \text{and} \qquad \overline{D}(P\mathcal{N}(\Gamma)) = \varprojlim D(\Gamma/N)$$

(see [12], Appendix I, No. 2). See also (5.C) below.

(2.E) Let A be a partially ordered set which is directed. Consider an inverse system consisting of groups Γ_α, with $\alpha \in A$, and homomorphisms $p_{\alpha,\beta} : \Gamma_\beta \longrightarrow \Gamma_\alpha$, with $\alpha, \beta \in A$ and $\alpha \leq \beta$. Let $\Gamma = \varprojlim \Gamma_\alpha$ denote the inverse limit. Even when the groups Γ_α are not all reduced to $\{1\}$ and the homomorphisms $p_{\alpha,\beta}$ are all onto, the natural homomorphisms $p_\alpha : \Gamma \longrightarrow \Gamma_\alpha$ need not be onto, and indeed the limit Γ can be reduced to one element; see [31]. (If A is the set of natural integers, with the usual order, it is straightforward to check that the p_α are onto.)

Because of this kind of phenomena, we have chosen here to define pro-$\mathcal{N}$-completions as appropriate topological completions. However, it would be possible and equivalent to use inverse limits systematically.

3 Universal property

(3.A) Let Γ, $\mathcal{N}$, and $P\mathcal{N}(\Gamma)$ be as in the previous section. Let H be a topological group. Assume that there is given a family $(\psi_N : H \longrightarrow \Gamma/N)_{N\in\mathcal{N}}$ of continuous homomorphisms such that, for $M, N \in \mathcal{N}$ with $N \subset M$, the diagram

$$\begin{tikzcd} H \arrow[rr, "\psi_N"] \arrow[dr, "\psi_M"'] & & \Gamma/N \arrow[dl, "p_{M,N}"] \\ & \Gamma/M & \end{tikzcd}$$

commutes. Then there exists a unique continuous homomorphism $\psi : H \longrightarrow P\mathcal{N}(\Gamma)$ such that the diagram

$$\begin{tikzcd} H \arrow[rr, "\psi"] \arrow[dr, "\psi_M"'] & & P\mathcal{N}(\Gamma) \arrow[dl, "p_M"] \\ & \Gamma/M & \end{tikzcd}$$

commutes for all $M \in \mathcal{N}$.

The pro-$\mathcal{N}$-completion $P\mathcal{N}(\Gamma)$ is characterized up to unique continuous homomorphism by this universal property.

Caveat. Even if ψ_N is onto for each $N \in \mathcal{N}$, ψ needs not be onto; see e.g. (6.A) below.

(3.B) Let $\mathcal{M}$ be a subset of $\mathcal{N}$ which is *cofinal*, namely such that any $N \in \mathcal{N}$ contains some M in $\mathcal{M}$. Then $\mathcal{CM}(\Gamma) = \mathcal{CN}(\Gamma)$, the topology defined on $\Gamma/\mathcal{CM}(\Gamma)$ by $\mathcal{M}$ coincides with that defined by $\mathcal{N}$, and the continuous homomorphism $P\mathcal{M}(\Gamma) \longrightarrow P\mathcal{N}(\Gamma)$ defined above is an isomorphism.

(3.C) Let $\mathcal{M}$ [respectively $\mathcal{N}$] be a directed family of normal subgroups of a group Γ [respectively Δ] and let $\psi : \Gamma \longrightarrow \Delta$ be a group homomorphism. Assume that, for each $N \in \mathcal{N}$, the family $\mathcal{M}_N = \{M \in \mathcal{M} \mid \psi(M) \subset N\}$ is cofinal in $\mathcal{M}$.

For a given $N \in \mathcal{N}$, the family of homomorphisms $\Gamma/M \longrightarrow \Delta/N$ induced by ψ (the family is indexed by $\mathcal{M}_N$) gives rise to a continuous homomorphism $P\mathcal{M}(\Gamma) \longrightarrow \Delta/N$. In turn, these give rise to a continuous homomorphism $P\mathcal{M}(\Gamma) \longrightarrow P\mathcal{N}(\Delta)$.

This will occur several times in Sect. 4.

4 Examples of directed sets of normal subgroups

(4.A) The set $\mathcal{F}$ of normal subgroups of finite index in a group Γ gives rise to the *profinite completion* $P\mathcal{F}(\Gamma)$ of Γ. There is a large literature on these completions, alluded to in the introduction.

For a prime number p, the set $\mathcal{F}_p$ of normal subgroups of index a power of p in a group Γ gives rise to the *pro-p-completion* $P_{\hat{p}}(\Gamma)$. See [18,49]. Since $\mathcal{F}_p \subset \mathcal{F}$, there is a canonical homomorphism

$$P\mathcal{F}(\Gamma) \longrightarrow P_{\hat{p}}(\Gamma)$$

by (3.C). The resulting homomorphism $P\mathcal{F}(\Gamma) \longrightarrow \prod_p P_{\hat{p}}(\Gamma)$ is sometimes an isomorphism, as it is the case for $\mathbf{Z}$, and sometimes not, as it is the case for $\bigoplus_{n \geq 5} \mathrm{Alt}(n)$, or for any non-trivial direct sum of non-abelian finite simple groups.

(4.B) The set $\mathcal{S}$ of normal subgroups with soluble quotients gives rise to the *true prosoluble completion* $P\mathcal{S}(\Gamma)$ of Γ; it is the main subject of the present note.

The *prosoluble completion* of the literature refers usually to the family $\mathcal{FS}$ of normal subgroups with *finite* soluble quotients (an exception is [17], where our $P\mathcal{S}(\Gamma)$ is called the "pro-solvable completion" of Γ, and where other variants appear). Since $\mathcal{FS} \subset \mathcal{F}$ and $\mathcal{FS} \subset \mathcal{S}$, there are canonical homomorphisms

$$P\mathcal{F}(\Gamma) \longrightarrow P\mathcal{F}\mathcal{S}(\Gamma) \qquad \text{and} \qquad P\mathcal{S}(\Gamma) \longrightarrow P\mathcal{F}\mathcal{S}(\Gamma)$$

by (3.C). See (6.A) and (6.B) for examples.

The adjective "true" should not mislead the reader: for some groups, for example for an infinite cyclic group, the true prosoluble completion is "much smaller" than the prosoluble completion.

(4.C) Similarly, we distinguish the *true pronilpotent completion* $P\mathcal{N}i(\Gamma)$ from the pronilpotent completion $P\mathcal{F}\mathcal{N}i(\Gamma)$ of the literature. There are again canonical homomorphisms

$$P\mathcal{F}(\Gamma) \longrightarrow P\mathcal{F}\mathcal{S}(\Gamma) \longrightarrow P\mathcal{F}\mathcal{N}i(\Gamma) \qquad \text{and} \qquad P\mathcal{S}(\Gamma) \longrightarrow P\mathcal{N}i(\Gamma) \longrightarrow P\mathcal{F}\mathcal{N}i(\Gamma)$$

by (3.C).

(4.D) If $\mathcal{N}$ is any of the classes appearing in Examples (4.A) to (4.C) above (see also (4.J) below), $P\mathcal{N}$ is a functor. This means that any homomorphism $\psi : \Gamma \longrightarrow \Delta$ factors through $\Gamma/\mathcal{C}\mathcal{N}(\Gamma) \longrightarrow \Delta/\mathcal{C}\mathcal{N}(\Delta)$ and then extends to a continuous homomorphism

$$P\mathcal{N}(\psi)\colon P\mathcal{N}(\Gamma) \longrightarrow P\mathcal{N}(\Delta).$$

As described in the introduction for $\mathcal{F}$ and $\mathcal{S}$, the *pro-$\mathcal{N}$-analogue of Grothendieck's problem* is to find examples of homomorphisms $\psi : \Gamma \longrightarrow \Delta$, with Γ, Δ residually $\mathcal{N}$ and possibly subjected to some extra conditions (such as finite generation or finite presentation), such that ψ *is not* an isomorphism and such that $P\mathcal{N}(\psi)$ *is one*.

(4.E) In a locally compact group which is totally discontinuous, any neighbourhood of the identity contains an open subgroup (Corollary 1 in Chapter 3, Sect. 4, No. 6 of [9]). It follows that a group which is profinite is residually finite.

Let p be a prime number. Let G be a pro-p-group, namely a profinite group in which any open subgroup is of index a power of p; assume [3] that G is finitely generated (we mean as a topological group, i.e. there exists a finite subset of G which generates a dense subgroup). It is a theorem of Serre that any subgroup of finite index in G is open (Theorem 4.2.5 in [54]). It follows that, if $P_{\hat{p}}(G)$ denotes the pro-p-completion of G viewed as an abstract group, the canonical homomorphism $\varphi_{\hat{p}} : G \longrightarrow P_{\hat{p}}(G)$ is a continuous isomorphism (see Proposition 1.1.2 in [54]).

Consider, in particular, a finitely generated group Γ which is residually a finite p-group and the embedding $\varphi_{\hat{p}} : \Gamma \longrightarrow P_{\hat{p}}(\Gamma)$ in its pro-p-completion. Then $P_{\hat{p}}(\varphi_{\hat{p}}) : P_{\hat{p}}(\Gamma) \longrightarrow P_{\hat{p}}(P_{\hat{p}}(\Gamma))$ is a continuous isomorphism, by which we will from now on identify $P_{\hat{p}}(P_{\hat{p}}(\Gamma))$ with $P_{\hat{p}}(\Gamma)$.

(4.F) By the recent solution, due to Nikolov and Segal, of a conjecture of Serre, any subgroup of finite index in a finitely generated profinite group is open [40–42]. As a consequence, the conclusion of (4.E) carries over from pro-p-completions to profinite completions; more precisely:

Consider a finitely generated group Γ which is residually finite and the embedding $\varphi_{\mathcal{F}} \colon \Gamma \longrightarrow P\mathcal{F}(\Gamma)$ in its profinite completion. Then $P\mathcal{F}(\varphi_{\mathcal{F}})\colon P\mathcal{F}(\Gamma) \longrightarrow P\mathcal{F}(P\mathcal{F}(\Gamma))$ is a continuous isomorphism, by which we will from now on identify $P\mathcal{F}(P\mathcal{F}(\Gamma))$ with $P\mathcal{F}(\Gamma)$.

(4.G) Say that a subgroup Δ of a profinite group G has the *congruence extension property* if any normal subgroup N of Δ is of the form $N = M \cap \Delta$ for some open normal subgroup M of G.

[3] This hypothesis cannot be deleted. Compare with [43] or Example 4.2.13 in [47], where an infinitely generated profinite group G which is not isomorphic to $P\mathcal{F}(G)$ is constructed.

Let p be a prime number, G a pro-p-group, and Δ a dense subgroup of G. Observe that Δ is residually a finite p-group (since G has this property). Assume, moreover, that G is finitely generated. Let $\psi : \Delta \longrightarrow G$ denote the inclusion and let $P_{\hat{p}}(\psi)\colon P_{\hat{p}}(\Delta) \longrightarrow G$ denote the homomorphism induced by ψ on the pro-p-completions (recall that we identify $P_{\hat{p}}(G)$ with G).

On the one hand, the homomorphism $P_{\hat{p}}(\psi)$ is onto. Indeed, for any open normal subgroup M of G, the composition of ψ with the projection $G \longrightarrow G/M$ is onto, by density of Δ in G. On the other hand, the following properties are equivalent:

(i) Δ has the congruence extension property;
(ii) the pro-p-topology and the topology induced by G coincide on Δ;
(iii) the homomorphism $P_{\hat{p}}(\psi)\colon P_{\hat{p}}(\Delta) \longrightarrow G$ is an isomorphism.

Indeed, (ii) implies (iii) because, for the topology induced by G, the completion of Δ coincides with its closure; and (iii) obviously implies (ii). As the topology of a topological group is determined by the neighbourhoods of the identity, (i) and (ii) are merely reformulations of each other.

Examples. Let $n \geq 2$ be an integer prime to p. Recall that the multiplication by n is invertible in the group $\mathbf{Z}_p$ of p-adic integers. Consider the situation where $G = \mathbf{Z}_p$ and $\Delta = \frac{1}{n}\mathbf{Z}$. The inclusion $\mathbf{Z} \hookrightarrow \frac{1}{n}\mathbf{Z}$, which is not an isomorphism, induces a continuous automorphism of $\mathbf{Z}_p$, which is the multiplication by $\frac{1}{n}$. *This shows that the pro-p-analogue of Grothendieck's problem has a straightforward solution.*

Consider an irrational p-adic integer x and let now Δ be the subgroup of $\mathbf{Z}_p$ generated by $\mathbf{Z}$ and x. Then $\Delta \approx \mathbf{Z}^2$, so that $P_{\hat{p}}(\Delta)$ is isomorphic to $\mathbf{Z}_p \oplus \mathbf{Z}_p$ and the homomorphism $P_{\hat{p}}(\Delta) \longrightarrow \mathbf{Z}_p$ induced by the inclusion $\Delta \hookrightarrow \mathbf{Z}_p$ is not an isomorphism.

(4.H) Let G be a finitely generated profinite group, and Δ a dense subgroup of G. Let $\psi\colon \Delta \longrightarrow G$ denote the inclusion. As in (4.G), the corresponding continuous homomorphism $P\mathcal{F}(\Delta) \longrightarrow G$ is always onto, and it is an isomorphism if and only if Δ has the congruence extension property.

Open problem. Let $\mathcal{G}$ be the first Grigorchuk group; it is an infinite group which is residually a finite 2-group, and all its proper quotients are finite 2-groups. This group was introduced in [21]; see also Chapter VIII in [30] and the Appendix. It follows that $\mathcal{G}$ embeds in its profinite completion, which coincides with its pro-2-completion, its true prosoluble completion, and its prosoluble completion. [Though it is a digression from our main theme, let us point out that $P_{\hat{2}}(\mathcal{G})$ coincides moreover with the closure $\bar{\mathcal{G}}$ of $\mathcal{G}$ in the compact automorphism group $\mathrm{Aut}\,(\mathcal{T})$ of the rooted dyadic tree $\mathcal{T}$ on which $\mathcal{G}$ acts in the usual way (Theorem 9 in [22]).]

The problem is to find an element g in the complement of $\mathcal{G}$ in $P\mathcal{F}(\mathcal{G})$ such that the subgroup Δ of $P\mathcal{F}(\mathcal{G})$ generated by $\mathcal{G}$ and g has the congruence extension property. This would provide one more example of a homomorphism $\psi\colon \mathcal{G} \longrightarrow \Delta$ which is not an isomorphism and which would be such that the homomorphism $P\mathcal{F}(\psi)$ is an isomorphism, namely one solution to the original Grothendieck problem of a different nature than the existing ones (from [13,44,45]).

In the Appendix, we provide an effective way to construct all elements $g \in \bar{\mathcal{G}}$. We also build elements $g \in \bar{\mathcal{G}}$ that do not belong to $\mathcal{G}$.

(4.I) There are other cases than those appearing above which are potentially interesting, of which we mention here two more.

For a group Γ, the following two properties are equivalent:

(i) for any pair N_1, N_2 of normal subgroups not reduced to $\{1\}$ in Γ, we have $N_1 \cap N_2 \neq \{1\}$;
(ii) there does *not* exist normal subgroups $N, N_1, N_2 \neq \{1\}$ of Γ such that $N = N_1 \times N_2$;

(we leave it as an exercise to the reader to check this). For Γ with these properties, the family $\mathcal{N}^{\neq 1}$ of all normal subgroups distinct from $\{1\}$ gives rise to the *pronormal topology* on Γ; see [20].

(4.J) The family $\mathcal{A}$ of normal subgroups with amenable quotients gives rise to the *proamenable completion* $P\mathcal{A}(\Gamma)$ of Γ. By (3.C), there are canonical homomorphisms from $P\mathcal{A}(\Gamma)$ to $P\mathcal{F}(\Gamma)$ and to $P\mathcal{S}(\Gamma)$. The related notion of residual amenability occurs for example in [16,19].

5 True prosoluble completions

(5.A) An obvious obstruction to the residual solubility of a group Γ is the existence of a perfect subgroup not reduced to one element. Any group Γ contains a unique maximal perfect subgroup, that we denote by P_Γ; this follows from the fact that a subgroup generated by two perfect subgroups is itself perfect. Observe that P_Γ is contained in the intersection $D^\infty(\Gamma)$ of all the groups in the derived series of Γ, but the inclusion can be strict. (In fact, P_Γ is the intersection of all the groups in the so-called transfinite derived series of Γ, but this transfinite series can be as long, without repetition, as the cardinality of Γ allows; see [39].)

(5.B) For a topological group G and an integer $n \geq 0$, we denote by $\overline{D^n}(G)$ the nth group of the topological derived series, defined inductively by $\overline{D^0}(G) = G$ and $\overline{D^{n+1}}(G) = \overline{[\overline{D^n}(G), \overline{D^n}(G)]}$, where $\overline{[A,B]}$ stands for the closure of the subgroup of G generated by commutators $a^{-1}b^{-1}ab$ with $a \in A$ and $b \in B$. We denote by $G_{(n)}$ the quotient $G/\overline{D^n}(G)$ and by $\sigma_{(n)} : G \longrightarrow G_{(n)}$ the canonical projection.

Recall that a topological group G is "topologically soluble", namely such that $\overline{D^n}(G) = \{1\}$ for n large enough, if and only if it is soluble, namely such that the nth term $D^n(G)$ of its ordinary derived series is reduced to $\{1\}$ for n large enough (see for example Chapter III, Sect. 9, No. 1 in [11]).

(5.C) For a group Γ and an integer $n \geq 0$, we claim that $P\mathcal{S}(\Gamma)_{(n)}$ is canonically isomorphic to $\Gamma_{(n)}$. This follows from contemplation of the commutative diagram

$$\begin{tikzcd}
\Gamma \arrow[r, "\sigma_{(n)}"] \arrow[d] & \Gamma_{(n)} \arrow[d, equal] \arrow[ddl, bend left=90, "\psi"] \\
P\mathcal{S}(\Gamma) \arrow[r, "P\mathcal{S}(\sigma_{(n)})"] \arrow[d] & \Gamma_{(n)} \arrow[d, equal] \\
P\mathcal{S}(\Gamma)_{(n)} \arrow[r, "P\mathcal{S}(\sigma_{(n)})_{(n)}"] & \Gamma_{(n)}
\end{tikzcd}$$

where the existence and canonicity of ψ follow from the universal property of the projection $\sigma_{(n)}$. In other words, as the morphism $\Gamma \longrightarrow P\mathcal{S}(\Gamma)_{(n)}$ has a range which is soluble of degree n, this morphism factors through $\Gamma_{(n)}$.

(5.D) In a group Γ, the family $\mathcal{S}$ of all normal subgroups with soluble quotients and the countable family $(D^n(\Gamma))_{n\geq 0}$ define the same procompletion $P\mathcal{S}(\Gamma)$, by Item (3.B).

It follows from (2.C) that the topology on $P\mathcal{S}(\Gamma)$ can be defined by a metric.

Recall that the family $\mathcal{S}$ can be uncountable. Hall has shown that this is the case for Γ a free group of rank 2; see [28], and the exposition in [14].

(5.E) For a topological group G and an integer $j \geq 1$, we denote by $\overline{C^j}(G)$ the jth group of the topological lower central series, defined inductively by $\overline{C^1}(G) = G$ and $\overline{C^{j+1}}(G) = \overline{[G, \overline{C^j}(G)]}$ for $j \geq 1$. Then G is nilpotent if and only if $\overline{C^j}(G) = \{1\}$ for j large enough. (Compare with (5.B).)

The nilpotent quotients $P\mathcal{N}i(\Gamma)/\overline{C^j}(P\mathcal{N}_i(\Gamma))$ and $\Gamma/C^j(\Gamma)$ are isomorphic for all $j \geq 1$. (Compare with (5.C).)

The topology on $P\mathcal{N}i(\Gamma)$ can be defined by a metric. (Compare with (5.D).)

6 Examples

(6.A) Let us show that the canonical homomorphism $P\mathcal{S}(\Gamma) \longrightarrow P\mathcal{FS}(\Gamma)$ needs not be onto.

First, consider an infinite cyclic group: $\Gamma = \mathbf{Z}$. Since $\mathbf{Z}$ is soluble, $\varphi_{\mathcal{S}} \colon \mathbf{Z} \longrightarrow P\mathcal{S}(\mathbf{Z})$ is an isomorphism onto. As any finite quotient of $\mathbf{Z}$ is soluble (indeed abelian), the prosoluble completion of $\mathbf{Z}$ coincides with its profinite completion. Hence

$$\mathbf{Z} = P\mathcal{S}(\mathbf{Z}) \not\approx P\mathcal{FS}(\mathbf{Z}) \approx P\mathcal{F}(\mathbf{Z}) \approx \prod_p \mathbf{Z}_p.$$

Here is another family of examples. For $k \geq 2$ and $d \geq 1$, denote by $F(k,d)$ the quotient of a non-abelian free group on k generators by the dth term of its derived series (the *free soluble group* of class d with k generators). This group is obviously soluble and infinite, and it is known to be residually finite (Theorem 6.3 in [26]). Hence $F(k,d) = P\mathcal{S}(F(k,d))$ embeds properly in its profinite completion $P\mathcal{F}(F(k,d))$, and the latter coincides with $P\mathcal{FS}(F(k,d))$.

(6.B) Similarly, the canonical homomorphisms $P\mathcal{S}(\Gamma) \longrightarrow P\mathcal{FS}(\Gamma)$ needs not be injective.

Consider a wreath product $\Gamma = S \wr T$ where S is soluble non-abelian and where T is soluble infinite. Theorem 3.1 in [26] establishes that Γ is not residually finite, so that in particular the morphism $\varphi_{\mathcal{FS}} \colon \Gamma \longrightarrow P\mathcal{FS}(\Gamma)$ has a kernel not reduced to $\{1\}$. But Γ is soluble, and therefore isomorphic to $P\mathcal{S}(\Gamma)$.

For a finitely generated group Γ, it is a theorem of P. Hall that $\Gamma/D^2(\Gamma)$ is residually finite [28], so that we have an embedding

$$\Gamma/D^2(\Gamma) \approx P\mathcal{S}(\Gamma)/\overline{D^2}(P\mathcal{S}(\Gamma)) \longrightarrow P\mathcal{F}\big(\Gamma/D^2(\Gamma)\big).$$

In particular, a finitely generated group Γ which is soluble of class 2 always embeds in $P\mathcal{F}(\Gamma)$.

However, there exists a finitely generated soluble group of class 3 which is non–Hopfian [29]. In particular, a finitely generated group Γ which is soluble of class at least 3 does not always embed in $P\mathcal{F}(\Gamma) = P\mathcal{FS}(\Gamma)$. A finitely *presented* soluble group which is non–Hopfian is described in [1].

(6.C) For each integer $k \geq 2$, let F_k denote the non-abelian free group on k generators. This is a residually soluble group (indeed it is residually a finite p-group for any prime p, by a result of Iwasawa, see No. 6.1.9 in [48]), and therefore embeds in its true prosoluble completion $P\mathcal{S}(F_k)$.

Since the abelianized groups $F_k/[F_k, F_k] \approx \mathbf{Z}^k$ are pairwise non-isomorphic, the true prosoluble completions $P\mathcal{S}(F_k)$ are pairwise non-isomorphic by (2.D) or (5.C).

(6.D) For $d \geq 3$, the group $SL_d(\mathbf{Z})$ is perfect (this is an immediate consequence of the following fact: any matrix in $SL_d(\mathbf{Z})$ is a product of elementary matrices; see for example Theorem 22.4 in [36]). It follows that $P\mathcal{S}(SL_d(\mathbf{Z}))$ is reduced to one element and that $P\mathcal{S}(GL_d(\mathbf{Z}))$ is the group of order two.

Consider, however, an integer $d \geq 2$, a prime p, and the congruence subgroup

$$\Gamma_d(p) = \mathrm{Ker}\,(SL_d(\mathbf{Z}) \longrightarrow SL_d(\mathbf{Z}/p\mathbf{Z}))\,,$$

which is a subgroup of finite index in $SL_d(\mathbf{Z})$. Then $\Gamma_d(p)$ is residually a finite p-group, and therefore embeds in $P\mathcal{S}(\Gamma_d(p))$; see for example Proposition 3.3.15 in [47], since their proof written for $d = 2$ carries over to any $d \geq 2$.

In particular, the property for a group Γ to embed up to finite kernel in its prosoluble completion is *not* stable by finite index.

(6.E) Groups which are residually soluble and not soluble include Baumslag–Solitar groups $\langle a, t \mid ta^p t^{-1} = a^q \rangle$ for $|p|, |q| \geq 2$ [46], positive one-relator groups [6], non-trivial free products[4] of soluble groups, and various just non-soluble groups [15]. Also, some free products of soluble groups amalgamated over a cyclic group are residually soluble and not soluble [32].

These are potential examples for further investigation of the properties of true prosoluble completions.

(6.F) For groups which are not residually soluble, it is a natural problem to find out properties of the true prosoluble kernel

$$\mathrm{Ker}(\Gamma \longrightarrow P\mathcal{S}(\Gamma)) = \bigcap_{n=0}^{\infty} D^n(\Gamma).$$

For example, consider the one-relator group[5]

$$\Gamma \;=\; \langle a, b; a = [a, a^b] \rangle$$

which appears in [5,6]. Baumslag has shown that the profinite kernel of Γ coincides with the derived group $D(\Gamma)$, which is also the smallest normal subgroup of Γ containing a, and which is not reduced to one element by the Magnus Freiheitssatz. As $D(\Gamma)$ is perfect, the true prosoluble kernel of Γ coincides also with $D(\Gamma)$. Since $\Gamma/D(\Gamma) \approx \mathbf{Z}$, we have $\mathbf{Z} \approx P\mathcal{S}(\Gamma) \not\approx P\mathcal{F}(\Gamma) \approx \prod_p \mathbf{Z}_p$; compare with (6.A).

Examples to investigate include:

- other one-relator groups which are not residually soluble;
- wreath products as in (6.B);

[4] Recall of a particular case of a result of Gruenberg, see Sec. 4 in [26]: any free product of residually soluble groups is residually soluble. By "non-trivial" free product, we mean that the free product has at least two factors, and that it is not the free product of two groups of order two.

[5] Recall that $[x, y] = x^{-1}y^{-1}xy$ and $x^y = y^{-1}xy$.

- free products of nilpotent groups with amalgamation (see for example Proposition 7 in [32]);
- parafree groups (see (7.B));
- meta-residually soluble groups, namely groups having a residually soluble normal subgroup with residually soluble quotient (see Theorem 2 in [33]); particular cases: free-by-free groups which are not residually soluble.

(6.G) Does there exist a discrete group Γ which on the one hand is residually finite and residually soluble, namely for which both homomorphisms $\Gamma \longrightarrow P\mathcal{F}(\Gamma)$, $\Gamma \longrightarrow P\mathcal{S}(\Gamma)$ are injective, and which on the other hand is such that the homomorphism $\Gamma \longrightarrow P\mathcal{F}\mathcal{S}(\Gamma)$ is NOT injective?

7 On the true prosoluble and the true pronilpotent analogues of Grothendieck's problem

(7.A) Consider as in (4.H) the Grigorchuk group $\mathcal{G}$ and its pro-2-completion $P_{\hat{2}}(\mathcal{G})$.

Hypothesis *We assume from now on that there exists a finitely generated subgroup Δ of $P_{\hat{2}}(\mathcal{G})$ which contains $\mathcal{G}$ properly and which has the congruence extension property.*

Observe that any finite quotient of Δ is a 2-group, as a consequence of the congruence extension property; thus, the canonical homomorphism from $P\mathcal{F}(\Delta)$ to $P_{\hat{2}}(\Delta)$ is an isomorphism. Let $\psi : \mathcal{G} \longrightarrow \Delta$ denote the inclusion. We know from (4.G) that $P_{\hat{2}}(\psi) \colon P_{\hat{2}}(\Delta) \longrightarrow P_{\hat{2}}(\mathcal{G})$ is an isomorphism.

Lemma *With the notation above, a quotient group of Δ is finite if and only if it is soluble.*

Proof Let N be a normal subgroup of Δ such that Δ/N is finite. We have already observed that the quotient Δ/N is a finite 2-group, in particular a nilpotent group, a fortiori a soluble group.

For the converse, we proceed by contradiction and assume that there exists a normal subgroup N in Δ such that the quotient Δ/N is soluble and infinite. Since Δ is finitely generated, Zorn's lemma implies that there exists a normal subgroup M in Δ containing N such that Δ/M is just infinite.

Let $\Delta/M = D^0 \supset D^1 \supset D^2 \supset \cdots$ denote the derived series of the just infinite soluble group Δ/M; denote by k the smallest integer such that D^{k+1} is of infinite index in Δ/M. Then $D^{k+1} = \{1\}$ because Δ/M is just infinite. The group D^k is abelian and of finite index in Δ/M; it is therefore finitely generated. The torsion subgroup of D^k is normal in Δ/M; it is of infinite index in D^k, and thus also in Δ/M; hence D^k has no torsion. Let Δ_1 denote the inverse image of D^k in Δ; it is a normal subgroup of finite index containing M. The quotient Δ_1/M is finitely generated and free abelian, say $\Delta_1/M \approx \mathbf{Z}^d$ for some $d \geq 1$.

Let Q_2 be the subgroup of Δ_1/M generated by the cubes, and denote by Δ_2 its inverse image in Δ_1; observe that Δ_2 is of finite index in Δ_1, and therefore also in Δ, and that $\Delta_1/\Delta_2 \approx (\mathbf{Z}/3\mathbf{Z})^d$. Let Δ_3 be the intersection of all the conjugates of Δ_2 in Δ; observe that Δ_3 is of finite index in Δ_2, and therefore in Δ, and that it is also normal in Δ. Then Δ/Δ_3 is a finite quotient of Δ of order

$$[\Delta : \Delta_1] \times [\Delta_1 : \Delta_2] \times [\Delta_2 : \Delta_3] = [\Delta : \Delta_1] \times 3^d \times [\Delta_2 : \Delta_3].$$

In particular, 3 divides the order of Δ/Δ_3.

We have obtained a contradiction because, since $P_{\hat{2}}(\Delta) \approx P\mathcal{F}(\Delta)$, any finite quotient of Δ is a 2-group. This ends the proof. $\square$

Consequence *Let $\mathcal{G}$ be the Grigorchuk group. If there would exist a dense subgroup $\psi : \Delta \longrightarrow P_{\hat{2}}(\mathcal{G})$ containing properly $\mathcal{G}$ and with the congruence extension property, then the morphism*

$$PS(\psi) \colon PS(\mathcal{G}) \longrightarrow PS(\Delta)$$

induced by ψ on the true prosoluble completions would be an isomorphism.

We have already mentioned that subgroups Δ of $P_{\hat{2}}(\mathcal{G})$ containing $\mathcal{G}$ properly can be constructed effectively, see the Appendix.

(7.B) Let us describe how Problem (i) from the introduction has been studied for true pronilpotent completions.

Recall that a group Γ is *parafree* if it is residually nilpotent and if there exists a free group F such that $F/C^j(F)$ and $\Gamma/C^j(\Gamma)$ are isomorphic for all $j \geq 1$. Nonfree parafree groups have been discovered by Baumslag [3]; later papers include [4,7].

Let Γ be a parafree group with finitely generated abelianization. Let F be as above; observe that F is finitely generated. Choose a subset T of Γ of which the canonical image freely generates the free abelian group $\Gamma/C^2(\Gamma)$; observe that T is finite. (Be careful: T needs not generate Γ.) Let S be a free set of generators of F such that the canonical image of S and T are in bijection with each other through the given isomorphism $F/C^2(F) \approx \Gamma/C^2(\Gamma)$, and let $\varphi : S \longmapsto T$ be a compatible bijection. Then φ extends to a homomorphism, again denoted by φ, from F to Γ, and this φ induces the given isomorphism from $F/C^2(F)$ onto $\Gamma/C^2(\Gamma)$. A group homomorphism with range a nilpotent group A is onto if and only if its composition with the abelianization $A \longrightarrow A/C^2(A)$ is onto (see e.g. [10], Corollary 4, page A I.70); it follows that the homomorphism $\varphi_{(j)}$ from $F/C^j(F)$ to $\Gamma/C^j(\Gamma)$ induced by φ is onto for all $j \geq 1$. Since the group $F/C^j(F) \approx \Gamma/C^j(\Gamma)$ is Hopfian for all $j \geq 1$ (any finitely generated residually finite group is Hopfian, by Mal'cev theorem [38]), it follows that $\varphi_{(j)}$ is an isomorphism for all $j \geq 1$. We have shown:

Observation *A residually nilpotent group Γ with finitely generated abelianization is parafree if and only if there exist a free group F of finite rank and a homomorphism $\varphi \colon F \longrightarrow \Gamma$ which induces an isomorphism $P\mathcal{N}i(\varphi) \colon P\mathcal{N}i(F) \longrightarrow P\mathcal{N}i(\Gamma)$ on the true pronilpotent completions.*

Note that, in case the set T is moreover generating for Γ, the group Γ itself is free; see Problem 2 on pp. 346–347 of [37]. However, as G. Baumslag discovered, there are pairs (Γ, F) as in the observation with Γ not free and generated by $k+1$ elements, and with F free of rank k, for each $k \geq 2$.

There is a related example on P. 173 of [52]. Let F_2 be the free group on two generators x and y. Set $y' = yxyx^{-1}y^{-1}$. Let Γ be the subgroup of F_2 generated by x and y'. Then Γ is a proper subgroup of F_2, because $y \notin \Gamma$ (though Γ is isomorphic to F_2). The inclusion of Γ in F_2 provides an isomorphism $\Gamma/C^j(\Gamma) \longrightarrow F_2/C^j(F_2)$ for all $j \geq 1$, and therefore an isomorphism from $P\mathcal{N}i(\Gamma)$ onto $P\mathcal{N}i(F_2)$.

(7.C) In [4], there are examples of pairs (F, Γ) of groups with the following properties: F is finitely generated and free, both F and Γ are residually soluble, the quotients

$F/D^j(F)$ and $\Gamma/D^j(\Gamma)$ are isomorphic for all $j \geq 0$, nevertheless F and Γ are *not* isomorphic (indeed Γ is not finitely generated).

However, nothing like the observation of Item (7.B) holds for the quotients by the groups of the derived series. We do not know if there exist a finitely generated free group F, a group Γ which is residually soluble and not free, and a homomorphism $\psi : F \longrightarrow \Gamma$ such that ψ induces an isomorphism $F/D^j(F) \longrightarrow \Gamma/D^j(\Gamma)$ for all $j \geq 1$, or equivalently such that $\mathcal{PS}(\psi) : \mathcal{PS}(F) \longrightarrow \mathcal{PS}(\Gamma)$ is an isomorphism.

Acknowledgements The authors acknowledge support from the *Swiss National Science Foundation*. The authors of the Appendix would like to acknowledge the warm hospitality and partial support of the Max-Planck-Institut für Mathematik in Bonn and NSF grant DMS 0600975.

Appendix: Construction of elements in the closure of Grigorchuk group

The group known as Grigorchuk group (also the first Grigorchuk group) was introduced in [21]. More information on this remarkable group can be found in [23,30]. Here we only introduce as much as is necessary to describe the elements in the closure $\bar{\mathcal{G}}$ of Grigorchuk group $\mathcal{G}$ in the pro-finite group Aut $(\mathcal{T})$ of binary rooted tree automorphisms. In fact, we describe constraints that need to be satisfied "near the top" of the portraits of the elements in $\mathcal{G}$ (and therefore in $\bar{\mathcal{G}}$ as well). These constraints, if satisfied by an element g in Aut $(\mathcal{T})$ at each of its sections (see below for details), guarantee that g belongs to the closure $\bar{\mathcal{G}}$. The constraints may be viewed as an effective version of the more conceptual description given by Grigorchuk in [23].

Grigorchuk group can be viewed as a group of automorphisms of the binary rooted tree $\mathcal{T}$. The vertices of the rooted binary tree $\mathcal{T}$ are in bijective correspondence with the finite words over $X = \{0, 1\}$. The empty word $\emptyset$ is the root, the set X^n of words of length n over X constitutes level n in the tree and every vertex u at level n has two children at level $n + 1$, namely $u0$ and $u1$. The group Aut $(\mathcal{T})$ of automorphisms of $\mathcal{T}$ decomposes algebraically as

$$\mathrm{Aut}\,(\mathcal{T}) = (\mathrm{Aut}\,(\mathcal{T}) \times \mathrm{Aut}\,(\mathcal{T})) \rtimes \mathrm{S}(2), \tag{1}$$

where $\mathrm{S}(2) = \{1, \sigma\} = \{(), (01)\}$ is the symmetric group of order 2 acting on Aut $(\mathcal{T}) \times$ Aut $(\mathcal{T})$ by permuting the coordinates. The normal subgroup Aut $(\mathcal{T}) \times$ Aut $(\mathcal{T})$ is the stabilizer of the first level of $\mathcal{T}$. The elements in Aut $(\mathcal{T})$ of the form $g = (g_0, g_1)$ act on $\mathcal{T}$ by

$$(0w)^g = 0w^{g_0}, \quad (1w)^g = 1w^{g_1},$$

while the elements of the form $g = (g_0, g_1)\sigma$ act by

$$(0w)^g = 1w^{g_0}, \qquad (1w)^g = 0w^{g_1},$$

for any word w over X. The automorphisms g_0 and g_1 in the *decomposition* $g = (g_0, g_1)\sigma^{\varepsilon}$ of g, where ε is 0 or 1, are called *sections* of g at the vertices 0 and 1, respectively. This definition is recursively extended to a notion of a section of g at any vertex of $\mathcal{T}$ by declaring $g_{\emptyset} = g$ and $g_{ux} = (g_u)_x$, for u a word over X and x a letter in X.

Grigorchuk group $\mathcal{G}$ is, by definition, the group generated by the automorphisms a, b, c and d of $\mathcal{T}$, whose decompositions, in the sense of (1), are given by

$$\begin{aligned} a &= (1,1)\sigma, \\ b &= (a,c), \\ c &= (a,d), \\ d &= (1,b). \end{aligned} \tag{2}$$

Therefore, the action of a, b, c, and d on $\mathcal{T}$ is given by

$$\begin{array}{ll} (0w)^a = 1w, & (1w)^a = 0w, \\ (0w)^b = 0w^a, & (1w)^b = 1w^c, \\ (0w)^c = 0w^a, & (1w)^c = 1w^d, \\ (0w)^d = 0w, & (1w)^d = 1w^b, \end{array}$$

for any word w over X. It is easy to establish that

$$a^2 = b^2 = c^2 = d^2 = 1, \quad bc = cb = d, \quad bd = db = c, \quad cd = dc = b.$$

These relations are called *simple relations* in $\mathcal{G}$. The *stabilizer* $\mathrm{Stab}_{\mathcal{G}}(X)$ *in* $\mathcal{G}$ *of level 1* in $\mathcal{T}$ is

$$\mathrm{Stab}_{\mathcal{G}}(X) = \langle b, c, d, aba, aca, ada \rangle,$$

and the decompositions of aba, aca and ada are given by

$$\begin{aligned} aba &= (c,a), \\ aca &= (d,a), \\ ada &= (b,1). \end{aligned} \tag{3}$$

The decomposition formulae given in (2) and (3) and the simple relation $aa = 1$ are sufficient to calculate the decomposition of any element in $\mathcal{G}$. For example,

$$abdabac = aba\ ada\ b\ aca\ a = (c,a)(b,1)(a,c)(d,a)(1,1)\sigma = (cbad, aca)\sigma.$$

Of course, we could make use of the other simple relations to write either

$$abdabac = \cdots = (cbad, aca)\sigma = (dad, aca)\sigma.$$

or

$$abdabac = acabac = aca\ b\ aca\ a = (d,a)(a,c)(d,a)(1,1)\sigma = (dad, aca)\sigma,$$

but this will not be necessary for our purposes (and would, in fact, be counterproductive in the proof of one of our lemmata).

Let g be an arbitrary element in $\mathrm{Aut}\,(\mathcal{T})$. The *portrait* of g is the binary rooted tree $\mathcal{T}$ with additional *decoration* on the vertices defined recursively as follows. If $g = (g_0, g_1)$ stabilizes level 1 in $\mathcal{T}$ then the portrait of g consists of the portrait of g_0 hanging below the vertex 0, the portrait of g_1 hanging below the vertex 1 and the root, which is decorated by 0. If $g = (g_0, g_1)\sigma$ does not stabilize level 1 (i.e., it is active at the root) the portrait looks the same as in the previous case, except that the root is decorated by 1. Thus, the portrait of g is the binary tree $\mathcal{T}$ with additional decoration $\alpha_u(g)$ on each vertex u, which is equal to 0 or 1 depending on whether g is active at the vertex u or not.

For every vertex xy on level 2, define

$$\beta_{xy}(g) = \alpha_{xy}(g) + \alpha_{x\bar{y}0}(g) + \alpha_{x\bar{y}1}(g),$$

where the addition is performed modulo 2 and $\bar{y}$ denotes the letter in $\{0, 1\}$ different from y. When g is assumed, the notation $\alpha_u(g)$ and $\beta_u(g)$ is simplified to α_u and β_u.

Theorem 1 *For any element g in Grigorchuk group $\mathcal{G}$ the portrait decoration satisfies the following constraints.*

(i) *If* $\alpha_0 = \alpha_1 = 0$ *then*

$$\beta_{00} = \beta_{11} = \beta_{01} = \beta_{10}.$$

(ii) *If* $\alpha_0 = 0$ *and* $\alpha_1 = 1$ *then*

$$\beta_{00} \neq \beta_{11} = \beta_{01} = \beta_{10}.$$

(iii) *If* $\alpha_0 = 1$ *and* $\alpha_1 = 0$ *then*

$$\beta_{00} = \beta_{11} = \beta_{01} \neq \beta_{10}.$$

(iv) *If* $\alpha_0 = \alpha_1 = 1$ *then*

$$\beta_{00} = \beta_{11} \neq \beta_{01} = \beta_{10}.$$

We say that an automorphism g in $\mathrm{Aut}\,(\mathcal{T})$ *simulates* $\mathcal{G}$ if its portrait decoration satisfies the constraints in Theorem 1. Recall that the pro-finite group $\mathrm{Aut}\,(\mathcal{T})$ is a metric space with a natural metric derived from the filtration of $\mathrm{Aut}\,(\mathcal{T})$ by its level stabilizers and defined as follows. The distance between two tree automorphisms g and h is

$$d(g,h) = \inf \left\{ \frac{1}{[\mathrm{Aut}\,(\mathcal{T}) : \mathrm{Stab}_{\mathrm{Aut}\,(\mathcal{T})}(X^n)]} \;\middle|\; g^{-1}h \in \mathrm{Stab}_{\mathrm{Aut}\,(\mathcal{T})}(X^n) \right\}.$$

In other words, if g and h agree on words of length n, but do not agree on words of length $n+1$, then the distance between g and h is $\frac{1}{2^{2^n-1}}$.

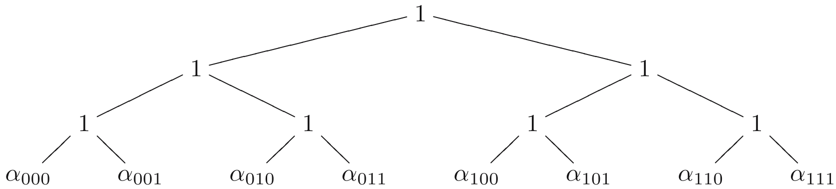
Theorem 2 *Let g be a binary tree automorphism. The following conditions are equivalent.*

(i) *g belongs to the closure $\bar{\mathcal{G}}$ of Grigorchuk group $\mathcal{G}$ in the pro-finite group* $\mathrm{Aut}\,(\mathcal{T})$*;*
(ii) *all sections of g simulate $\mathcal{G}$;*
(iii) *the distance from any section of g to $\mathcal{G}$ in the metric space* $\mathrm{Aut}\,(\mathcal{T})$ *is at most* $\frac{1}{2^{15}}$.

The proofs will follow from some combinatorial observations on the structure of words representing elements in $\mathcal{G}$. Before the proofs, we consider some examples.

Example 1 We show how Theorem 1 and Theorem 2 can be used to construct elements in the closure $\bar{\mathcal{G}}$.

The constraints in Theorem 1 imply that exactly 2^{12} different portrait decorations are possible on levels 0 through 3 for elements in $\mathcal{G}$. Indeed, assume that the portrait decoration is already freely chosen on levels 0 through 2. In particular, α_0 and α_1 are known. There are 8 vertices on level 3, but according to the constraints in Theorem 1 we may choose the decoration freely only on 5 of them. Namely, as soon as we chose the decoration for two vertices with common parent, the values of β_{00}, β_{01}, β_{10} and β_{11} are uniquely determined and we may freely choose only the decoration on one of the vertices in each of the 3 remaining pairs of vertices with common parent, while the other is forced on us. For example, let us set $\alpha_u = 1$ for all u on level 0 through level 2.



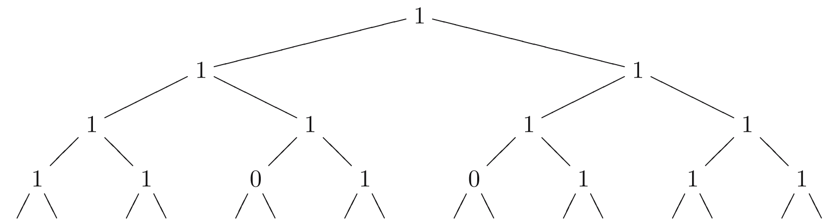
Further, for all vertices at level 3 whose label ends in 1 choose $\alpha_{u1} = 1$. Finally, choose $\alpha_{110} = 1$. At this moment, after making 12 free choices, we have $\beta_{10} = 1$ and, according to Theorem 1, we must have $\beta_{01} = 1$, $\beta_{00} = \beta_{11} = 0$. In accordance with the other choices already made on level 3, we must then have

$$\alpha_{000} = 1, \quad \alpha_{010} = 0, \quad \alpha_{100} = 0.$$

We may now continue building a portrait of an element in $\bar{\mathcal{G}}$ by extending (independently!) the left half and the right half of the portrait one more level by following only the constraints imposed by Theorem 1.

As a general strategy (one can certainly choose a different one, guided by any suitable purpose), we choose arbitrarily the decoration on all vertices whose label ends in 1 or in 110, and then fill in the decoration on the remaining vertices following Theorem 1.

If we extend our example one more level by decorating by 1 all vertices whose label ends in 1 or in 110, we obtain



Continuing in the same fashion (choosing 1 whenever possible) we arrive at the portrait description of the element f in $\bar{\mathcal{G}}$ defined by the following decomposition formulae

$$\begin{aligned}
f &= (\ell, r)\,\sigma, \\
\ell &= (r, m)\,\sigma, \\
r &= (m, r)\,\sigma, \\
m &= (n, f)\,\sigma, \\
n &= (r, m).
\end{aligned}$$

There are many ways to see that f (or any section of f) does not belong to $\mathcal{G}$. Perhaps the easiest way is to observe that f is not a bounded automorphism, while all elements in $\mathcal{G}$ are bounded automorphisms of $\mathcal{T}$. Recall that, by definition, an automorphism g of $\mathcal{T}$ is *bounded* if the sum $\sum_{u\in X^n}\alpha_u(g)$ is uniformly bounded, for all n. Note that f is defined by a 5-state automaton which is not bounded, while the automaton defining $\mathcal{G}$ is bounded. For more on groups of automorphisms generated by automata see [24], and for bounded automorphisms and bounded automata see [51].

To aid construction of elements in $\bar{\mathcal{G}}$ we provide the following table of trees indicating the 8 possibilities for the values of α_0, α_1, β_{00}, β_{01}, β_{10}, and β_{11}.

0 0	0 1	1 0	1 1
0 0 0 0	1 0 0 0	0 0 1 0	0 1 1 0
0 0	0 1	1 0	1 1
1 1 1 1	0 1 1 1	1 1 0 1	1 0 0 1

In each tree, α_0, α_1, β_{00}, β_{01}, β_{10}, and β_{11} are indicated in their respective positions. To use the table, choose values for α_0, α_1 and any one of β_{00}, β_{01}, β_{10}, and β_{11}. The unique tree in the above table that agrees with the chosen values provides the unique values for the remaining 3 parameters among β_{00}, β_{01}, β_{10}, and β_{11}. For example, if $\alpha_0=\alpha_1=1$ and $\beta_{11}=0$, the correct pattern in the table is the one in the right upper corner, indicating that $\beta_{00}=0$, $\beta_{01}=1$ and $\beta_{10}=1$.

The following example provides additional ways to build elements in $\bar{\mathcal{G}}$, which does not rely on Theorem 1 and Theorem 2, but, rather, on the branch structure of $\mathcal{G}$ (see [22] for more details). This approach does not produce all elements in $\bar{\mathcal{G}}$, but does produce some that are easy to describe.

Example 2 Infinitely many elements in $\bar{\mathcal{G}}$ that are not in $\mathcal{G}$ are contained in the following isomorphic copy of $K=\langle a^{-1}b^{-1}ab\rangle^{\mathcal{G}}\le\mathcal{G}$ (recall that $\mathcal{G}$ is a regular branch group over the normal closure K of the commutator $[a,b]$ in $\mathcal{G}$; see [22] or [30] for details). For each element $k\in K$ define an element $\bar{k}$ in $\bar{\mathcal{G}}$ by

$$\bar{k}=(k,\bar{k}).$$

The group $\bar{K}=\{\bar{k}\mid k\in K\}$ is canonically isomorphic to K, but the intersection $\mathcal{G}\cap\bar{K}$ is trivial. Indeed, the only non-trivial finitary element (element whose activity is trivial below some level) in $\mathcal{G}$ is a. Thus K does not contain any non-trivial finitary elements. On the other hand, the total activity of the element $\bar{k}$ at level n is the sum of the total activities of the element k at levels 0 through $n-1$. Since K has no non-trivial finitary elements, it follows that, for non-trivial k, the element $\bar{k}$ has unbounded activity. Thus $\mathcal{G}\cap\bar{K}=\{1\}$.

More generally, an easy way to construct some (certainly not all) elements in $\bar{\mathcal{G}}$ is to choose an infinite set of independent vertices V in $\mathcal{T}$ (no vertex is below any other vertex) and associate to each such vertex an arbitrary element of K. The automorphism of $\mathcal{T}$ that is inactive at every vertex that does not have a prefix in V and whose sections at the vertices of V are the assigned elements from K is an element of $\bar{\mathcal{G}}$. Indeed, K is included in the rigid stabilizer of any vertex. This implies that for any finite n-tuple of independent vertices and any choice of an n-tuple of elements from K, there exists an element in K that uses the chosen n elements as the sections at the chosen n vertices. When we construct an element g by selecting (countably) infinitely many independent vertices and infinitely many corresponding elements in K, we cannot claim that g is equal to an element of K, but, obviously, there is a sequence of elements in K that converges to g.

We now prove Theorem 1 and Theorem 2.

Let W be a word over $\{a, b, c, d\}$. The letters in $B = \{b, c, d\}$ are called B-letters. For a subset C of B the letters in C are called C-letters. Denote by $N_C(W)$ the number of C-letters occurring in W. An occurrence of a B-letter ℓ is called even or odd depending on whether an even or odd number of a's appear before ℓ in W. For a parity $p \in \{0, 1\}$ and a subset C of B, denote by $N^p_C(W)$ the number of C-letters of parity p in W. For parities $p, q \in \{0, 1\}$, denote by $N^{p,q}_{b,c}(W)$ the number of $\{b, c\}$-letters ℓ of parity p in W such that the number of $\{b, c\}$-letters of parity $\bar{p}$ that appear before ℓ in W has parity q (here parity $\bar{p}$ denotes the parity different from p). For example,

$$\begin{aligned} N^1_{b,c}(a\mathbf{bc}aadabdbcad\mathbf{cb}d\mathbf{b}abdbc) &= 5, \\ N^{1,1}_{b,c}(abcaadabdbcad\mathbf{cb}d\mathbf{b}abdbc) &= 3, \\ N^{1,0}_{b,c}(a\mathbf{bc}aadabdbcadcbdbabdbc) &= 2, \end{aligned}$$

where, in all examples, the letters that are counted are indicated in boldface. When W is assumed, the notation $N^p_C(W)$ and $N^{p,q}_{b,c}(W)$ is simplified to N^p_C and $N^{p,q}_{b,c}$.

Lemma 1 *For any word W over $\{a, b, c, d\}$ representing an element g in $\mathcal{G}$*

$$N^{1,0}_{b,c} = \beta_{00}, \quad N^{0,1}_{b,c} = \beta_{11}, \quad N^{1,1}_{b,c} = \beta_{01}, \quad N^{0,0}_{b,c} = \beta_{10},$$

where all equalities are taken modulo 2.

Proof Let the words W_u, for u a word over X of length at most 3, represent the sections of g at the corresponding vertices, these words being obtained by decomposition from W, without applying any simple relations other than $aa = 1$ (i.e., no relations involving B-letters are applied).

We have (modulo 2)

$$\begin{aligned} \beta_{00} &= \alpha_{00} + \alpha_{010} + \alpha_{011} = \alpha_{00} + N_{b,c}(W_{01}) \\ &= N^0_{b,c}(W_0) + N^0_{b,d}(W_0) = N^0_{c,d}(W_0) = N^{1,0}_{b,c}(W). \end{aligned}$$

The other equalities are obtained in an analogous fashion. □

Lemma 2 *Let W be a word over $\{a, b, c, d\}$. Modulo 2 we have*

(i) *if $N^0_{b,c} = 0$, then $N^{1,1}_{b,c} = N^{0,1}_{b,c} = N^{0,0}_{b,c}$;*
(ii) *if $N^0_{b,c} = 1$, then $N^{1,0}_{b,c} = N^{0,1}_{b,c} \neq N^{0,0}_{b,c}$;*
(iii) *if $N^1_{b,c} = 0$, then $N^{0,1}_{b,c} = N^{1,1}_{b,c} = N^{1,0}_{b,c}$;*
(iv) *if $N^1_{b,c} = 1$, then $N^{0,0}_{b,c} = N^{1,1}_{b,c} \neq N^{1,0}_{b,c}$.*

Proof (i) Assume $N^0_{b,c}$ is even.

The structure of the word W can be represented schematically by

$$W = n_1\ \ell_1\ n'_1\ \ell_2\ n_2\ \ell_3\ n'_2\ \dots\ n_k\ \ell_{2k-1}\ n'_k\ \ell_{2k}\ n_{k+1},$$

where $\ell_i, i = 1, \dots, 2k$ represent all the even occurrences of $\{b, c\}$-letters in W and the numbers n_i, n'_i represent the number of odd occurrences of $\{b, c\}$-letters between the consecutive even occurrences of $\{b, c\}$-letters.

Then (modulo 2)

$$N^{1,1}_{b,c} = \sum_{i=1}^{k} n'_i = |\{i \mid 1 \leq i \leq k,\ n'_i \text{ is odd }\}| = N^{0,1}_{b,c}.$$

Indeed, for $i = 1, \dots, k$, the n'_i odd occurrences of $\{b, c\}$-letters between ℓ_{2i-1} and ℓ_{2i} are preceded by an odd number (exactly $2i - 1$) of even occurrences of $\{b, c\}$-letters. Thus $N^{1,1}_{b,c} = \sum_{i=1}^{k} n'_i$. On the other hand, for $i = 1, \dots, k$, whenever n'_i is odd exactly one of ℓ_{2i-1} and ℓ_{2i} is preceded by an odd number of odd occurrences of $\{b, c\}$-letters, while whenever n'_i is even either both or none of ℓ_{2i-1} and ℓ_{2i} are preceded by an odd number of odd occurrences of $\{b, c\}$-letters. Thus $|\{i \mid 1 \leq i \leq k,\ n'_i \text{ is odd }\}| = N^{0,1}_{b,c}$ modulo 2.

Since $N^{0,0}_{b,c} + N^{0,1}_{b,c} = N^{0}_{b,c}$ is even, we clearly have $N^{0,0}_{b,c} = N^{0,1}_{b,c}$, modulo 2.

(ii) Assume $N^{0}_{b,c}$ is odd.

The structure of the word W can be represented schematically by

$$W = n_1\ \ell_1\ n'_1\ \ell_2\ n_2\ \ell_3\ n'_2\ \dots\ n'_{k-1}\ \ell_{2k-2}\ n_k\ \ell_{2k-1}\ n'_k,$$

where $\ell_i, i = 1, \dots, 2k-1$ represent all the even occurrences of $\{b, c\}$-letters in W and the numbers n_i, n'_i represent the number of odd occurrences of $\{b, c\}$-letters between the consecutive even occurrences of $\{b, c\}$-letters.

Then (modulo 2)

$$N^{1,0}_{b,c} = \sum_{i=1}^{k} n_i = |\{i \mid 1 \leq i \leq k,\ n_i \text{ is odd }\}| = N^{0,1}_{b,c}.$$

Indeed, for $i = 2, \dots, k$, the n_i odd occurrences of $\{b, c\}$-letters between ℓ_{2i-2} and ℓ_{2i-1} are preceded by an even number (exactly $2i - 2$) of even occurrences of $\{b, c\}$-letters. In addition, the n_1 odd occurrences of $\{b, c\}$-letters from the beginning of W are preceded by no even occurrences of $\{b, c\}$-letters. Thus $N^{1,0}_{b,c} = \sum_{i=1}^{k} n_i$. On the other hand, for $i = 2, \dots, k$, whenever n_i is odd exactly one of ℓ_{2i-2} and ℓ_{2i-1} is preceded by an odd number of odd occurrences of $\{b, c\}$-letters, while whenever n_i is even either both or none of ℓ_{2i-2} and ℓ_{2i-1} are preceded by an odd number of odd occurrences of $\{b, c\}$-letters. In addition, whether ℓ_1 is preceded by an even or odd number of odd occurrences of $\{b, c\}$-letters depends on the parity of n_1. Thus $|\{i \mid 1 \leq i \leq k,\ n_i \text{ is odd }\}| = N^{0,1}_{b,c}$ modulo 2.

Since $N^{0,0}_{b,c} + N^{0,1}_{b,c} = N^{0}_{b,c}$ is odd, we clearly have $N^{0,0}_{b,c} \neq N^{0,1}_{b,c}$, modulo 2.

(iii) and (iv) are analogous to (i) and (ii). □

Proof of Theorem 1 Follows directly from Lemma 1, Lemma 2, and the observations $\alpha_0 = N^{0}_{b,c}$ and $\alpha_1 = N^{1}_{b,c}$ modulo 2. □

Proof of Theorem 2 We use the following (modification of the) description of the elements in $\bar{\mathcal{G}}$ provided in [23]. A binary tree automorphism g belongs to $\bar{\mathcal{G}}$ if and only if, for each section g_u of g, the portrait of g_u agrees with the portrait of some element in $\mathcal{G}$ up to and including level 3.

(i) is equivalent to (iii). Portraits of two automorphisms agree at least up to level 3 if and only if their actions on the tree agree at least up to level 4, which, in turn, is equivalent to the condition that the distance between the two automorphisms is at most $\frac{1}{2^{2^4-1}} = \frac{1}{2^{15}}$.

(i) implies (ii). If g is in $\bar{\mathcal{G}}$, then the portrait of each section g_u of g agrees with the portrait of some element in $\mathcal{G}$ up to and including level 3. The portrait decorations of the elements in $\mathcal{G}$ must satisfy the constraints in Theorem 1, and therefore each section g_u simulates $\mathcal{G}$.

(ii) implies (i). It is known that $[\mathcal{G} : \mathrm{Stab}_{\mathcal{G}}(X^4)] = 2^{12}$. Thus, for elements in $\mathcal{G}$, there are exactly 2^{12} possible portrait decorations on level 0 through 3. The constraints of Theorem 1 provide for exactly 2^{12} different decorations of the appropriate size (see the discussion in Example 1). Thus if a tree automorphism simulates $\mathcal{G}$, then its portrait agrees with the portrait of an actual element in $\mathcal{G}$ up to and including level 3. □

As another application, we offer a proof of the following result, obtained by Grigorchuk in [22].

Theorem 3 *The Hausdorff dimension of* $\bar{\mathcal{G}}$ *in* Aut $(\mathcal{T})$ *is* $\frac{5}{8}$.

Proof It is known that the Hausdorff dimension can be calculated as the limit

$$\liminf_{n\to\infty} \frac{\log[\bar{\mathcal{G}} : \mathrm{Stab}_{\bar{\mathcal{G}}}(X^n)]}{\log[\mathrm{Aut}\,(\mathcal{T}) : \mathrm{Stab}_{\mathrm{Aut}\,(\mathcal{T})}(X^n)]},$$

comparing the relative sizes of the level stabilizers of $\bar{\mathcal{G}}$ and Aut $(\mathcal{T})$ (see [2]). Applying the strategy of construction of elements in $\bar{\mathcal{G}}$ indicated in Example 1, it follows that, in the portrait of an element g in $\bar{\mathcal{G}}$, 5 out of 8 vertices at level 3 and below can have any decoration we choose (0 or 1) and the other three have uniquely determined decoration. Thus, the limit determining the Hausdorff dimension of $\bar{\mathcal{G}}$ is $\frac{5}{8}$. □

References

1. Abels, H.: An example of a finitely presented solvable group. In: Wall, C.T.C. (ed.). Homological Group Theory, pp. 205–211. Durham 1977, Cambridge Univ. Press (1979)
2. Barnea, Y., Shalev, A.: Hausdorff dimension, pro-*p* groups, and Kac-Moody algebras. Trans. Amer. Math. Soc. **349**(12), 5073–5091 (1997)
3. Baumslag, G.: Some groups that are just free. Bull. Amer. Math. Soc. **73**, 621–622 (1967)
4. Baumslag, G.: More groups that just about free. Bull. Amer. Math. Soc. **74**, 752–754 (1968)
5. Baumslag, G.: A non-cyclic one-relator group all of whose finite quotients are cyclic. J. Austral. Math. Soc. **10**, 497–498 (1969)
6. Baumslag, G.: Positive one–relator groups. Trans. Amer. Math. Soc. **156**, 165–183 (1971)
7. Baumslag, G.: Parafree groups. Bartholdi, L., Ceccherini-Silberstein, T., Smirnova-Nagnibeda, T., Zuk, A. (eds.) Infinite Groups: Geometric, Combinatorial and Dynamical Aspects, pp. 1–15. Series: Progress in Mathematics, vol. 248. Birkhäuser (2005)
8. Birkhoff, G.: More–Smith convergence in general topology. Ann. Math. **38**, 39–56 (1937)
9. Bourbaki N.: Topologie générale, chapitres 3 et 4, troisième édition. Hermann (1960)
10. Bourbaki, N.: Algèbre, chapitres 1 à 3. Diffusion C.C.L.S., Paris (1970)
11. Bourbaki, N.: Groupes et algèbres de Lie, chapitres 2 et 3. Hermann (1972)
12. Bourbaki, N.: Groupes et algèbres de Lie, chapitre 9. Masson (1982)
13. Bridson M., Grunewald, F.J.: Grothendieck's problems concerning profinite completions and representations of groups. Ann. Math. **160**, 359–373 (2004)

14. Brunner, A.M., Burns, R.G., Wiegold, J.: On the number of quotients, of one way or another,of the modular group. Math. Sci. **4**, 93–98 (1979)
15. Brunner, A.M., Sidki, S., Vieira, A.C.: A just nonsolvable torsion-free group defined on the binary tree. J. Algebra **211**, 99–114 (1999)
16. Clair, B.: Residual amenability and the approximation of L^2-invariants. Mich. Math. J. **46**, 331–346 (1999)
17. Cochran, T., Harvey, S.: Homology and derived series of groups. arXiv:math.GT/0407203
18. Dixon, J.D., du Sautoy, M., Mann, A., Segal, D.: Analytic Pro-p Groups. Cambridge University Press (1991)
19. Elek G., Szabo, E.: On sofic groups. arXiv:math.GR/0305352
20. Glasner, Y., Souto, J., Storm, P.: Finitely generated subgroups of lattices in $PSL_2\mathbf{C}$. arXiv:math.GT/0504441
21. Grigorchuk, R.I.: Burnside's problem on periodic groups. Funct. Anal. Appl. **14**, 41–43 (1980)
22. Grigorchuk, R.I.: Just infinite branch groups. In: du Sautoy, M., Segal, D., Shalev, A. (eds.) New Horizons in Pro-p-groups, pp. 121–179. Brikhäuser (2000)
23. Grigorchuk, R.I.: Solved and unsolved problems around one group. Prog. Math. 117–217 (2005)
24. Grigorchuk, R.I., Nekrashevich, V.V., Sushchanskiĭ, V.I.: Automata, dynamical systems, and groups. Tr. Mat. Inst. Steklova **231**, 134–214 (2000)
25. Grothendieck, A.: Représentations linéaires et compactification profinie des groupes discrets. Manuscripta Math. **2**, 375–396 (1970)
26. Gruenberg, L.: Residual properties of infinite soluble groups. Proc. London Math. Soc. **7**, 29–62 (1957)
27. Hall, M.: A topology for free groups and related groups. Ann. Math. **52**, 127–139 (1950)
28. Hall, P.: On the finiteness of certain soluble groups. Proc. London Math. Soc. **9**, 595–622 (1959) [= Collected Works, 515–544]
29. Hall, P.: The Frattini subgroups of finitely generated groups. Proc. London Math. Soc. **11**, 327–352 (1961) [= Collected Works, 581–608]
30. de la Harpe, P.: Topics in Geometric Group Theory. The University of Chicago Press (2000)
31. Higman, G., Stone, A.H.: On inverse systems with trivial limits. J. London Math. Soc. **29**, 233–236 (1954)
32. Kahrobaei, D.: On the residual solvability of generalized free products of finitely generated nilpotent groups. arXiv:math.GR/0510465
33. Kahrobaei, D.: Are doubles of residually solvable groups. residually solvable? Preprint
34. Kassabov, M., Nikolov, N.: Cartesian products as profinite completions. arXiv:math.GR/ 0602446
35. Kelley, J.L.: General Topology. Van Nostrand (1955)
36. MacDuffee, C.C.: The Theory of Matrices. Chelsea Publ. Comp. (1946)
37. Magnus, W., Karras, A., Solitar, D.: Combinatorial Group Theory. J. Wiley (1966)
38. Mal'cev, A.I.: On the faithful representation of infinite groups by matrices. Amer. Math. Soc. Transl. **45**(2), [Russian original: Mat. SS.(N.S.) 8(50) (1940), pp. 405–422]
39. A.I. Mal'cev.: Generalized nilpotent algebras and their associated groups. Mat. Sbornik N.S. **25**(67), 347–366 (1949)
40. Nikolov, N., Segal, D.: Finite index subgroups in profinite groups. C.R. Acad. Sci. Paris, Sér. I **337**, 303–308 (2003)
41. Nikolov, N., Segal, D.: On finitely generated profinite groups I: strong completeness and uniform bounds. arXiv:math.GR/0604399
42. Nikolov, N., Segal, D.: On finitely generated profinite groups II: products in quasisimple groups. arXiv:math.GR/0604400
43. Peterson, L.H.: Discontinuous characters and subgroups of finite index. Pacific J. Math. **44**, 683–691 (1973)
44. Platonov, V.P., Tavgen, O.I.: On Grothendieck's problem of profinite completions of groups. Soviet Math. Dokl. **33**, 822–825 (1986)
45. Platonov, V.P., Tavgen, O.I.: Grothendieck's problem on profinite completions and representations of groups. K-theory **4**, 89–101 (1990)
46. Raptis, E., Varsos, D.: Residual properties of HNN-extensions with base group an abelian group. J. Pure Appl. Algebra **59**, 285–290 (1989)
47. Ribes, L., Zalesskii, P.: Profinite Groups. Springer (2000)
48. Robinson, D.J.S.: A course in the theory of groups. Springer (1982)
49. du Sautoy, M., Segal, D., Shalev, A.: New Horizons in Pro-p-Groups. Birkhäuser (2000)

50. Serre, J.–P. Cohomologie Galoisienne. Lecture Notes in Math. 5, Springer (1973) [cinquième édition 1994]
51. Sidki, S.: Finite automata of polynomial growth do not generate a free group. Geom. Dedicata **108**, 193–204 (2004)
52. Stallings, J.: Homology of central series of groups. J. of Algebra **2**, 170–181 (1965)
53. Weil, A: L'intégration dans les groupes topologiques et ses applications. Hermann (1940)
54. Wilson, J.: Profinite Groups. Clarendon Press (1998)