Contemporary Mathematics

A dichotomy for finitely generated subgroups of word hyperbolic groups

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Abstract. Given $L > 0$ elements in a word hyperbolic group G, there exists a number $M = M(G, L) > 0$ such that at least one of the assertions is true: (i) these elements generate a free and quasiconvex subgroup of G; (ii) they are Nielsen equivalent to a system of L elements containing an element of length at most M up to conjugation in G. The constant M is given explicitly. The result is generalized to groups acting by isometries on Gromov hyperbolic spaces. For proof we use a graph method to represent finitely generated subgroups of a group.

1. Introduction

Let H be a subgroup of a word hyperbolic group G. It is known that either H is elementary (that is, it contains a cyclic subgroup of finite index) or H contains a non-abelian free subgroup of rank two. In the case G is torsion-free, there are, up to conjugacy, finitely many Nielsen equivalence classes of non-free subgroups of G generated by two elements [**3**].

Our main result gives a sufficient condition for H to be free and quasiconvex in G. It is an improvement of a result due to Gromov [**4**, 5.3.A].

Theorem 1. *For any $\delta \geq 0$ and an integer $L > 0$ there exists a number $M = M(\delta, L) > 0$ with the following property.*

Let G be a δ-hyperbolic group with respect to a finite generating set $\mathcal{X}$ and H be a subgroup of G generated by $h_1, \cdots, h_L$. Then at least one of the following assertions is true.

(i) *H is free on $h_1, \cdots, h_L$ and quasiconvex in G;*
(ii) *The tuple $(h_1, \ldots, h_L)$ is Nielsen equivalent to an L-tuple $(h'_1, \ldots, h'_L)$ with h'_1 conjugate to an element in G of word length at most M with respect to $\mathcal{X}$.*

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The constant $M = M(\delta, L)$ can be calculated explicitly. Then, as an immediate consequence of Theorem 1 we obtain Gromov's result. Let

$$\mathsf{Rad}\ H = 1/2 \inf_{h \in H \setminus \{\mathsf{id}\}} |[h]|,$$

where $|[h]| = \inf_{h'} |h'|$ for all $h' \in G$ conjugate to h and $|h|$ is the word length of h. This is called the injectivity radius of H.

THEOREM 2. (cf. [**4**, 5.3.A]) Let G be a δ-hyperbolic group and H be a subgroup of G generated by L elements. If

$$\mathsf{Rad}\ H \geq \frac{1}{2}\left(\frac{14400(\delta+1)}{\ln 2}L^2\right)^{1+1/2},$$

then H is free and quasiconvex[1] in G.

The following result is a natural generalization of Theorem 1 to groups acting on hyperbolic spaces.

THEOREM 3. *For any $\delta \geq 0$ and an integer $L > 0$ there exists a number $M = M(\delta, L) > 0$ with the following property.*

Let G be a group on L generators $g_1, \cdots, g_L$ acting on a δ-hyperbolic space (X, d) by isometries. Then at least one of the following assertions is true.

(i) *G is free on $g_1, \cdots, g_L$ and for every $x \in X$ the map $G \to X$ which assigns to each $g \in G$ the element $gx \in X$ is a quasi-isometric embedding;*
(ii) *The tuple $(g_1, \ldots, g_L)$ is Nielsen equivalent to an L-tuple $(g_1', \ldots, g_L')$ with $d(g_1'y, y) < M$ for some $y \in X$.*

The constant $M = M(\delta, L)$ can be calculated explicitly.

For proof of Theorem 1 a representation of finitely generated subgroups of a group by labelled graphs is used [**1**]. The technique is of independent interest. In particular, transformations of a labelled graph defined below can be viewed as a generalization of free reductions and Nielsen reductions of tuples of group elements.

Theorem 3 is obtained with essentially the same methods, except that, in our arguments, instead of the word length metric, we refer to the metric on G induced from the group action.

I was informed that Kapovich and Weidmann showed both results (without an explicit estimate on the constant) independently of my work and by different methods [**6**].

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[1] This lower bound is not optimal. For example, one can take $\frac{1}{2}\left(\frac{14400(\delta+1)}{\ln 2}L^2\right)^{1+\varepsilon}$ with $\varepsilon > 0$ or a better constant satisfying condition (4) below.

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