COARSE NON-AMENABILITY AND COVERS WITH SMALL EIGENVALUES

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ABSTRACT. Given a closed Riemannian manifold M and a (virtual) epimorphism $\pi_1(M) \twoheadrightarrow \mathbb{F}_2$ of the fundamental group onto a free group of rank 2, we construct a tower of finite sheeted regular covers $\{M_n\}_{n=0}^{\infty}$ of M such that $\lambda_1(M_n) \to 0$ as $n \to \infty$. This is the first example of such a tower which is not obtainable up to uniform quasi-isometry (or even up to uniform coarse equivalence) by the previously known methods where $\pi_1(M)$ is supposed to surject onto an amenable group.

1. INTRODUCTION

Let M be a closed (that is, compact and without boundary) Riemannian manifold with fundamental group $\pi_1(M)$. A residually finite group G, a surjective homomorphism $\pi_1(M) \twoheadrightarrow G$ and a nested sequence of finite index normal subgroups of G with trivial intersection gives rise to a tower of finite sheeted regular covers of M; conversely, every tower of finite sheeted regular covers arises in this manner. In summary, writing $G_0 = G$ and $M_0 = M$, we have:

$$(*) \qquad \begin{aligned} & G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \cdots, \quad \text{with } \bigcap_{n=0}^{\infty} G_n = \{1\}, \\ & \pi_1(M_n) \twoheadrightarrow G_n, \text{ and finite groups } \Gamma_n := G/G_n, \end{aligned} \qquad \begin{tikzcd} \vdots \arrow[d, dotted] & \vdots \arrow[d, dotted] \\ \Gamma_2 \arrow[d] & M_2 \arrow[d] \\ \Gamma_1 \arrow[d] & M_1 \arrow[d] \\ \Gamma_0 & M_0. \end{tikzcd}$$

In the context of spectral geometry of towers of covers one studies the asymptotic behavior of the first non-zero eigenvalues $\lambda_1(M_n)$ of the Laplacian, that is, of the Laplace-Beltrami operator of the individual Riemannian manifolds M_n. In particular, the following questions are classical:

(a) Does there exist a tower with $\lambda_1(M_n) \geqslant c > 0$ uniformly over n?
(b) Does there exist a tower with $\lambda_1(M_n) \to 0$ as $n \to \infty$?

In this note we are concerned with (b). The earliest positive result on this question is due to Randol, who studied the case of cyclic covers using the trace formula [9]. Subsequent results of

2000 *Mathematics Subject Classification.* Primary 20F65; Secondary 20F69, 58G25, 20F34.
Key words and phrases. Amenability, coarse embeddings, expander graphs, graph coverings.
The research of the first author was partially supported by the ERC grant ANALYTIC no. 259527 and by the Swiss NSF, Sinergia Grant CRSI22_130435. The second author was partially supported by NSF Grant DMS-0349367.

Brooks [3, 2] and Burger [4] were obtained by relating the eigenvalues $\lambda_1(M_n)$ to combinatorial properties of the Cayley graphs of finite groups of deck transformations Γ_n. Similar results are due to Sunada [12].

In all cases, the method to build a tower of covers satisfying (b) rests on choosing an *amenable* group G for the construction $(*)$. Our main result is that it is possible to obtain such a tower when G is the free group on two generators. In the statement, $H^{(2)}$ denotes the subgroup of the discrete group H generated by the squares of its elements.

Theorem. *Let M be a closed Riemannian manifold, whose fundamental group admits a virtual*[1] *surjective homomorphism onto the free group of rank 2. Taking the nested sequence of subgroups in $(*)$ to be the sequence of iterated squares in the free group*

$$(**)\qquad G_0 = \mathbb{F}_2, \quad G_1 = \mathbb{F}_2^{(2)}, \quad G_2 = (\mathbb{F}_2^{(2)})^{(2)}, \quad G_3 = ((\mathbb{F}_2^{(2)})^{(2)})^{(2)}, \quad \cdots$$

we obtain a tower of covers of M for which $\lambda_1(M_n) \to 0$ as $n \to \infty$. This tower is not obtainable up to uniform quasi-isometry (or even uniform coarse equivalence) by the construction $(*)$ with an amenable G.

Observe here that each G_n is normal, even characteristic, in $\mathbb{F}_2$.

The hypothesis of the theorem means that the fundamental group is *large* (the terminology is due to Gromov [5]). It applies to many hyperbolic manifolds [6], in particular, to a closed orientable surface of genus at least two – the fundamental group of such a manifold surjects onto $\mathbb{F}_2$.

We conclude the introduction by remarking that in more modern terminology the classical problems above concerning the construction $(*)$ can be rephrased in terms of *Property* τ: (a) asks for G to have Property τ with respect to the family of subgroups $(G_n)_{n\geqslant 0}$, whereas (b) asks, after perhaps passing to a subsequence, for G to *not* have Property τ with respect to the $(G_n)_{n\geqslant 0}$. This is explained in the work of Burger and Brooks cited above. Thus, the first assertion in the theorem is essentially equivalent to the assertion that $\mathbb{F}_2$ does not have Property τ with respect to the subgroups appearing in $(**)$. For the definition and relevant facts about Property τ see [7].

2. EIGENVALUES

A *graph* is a collection of *vertices* and *edges*. With a small number of exceptions, we permit neither multiple edges nor loops, so that an edge is uniquely determined by its incident vertices. Our graphs are unoriented. The *Cheeger constant* of a finite graph Γ is

$$h(\Gamma) = \inf \frac{\#E(A,B)}{\min\{\#A, \#B\}}, \tag{2.1}$$

where the infimum is taken over all decompositions of the vertex set of Γ as a disjoint union $A \sqcup B$ and where, for such a decomposition, $E(A,B)$ denotes the set of edges with one incident vertex in A and the other in B.

We shall make use of the following result of Brooks which, in the notation of $(*)$, relates the eigenvalues of the M_n to the Cheeger constants of the Cayley graphs of the Γ_n computed with

[1] A virtual homomorphism is a homomorphism of a finite-index subgroup.

respect to the canonical images of generators of G and denoted, by an abuse of notation, again by Γ_n. We shall require only the forward implication, which is the content of [3, Lemma 1].

Theorem (Brooks)**.** *In the notation of* $(*)$ *we have* $h(\Gamma_n) \to 0$ *precisely when* $\lambda_1(M_n) \to 0$. □

Thus, the first statement in theorem of the introduction is reduced to the following:

1. **Proposition.** *Let* $G = \mathbb{F}_2$ *be the free group of rank 2. Consider the tower of iterated squares* $(**)$ *and the corresponding quotients:*

$$\Gamma_0 = \{1\} \leftarrow \Gamma_1 = \mathbb{F}_2/\mathbb{F}_2^{(2)} \leftarrow \Gamma_2 = \mathbb{F}_2/(\mathbb{F}_2^{(2)})^{(2)} \leftarrow \cdots$$

Abusing notation, view each Γ_n *as a Cayley graph with respect to the images of the standard free generators of* $\mathbb{F}_2$*. Then we have* $h(\Gamma_n) \to 0$ *as* $n \to \infty$.

In preparation for the proof we recall the construction of the $\mathbb{Z}/2$-*homology cover* of a finite graph Σ. Fix a maximal tree T in Σ and let $e_1, \dots, e_r$ be the edges of Σ *not* in T. The vertex and edge sets of the $\mathbb{Z}/2$-homology cover $\widetilde{\Sigma}$ are

$$\widetilde{V} = V \times \oplus_1^r \mathbb{Z}/2, \quad \widetilde{E} = E \times \oplus_1^r \mathbb{Z}/2,$$

where E and V denote the vertex and edge sets of Σ. Let $e \in E$ and let $v, w \in V$ be the vertices incident with e. Consider the edge $(e, \alpha) \in \widetilde{E}$. Incidence is defined in two cases:

$$(e,\alpha) \text{ contains } \begin{cases} (v,\alpha) \text{ and } (w,\alpha), & \text{when } e \text{ belongs to the maximal tree } T \\ (v,\alpha) \text{ and } (w, \alpha + \overline{e}_j), & \text{when } e = e_j, \text{ for some } 1 \leqslant j \leqslant r. \end{cases}$$

Here $\overline{e}_j = (\dots, 1, \dots)$ is the standard basis vector with a single 1 in the j-the position and 0's elsewhere. Strictly speaking, when defining incidence it is necessary to *direct* the edges e_j. It is quickly verified however that, while the edges are parameterized in a different manner, the underlying *undirected* graph is independent of the choice. We shall not dwell on this aspect.

Remark. The construction given here of the $\mathbb{Z}/2$-homology cover is a special case of the classical construction of a finite sheeted regular cover of Σ corresponding to a given normal subgroup of finite index in $\pi_1(\Sigma)$, see, for example, [11, Ch. 2]. Indeed, with $e_1, \dots, e_r$ as above, and after directing each e_j, we identify

$$\pi_1(\Sigma) \cong \mathbb{F}_r = \langle\, e_1, \dots, e_r \,\rangle.$$

Then the cover corresponding to the kernel of the epimorphism

$$\pi_1(\Sigma) \cong \mathbb{F}_r \twoheadrightarrow \mathbb{F}_r/\mathbb{F}_r^{(2)} \cong \oplus_1^r \mathbb{Z}/2 \quad \text{defined by} \quad e_j \mapsto \overline{e}_j,$$

is the $\mathbb{Z}/2$-homology cover.

2. **Lemma.** *Let* Σ *be a finite graph, with vertex set* V*; let* $\widetilde{\Sigma}$ *be its* $\mathbb{Z}/2$*-homology cover. We have*

$$h(\widetilde{\Sigma}) \leqslant \frac{2}{\#V}.$$

Proof. We employ the notation introduced above for $\widetilde{\Sigma}$. We shall exhibit a decomposition of the vertex set $\widetilde{V} = A \sqcup B$ for which the quotient in (2.1) is bounded by $2/\#V$. Let

$$A = \{ (v, \alpha) \in \widetilde{V} : \alpha = (*, \ldots, *, 0) \}, \quad B = \{ (w, \beta) \in \widetilde{V} : \beta = (*, \ldots, *, 1) \},$$

each of which contains exactly $2^{r-1}\,\#V$ vertices. The edges in $\widetilde{E}$ with one vertex in A and the other in B are exactly those of the form (e_r, γ), for arbitrary $\gamma \in \oplus_1^r \mathbb{Z}/2$; thus $E(A, B)$ contains exactly 2^r edges. □

Proof of Proposition 1. The Cayley graph Γ_n is the n-th iterated $\mathbb{Z}/2$-homology cover of the "figure 8". Since the number of vertices in Γ_n tends to infinity, the result follows from the previous lemma. □

Remark. A more detailed analysis gives information on the *rate* of the convergence $h(\Gamma_n) \to 0$. Indeed, let V_n be the set of vertices and E_n the set of edges of (the Cayley graph of) Γ_n. We have

$$\frac{\#V_{n+1}}{\#V_n} = \frac{\#E_{n+1}}{\#E_n} = 2^{\operatorname{rk} \pi_1(\Gamma_n)}.$$

Now, the rank of the fundamental group $\pi_1(\Gamma_n)$ is the number of edges *not* belonging to a fixed maximal tree in Γ_n. Since $\#E_n = 2 \cdot \#V_n$, the rank of $\pi_1(\Gamma_n)$ is $\#V_n + 1$. Thus, we get the recursive formula

$$\#V_{n+1} = \#V_n \cdot 2^{\#V_n + 1}.$$

In particular, $\#V_n$ grows faster than an iterated exponential and, according to the previous lemma, the Cheeger constant $h(\Gamma_{n+1})$ decays as the reciprocal of $\#V_n$.

3. Non uniform coarse equivalence

We shall now show that the tower constructed in the previous section cannot be duplicated beginning with an amenable group in $(*)$, thus completing the proof of the theorem in the introduction.

Two families $(X_n)_{n\geqslant 0}$ and $(Y_n)_{n\geqslant 0}$ of metric spaces are *uniformly quasi-isometric* if there exist functions $f_n : X_n \to Y_n$ and constants $C \geqslant 1$ and $D \geqslant 0$ such that for all $x, y \in X_n$ and $z \in Y_n$, we have

- $C^{-1}d(x, y) - D \leqslant d(f_n(x), f_n(y)) \leqslant Cd(x, y) + D$,
- $d(z, f_n(X_n)) \leqslant D$.

The families $(X_n)_{n\geqslant 0}$ and $(Y_n)_{n\geqslant 0}$ are *uniformly coarsely equivalent* if there exist functions $f_n : X_n \to Y_n$ with the following two properties:

- $\forall A\ \exists B$ such that $\forall n\ \forall x, y \in X_n$ we have $d(x, y) \leqslant A \Rightarrow d(f_n(x), f_n(y)) \leqslant B$,
- $\forall A\ \exists B$ such that $\forall n\ \forall x, y \in X_n$ we have $d(x, y) \geqslant B \Rightarrow d(f_n(x), f_n(y)) \geqslant A$.

If two families are uniformly quasi-isometric then they are uniformly coarsely equivalent. Observe that these notions apply to individual spaces, which we regard as trivial families containing a single space. We say, for example, two spaces are coarsely equivalent.

3. **Proposition** (The Uniform Švarc-Milnor Lemma)**.** *Continue with the notation of* $(*)$. *Equip each* Γ_n *with the word metric associated to a fixed finite generating set for* G*; equip each* M_n *with the path metric associated to its Riemannian structure. The families* $(\Gamma_n)_{n\geqslant 0}$ *and* $(M_n)_{n\geqslant 0}$ *are uniformly quasi-isometric.*

Remark. In the situation of $(*)$ the group G is indeed finitely generated. Further, the statement in the proposition is independent of the choice of generators for G.

Proof of Proposition 3. The result follows from the Švarc-Milnor Lemma [1, Prop. I.8.19], observing that the inherent quasi-isometry constants (see the proof of the Lemma) depend only on the diameter of a fundamental domain for the action. In detail, Γ_n is the group of deck transformations of the cover M_n of M, whereas G is the group of deck transformations of the cover corresponding to the kernel of the surjective homomorphism $\pi_1(M) \twoheadrightarrow G$. Further, the image in M_n of a bounded fundamental domain for the action of G is a fundamental domain for the action of Γ_n, of no greater diameter. □

Thus, the second statement in the theorem of the introduction is reduced to the following:

4. **Proposition.** *Consider the tower of iterated squares* $(**)$ *of the free group* $\mathbb{F}_2$ *and the corresponding quotients*

$$\Gamma_0 = \{1\} \leftarrow \Gamma_1 = \mathbb{F}_2/\mathbb{F}_2^{(2)} \leftarrow \Gamma_2 = \mathbb{F}_2/(\mathbb{F}_2^{(2)})^{(2)} \leftarrow \cdots$$

Then the family $(\Gamma_n)_{n\geqslant 0}$ *is not uniformly coarsely equivalent to any family of quotients of an amenable group.*

Let G be a finitely generated discrete group, and let ℓ be the word length associated to a fixed finite and symmetric set of generators. Of the many equivalent definitions of amenabilitiy we shall work with *Reiter's condition* – G is *amenable* if for every $\varepsilon > 0$ and for every $R > 0$ there exists a finitely supported $\xi \in \ell^1(G)$ such that $\xi \geqslant 0$, $\|\xi\| = 1$ and

$$\ell(g) \leqslant R \Rightarrow \|g \cdot \xi - \xi\| < \varepsilon, \tag{3.1}$$

where the action of G on $\ell^1(G)$ is defined by $g \cdot \xi(h) = \xi(g^{-1}h)$.

Our main tool to prove Proposition 4 is the use of Property A, a weak form of amenability introduced by Yu in the context of the Baum-Connes conjecture in topology [14].

Let X be a discrete metric space of *bounded geometry* – that is, the number of points in a ball of fixed radius is bounded, the bound depending only on the radius of the ball and not on its center. Of the many equivalent definitions of Property A we choose the one most closely related to Reiter's condition – X has *Property* A if for every $\varepsilon > 0$ and $R > 0$ there exists an $S > 0$ and for each $x \in X$ a function $\xi_x \in \ell^1(X)$ such that $\xi_x \geqslant 0$, $\|\xi_x\| = 1$ and

$$d(x,y) \leqslant R \Rightarrow \|\xi_x - \xi_y\| < \varepsilon,$$
$$\xi_x(y) \neq 0 \Rightarrow d(x,y) \leqslant S.$$

The analogy with amenability being clear, we say that a metric space having Property A is *coarsely amenable* whereas one not having Property A is *coarsely non-amenable*.

Finally, a metric space X is the *coarse union* of its subspaces X_n if $X = \sqcup X_n$ (disjoint union), and if $d(X_n, X_m) \to \infty$ as $n + m \to \infty$. If the X_n are metric spaces each having finite diameter, then there exists a metric space X which is the coarse union of (isometric copies of) the X_n. Further, any two such unions are coarsely equivalent. Moreover, if Y is the coarse union of the Y_n then X and Y are coarsely equivalent when the X_n and Y_n are uniformly coarsely equivalent.

We require the following slight generalization of [10, Prop. 11.39]. We include a proof which is both different from other proofs in the literature and convenient for our result.

5. **Proposition.** *Let G be a finitely generated amenable group. Every quotient of G is amenable; the coarse union of any family of finite quotients of G is coarsely amenable.*

Proof. Let H be a quotient of G and identify H with a set of cosets $\{ gK \}$. Fix a finite and symmetric set of generators for G and equip G with the associated word length; equip H with the word length associated to the induced generators. With these conventions

$$\ell_H(x) \leqslant R \iff \exists\, g \in x \text{ such that } \ell_G(g) \leqslant R$$

and, in particular, the map $G \twoheadrightarrow H$ is contractive. Given $\varepsilon > 0$ and $R > 0$, obtain $\xi \in \ell^1(G)$ as in (3.1). Define

$$\eta(x) = \sum_{g \in x} \xi(g), \tag{3.2}$$

so that $\eta \geqslant 0$ and $\|\eta\| = 1$. Further, when $z \in H$ has length at most R we obtain $g \in G$ of length at most R such that $z = gK$. We then calculate

$$\|z \cdot \eta - \eta\| = \sum_{x \in H} |\eta(g^{-1}x) - \eta(x)| \leqslant \sum_{x \in H} \sum_{h \in x} |\xi(g^{-1}h) - \xi(h)| = \|g \cdot \xi - \xi\| < \varepsilon.$$

We conclude that H is amenable.

When dealing with a coarse union the essential observation is that, in the previous argument, if ξ is supported on the elements of length at most S then the same is true of η. Thus, let $\{H_n\}$ be a family of finite quotients of G, each equipped with a length function as above, and let X be a coarse union of the H_n. Given $\varepsilon > 0$ and $R > 0$ proceed as above – obtain a Reiter function ξ for G and define η_n as in (3.2). For $x \in X$ define

$$\xi_x = \begin{cases} \chi_N, & x \in H_n,\, n \leqslant N \\ x \cdot \eta_n, & x \in H_n,\, n > N, \end{cases}$$

where N is chosen large enough so that for $n > N$ the distance between H_n and any other H_m is at least R and where χ_N is the normalized characteristic function of $H_1 \cup \cdots \cup H_N$. Finally, choose S larger than the diameter of $H_1 \cup \cdots \cup H_N$ and large enough so that ξ is supported on elements of length at most S in G. The required properties are easily verified. $\square$

Proof of Proposition 4. The iterated squares are proper characteristic subgroups of the free group, hence, by Levi's theorem [8, Ch.I, Prop. 3.3], they have trivial intersection, $\cap \mathbb{F}_2^{(2)\ldots(2)} = \{ 1 \}$. Thus, the coarse union of the metric spaces Γ_n is an example of a coarsely non-amenable *box space*. See

[10, Def. 11.24 and Prop. 11.39]. (This statement is the converse of the previous proposition, and can also be proved by modifying the above argument.) This finishes the proof as coarse amenability is invariant under coarse equivalence, see, for example, [13, Prop. 4.2]. □

We conclude with two remarks. First, we have used a very crude invariant from coarse geometry to distinguish towers constructed from the sequence of iterated squares ($**$) from those constructed beginning with an amenable group in ($*$) – the former are coarsely non-amenable while the latter are coarsely amenable. More refined invariants would be needed to establish the existence of coarsely inequivalent towers constructed as in ($*$) from a given non-amenable group.

Second, our construction involving the iterated squares ($**$) is particular to the free group. It would be interesting to remove the hypothesis of 'largeness' from our theorem.

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