QUANTIFYNG METRIC APPROXIMATIONS OF DISCRETE GROUPS

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Abstract. We introduce and systematically study a profile function whose asymptotic behavior quantifies the dimension or the size of a metric approximation of a finitely generated group G by a family of groups $\mathcal{F} = \{(G_\alpha, d_\alpha, k_\alpha, \varepsilon_\alpha)\}_{\alpha\in I}$, with each group G_α equipped with a bi-invariant metric d_α and a dimension k_α, for strictly positive real numbers ε_α such that $\inf_\alpha \varepsilon_\alpha > 0$. Through the notion of residually amenable profile we introduce, our approach generalizes classical isoperimetric (or Følner) profiles of amenable groups and recently introduced functions quantifying residually finite groups. Our viewpoint is much more general and covers hyperlinear and sofic approximations as well as many other metric approximations such as weakly sofic, weakly hyperlinear, and linear sofic approximations.

1. Introduction

Approximation is ubiquitous in mathematics. In the theory of groups, it is particularly natural to approximate infinite groups by finite ones. A fundamental realization of this idea has lead Mal'cev (1940's) and P. Hall (1955) to the notion of a residually finite group: a group where the algebraic structure on any finite fixed set of elements is exactly as if these elements were in a suitable finite quotient of the group.

Once a concept of approximation is coined, a crucial question is how to compare distinct approximations of the same object, and, in particular, how to quantify the way an object is approximated. For residually finite groups, there are two main ways of quantifying the approximation of an infinite group by finite ones. The first way is to compute how many subgroups of a given finite index the group possesses. This is a classical subject of research on the subgroup growth, initiated by M. Hall (1949), which allows to enumerate how the group can be approximated by a finite quotient of a prescribed cardinality. The second way of quantifying is to compute the minimal cardinality among all possible finite quotients that detect the algebraic structure of the fixed finite set of elements of the residually finite group. This viewpoint is more recent and it is about the so-called full residual finiteness growth, see below for the definition.

In this paper, we push this second idea of quantifying of approximations of infinite groups significantly beyond the class of residually finite groups and apply it to much more general *metric* approximations of infinite groups in contrast to classical *algebraic* approximations. Metric approximations are approximations by groups equipped with bi-invariant metrics (see the next section for precise definitions) and they are very natural to study. Intuitively, we require that the algebraic operation on a finite set of group elements of the approximated group is almost as if these elements were in the approximating group, where 'almost' refers to the fixed bi-invariant

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metric. This simple idea has gained a major importance following Gromov's introduction of *sofic* groups (= groups metrically approximated by symmetric groups of finite degrees, endowed with the normalized Hamming distance) and his settlement, for sofic groups, of Gottschalk's surjunctivity conjecture (1973) in topological dynamics. Another renowned example of metric approximation is that by unitary groups of finite rank, endowed with the normalized Hilbert-Schmidt distance. This defines the class of *hyperlinear* groups, appeared in the context of Connes' embedding problem (1972) in operator algebra.

We encompass both sofic and hyperlinear groups as well as their generalizations such as linear sofic groups, weakly sofic groups, and weakly hyperlinear groups into a general framework of metric approximations by groups with, in addition to a prescribed bi-invariant metric, a dimension or a size, associated with each of the approximating groups. For instance, the dimension of a finite symmetric group is chosen to be its degree, of a unitary group – its rank, of a finite group – its cardinality, etc. Our general quantification function, called *metric profile*, is then defined to be, given a finite set of group elements in the approximated group, e.g. the ball of finite radius with respect to the word length metric, the minimal dimension among all possible metric approximations which 'almost' preserve the algebraic structure of this finite set. Viewed within sofic groups, our approach is orthogonal to the recently emerged theory of sofic entropy started in the seminal work of L. Bowen (such a theory is not yet available for an a priori wider class of hyperlinear groups). Restricted to residually finite groups, the contrast between Bowen's viewpoint and our approach is exactly the distinction between the subgroup growth of a group and the full residual finiteness growth, respectively.

Since metric approximations generalize classical algebraic approximations, the previously known functions, quantifying 'exact' approximations (versus 'almost' ones), occur to be upper bounds for our metric profile. For example, a knowledge about the full residual finiteness growth of a residually finite group gives an estimate on the sofic and on the hyperlinear profiles of such a group. If the approximating groups are amenable, then besides a chosen dimension, they carry an associated isoperimetric function, the famous Følner function. We make use of this classical function and of our metric profile philosophy to define the *residually amenable profile* for every residually amenable group (and more generally, for every group locally embeddable into amenable ones). This allows to extend a classical study of Følner functions of amenable groups to non-amenable groups metrically approximable by amenable ones.

A main aim of this paper is to provide a necessary theoretical base for a further more specific quantitative analysis of metric approximations of concrete discrete groups. We meticulously compare our metric profile with previously investigated quantifying functions alluded to above. Since the classes of groups we study are preserved under several group-theoretical operations such as taking subgroups, direct and free products, extensions by amenable groups, restricted wreath products, etc., we also provide the corresponding estimates on the suitable metric profiles. On the way, we collect some crucial examples and finally formulate a number of open problems.

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