ON THE CAYLEY GRAPH OF A GENERIC FINITELY PRESENTED GROUP.

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Abstract. We prove that in a certain statistical sense the Cayley graph of almost every finitely presented group with $m \geq 2$ generators contains a subdivision of the complete graph on $l \leq 2m+1$ vertices. In particular, this Cayley graph is non planar. We also show that some group constructions preserve the planarity.

1. Introduction

To any finite presentation of a group in terms of generators and defining relations there is an associated Cayley graph. This graph depends on the choice of the group generating set. So, in general, the same group has completely different Cayley graphs (from the graph theory viewpoint). In particular, it is not hard to find a group and two different sets of generators such that the Cayley graph with respect to one generating set is planar and not planar with respect to the other. As an example, take the cyclic group $\mathbb{Z}/5\mathbb{Z}$ of order five and two generating sets. The first one consisting of a single non trivial element and the other one consisting of all non trivial elements. Then with respect to the first generating set, the Cayley graph is a cycle, so it is planar, but with respect to the second one it is not, as it is the complete graph on five vertices.

On the other hand, the existence of planar Cayley graph may give an information about algebraic structure of a group. In fact, groups having planar Cayley graphs are rather scarce. In 1896, applying Cayley's method of the graphical presentation of a group, in particular, to rotation groups of the regular three- and four-dimensional bodies, Maschke classified all finite groups with planar Cayley graphs [15]. These are the finite subgroups of the special orthogonal group $SO(3)$ (i.e. cyclic, dihedral, and the rotational symmetry groups of the regular solids) and their direct products with the group of order 2. It is worth to notice that for finite graphs the way given in [15] to define the planarity is very natural and it is not ambiguous. The situation for infinite graphs is more complicated. In this case there are two nonequivalent definitions of planarity.

Definition 1.

(I) *A graph is planar without accumulation points if there is an embedding of the graph in $\mathbb{R}^2$ such that there are no accumulation points for its set of vertices.*

(II) *A graph is planar if there is an embedding of the graph in $\mathbb{R}^2$.*

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Infinite groups admitting planar Cayley graphs without accumulation points of vertices were treated by Levinson. In [14], he and Rapaport find all "special planar" presentations. "Special planar" means that the Cayley graph can be chosen point-symmetric (with the same counterclockwise succession of the edges at each vertex) and locally finite (without accumulation points of vertices). They gave also some conditions on the set of defining relations which are necessary to make the presentation special planar. In [13], Levinson produces moreover an algorithm to decide whether or not a Cayley graph of a group with solvable word problem is planar without accumulation points of vertices.

A different geometrical approach of group planarity was initiated by Poincaré [17]. It gave rise to the following question : Which groups have a planar Cayley complex ? Now there exists a complete classification of such groups. These are Fuchsian groups and free products of countably many cyclic groups [11, Prop. III.5.4], [20, Ch. 4]. Planarity of the Cayley complex implies planarity of the Cayley graph, but the converse is not true. An example of a planar group without planar Cayley complex is the free product of $\mathbb{Z}^2$ by $\mathbb{Z}$, see Figure 1.

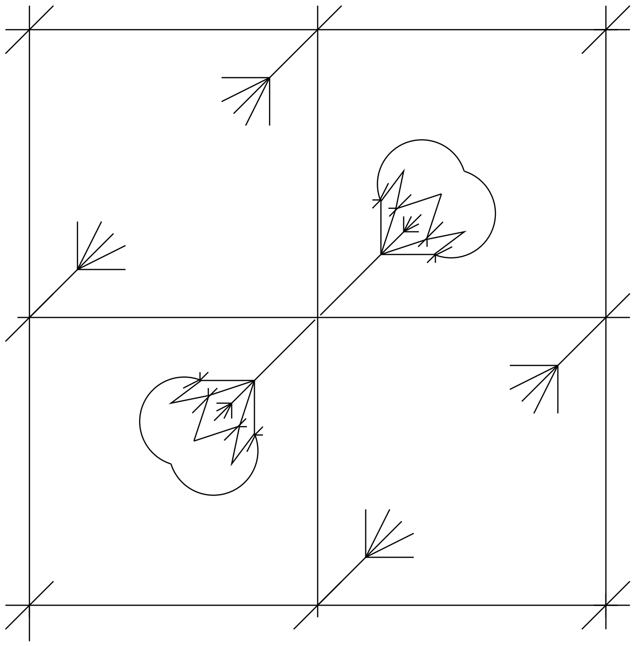

FIGURE 1. Part of ball of radius 3 in $\mathbb{Z}^2 * \mathbb{Z}$

To allow accumulation points in the definition of planarity gives more freedom, so we ask ourselves whether or not the planarity with accumulation points of vertices is frequent. The aim of the present paper is to show that for any $m \geq 2$ and $l \leq 2m+1$ one can find a subdivision (see Section 2.1 for the definition of subdivision) of the complete graph K_l (and hence of every finite graph on at most l vertices) in the Cayley graph of almost every finite presentation of a group with m generators and long enough defining relations. In particular, the Cayley graph of such a *generic* group is non planar (even with the above mentioned relaxation of the notion of planarity).

More precisely, for any fixed m and n, let $N = N(m, n, t)$ denote the number of all group presentations

$$G = \langle x_1, \ldots, x_m \mid r_1 = 1, \ldots, r_n = 1 \rangle, \tag{1}$$

where $\{r_1, \ldots, r_n\}$ are cyclically reduced words in the alphabet $X_m = \{x_1^{\pm 1}, \ldots, x_m^{\pm 1}\}$ of length $|r_i| \leq t$. Let $N_l = N_l(m, n, t)$ denote the number of all such group presentations whose Cayley graphs contain a subdivision of the complete graph K_l on l vertices.

Theorem 1. *For any $m \geq 2, n > 0$ and $l \leq 2m + 1$*

$$\lim_{t \to \infty} N_l/N = 1.$$

Moreover, there is a real number $c > 0$ depending on m and n such that

$$1 - N_l/N < \exp(-ct) \textit{ for all } t > 0.$$

Let $N_{np} = N_{np}(m, n, t)$ denote the number of all group presentations (1) with non planar Cayley graphs. The previous theorem together with Theorem 5 below imply immediately the following result.

Theorem 2. *For any $m \geq 2$ and $n > 0$*

$$\lim_{t \to \infty} N_{np}/N = 1.$$

Moreover, there is a real number $c > 0$ depending on m and n such that

$$1 - N_{np}/N < \exp(-ct) \textit{ for all } t > 0.$$

By a result of Levinson [12], the Cayley graph of an infinite finitely generated group has the genus either 0 (hence the group is planar) or the infinity. Thus the previous theorem implies that the genus of a generic finitely presented group (that is the minimum of genus of its Cayley graphs taken over all generating sets, see Section 2.1) is the infinity.

To prove Theorem 1 we use firstly that the metric small cancellation condition $C'(\lambda)$ with $\lambda > 0$ is verified for a generic (in the above defined statistical sense) finitely presented group, see, for example, [1, 2]. Then we apply the technique of small cancellation theory to find the complete graph K_l with $l \leq 2m + 1$ in the Cayley graph of a generic group presentation with m generators.

A probabilistic point of view on the notion of a generic group was first considered by Gromov [5], see also [6, 7] and [9, Problem 11.75] for an independent definition by Ol'shanskii. In [5] Gromov announced that word hyperbolicity of a finitely generated group is a generic property, in a sense slightly stronger than the definition above. A proof of this result was given by Ol'shanskii [16], see also Champetier [2] for more results in case of two defining relations. A survey on the "random" viewpoint in geometric group theory is recently presented by Ghys in [4].

The model of a generic group (via the asymptotic density) defined above is closely related to the density model developed by Gromov in [6]. This model depends on a density parameter d with $0 \leq d \leq 1$. It consists in choosing at random roughly $(2m-1)^{d\ell}$ words of a given length ℓ in the alphabet of m letters (more precisely the number of chosen words of length ℓ is between $C_1(2m-1)^{d\ell}$ and $C_2(2m-1)^{d\ell}$ for two given constants $C_1 < C_2$), then defining a group presentation on m generators where these words are defining relations, and finally letting $\ell \to \infty$. Under our model, one can assume that in a generic presentation all defining relations have almost the same length t. Indeed, the proportion of reduced words of length at most $t(1-\varepsilon)$ among all reduced words of length at most t decrease exponentially as $(2m-1)^{-t\varepsilon}$. Hence the share of corresponding group presentations is exponentially small. Thus the model we are using in the paper is the case of the density model with the density parameter $d = 0$. It is worth noticing that our main result, Theorem 1, remains true for all groups of density $d < 1/12$ as such groups are known to satisfy the small cancellation condition $C'(\lambda)$ with $\lambda < 2d$ [6, Ch. 9.B]. Hence our arguments (see our proof below) work in this case as well.

We also describe some group constructions preserving planarity.

Theorem 3.

(i) *Planarity is preserved under free products of groups.*

(ii) *For $j = 1, 2$, let G_j be a group generated by a finite set X_j containing a generator s_j of order 2, assume that the Cayley graph $C(G_i, X_i)$ is planar. Let G be the amalgamated product of G_1 and G_2 along $\{1, s_1\} = \{1, s_2\} \equiv \mathbb{Z}/2\mathbb{Z}$. Then G is a planar group. More precisely, $C(G, X_1 \cup X_2)$ is a planar graph.*

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