CONSTRUCTION OF ELEMENTS IN THE CLOSURE OF GRIGORCHUK GROUP

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Abstract. We describe constraints that need to be satisfied "near the top" of the portraits of the elements in Grigorchuk group. These constraints, if satisfied by the portraits of all sections of some binary tree automorphism, guarantee that this automotphism belongs to the closure of Grigorchuk group in the pro-finite group of binary tree automorphisms. This answers a question of Grigorchuk.

The group known as Grigorchuk group (also the first Grigorchuk group) was introduced in [Grigo–80]. More information on this remarkable group can be found in [Grigo–05] and in [Harpe–00]. Here we only introduce as much as is necessary to describe the elements in the closure $\bar{\mathcal{G}}$ of Grigorchuk group $\mathcal{G}$ in the pro-finite group $Aut(\mathcal{T})$ of binary rooted tree automorphisms. In fact, we describe constraints that need to be satisfied "near the top" of the portraits of the elements in $\mathcal{G}$ (and therefore in $\bar{\mathcal{G}}$ as well). These constraints, if satisfied by an element g in $Aut(\mathcal{T})$ at each of its sections (see below for details), guarantee that g belongs to the closure $\bar{\mathcal{G}}$. The constraints may be viewed as an effective version of the more conceptual description given by Grigorchuk in [Grigo–05].

Grigorchuk group can be viewed as a group of automorphisms of the binary rooted tree $\mathcal{T}$. The vertices of the rooted binary tree $\mathcal{T}$ are in bijective correspondence with the finite words over $X = \{0, 1\}$. The empty word $\emptyset$ is the root, the set X^n of words of length n over X constitutes level n in the tree and every vertex u at level n has two children at level $n+1$, namely $u0$ and $u1$. The group $Aut(\mathcal{T})$ of automorphisms of $\mathcal{T}$ decomposes algebraically as

$$Aut(\mathcal{T}) = (Aut(\mathcal{T}) \times Aut(\mathcal{T})) \rtimes \mathrm{S}(2), \tag{1}$$

where $\mathrm{S}(2) = \{1, \sigma\} = \{(), (01)\}$ is the symmetric group of order 2 acting on $Aut(\mathcal{T}) \times Aut(\mathcal{T})$ by permuting the coordinates. The normal subgroup $Aut(\mathcal{T}) \times Aut(\mathcal{T})$ is the stabilizer of the first level of $\mathcal{T}$. The elements in $Aut(\mathcal{T})$ of the form $g = (g_0, g_1)$ act on $\mathcal{T}$ by

$$(0w)^g = 0w^{g_0}, \qquad (1w)^g = 1w^{g_1},$$

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while the elements of the form $g = (g_0, g_1)\sigma$ act by

$$(0w)^g = 1w^{g_0}, \qquad (1w)^g = 0w^{g_1},$$

for any word w over X. The automorphisms g_0 and g_1 in the decomposition $g = (g_0, g_1)\sigma^\varepsilon$ of g, where ε is 0 or 1, are called *sections* of g at the vertices 0 and 1, respectively. This definition is recursively extended to a notion of a section of g at any vertex of $\mathcal{T}$ by declaring $g_\emptyset = g$ and $g_{ux} = (g_u)_x$, for u a word over X and x a letter in X.

Grigorchuk group $\mathcal{G}$ is, by definition, the group generated by the automorphisms a, b, c and d of $\mathcal{T}$, whose decompositions, in the sense of (1), are given by

$$\begin{aligned} a &= (1,1)\sigma, \\ b &= (a,c), \\ c &= (a,d), \\ d &= (1,b). \end{aligned} \tag{2}$$

Therefore, the action of a, b, c, and d on $\mathcal{T}$ is given by

$$\begin{aligned} (0w)^a &= 1w, & (1w)^a &= 0w, \\ (0w)^b &= 0w^a, & (1w)^b &= 1w^c, \\ (0w)^c &= 0w^a, & (1w)^c &= 1w^d, \\ (0w)^d &= 0w, & (1w)^d &= 1w^b, \end{aligned}$$

for any word w over X. It is easy to establish that

$$a^2 = b^2 = c^2 = d^2 = 1, \qquad bc = cb = d, \qquad bd = db = c, \qquad cd = dc = b.$$

These relations are called simple relations in $\mathcal{G}$. The stabilizer $Stab_{\mathcal{G}}(X)$ in $\mathcal{G}$ of level 1 in $\mathcal{T}$ is

$$Stab_{\mathcal{G}}(X) = \langle b, c, d, aba, aca, ada \rangle,$$

and the decompositions of aba, aca and ada are given by

$$\begin{aligned} aba &= (c,a), \\ aca &= (d,a), \\ ada &= (b,1). \end{aligned} \tag{3}$$

The decomposition formulae given in (2) and (3) and the simple relation $aa = 1$ are sufficient to calculate the decomposition of any element in $\mathcal{G}$. For example,

$$abdabac = aba\ ada\ b\ aca\ a = (c,a)(b,1)(a,c)(d,a)(1,1)\sigma = (cbad, aca)\sigma.$$

Of course, we could make use of the other simple relations to write either

$$abdabac = \cdots = (cbad, aca)\sigma = (dad, aca)\sigma.$$

or

$$abdabac = acabac = aca\ b\ aca\ a = (d,a)(a,c)(d,a)(1,1)\sigma = (dad, aca)\sigma,$$

but this will not be necessary for our purposes (and would, in fact, be counterproductive in the proof of one of our lemmata).

Let g be an arbitrary element in $Aut(\mathcal{T})$. The *portrait* of g is the binary rooted tree $\mathcal{T}$ with additional *decoration* on the vertices defined recursively as follows. If $g = (g_0, g_1)$ stabilizes level 1 in $\mathcal{T}$ then the portrait of g consists of the portrait of g_0 hanging below the vertex 0, the portrait of g_1 hanging below the vertex 1 and the root, which is decorated by 0. If $g = (g_0, g_1)\sigma$ does not stabilize level 1 (i.e., it is active at the root) the portrait looks the same as in the previous case, except that the root is decorated by 1. Thus, the portrait of g is the binary tree $\mathcal{T}$ with additional decoration $\alpha_u(g)$ on each vertex u, which is equal to 0 or 1 depending on whether g is active at the vertex u or not.

For every vertex xy on level 2, define

$$\beta_{xy}(g) = \alpha_{xy}(g) + \alpha_{x\bar{y}0}(g) + \alpha_{x\bar{y}1}(g),$$

where the addition is performed modulo 2 and $\bar{y}$ denotes the letter in $\{0,1\}$ different from y. When g is assumed, the notation $\alpha_u(g)$ and $\beta_u(g)$ is simplified to α_u and β_u.

Theorem 1. *For any element g in Grigorchuk group $\mathcal{G}$ the portrait decoration satisfies the following constraints.*

(i) *If $\alpha_0 = \alpha_1 = 0$ then*

$$\beta_{00} = \beta_{11} = \beta_{01} = \beta_{10}.$$

(ii) *If $\alpha_0 = 0$ and $\alpha_1 = 1$ then*

$$\beta_{00} \neq \beta_{11} = \beta_{01} = \beta_{10}.$$

(iii) *If $\alpha_0 = 1$ and $\alpha_1 = 0$ then*

$$\beta_{00} = \beta_{11} = \beta_{01} \neq \beta_{10}.$$

(iv) *If $\alpha_0 = \alpha_1 = 1$ then*

$$\beta_{00} = \beta_{11} \neq \beta_{01} = \beta_{10}.$$

We say that an automorphism g in $Aut(\mathcal{T})$ *simulates* $\mathcal{G}$ if its portrait decoration satisfies the constraints in Theorem 1. Recall that the pro-finite group $Aut(\mathcal{T})$ is a metric space with a natural metric derived from the filtration of $Aut(\mathcal{T})$ by its level stabilizers and defined as follows. The distance between two tree automorphisms g and h is

$$d(g,h) = \inf\left\{ \frac{1}{[Aut(\mathcal{T}) : Stab_{Aut(\mathcal{T})}(X^n)]} \mid g^{-1}h \in Stab_{Aut(\mathcal{T})}(X^n) \right\}.$$

In other words, if g and h agree on words of length n, but do not agree on words of length $n+1$, then the distance between g and h is $\frac{1}{2^{2^n-1}}$.

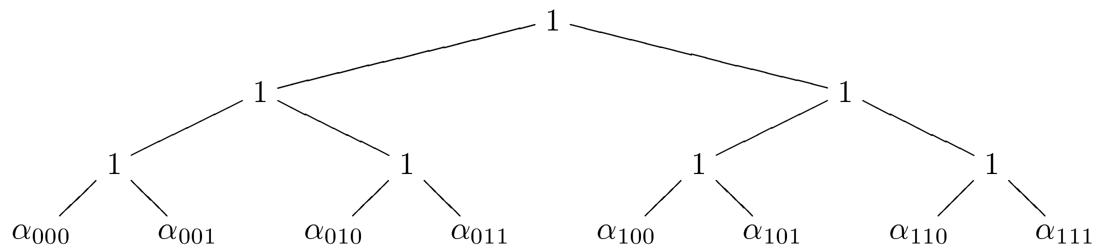
Theorem 2. *Let g be a binary tree automorphism. The following conditions are equivalent.*

(i) *g belongs to the closure $\bar{\mathcal{G}}$ of Grigorchuk group $\mathcal{G}$ in the pro-finite group $\mathrm{Aut}(\mathcal{T})$.*
(ii) *all sections of g simulate $\mathcal{G}$.*
(iii) *the distance from any section of g to $\mathcal{G}$ in the metric space $\mathrm{Aut}(\mathcal{T})$ is at most $\frac{1}{2^{15}}$.*

The proofs will follow from some combinatorial observations on the structure of words representing elements in $\mathcal{G}$. Before the proofs, we consider some examples.

Example 1. We show how Theorem 1 and Theorem 2 can be used to construct elements in the closure $\bar{\mathcal{G}}$.

The constraints in Theorem 1 imply that exactly 2^{12} different portrait decorations are possible on levels 0 through 3 for elements in $\mathcal{G}$. Indeed, assume that the portrait decoration is already freely chosen on levels 0 through 2. In particular, α_0 and α_1 are known. There are 8 vertices on level 3, but according to the constraints in Theorem 1 we may choose the decoration freely only on 5 of them. Namely, as soon as we chose the decoration for two vertices with common parent, the values of β_{00}, β_{01}, β_{10} and β_{11} are uniquely determined and we may freely choose only the decoration on one of the vertices in each of the 3 remaining pairs of vertices with common parent, while the other is forced on us. For example, let us set $\alpha_u = 1$ for all u on level 0 through level 2.



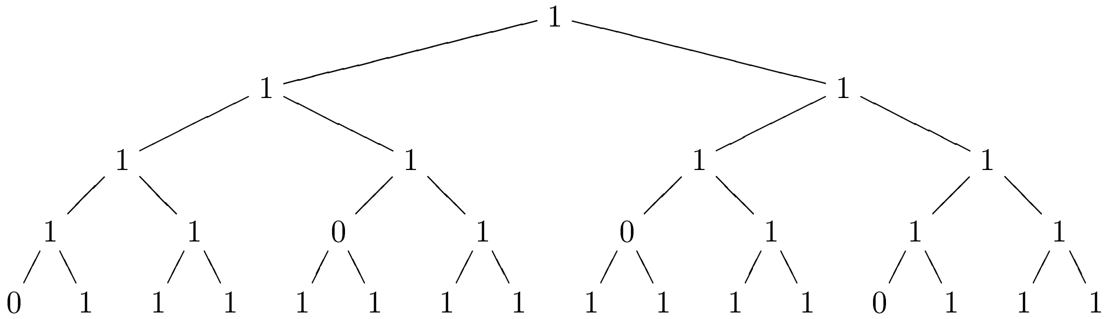
Further, for all vertices at level 3 whose label ends in 1 choose $\alpha_{u1} = 1$. Finally, choose $\alpha_{110} = 1$. At this moment, after making 12 free choices, we have that $\beta_{10} = 1$ and according to Theorem 1 we must have $\beta_{01} = 1$, $\beta_{00} = \beta_{11} = 0$. In accordance with the other choices already made on level 3, we must then have

$$\alpha_{000} = 1, \qquad \alpha_{010} = 0, \qquad \alpha_{100} = 0.$$

We may now continue building a portrait of an element in $\bar{\mathcal{G}}$ by extending (independently!) the left half and the right half of the portrait one more level by following only the constraints imposed by Theorem 1.

As a general strategy (one can certainly choose a different one, guided by any suitable purpose), we choose arbitrarily the decoration on all vertices whose label ends in 1 or in 110, and then fill in the decoration on the remaining vertices following Theorem 1.

If we extend our example one more level by decorating by 1 all vertices whose label ends in 1 or in 110, we obtain

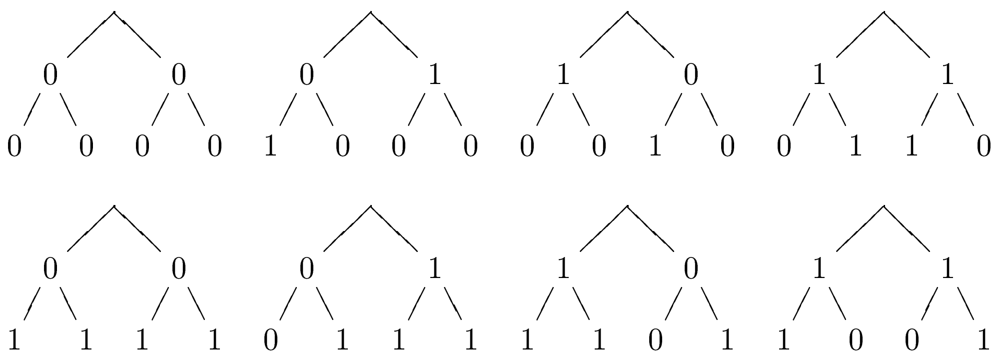


Continuing in the same fashion (choosing 1 whenever possible) we arrive at the portrait description of the element f in $\bar{\mathcal{G}}$ defined by the following decomposition formulae

$$\begin{aligned} f &= (\ell, r) \;\; \sigma, \\ \ell &= (r, m) \;\; \sigma, \\ r &= (m, r) \;\; \sigma, \\ m &= (n, f) \;\; \sigma, \\ n &= (r, m). \end{aligned}$$

There are many ways to see that f (or any section of f) does not belong to $\mathcal{G}$. Perhaps the easiest way is to observe that f is not a bounded automorphism, while all elements in $\mathcal{G}$ are bounded automorphisms of $\mathcal{T}$. Recall that, by definition, an automorphism g of $\mathcal{T}$ is *bounded* if the sum $\sum_{u \in X^n} \alpha_u(g)$ is uniformly bounded, for all n. Note that f is defined by a 5-state automaton, which is not bounded, while the automaton defining $\mathcal{G}$ is bounded. For more on groups of automorphisms generated by automata see [GrigoNS–00], and for bounded automorphisms and bounded automata see [Sidki–04].

To aid construction of elements in $\bar{\mathcal{G}}$ we provide the following table of trees indicating the 8 possibilities for the values of α_0, α_1, β_{00}, β_{01}, β_{10}, and β_{11}.



In each tree, α_0, α_1, β_{00}, β_{01}, β_{10}, and β_{11} are indicated in their respective positions. To use the table, choose values for α_0, α_1 and any one of β_{00}, β_{01}, β_{10}, and β_{11}. The unique tree in the above table that agrees with the chosen values provides the unique values for the remaining 3 parameters among β_{00}, β_{01}, β_{10}, and β_{11}. For example, if $\alpha_0 = \alpha_1 = 1$ and $\beta_{11} = 0$, the correct pattern in the table is the one in the right upper corner, indicating that $\beta_{00} = 0$, $\beta_{01} = 1$ and $\beta_{10} = 1$.

The following example provides additional ways to build elements in $\bar{\mathcal{G}}$, which does not rely on Theorem 1 and Theorem 2, but, rather, on the branch structure of $\mathcal{G}$ (see [Grigo-00] for more details). This approach does not produce all elements in $\bar{\mathcal{G}}$, but does produce some that are easy to describe.

Example 2. Infinitely many elements in $\bar{\mathcal{G}}$ that are not in $\mathcal{G}$ are contained in the following isomorphic copy of $K = [a, b]^{\mathcal{G}} \leq \mathcal{G}$ (recall that $\mathcal{G}$ is a regular branch group over the normal closure K of $[a, b]$ in $\mathcal{G}$; see [Grigo–00] or [Harpe–00] for details). For each element $k \in K$ define an element $\bar{k}$ in $\bar{\mathcal{G}}$ by

$$\bar{k} = (k, \bar{k}).$$

The group $\bar{K} = \{\bar{k} \mid k \in K\}$ is canonically isomorphic to K, but the intersection $\mathcal{G} \cap \bar{K}$ is trivial.

More generally, an easy way to construct some (certainly not all) elements in $\bar{\mathcal{G}}$ is to choose an infinite set of independent vertices V in $\mathcal{T}$ (no vertex is below some other vertex) and associate to each such vertex an arbitrary element of K. The automorphism of $\mathcal{T}$ that is inactive at every vertex that does not have a prefix in V and whose sections at the vertices of V are the assigned elements from K is an element of $\bar{\mathcal{G}}$.

We now prove Theorem 1 and Theorem 2.

Let W be a word over $\{a, b, c, d\}$. The letters in $B = \{b, c, d\}$ are called B-letters. For a subset C of B the letters in C are called C-letters. Denote by $N_C(W)$ the number of C-letters occurring in W. An occurrence of a B-letter ℓ is called even or odd depending on whether an even or odd number of a's appear before ℓ in W. For a parity $p \in \{0, 1\}$ and a subset C of B, denote by $N^p_C(W)$ the number of C-letters of parity p in W. For parities $p, q \in \{0, 1\}$, denote by $N^{p,q}_{b,c}(W)$ the number of $\{b, c\}$-letters ℓ of parity p in W such that the number of $\{b, c\}$-letters of parity $\bar{p}$ that appear before ℓ in W has parity q (here parity $\bar{p}$ denotes the parity different from p). For example,

$$N^1_{b,c}(a\mathbf{bc}aadabdbcad\mathbf{cb}d\mathbf{b}abdbc) = 5,$$
$$N^{1,1}_{b,c}(abcaadabdbcad\mathbf{cb}d\mathbf{b}abdbc) = 3,$$
$$N^{1,0}_{b,c}(a\mathbf{bc}aadabdbcadcbdbabdbc) = 2,$$

where, in all examples, the letters that are counted are indicated in boldface. When W is assumed, the notation $N^p_C(W)$ and $N^{p,q}_{b,c}(W)$ is simplified to N^p_C and $N^{p,q}_{b,c}$.

Lemma 1. *For any word W over $\{a, b, c, d\}$ representing an element g in $\mathcal{G}$*

$$N^{1,0}_{b,c} = \beta_{00}, \qquad N^{0,1}_{b,c} = \beta_{11}, \qquad N^{1,1}_{b,c} = \beta_{01}, \qquad N^{0,0}_{b,c} = \beta_{10},$$

where all equalities are taken modulo 2.

Proof. Let the words W_u, for u a word over X of length at most 3, represent the sections of g at the corresponding vertices, and let these words be obtained by decomposition from

W, without applying any simple relations other than $aa = 1$ (i.e., no relations involving B-letters are applied).

We have (modulo 2)

$$\begin{aligned}\beta_{00} &= \alpha_{00} + \alpha_{010} + \alpha_{011} = \alpha_{00} + N_{b,c}(W_{01}) = \\ &= N^0_{b,c}(W_0) + N^0_{b,d}(W_0) = N^0_{c,d}(W_0) = N^{1,0}_{b,c}(W).\end{aligned}$$

The other equalities are obtained in an analogous fashion.

Lemma 2. *Let W be a word over $\{a, b, c, d\}$. Modulo 2 we have*

(i) *if $N^0_{b,c} = 0$, then $N^{1,1}_{b,c} = N^{0,1}_{b,c} = N^{0,0}_{b,c}$.*
(ii) *if $N^0_{b,c} = 1$, then $N^{1,0}_{b,c} = N^{0,1}_{b,c} \neq N^{0,0}_{b,c}$.*
(iii) *if $N^1_{b,c} = 0$, then $N^{0,1}_{b,c} = N^{1,1}_{b,c} = N^{1,0}_{b,c}$.*
(iv) *if $N^1_{b,c} = 1$, then $N^{0,0}_{b,c} = N^{1,1}_{b,c} \neq N^{1,0}_{b,c}$.*

Proof. (i) Assume $N^0_{b,c}$ is even.

The structure of the word W may be represented schematically by

$$W = n_1\ \ell_1\ n'_1\ \ell_2\ n_2\ \ell_3\ n'_2\ \dots\ n_k\ \ell_{2k-1}\ n'_k\ \ell_{2k}\ n_{k+1},$$

where ℓ_i, $i = 1, \dots, 2k$ represent all the even occurrences of $\{b, c\}$-letters in W and the numbers n_i, n'_i represent the number of odd occurrences of $\{b, c\}$-letters between the consecutive even occurrences of $\{b, c\}$-letters.

Then (modulo 2)

$$N^{1,1}_{b,c} = \sum_{i=1}^{k} n'_i = |\{i \mid 1 \le i \le k,\ n'_i \text{ is odd }\}| = N^{0,1}_{b,c}.$$

Indeed, for $i = 1, \dots, k$, the n'_i odd occurrences of $\{b, c\}$-letters between ℓ_{2i-1} and ℓ_{2i} are preceded by an odd number (exactly $2i - 1$) of even occurrences of $\{b, c\}$-letters. Thus $N^{1,1}_{b,c} = \sum_{i=1}^{k} n'_i$. On the other hand, for $i = 1, \dots, k$, whenever n'_i is odd exactly one of ℓ_{2i-1} and ℓ_{2i} is preceded by an odd number of odd occurrences of $\{b, c\}$-letters, while whenever n'_i is even either both or none of ℓ_{2i-1} and ℓ_{2i} are preceded by an odd number of odd occurrences of $\{b, c\}$-letters. Thus $|\{i \mid 1 \le i \le k,\ n'_i \text{ is odd }\}| = N^{0,1}_{b,c}$ modulo 2.

Since $N^{0,0}_{b,c} + N^{0,1}_{b,c} = N^0_{b,c}$ is even, we clearly have $N^{0,0}_{b,c} = N^{0,1}_{b,c}$, modulo 2.

(ii) Assume $N^0_{b,c}$ is odd.

The structure of the word W may be represented schematically by

$$W = n_1\ \ell_1\ n'_1\ \ell_2\ n_2\ \ell_3\ n'_2\ \dots\ n'_{k-1}\ \ell_{2k-2}\ n_k\ \ell_{2k-1}\ n'_k.$$

where ℓ_i, $i = 1, \dots, 2k - 1$ represent all the even occurrences of $\{b, c\}$-letters in W and the numbers n_i, n'_i represent the number of odd occurrences of $\{b, c\}$-letters between the consecutive even occurrences of $\{b, c\}$-letters.

Then (modulo 2)

$$N_{b,c}^{1,0} = \sum_{i=1}^{k} n_i = |\{i \mid 1 \le i \le k,\ n_i \text{ is odd }\}| = N_{b,c}^{0,1}.$$

Indeed, for $i = 2, \dots, k$, the n_i odd occurrences of $\{b, c\}$-letters between ℓ_{2i-2} and ℓ_{2i-1} are preceded by an even number (exactly $2i-2$) of even occurrences of $\{b, c\}$-letters. In addition, the n_1 odd occurrences of $\{b, c\}$-letters from the beginning of W are preceded by no even occurrences of $\{b, c\}$-letters. Thus $N_{b,c}^{1,0} = \sum_{i=1}^{k} n_i$. On the other hand, for $i = 2, \dots, k$, whenever n_i is odd exactly one of ℓ_{2i-2} and ℓ_{2i-1} is preceded by an odd number of odd occurrences of $\{b, c\}$-letters, while whenever n_i is even either both or none of ℓ_{2i-2} and ℓ_{2i-1} are preceded by an odd number of odd occurrences of $\{b, c\}$-letters. In addition, whether ℓ_1 is preceded by an even or odd number of odd occurrences of $\{b, c\}$-letters depends on the parity of n_1. Thus $|\{i \mid 1 \le i \le k,\ n_i \text{ is odd }\}| = N_{b,c}^{0,1}$ modulo 2.

Since $N_{b,c}^{0,0} + N_{b,c}^{0,1} = N_{b,c}^{0}$ is odd, we clearly have $N_{b,c}^{0,0} \neq N_{b,c}^{0,1}$, modulo 2.

(iii) and (iv) are analogous to (i) and (ii).

Proof of Theorem 1. Follows directly from Lemma 1, Lemma 2, and the observations $\alpha_0 = N_{b,c}^{0}$ and $\alpha_1 = N_{b,c}^{1}$ modulo 2.

Proof of Theorem 2. We use the following (modification of the) description of the elements in $\bar{\mathcal{G}}$ provided in [grigorchuk:unsolved]. A binary tree automorphism g belongs to $\bar{\mathcal{G}}$ if and only, for each section g_u of g, the portrait of g_u agrees with the portrait of some element in $\mathcal{G}$ up to and including level 3.

(i) is equivalent to (iii). Portraits of two automorphisms agree at least up to level 3 if and only if their actions on the tree agree at least up to level 4, which, in turn, is equivalent to the condition that the distance between the two automorphisms is at most $\frac{1}{2^{2^4-1}} = \frac{1}{2^{15}}$.

(i) implies (ii). If g is in $\bar{\mathcal{G}}$, then the portrait of each section g_u of g agrees with the portrait of some element in $\mathcal{G}$ up to and including level 3. The portrait decorations of the elements in $\mathcal{G}$ must satisfy the constraints in Theorem 1, and therefore each section g_u simulates $\mathcal{G}$.

(ii) implies (i). It is known that $[\mathcal{G} : Stab_{\mathcal{G}}(X^4)] = 2^{12}$. Thus, for elements in $\mathcal{G}$, there are exactly 2^{12} possible portrait decorations on level 0 through 3 . The constraints of Theorem 1 provide for exactly 2^{12} different decorations of the appropriate size (see the discussion in Example 1). Thus if a tree automorphism simulates $\mathcal{G}$, then its portrait agrees with the portrait of an actual element in $\mathcal{G}$ up to and including level 3.

As another application, we offer a proof of the following result, obtained by Grigorchuk in [Grigo–00].

Theorem 3. *The Hausdorff dimension of $\bar{\mathcal{G}}$ in $Aut(\mathcal{T})$ is $\frac{5}{8}$.*

Proof. It is known that the Hausdorff dimension can be calculated as the limit

$$\liminf_{n\to\infty} \frac{\log[\bar{\mathcal{G}} : Stab_{\bar{\mathcal{G}}}(X^n)]}{\log[Aut(\mathcal{T}) : Stab_{Aut(\mathcal{T})}(X^n)]},$$

comparing the relative sizes of the level stabilizers of $\bar{\mathcal{G}}$ and $Aut(\mathcal{T})$ (see [BarSh–97]). Applying the strategy of construction of elements in $\bar{\mathcal{G}}$ indicated in Example 1, it follows that, in the portrait of an element g in $\bar{\mathcal{G}}$, 5 out of 8 vertices at level 3 and below can have any decoration we choose (0 or 1) and the other three have uniquely determined decoration. Thus, the limit determining the Hausdorff dimension of $\bar{\mathcal{G}}$ is $\frac{5}{8}$.

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