ON QUASICONVEX SUBGROUPS OF WORD HYPERBOLIC GROUPS

G. N. ARZHANTSEVA

Abstract. We prove that a quasiconvex subgroup H of infinite index of a torsion free word hyperbolic group can be embedded in a larger quasiconvex subgroup which is the free product of H and an infinite cyclic group. Some properties of quasiconvex subgroups of word hyperbolic group are also discussed.

1. Introduction

Word hyperbolic groups were introduced by M. Gromov as a geometric generalization of certain properties of discrete groups of isometries of hyperbolic spaces $\mathbb{H}^n$. Finite groups, finitely generated free groups, classical small cancellation groups and groups acting discretely and cocompactly on hyperbolic spaces are basic examples of word hyperbolic groups. Any word hyperbolic group is finitely presented. Finite extensions and free products of finitely many word hyperbolic groups are also word hyperbolic. A large number of results on word hyperbolic groups as well as conjectures and research problems are contained in the original article [9].

In this paper, we study properties of *quasiconvex* subgroups of word hyperbolic groups (see the next section for the definition). Our main result gives in fact a method for constructing quasiconvex subgroups of word hyperbolic groups.

Theorem 1. *Let G be a non-elementary torsion-free word hyperbolic group and H be a quasiconvex subgroup of G of infinite index. Then there exists an element $g \in G$ such that the subgroup $sgp\langle H, g\rangle$ generated by H and g is the free product $H * \langle g\rangle$ and is quasiconvex in G.*

The statement of the theorem was formulated by M.Gromov in [9, 5.3.C], with a very general sketch of a proof. We follow in part Gromov's approach.

There are two parts in our proof of Theorem 1: first we find an element $g \in G$ such that the subgroup $sgp\langle H, g\rangle$ is a free product, and then we prove that this subgroup is quasiconvex in G. For the first part, we choose a double coset HxH whose shortest representative x is sufficiently long, as a word in the generators of G. This is possible as we prove that the number of double cosets of a word hyperbolic group G modulo a quasiconvex subgroup H of infinite index is also infinite (Proposition 1). For g, we take x^M for M large enough.

The fact that, for the chosen g, the subgroup $sgp\langle H, g\rangle$ is quasiconvex is not trivial even if we know that it decomposes into the free product $H * \langle g\rangle$ of a quasiconvex subgroup H and a cyclic subgroup $\langle g\rangle$ and any cyclic subgroup of a word hyperbolic group is quasiconvex. In general, a subgroup of a word hyperbolic group, which is

1991 *Mathematics Subject Classification.* 20F32, 20E07, 20E06.

Key words and phrases. Word hyperbolic groups, quasiconvex subgroups, commensurator.

The work has been supported in part by the Swiss National Science Foundation.

a free product of two quasiconvex, even cyclic, subgroups need not be quasiconvex. For example, let $G = \langle a, t \mid at^{-1}ata^2t^{-2}a^{-1}t^2 = 1\rangle$ and let M be the so-called Moldavansky subgroup, that is, $M = sgp\langle a, t^{-1}at, t^{-2}at^2\rangle$. It is known [10] that G is a torsion-free non-elementary word hyperbolic group and M is a non-quasiconvex free subgroup of rank 2.

Note that under the assumptions of Theorem 1, we can construct an infinite sequence $H = F_0 < F_1 < \cdots < F_k < \cdots$ of subgroups of G starting with H where F_i is the free product of H and a free group of rank i. To do this, we only need to notice that the subgroup $H * \langle g\rangle$ in Theorem 1 will have infinite index in G if we replace g by any proper power of it. In particular, taking $H = 1$ we get an ascending sequence of quasiconvex free subgroups of G of ascending rank. As an immediate consequence of Theorem 1 we get also that, for given G and H, there are infinitely many elements g satisfying the conclusion of the theorem.

Theorem 1 is proved in Sections 1–4. In Section 5, we give a short proof of the result due to I.Kapovich and H.Short that an infinite quasiconvex subgroup of a word hyperbolic group has a finite index in its commensurator. We give also some corollaries to this result.

Acknowledgments. I am grateful to my supervisor A. Yu. Ol'shanskii for suggesting the problem and for helpful discussions. I thank also to P. de la Harpe, I. G. Lysionok, and the referee for useful comments.

References

[1] G. Baumslag, C.F. Miller III and H. Short, *Unsolvable problems about small cancellation and word hyperbolic groups*, Bull. London Math Soc. **26**(1) (1994), 97-101.

[2] N. Bourbaki, *Éléments de mathématique. Groupes et algèbres de Lie. Chapitres 4, 5 et 6.*, Masson, Paris, 1981.

[3] N. Brady, *Branched covering of cubical complexes and subgroups of hyperbolic groups*, J. London Math. Soc. (1999), to appear.
University of Utah, 1995, preprint.

[4] M. Coornaert, T. Delzant, A. Papadopoulos, *Notes sur les groupes hyperboliques de Gromov,* Lecture Notes in Math. **1441**, Springer, 1990.

[5] L. Corwin, *Induced representations of discrete groups,* Proc. Amer. Math. Soc. **47** (1975), 279–287.

[6] E. Ghys and P. de la Harpe (editors), *Sur les groupes hyperboliques d'après Mikhael Gromov*, Progress in Math., **83**, Birkhäuser, Boston, Ma., 1990.

[7] R. Gitik, M. Mitra, E. Rips and M. Sageev, *Widths of subgroups*, Trans. Amer. Math. Soc., **350**(1) (1998), 321-329.

[8] L. Greenberg,*Commensurable groups of Moebius transformations*, Discontinuous groups and Riemann surfaces (Proc. Conf., Univ. Maryland, College Park, Md., 1973), Ann. of Math. Studies, **79**, Princeton Univ. Press, 1974, 227-237.

[9] M. Gromov, *Hyperbolic groups*, in: Essays in group theory, ed. S.M. Gersten, M.S.R.I. Publ. **8**, Springer, 1987, 75-263.

[10] I. Kapovich, *Howson property and one-relator groups*, Communications in algebra, **27**(3) (1999), 1057-1072.

[11] I. Kapovich and H. Short, *Greenberg's theorem for quasiconvex subgroups of word hyperbolic groups*, Can. J. Math. **48**(6) (1996), 1224-1244.

[12] G. W. Mackey, *The theory of unitary group presentations*, The University of Chicago Press, 1976.

[13] G. A. Margulis, *Arithmetic properties of discrete subgroups*, Russian Math.Surveys **29**(1) (1974), 107-156.

[14] M. L. Mihalik, W. Towle, *Quasiconvex subgroups of negatively curved groups*, J. Pure and Applied Algebra, **95**(1994), 297-301.

[15] A. Yu. Ol'shanskii, *Periodic quotients of hyperbolic groups,* Mat. Zbornik, **182(4)**(1991), 543-567 (in Russian), English translation in Math. USSR Sbornik **72**(2) (1992).
[16] A. Yu. Ol'shanskii, *On residualing homomorphisms and G -subgroups of hyperbolic groups,* Int. J. Algebra and Comput. **3**(4) (1993), 365-409.
[17] E. Rips, *Subgroups of small cansellation groups*, Bull. London Math Soc. **14**(1982), 45-47.
[18] H. B. Short, *Quasiconvexity and a Theorem of Howson's*, Group theory from a Geometric Viewpoint, Proc. ICTP Trieste, World Scientific, Singapore, 1991, 168-176.
[19] E. Swenson, *Limit Sets in the Boundary of Negatively Curved Groups,* 1994, preprint.

Section de Mathématiques, Université de Genève, CP 240, 1211 Genève 24, Switzerland

E-mail address: `Goulnara.Arjantseva@math.unige.ch`