A PROPERTY OF SUBGROUPS OF INFINITE INDEX IN A FREE GROUP

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Abstract. We prove that if H is a finitely generated subgroup of infinite index in a free group F_m, then, in a certain statistical meaning, the normal subgroup generated by "randomly" chosen elements $r_1, \ldots, r_n$ of F_m has trivial intersection with H.

1. Introduction

Let F_m be a free group with free generators $x_1, \ldots, x_m$ and H a finitely generated subgroup of F_m. It is known [3] that if H contains a non-trivial normal subgroup of F_m then H has finite index in F_m. Karrass and Solitar proved in [4] that if H has non-trivial intersection with every non-trivial normal subgroup of F_m, then H has finite index in F_m. This is a stronger result, since any two non-trivial normal subgroups of a free group have non-trivial intersection. On the other hand, it is obvious that if H has finite index in F_m then H has non-trivial intersection with each of the non-trivial subgroups of F_m. Thus if H is a finitely generated subgroup of F_m then H is of infinite index if and only if there is a normal subgroup K of F_m such that $K \cap H = \{1\}$. In the present paper, we study this property of subgroups of free groups from a statistical point of view. We prove that if H is a finitely generated subgroup of F_m of infinite index, then a randomly chosen normal subgroup K of F_m has trivial intersection with H with the probability tending to 1 as the lengths of the elements whose normal closure is K tend to infinity. In other words, this gives a "generic property" of normal subgroups of a free group: for a fixed H, a generic normal subgroup of F_m trivially intersects with H.

For a precise formulation, let $N = N(m, n, t)$ denote the number of all n-tuples $\{r_1, \ldots, r_n\}$ of cyclically reduced words in the alphabet $X_m = \{x_1^{\pm 1}, \ldots, x_m^{\pm 1}\}$ of length $|r_i| \leq t$ and let $N_H = N_H(m, n, t)$ denote the number of all such n-tuples $\{r_1, \ldots, r_n\}$ whose normal closure in F_m has trivial intersection with H. Our main result is the following theorem.

Theorem 1. *For any fixed n and m, if H is a finitely generated subgroup of infinite index in F_m, then*

$$\lim_{t \to \infty} N_H/N = 1.$$

Moreover, there is a real number $c > 0$ depending on m, n and H such that $1 - N_H/N < \exp(-ct)$ for all $t > 0$.

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Note that the normal closure of the elements $r_1, \ldots, r_n$ depends only on their conjugacy classes. So the requirement for r_i to be cyclically reduced is not restrictive in this sense.

In the proof of Theorem 1, we will use presentations of finitely generated subgroups of F_m by finite connected labelled graphs. Such presentations have been introduced by J. Stallings and have been used to study subgroups of free groups, free inverse monoids, and "statistical" properties of finitely presented groups (for details see [7], [6] and [1]).

Using Theorem 1 we get a generic property of finitely presented groups. Recall the definition of a generic property. For any $m \geq 2$ and $n \geq 1$ we consider all group presentations

$$G = \langle x_1, \ldots, x_m \mid r_1 = 1, \ldots, r_n = 1 \rangle, \tag{1}$$

where $r_1, \ldots, r_n$ are cyclically reduced words in the alphabet X_m of length $|r_i| \leq t$. Note that the number of all such presentations (1) is exactly $N = N(m, n, t)$. Let $N_{\mathcal{P}}$ be the number of presentations (1) with the group G having a property $\mathcal{P}$. A property $\mathcal{P}$ of m-generated groups is said to be *generic* if $\lim_{t \to \infty} N_{\mathcal{P}}/N = 1$ for any n. We say that the genericity is exponential if this ratio tends to 1 faster than some function $1 - \exp(-ct)$ with $c > 0$. In this paper, we will consider exponential genericity only. For more information and examples of generic properties see [1], [2].

Theorem 2. *Let $h_1, h_2, \ldots, h_k$ be elements of the free group F_m and $r \geq 1$. Then the following conditions are equivalent:*

(i) *the elements $h_1, h_2, \ldots, h_k$ generate a free subgroup of rank r of infinite index in F_m;*

(ii) *for a generic group G, the images of the elements h_i under the natural epimorphism $F_m \to G$ generate a free subgroup of rank r in G;*

(iii) *for a generic group G, the images of the elements h_i under the natural epimorphism $F_m \to G$ generate a free subgroup of rank r of infinite index in G.*

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