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SOLVABLE GROUPS WITH POLYNOMIAL DEHN FUNCTIONS.

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ABSTRACT. Given a finitely presented group H, finitely generated subgroup B of H, and a monomorphism $\psi : B \to H$, we obtain an upper bound of the Dehn function of the corresponding HNN-extension $G = \langle H, t \mid t^{-1}Bt = \psi(B)\rangle$ in terms of the Dehn function of H and the distortion of B in G. Using such a bound, we construct first examples of non-polycyclic solvable groups with polynomial Dehn functions. The constructed groups are metabelian and contain the solvable Baumslag-Solitar groups. In particular, this answers a question posed in [6].

1. INTRODUCTION

Recall that the Denh function of a finitely presented group $G = \langle \mathcal{Y} \mid \mathcal{R}\rangle$ is the smallest function $\delta : \mathbb{N} \to \mathbb{N}$ with the following property. For any word w over $\mathcal{Y} \cup \mathcal{Y}^{-1}$ representing the identity in G, there exists a van Kampen diagram over the presentation of G with the number of 2–cells at most $\delta(|w|)$, where $|w|$ denotes the length of w. As is well–known, the asymptotic behavior of this function is essentially independent of the choice of the finite presentation of G. The extensive study of Dehn functions is inspired by a close connection with algorithmic problems and algebraic properties of groups. For more information and backgrounds see, for example, [19], [16], and [6].

The aim of the present paper is to investigate the Dehn functions of finitely presented HNN-extensions and construct non–polycyclic solvable groups with polynomial Dehn functions. In this section we formulate and discuss the main results of the paper and refer the reader to Section 2 for precise definitions. To formulate our results, we need an auxiliary notion. A function $f : \mathbb{N} \to \mathbb{N}$ is said to be *superadditive* if $f(a + b) \geq f(a) + f(b)$ for any $a, b \in \mathbb{N}$. Given an arbitrary function $f : \mathbb{N} \to \mathbb{N}$, we put

$$\tilde{f}(n) = \max_{i=1,\ldots,n} \left(\max_{a_1+\cdots+a_i=n,\ a_i\in\mathbb{N}} (f(a_1) + \cdots + f(a_i)) \right) \tag{1.1}$$

In fact, $\tilde{f}$ is the smallest superadditive function such that $\tilde{f}(n) \geq f(n)$ for all n. For convenience, we will assume all Dehn functions under consideration to be at least linear. Besides, for any two elements a, t of a group G, we will denote by a^t the element $t^{-1}at$ and by $[a, t]$ the commutator $a^{-1}t^{-1}at$.

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The following theorem plays an auxiliary role in our paper although it can be of independent interest.

Theorem 1.1. *Let $H = \langle \mathcal{X} \mid \mathcal{R} \rangle$ be a finitely presented group, $B = gp\langle b_1, \ldots, b_s \rangle$ be a finitely generated subgroup of H with $\{b_1, \ldots, b_s\} \in \mathcal{X}$, and $\psi : B \to H$ be a monomorphism. Let G be the corresponding HNN-extension of H with stable letter t, i.e.,*

$$G = \langle H, t \mid b_i^t = \psi(b_i),\ i = \overline{1, s} \rangle. \tag{1.2}$$

Then the Dehn function of G satisfies

$$\delta_G(n) \preccurlyeq n\delta_H(\tilde{\Delta}_B^G(n)), \tag{1.3}$$

where $\Delta_B^G(n)$ is the distortion function of B in G.

In particular, Theorem 1.1 implies

Corollary 1.2. *Under the hypothesis of Theorem 1.1, if the natural inclusion $B \to G$ is a quasi–isometry then*

$$\delta_G(n) \preccurlyeq n\delta_H(n).$$

Let us mention some known results in this direction. In [8], Brick investigated the Dehn functions of HNN-extensions with finite associated subgroups. Bridson and Gersten studied the Dehn function of abelian–by–cyclic groups in [9]. In the case of trivial HNN-extensions, that is

$$G = \langle H, t \mid b_i^t = b_i,\ i = \overline{1, s} \rangle, \tag{1.4}$$

the upper bound of the Dehn function of G similar to (1.3) can be found in [11, Ch. III.Γ, Proposition 6.20]. Moreover, in this particular case one can give a lower estimate of the Dehn function of G in the following way [10]

$$\max\{\delta_H(n), n\Delta_B^H(n)\} \preccurlyeq \delta_G(n). \tag{1.5}$$

The Dehn functions of certain HNN-extensions and amalgamated products of word hyperbolic groups when associated (amalgamated) subgroups satisfy geometric conditions were studied by Kharlampovich and Myasnikov in [23], see also [4].

In a general setting the Dehn functions of amalgamated products and HNN-extensions were investigated by Bernasconi in [3]. However, the results from [3] are not convenient for our goal. We would like to mention that in terms of Theorem 1.1, Bernasconi shown the following estimate ([3], Corollary 5).

$$\delta_G(n) \preceq n\delta_H(\lambda^{(n)}(n)), \tag{1.6}$$

where $\lambda(n) \sim \max\{\Delta_B^H(n), \Delta_{\psi(B)}^H(n)\} + n$ and $\lambda^{(k)}(n)$ is inductively defined by $\lambda^{(1)}(n) = \lambda(n)$, $\lambda^{(i+1)}(n) = \lambda(\lambda^{(i)}(n))$. This estimate was obtained using the algorithm of reducing a word to a normal form in the HNN-extension and it is useful in a number of cases; on the other hand, in the best case when the embedding of B in H is a quasi-isometry, we obtain the exponetial-type bound $\delta_G(n) \preceq n\delta_H(2^n)$. Thus one can not use this result to construct groups with polynomial Dehn functions. This is the reason why we need Corollary 1.2. Notice that, in contrast to the above–mentioned result of Bernasconi, Theorem 1.1 can not be proved by using the usual algorithm for reducing a word to a normal form.

Now we are going to discuss the possibility of the lower bound for the Dehn functions of HNN-extensions. In case of nontrivial HNN-extensions it can turn out that $\delta_G(n)$ is much smaller than $\delta_H(n)$. In particular, Brady [7] gave an example of a hyperbolic group $G = H \rtimes_\psi \mathbb{Z}$ where H is finitely presented but not of type FP_3. Since any word hyperbolic group is of type FP_∞, H is not word hyperbolic and, in particular, $\delta_H(n) \succeq n^2$ although $\delta_G(n) \preccurlyeq n$ [18], [28]. Moreover, in general, it is impossible to give a lower bound for $\delta_G(n)$ in terms of the distortion of B. It will be shown in Section 3 that there is an HNN-extension presented by (1.2) such that G is word hyperbolic but $\Delta_B^H(n)$ growths faster than any recursive function. However, the lower bound can be obtained in some nontrivial partial cases [17].

The following theorem shows how one can bound the Dehn function of the HNN-extension given by (1.2) having no information about the distortion of B in G (or B in H).

Theorem 1.3. *Let $H = \langle \mathcal{X} \mid \mathcal{R}\rangle$ be a finitely presented group, $B = gp\langle b_1, \dots, b_s\rangle$ be a finitely generated normal subgroup of H, and $\psi : B \to B$ be an automorphism. Let G be the corresponding HNN-extension defined by (1.2). Then the Dehn function of G satisfies*

$$\delta_G(n) \preccurlyeq \delta_H(2^{\tilde{\delta}_{H/B}(n)}).$$

In the present paper, using Corollary 1.2, we construct first examples of finitely presented metabelian groups with polynomial Dehn functions. Note that all finitely presented solvable groups that are known to have polynomial Dehn functions are nilpotent [19, §5], [32] or polycyclic [12], [13]. The following theorem provides a source of non–polycyclic examples of solvable groups with polynomial isoperimetric function. We recall that the Baumslag–Solitar groups are given by the presentations

$$BS(p,q) = \langle a, t, \mid (a^p)^t = a^q\rangle, \quad p, q \in \mathbb{N}. \tag{1.7}$$

It is well–known that for all $q > 1$ the group $BS(1,q)$ is metabelian, non–polycyclic, and has the exponential Dehn function [16].

Theorem 1.4. *For any integer $q > 1$, there exists a finitely presented metabelian group $G(q)$ such that the following conditions hold.*

(i) *$BS(1,q)$ can be quasi-isometrically embedded into $G(q)$;*
(ii) *$G(q)$ has at most cubic Dehn function.*

First examples of finitely presented groups $H(q)$ with the Dehn functions $\delta_{H(q)}(n) \simeq n^{10}$ containing $BS(1,q)$ were obtained by Ol'shanskii and Sapir in [30]. However, the groups constructed in [30] are very distant from solvable ones. To prove Theorem 1.4 we first give examples of polycyclic groups with quadratic Dehn functions using some results of Drutu [12], [13]. Then we produce an HNN-extension of such a group that contains $BS(1,q)$ and use Corollary 1.2 to estimate the Dehn function of the obtained group. It is worth to notice also that the problem to find a group $G(q)$ such as in Theorem 1.4 was suggested in [6]. This question goes up to the problem of simulating of Turing mashines in groups with a good control of Dehn functions using ideas of the Novikov-Boone construction [35]. We refer the interested reader to the discussion in [31, Section 2.2].

In connection with Theorem 1.4, we recall that the word problem of a group G is in NP (solvable in polynomial time by a non–deterministic Turing machine) if and only if G can be (quasi-isometrically) embedded into a finitely presented group with polynomial Dehn function [6]. It is known that any finitely generated metabelian group can be represented by matrices over a finite product of fields [24],[33],[37]. It follows that the word problem is in NP for any finitely generated metabelian group. On the other hand, any finitely generated metabelian group admits an embedding into a finitely presented one [2]. Thus it is naturally to ask

Question. *Is any finitely generated metabelian group embeddable into a finitely presented metabelian group with polynomial Dehn function ?*

Theorem 1.4 provides a nontrivial reason to expect an affirmative answer to the above question.

The paper is organized as follows. In the next section we collect all necessary definitions and notation. Theorem 1.1 and Theorem 1.3 will be proved in Section 3. Finally, the construction of groups $G(q)$ and the proof of Theorem 1.4 will be given in Section 4.

discussions and explanations concerning the construction of polycyclic groups with quadratic Dehn functions. We thank also to the referee who paid our attention to A. Bernasconi's work.

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