UNIFORM NON-AMENABILITY

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Abstract. For any finitely generated group G an invariant $\mathrm{Føl}\, G \geq 0$ is introduced which measures the "amount of non-amenability" of G. If G is amenable, then $\mathrm{Føl}\, G = 0$. If $\mathrm{Føl}\, G > 0$, we call G *uniformly non-amenable*. We study the basic properties of this invariant; for example, its behaviour when passing to subgroups and quotients of G. We prove that the following classes of groups are uniformly non-amenable: non-abelian free groups, non-elementary word-hyperbolic groups, large groups, free Burnside groups of large enough odd exponent, and groups acting acylindrically on a tree. Uniform non-amenability implies uniform exponential growth. We also exhibit a family of non-amenable groups (in particular including all non-solvable Baumslag-Solitar groups) which are not uniformly non-amenable, that is, they satisfy $\mathrm{Føl}\, G = 0$. Finally, we derive a relation between our uniform Følner constant and the uniform Kazhdan constant with respect to the left regular representation of G.

Introduction

Amenability is a fundamental concept with many apparently unrelated but logically equivalent formulations in different branches of mathematics, such as measure theory, representation theory, geometry, and algebra. Following the work of Følner, the geometric notion of amenability can be paraphrased as follows:

A space is amenable if it can be exhausted by a family of sets A_n of finite volume, with boundaries $\mathrm{Bdy} A_n$, also of finite volume, such that the ratio Volume $(\mathrm{Bdy} A_n)$/Volume (A_n) tends to 0 as $n \to \infty$.

For a finitely generated group G the volume of a subset A of G is simply set to be its cardinality $\#A$. The boundary ∂A of A can be defined in several different manners, each of them usually dependent on the choice of a finite generating set X of G. In this paper, we use

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the *inner boundary* of A (we will simply say "boundary")

$$\partial_X A = \{a \in A \mid ax \notin A \text{ for some } x \in X^{\pm 1}\}.$$

This seems to be the most natural definition from the viewpoint of geometric group theory (as it measures nicely the isoperimetric quality of the set A), and is used in this and related contexts by many authors. For the purpose of deciding whether a group is amenable, that is, whether there exists a family of Følner sets $A_n \subset G$ with

$$\lim_{n\to\infty} \frac{\#\partial A_n}{\#A_n} = 0\,,$$

all the competing definitions for the boundary ∂A_n turn out to be equivalent: one can consider the so-called Cheeger boundary, the exterior boundary, the above defined interior boundary, etc. One of these alternative boundary notions determines the Kazhdan constant $K(\lambda_G, G, X)$ of G with respect to the left regular representation λ_G of G, which will be defined in Section 2.

This paper, however, focuses on *non-amenable groups*, and more precisely, on the amount of non-amenability a group possesses. In order to measure non-amenability, one quickly realizes that knowledge of any lower bound on $\frac{\#\partial_X A}{\#A}$ for all finite $A \subset G$ is worthless, unless it is *uniform* with respect to all finite generating systems X of G. Thus we define the *uniform Følner constant*

$$\text{Føl}\, G = \inf_X \inf_A \frac{\#\partial_X A}{\#A}\,,$$

where A ranges over all non–empty finite subsets of G and X ranges over all finite generating sets of G. Note that $\text{Føl}\, G$ is, in a rather subtle way, sensitive to the choice made among the competing definitions for the boundary of A.

In this paper, we prove in particular that, if G is a group generated by n elements, then one has:

$$0 \leq \text{Føl}\, G \leq \frac{2n-2}{2n-1}.$$

The lower bound is achieved if G is amenable; correspondingly we call groups G with non-zero Følner invariant, $\text{Føl}\, G > 0$, *uniformly non-amenable.*

In Section 13 we show that there exist non-amenable groups which are *not* uniformly non-amenable. They deserve special interest, as, by a result of Osin [23], the much sought for examples of groups with exponential but not uniform exponential growth must be either amenable but not elementary amenable, or else non-amenable, while we show in Section 1 that the uniform growth rate $\omega(G)$ is bounded below by $\frac{1}{1-\text{Føl}\,G}$. Together with the above mentioned Kazhdan constant, also uniformized over all finite generating systems, the situation can be summarized as follows (compare Propositions 1.4. and 2.4.): Every finitely generated group G satisfies

$$0 \leq \frac{1}{2}\, K(\lambda_G, G)^2 \leq \text{Føl}\, G \leq 1 - \frac{1}{\omega(G)}.$$

The maximal value, $\text{Føl}\, G = (2n-2)/(2n-1)$, is achieved for G free of rank n, and by no other group generated by n elements, see Section 5. For surface groups S_g we give fairly close lower and upper bounds for $\text{Føl}\, S_g$ in Section 6. A less precise lower bound is computed in Section 8 for virtually free groups. While the computation for free groups follows directly from geometric arguments in the Cayley graph, our proof for surface groups is the model for a much larger class of groups, for which uniform non-amenability is summarized in Theorem below.

It relies on two basic good properties of the uniform Følner constant (see Sections 4 and 7):

(1) $\text{Føl}\, G \geq \text{Føl}\, G/N$ for any normal subgroup N of G, and
(2) $\text{Føl}_X G \geq \frac{1}{C}\, \text{Føl}_Y H$ for any subgroup H of G,

where the constant C depends only on the cardinality of the generating system Y of H and the maximal length of the elements of Y with respect to the generating system $X^{\pm 1}$ of G (the terminology is given in Section 1 below).

With these tools we can establish the following theorem. Our proofs are inspired by the previously known results on uniform growth.

Theorem
The following classes of groups G are uniformly non-amenable:

(1) *non-elementary word-hyperbolic groups;*
(2) *large groups (i.e. groups containing a finite index subgroup that surjects onto a non-abelian free group);*
(3) *free Burnside groups of large enough odd exponent;*
(4) *groups which act acylindrically on a simplicial tree without global fixed points.*

Finally, we would like to direct the readers attention to recent work of [24] by Osin, which, in a late state of our work, was communicated to us by de la Harpe. The last section of our paper has been influenced by looking at this paper and at its “predecessor” by Meier [20]. We would also like to thank the referees for their helpful remarks.

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