Advanced Complex Analysis

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Preface

These are lecture notes for the course *Advanced complex analysis* which I held in Vienna in Fall 2016 and 2017 (three semester hours). I am grateful to Gerald Teschl, who based his *Advanced complex analysis* course on these notes in Fall 2019, for corrections and suggestions that improved the presentation.

We follow quite closely the presentation of [**11**]. In the following the primary sources for the single chapters are briefly indicated.

chapter 1: [**11**].
chapter 2: [**1**], [**3**], [**8**], [**11**], [**13**].
chapter 3: [**8**], [**10**], [**11**], [**13**].
chapter 4: [**8**], [**11**].
chapter 5: [**8**], [**11**], [**13**], [**14**].
chapter 6: [**1**], [**3**], [**4**], [**5**], [**8**], [**11**], [**13**], [**14**].
chapter 7: [**8**], [**11**].
chapter 8: [**1**], [**3**], [**6**], [**7**], [**11**].

Contents

Notation

A domain is a nonempty open subset $U \subseteq \mathbb{C}$. A connected domain is called a region. We denote by $D_r(c) = \{z \in \mathbb{C} : |z - c| < r\}$ the open disk of radius r and center c. $\overline{D}_r(c)$ denotes the closed disk and $\partial D_r(c)$ its boundary; if not stated otherwise, it is always assumed to be oriented counterclockwise. By $\mathbb{D}$ we denote the unit disk $\mathbb{D} = D_1(0)$, by $\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ the upper half plane. The Riemann sphere $\mathbb{C} \cup \{\infty\}$ is denoted by $\widehat{\mathbb{C}}$. We use $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and $\mathbb{C}^*_a = \mathbb{C} \setminus \{a\}$, for $a \in \mathbb{C}$, as well as $\mathbb{D}^* := \mathbb{D} \setminus \{0\}$ and $D^*_r(a) := D_r(a) \setminus \{a\}$. If V is a relatively compact open subset of U we write $V \Subset U$.

Let $U \subseteq \mathbb{C}$ be a domain. If $K \subseteq U$ is compact and f is continuous on K, i.e., $f \in C(K)$, then we write $|f|_K := \sup_{z \in K} |f(z)|$. By $\mathcal{H}(U)$ we denote the set of all holomorphic functions $f : U \to \mathbb{C}$. And $\mathcal{O}(K)$ denotes the set of all $f \in C(K)$ such that f is the restriction to K of a function which is holomorphic on an open neighborhood of K. By $\operatorname{Aut}(U)$ we denote the set of automorphisms of U.

We recall that the Cayley mapping

$$h : \mathbb{H} \to \mathbb{D},\ z \mapsto \frac{z - i}{z + i}, \quad h^{-1} : \mathbb{D} \to \mathbb{H}, z \mapsto i\frac{1 + z}{1 - z},$$

is a biholomorphism.

We denote by $|\gamma| := \operatorname{im}(\gamma)$ the image of a curve $\gamma : [0, 1] \to \mathbb{C}$; it is a compact subset of $\mathbb{C}$. For $a, b \in \mathbb{C}$ we write $[a, b]$ for the oriented line segment for a to b, i.e., $(1 - t)a + tb$, $t \in [0, 1]$.

$u_n \downarrow u$ means that u_n is a sequence of real valued functions such that $u_n \geq u_{n+1}$ and $u_n \to u$ pointwise.

Let X be a Riemann surface. Then $\mathcal{O}_X$ is the sheaf of germs of holomorphic functions on X, and $\mathcal{O}_{X,x}$ is the ring of germs of holomorphic functions at $x \in X$. In the case of $X = \mathbb{C}$ we just write $\mathcal{O} = \mathcal{O}_{\mathbb{C}}$ and $\mathcal{O}_a = \mathcal{O}_{\mathbb{C},a}$ if $a \in \mathbb{C}$.

CHAPTER 1

Analytic continuation

1. Covering spaces

A mapping $p : X' \to X$ between topological spaces is a **local homeomorphism** if for each $a' \in X'$ there is an open neighborhood U' of a' in X' such that $p(U') = U$ is open and $p|_{U'}$ is a homeomorphism onto U.

Let Y be a topological space and let $f : Y \to X$ be continuous. A **lifting** of f to X' over p is a continuous mapping $f' : Y \to X'$ such that $p \circ f' = f$.

Lemma 1.1 (uniqueness of liftings)**.** *Let X, X' be Hausdorff spaces and let $p : X' \to X$ be a local homeomorphism. Let Y be a connected Hausdorff space. Let $f : Y \to X$ be continuous and assume that f_1, f_2 are liftings of f. If there exists $y_0 \in Y$ such that $f_1(y_0) = f_2(y_0)$, then $f_1 = f_2$.*

Proof. Let $A = \{y \in Y : f_1(y) = f_2(y)\}$. Then $y_0 \in A$ and A is closed, since X' is Hausdorff (X' is Hausdorff if and only if the diagonal $\Delta \subseteq X' \times X'$ is closed, A is the preimage of Δ under (f_1, f_2)). We claim that A is also open. For, let $y \in A$ and $a' = f_1(y) = f_2(y)$. There is an open neighborhood U' of a' such that $p(U') = U$ is open and $p|_{U'}$ is a homeomorphism onto U. Since f_1, f_2 are continuous, there is a neighborhood V of y such that $f_1(V) \subseteq U'$, $f_2(V) \subseteq U'$. For every $v \in V$, $p(f_1(v)) = f(v) = p(f_2(v))$, and thus, since $p|_{U'}$ is injective, $f_1 = f_2$ on V. That is $V \subseteq A$, and A is open. □

A Hausdorff topological space X is an n-dimensional **manifold** if every point $a \in X$ has an open neighborhood U which is homeomorphic to an open set in $\mathbb{R}^n$.

Let X, X' be manifolds and $p : X' \to X$ a continuous mapping. Then p is called a **covering map**, and X' a **covering** of X, if every $a \in X$ has a neighborhood U with the following property: $p^{-1}(U)$ is a disjoint union of open sets $U'_j \subseteq X'$, $j \in J$, such that $p|_{U'_j}$ is a homeomorphism onto U for each $j \in J$. Clearly, a covering map is a local homeomorphism.

Lemma 1.2 (curve lifting property of coverings)**.** *Each curve $\gamma : [0,1] \to X$ in the base space of a covering map $p : X' \to X$ with $\gamma(0) = a$ can be lifted uniquely to a curve $\gamma' : [0,1] \to X'$ with $\gamma'(0) = a'$, where $a' \in p^{-1}(a)$.*

Proof. By Lemma 1.1, it suffices to show the existence of γ'.

Since $[0,1]$ is compact, there is a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ and open sets $U_j \subseteq X$, $1 \le j \le n$, such that $\gamma([t_{j-1}, t_j]) \subseteq U_j$, $p^{-1}(U_j)$ is a disjoint union of open sets $U'_{jk} \subseteq X'$, and $p|_{U'_{jk}} : U'_{jk} \to U_j$ is a homeomorphism. We show by induction on j the existence of a lifting γ'_j on $[0, t_j]$ with $\gamma'_j(0) = a'$. There is nothing to prove for $j = 0$. Suppose that $j \ge 1$ and that γ'_{j-1} is already constructed. Set $x'_{j-1} := \gamma'_{j-1}(t_{j-1})$. Then $p(x'_{j-1}) = \gamma(t_{j-1}) \in U_j$ and x'_{j-1} lies in U'_{jk} for some k. Setting

$$\gamma'_j(t) := \begin{cases} \gamma'_{j-1}(t) & \text{if } t \in [0, t_{j-1}], \\ p|_{U'_{jk}}^{-1}(\gamma(t)) & \text{if } t \in [t_{j-1}, t_j], \end{cases}$$

yields a lifting on $[0, t_j]$. $\square$

We mention the following converse without proof.

Theorem 1.3. *Let $p : X' \to X$ be a local homeomorphism between manifolds. Then p is a covering map if and only if it has the curve lifting property.*

Example 1.4. Let $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$. The mapping $\exp : \mathbb{C} \to \mathbb{C}^*$ is a covering map. In fact, $U := \{z \in \mathbb{C} : \alpha < \operatorname{Im} z < \alpha + 2\pi\}$, for $\alpha \in \mathbb{R}$, is mapped to $\exp U = \mathbb{C}^* \setminus \{re^{i\alpha} : r > 0\}$, and $\exp^{-1}(\exp U) = \bigcup_{n \in \mathbb{Z}} (U + 2\pi i n)$.

For every positive integer n, the mapping $\mathbb{C}^* \to \mathbb{C}^*$, $z \mapsto z^n$, is a covering map.

Exercise 1. Let n be a positive integer. Prove that $\mathbb{C}^* \to \mathbb{C}^*$, $z \mapsto z^n$, is a covering map. Determine the lifting $\tilde{\gamma}$ of $\gamma(t) = e^{2\pi i t}$, $t \in [0, 1]$, with $\tilde{\gamma}(0) = 1$.

2. The sheaf of germs of holomorphic functions

Let $a \in \mathbb{C}$. Consider the set of pairs (U, f), where $U \subseteq \mathbb{C}$ is an open set containing a and $f \in \mathcal{H}(U)$. We define an equivalence relation $\sim$ on this set by $(U, f) \sim (V, g)$ if there exists an open set W with $a \in W \subseteq U \cap V$ such that $f|_W = g|_W$. An equivalence class is called a **germ of a holomorphic function** at a. The equivalence class of (U, f) is denoted by f_a; we say that f_a is the germ of f at a. The set of all such germs at a is denoted by $\mathcal{O}_a$. The **value** of the germ $f_a \in \mathcal{O}_a$ is defined by $\operatorname{ev}_a(f_a) = f_a(a) := f(a)$, where (U, f) is any representative of f_a.

Lemma 2.1. *Addition and multiplication of functions induces the structure of commutative ring on $\mathcal{O}_a$. $\mathcal{O}_a$ is a complex vector space. The non-units of $\mathcal{O}_a$ form a maximal ideal $\mathfrak{m}_a = \{f_a \in \mathcal{O}_a : f_a(a) = 0\}$ in $\mathcal{O}_a$. We have $\mathcal{O}_a/\mathfrak{m}_a \cong \mathbb{C}$.*

Exercise 2. Prove Lemma 2.1.

Consider the disjoint union $\mathcal{O} := \bigsqcup_{a \in \mathbb{C}} \mathcal{O}_a$. We introduce a topology on $\mathcal{O}$ as follows. Let $f_a \in \mathcal{O}_a$ and let (U, f) be a representative of f_a. Set

$$N(U, f) := \{f_z \in \mathcal{O}_z : f_z \text{ is the germ at } z \in U \text{ defined by } (U, f)\}. \tag{2.1}$$

We require that the sets $N(U, f)$, where (U, f) runs over all representatives of f_a, form a fundamental system of neighborhoods of f_a.

Consider the mapping $\pi : \mathcal{O} \to \mathbb{C}$ given by $\pi(f_a) = a$ if $f_a \in \mathcal{O}_a$. Then $(\mathcal{O}, \pi)$ is called the **sheaf of germs of holomorphic functions** on $\mathbb{C}$.

Lemma 2.2. *$\mathcal{O}$ is a Hausdorff space.*

Proof. Let $f_a \in \mathcal{O}_a$, $g_b \in \mathcal{O}_b$, and suppose that $f_a \neq g_b$. Let (U, f), (V, g) be representatives of f_a, g_b, respectively.

If $a \neq b$, there are neighborhoods $U' \subseteq U$, $V' \subseteq V$ of a, b, respectively, such that $U' \cap V' = \emptyset$. Then $N(U', f)$, $N(V', g)$ are disjoint neighborhoods of f_a, g_b.

If $a = b$, let $D \subseteq U \cap V$ be a disk centered at a. Then $N(D, f) \cap N(D, g) = \emptyset$. Indeed, if $h_z \in N(D, f) \cap N(D, g)$, then (U, f) and (V, g) both define the germ h_z at z. So there is a neighborhood $W \subseteq D$ of z such that $f|_W = g|_W$. By the identity theorem, $f = g$ on D, in particular, $f_a = g_a$, a contradiction. $\square$

Lemma 2.3. *$\pi : \mathcal{O} \to \mathbb{C}$ is continuous and a local homeomorphism. Thus $\mathcal{O}$ is a two-dimensional manifold.*

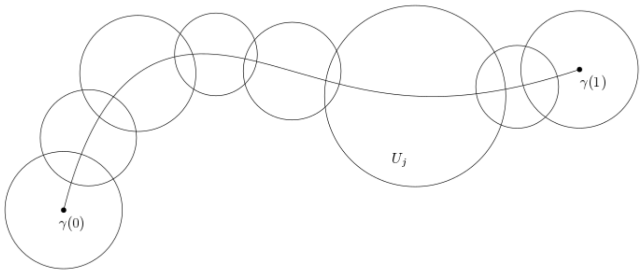
Proof. Let $f_a \in \mathcal{O}_a$ and let (U, f) be a representative of f_a. Note that $\pi(N(U, f)) = U$. If $V \subseteq \mathbb{C}$ is an open set containing a, then $\pi(N(U \cap V, f)) = U \cap V \subseteq V$, so that π is continuous. Moreover, $\pi(N(U, f)) = U$ also implies that π is open. The restriction $\pi|_{N(U,f)}$ is injective and has the inverse $z \mapsto f_z$, so $\pi|_{N(U,f)}$ is a homeomorphism onto U. □

Let $f_a \in \mathcal{O}_a$ and let $\gamma : [0, 1] \to \mathbb{C}$ be a curve with $\gamma(0) = a$. An **analytic continuation of** f_a **along** γ is a lifting $\tilde{\gamma}$ of γ over $\pi : \mathcal{O} \to \mathbb{C}$ such that $\tilde{\gamma}(0) = f_a$.

This means that for each $t_0 \in [0, 1]$ there is a neighborhood I of t_0 in $[0, 1]$, an open set $U \subseteq \mathbb{C}$ with $\gamma(I) \subseteq U$, and $f \in \mathcal{H}(U)$ such that $f_{\gamma(t)} = \tilde{\gamma}(t)$ for all $t \in I$. In fact, let $t_0 \in [0, 1]$ and suppose $N(U, f)$ is a neighborhood of $\tilde{\gamma}(t_0)$ in $\mathcal{O}$. Then there is a neighborhood I of t_0 in $[0, 1]$ such that $\tilde{\gamma}(I) \subseteq N(U, f)$. Thus $\gamma(I) \subseteq U$ and $\tilde{\gamma}(t) = f_{\gamma(t)}$.

Since $[0, 1]$ is compact, this condition is equivalent to the following: there exist a partition $0 = t_0 < t_1 < \cdots < t_n = 1$, domains $U_j \subseteq \mathbb{C}$ with $\gamma([t_{j-1}, t_j]) \subseteq U_j$, and $f_j \in \mathcal{H}(U_j)$ such that

(1) f_a is the germ of f_1 at a,
(2) $f_j|_{V_j} = f_{j+1}|_{V_j}$, where V_j is the connected component of $U_j \cap U_{j+1}$ that contains $\gamma(t_j)$.



Lemma 2.4 (permanence of relations)**.** *Let $\gamma : [0, 1] \to \mathbb{C}$ be a curve with $\gamma(0) = a$. Let $f_a, g_a \in \mathcal{O}_a$ and let P be a polynomial in two variables. Suppose that f_a, g_a can be continued analytically along γ and that $P(f_a, g_a) = 0$. Then, if $F(t)$,$G(t)$ denote the germs at $\gamma(t)$ obtained by analytic continuation of f_a, g_a, respectively, along γ, we have $P(F(t), G(t)) = 0$ for all $0 \le t \le 1$.*

Proof. Let $D \subseteq \mathbb{C}$ be a disk, and let $\varphi, \psi \in \mathcal{H}(D)$. If there is $z \in D$ such that $P(\varphi_z, \psi_z) = 0$ then $P(\varphi, \psi) \equiv 0$ on D, by the identity theorem. Thus the set $\{t \in [0, 1] : P(F(t), G(t)) = 0\}$ is open in $[0, 1]$. Clearly, it is also closed in $[0, 1]$ and contains 0, thus it is all of $[0, 1]$. □

Exercise 3. Show that the mapping $\pi : \mathcal{O} \to \mathbb{C}$ does not have the curve lifting property and hence is not a covering map. Hint: Consider the germ φ at 1 of the function $z \mapsto 1/z$, and show that the curve $\gamma : [0, 1] \to \mathbb{C}$, $\gamma(t) = 1 - t$, does not admit a lifting $\tilde{\gamma}$ to $\mathcal{O}$ with $\tilde{\gamma}(0) = \varphi$. Use Lemma 2.4.

Exercise 4. Let $f \in \mathcal{H}(\mathbb{C})$. Show that $N(\mathbb{C}, f)$ is the connected component in $\mathcal{O}$ of the germ f_0 at 0 of f. Hint: Use that an open subset X in the manifold $\mathcal{O}$ is connected if and only if X is pathwise connected.

3. Integration along curves

Germs of holomorphic functions can be differentiated. We define a mapping $d : \mathcal{O} \to \mathcal{O}$ as follows. Let $f_a \in \mathcal{O}_a$ and let (U, f) be a representative of f_a. Then $df_a := (f')_a$ is the germ at a of (U, f'), where f' is the derivative of f.

Proposition 3.1. *$d : \mathcal{O} \to \mathcal{O}$ is a covering map.*

Proof. Let $f_a \in \mathcal{O}_a$ and let (U, f) be a representative of f_a. Let $D \subseteq U$ be a disk centered at a. Let F be a primitive of f on D. We claim that

$$d^{-1}(N(D, f)) = \bigcup_{c\in\mathbb{C}} N(D, F + c). \tag{3.1}$$

We clearly have $dN(D, F + c) = N(D, f)$. For the other inclusion, let $z \in D$, $g_z \in \mathcal{O}_z$, and $dg_z = f_z$. Let (W, g) be a representative of g_z, where $W \subseteq D$ is a connected neighborhood of z. Then $g' = f$ near z, hence on W, and consequently, $(g - F)' = 0$ on W. So there is a constant $c \in \mathbb{C}$ such that $g = F + c$ on W, and $g_z \in N(D, F + c)$. This shows (3.1). Clearly, the union is disjoint.

Let us prove that $d|_{N(D,F+c)}$ is a homeomorphism onto $N(D, f)$ for each $c \in \mathbb{C}$. It suffices to check that $d|_{N(D,F+c)}$ is injective, which is obvious because d takes distinct elements of $N(D, F + c)$ to germs at different points of D. $\square$

Let $U \subseteq \mathbb{C}$ be a domain, $f \in \mathcal{H}(U)$, and $\gamma : [0, 1] \to U$ a curve in U. A **primitive of f along** γ is by definition a lifting over $d : \mathcal{O} \to \mathcal{O}$ of the curve $\Gamma : [0, 1] \to \mathcal{O}$ given by $\Gamma(t) := f_{\gamma(t)}$. It exists, by the curve lifting property of coverings 1.2, since $d : \mathcal{O} \to \mathcal{O}$ is a covering map, by Proposition 3.1.

If F_1, F_2 are two primitives of f along γ, then there is a constant c such that $F_1(t) = F_2(t) + c$ for all $t \in [0, 1]$. This follows from the uniqueness of liftings 1.1, since $F_1(0)$ and $F_2(0)$ are both primitives of f in a neighborhood of $\gamma(0)$, and hence $F_1(0) = F_2(0) + c$ for some $c \in \mathbb{C}$.

Proposition 3.2. *Let $U \subseteq \mathbb{C}$ be a domain, $f \in \mathcal{H}(U)$, and $\gamma : [0, 1] \to U$ piecewise C^1. Let $F : [0, 1] \to \mathcal{O}$ be a primitive of f along γ. Then*

$$\int_\gamma f\, dz = F(1)(\gamma(1)) - F(0)(\gamma(0)).$$

Proof. We define a mapping $G : [0, 1] \to \mathcal{O}$ as follows. Let $t \in [0, 1]$, and let D be a disk centered at $\gamma(t)$ and contained in U. Let h be the primitive of f on D for which

$$h(\gamma(t)) = \int_0^t f(\gamma(s))\gamma'(s)\, ds.$$

Let $G(t)$ be the germ at $\gamma(t)$ of h, i.e., $G(t) := h_{\gamma(t)}$. Then $dG(t) = f_{\gamma(t)}$. We will show that G is a lifting of Γ, i.e., that G is continuous. This will imply the assertion: there is a constant c such that $F(t) = G(t) + c$ for all $t \in [0, 1]$, and so

$$\begin{aligned} F(1)(\gamma(1)) - F(0)(\gamma(0)) &= G(1)(\gamma(1)) - G(0)(\gamma(0)) \\ &= \int_0^1 f(\gamma(s))\gamma'(s)\, ds = \int_\gamma f\, dz. \end{aligned}$$

Let us prove that G is continuous. Let $t_0 \in [0, 1]$, and let D be a small disk with center $\gamma(t_0)$. Let $h \in \mathcal{H}(D)$ be such that (D, h) is a representative of the germ $G(t_0)$. Then, by definition, $h(\gamma(t_0)) = \int_0^{t_0} f(\gamma(s))\gamma'(s)\, ds$. Let $\epsilon > 0$ be such that

$\gamma(t) \in D$ if $|t - t_0| < \epsilon$. For such t,

$$h(\gamma(t)) - h(\gamma(t_0)) = \int_{t_0}^{t} \frac{d}{ds} h(\gamma(s))\, ds = \int_{t_0}^{t} f(\gamma(s))\gamma'(s)\, ds,$$

since $h' = f$ on D. Thus, $h(\gamma(t)) = \int_0^t f(\gamma(s))\gamma'(s)\, ds$, so that $G(t) = h_{\gamma(t)}$ for $|t - t_0| < \epsilon$. In particular, $G(t) \in N(D, h)$, hence G is continuous. ☐

If $\gamma : [0,1] \to U$ is any curve, i.e., just continuous and not necessarily piecewise C^1, we may use this as the definition of $\int_\gamma f\, dz$: if $f \in \mathcal{H}(U)$ and $F : [0,1] \to \mathcal{O}$ is a primitive of f along γ, then we define

$$\int_\gamma f\, dz := F(1)(\gamma(1)) - F(0)(\gamma(0)). \tag{3.2}$$

Corollary 3.3. *Let $U \subseteq \mathbb{C}$ be a domain and $f \in \mathcal{H}(U)$. If f has a primitive on U, then for any closed curve γ in U,*

$$\int_\gamma f\, dz = 0.$$

Proof. Let h be a primitive of f on U. Then $F : [0,1] \to \mathcal{O}$ defined by $F(t) := h_{\gamma(t)}$ is a primitive of f along γ. So $\int_\gamma f\, dz = h(\gamma(1)) - h(\gamma(0)) = 0$ since $\gamma(1) = \gamma(0)$. ☐

4. The monodromy theorem

Let X be a manifold. Let $\gamma_i : [0,1] \to X$, $i = 0,1$, be curves in X. We say that γ_0 and γ_1 are **homotopic** if there is a continuous mapping $H : [0,1] \times [0,1] \to X$, $H(s,t) = H_s(t) = H^t(s)$, such that $H_0 = \gamma_0$ and $H_1 = \gamma_1$. The mapping H is called a **homotopy**. It defines a one-parameter family of curves $\gamma_s := H_s$ in X which connects γ_0 and γ_1; we will also write $H = \{\gamma_s\}_{s\in[0,1]}$.

Suppose that $H_0(0) = H_1(0) = a$. We say that the homotopy H **fixes** a if $H(s,0) = a$ for all $s \in [0,1]$. Provided that also $H_0(1) = H_1(1) = b$, we say that H **fixes the endpoints** if $H(s,0) = a$ and $H(s,1) = b$ for all $s \in [0,1]$.

Theorem 4.1 (general monodromy theorem)**.** *Let X, X' be manifolds and $p : X' \to X$ a local homeomorphism. Let $a' \in X'$ and $a = p(a')$. Let $H : [0,1]^2 \to X$ be a homotopy between γ_0 and γ_1 fixing the starting point $a = \gamma_0(0) = \gamma_1(0)$. Suppose that each curve $\gamma_s := H_s$, $s \in [0,1]$, has a lifting γ'_s over $p : X' \to X$ which starts at a'. Then $H'(s,t) := \gamma'_s(t)$ is a homotopy between γ'_0 and γ'_1.*

Proof. We must show continuity of $H' : [0,1]^2 \to X'$. Let $I := [0,1]$.

Fix $(s_0, t_0) \in I^2$. Since γ'_{s_0} is continuous and hence $\gamma'_{s_0}(I)$ is compact, we may choose open sets $U'_0, \ldots, U'_n$ in X' and points $0 = \tau_0 < \tau_1 < \cdots < \tau_n = 1$ such that $p|_{U'_j} =: p_j$ is a homeomorphism onto an open set U_j in X and $\gamma'_{s_0}([\tau_j, \tau_{j+1}]) \subseteq U'_j$, $j = 0,1,\ldots,n-1$. We may assume without loss of generality that t_0 is an interior point of some $[\tau_{j_0}, \tau_{j_0+1}]$, unless t_0 is 0 or 1.

By the continuity of H, there exists $\epsilon > 0$ such that $\gamma_s(t) \in U_j$ for $|s - s_0| < \epsilon$, $s \in I$, $t \in [\tau_j, \tau_{j+1}]$, and $j = 0,1,\ldots,n-1$. We will prove that, for $|s - s_0| < \epsilon$, $s \in I$, $t \in [\tau_j, \tau_{j+1}]$, and $j = 0,1,\ldots,n-1$,

$$\gamma'_s(t) = p_j^{-1}(\gamma_s(t)). \tag{4.1}$$

This implies that H' is continuous at (s_0, t_0), since (s_0, t_0) is an interior point (relative to I^2) of the set $\{s \in I : |s - s_0| < \epsilon\} \times [\tau_{j_0}, \tau_{j_0+1}]$.

We show (4.1) by induction on j. Let $j = 0$. Fix $s \in I$ with $|s - s_0| < \epsilon$. The curves γ'_s and $p_0^{-1} \circ \gamma_s$ are both liftings of γ_s on the interval $[\tau_0, \tau_1]$, and

$\gamma_s'(0) = a' = (p_0^{-1} \circ \gamma_s)(0)$ (because $a' = \gamma_{s_0}'(0) \in U_0'$). By uniqueness of liftings 1.1, (4.1) holds for $j = 0$.

Suppose that (4.1) has been proved for all $0 \leq j < k$. For fixed s, the curves γ_s' and $p_k^{-1} \circ \gamma_s$ are both liftings of γ_s on the interval $[\tau_k, \tau_{k+1}]$. By Lemma 1.1, it is enough to prove

$$\gamma_s'(\tau_k) = p_k^{-1}(\gamma_s(\tau_k)) \quad \text{for } |s - s_0| < \epsilon,\ s \in I. \tag{4.2}$$

By induction hypothesis, (4.1) for $j = k - 1$ and $t = \tau_k$ gives

$$\gamma_s'(\tau_k) = p_{k-1}^{-1}(\gamma_s(\tau_k)) \quad \text{for } |s - s_0| < \epsilon,\ s \in I. \tag{4.3}$$

In particular, for $s = s_0$,

$$p_k^{-1}(\gamma_{s_0}(\tau_k)) = \gamma_{s_0}'(\tau_k) = p_{k-1}^{-1}(\gamma_{s_0}(\tau_k)),$$

since $\gamma_{s_0}'(\tau_k) \in U_{k-1}' \cap U_k'$. Thus, $s \mapsto p_{k-1}^{-1}(\gamma_s(\tau_k))$ and $s \mapsto p_k^{-1}(\gamma_s(\tau_k))$ are both liftings of $s \mapsto \gamma_s(\tau_k)$, for $|s - s_0| < \epsilon$, $s \in I$, and they coincide for $s = s_0$. By Lemma 1.1, $p_{k-1}^{-1}(\gamma_s(\tau_k)) = p_k^{-1}(\gamma_s(\tau_k))$ for all $|s - s_0| < \epsilon$, $s \in I$, which together with (4.3) implies (4.2) and hence (4.1) for $j = k$. □

Corollary 4.2. *Let X, X' be manifolds and $p : X' \to X$ a local homeomorphism. Let $a' \in X'$, $a = p(a')$, and $b \in X$. Let $H : [0,1]^2 \to X$ be a homotopy between γ_0 and γ_1 fixing $a = \gamma_0(0) = \gamma_1(0)$ and $b = \gamma_0(1) = \gamma_1(1)$. Suppose that each curve $\gamma_s := H_s$, $s \in [0,1]$, has a lifting γ_s' over $p : X' \to X$ which starts at a'. Then the endpoints of γ_0' and γ_1' coincide, and $\gamma_s'(1)$ is independent of s.*

Proof. By Theorem 4.1, the mapping $s \mapsto \gamma_s'(1)$ is continuous. Thus it is a lifting of the constant curve $s \mapsto \gamma_s(1) = b$, and so it is itself constant, by Lemma 1.1. □

Theorem 4.3 (classical monodromy theorem)**.** *Let γ_0,γ_1 be curves in $\mathbb{C}$ with the same endpoints $a = \gamma_0(0) = \gamma_1(0)$, $b = \gamma_0(1) = \gamma_1(1)$, and let $H = \{\gamma_s\}_{s\in[0,1]}$ be a homotopy between γ_0 and γ_1 fixing the endpoints. Let $f_a \in \mathcal{O}_a$ and suppose that f_a can be continued analytically along γ_s for all $s \in [0,1]$. Then analytic continuation of f_a along γ_0 and γ_1 result in the same germ at b.*

Proof. Apply Corollary 4.2 to $\pi : \mathcal{O} \to \mathbb{C}$. □

Next we will derive several applications of the monodromy theorem.

Theorem 4.4 (homotopy form of Cauchy's theorem)**.** *Let $U \subseteq \mathbb{C}$ be a domain. Let $\gamma_0, \gamma_1 : [0,1] \to U$ be curves in U with the same endpoints, $a = \gamma_0(0) = \gamma_1(0)$ and $b = \gamma_0(1) = \gamma_1(1)$. Suppose that there is a homotopy between γ_0, γ_1 in U fixing the endpoints. Then, for each $f \in \mathcal{H}(U)$,*

$$\int_{\gamma_0} f\, dz = \int_{\gamma_1} f\, dz.$$

Proof. Let $H : [0,1]^2 \to U$ be a homotopy between γ_0, γ_1 in U fixing the endpoints. Then $K : [0,1]^2 \to \mathcal{O}$, where $K_s(t)$ is the germ at $H_s(t)$ of (U, f) is a homotopy between K_0 and K_1 fixing the endpoints f_a and f_b. Let F_a be the germ at a of some primitive of f in a neighborhood of a. Since $d : \mathcal{O} \to \mathcal{O}$ is a covering map, by Proposition 3.1, K_s has a lifting $K_s' : [0,1] \to \mathcal{O}$ over d such that $K_s'(0) = F_a$, for all $s \in [0,1]$. By Corollary 4.2, $K_0'(1) = K_1'(1)$, and hence

$$\int_{\gamma_0} f\, dz = K_0'(1)(\gamma_0(1)) - K_0'(0)(\gamma_0(0)) = K_1'(1)(\gamma_1(1)) - K_1'(0)(\gamma_1(0)) = \int_{\gamma_1} f\, dz,$$

since $K_0'(0)(\gamma_0(0)) = K_1'(0)(\gamma_1(0)) = F_a(a)$ and $\gamma_0(1) = \gamma_1(1) = b$. □

It is not hard to show that any curve $\gamma : [0,1] \to U$ is homotopic in U to a piecewise C^1-curve $\tilde{\gamma} : [0,1] \to U$ (by a homotopy fixing the endpoints). So, by Theorem 4.4, it is no loss of generality to assume that γ is piecewise C^1, if one deals with path integrals $\int_\gamma f\,dz$ of holomorphic functions f.

A pathwise connected Hausdorff space X is said to be **simply connected** if for any two curves γ_0, γ_1 in X with the same endpoints there is a homotopy in X between γ_0 and γ_1 fixing the endpoints. If γ_0 is homotopic in X to a constant curve γ_1 (i.e., a point), we say that γ_0 is **null-homotopic** in X.

Corollary 4.5. *If $U \subseteq \mathbb{C}$ is a simply connected domain, then for each $f \in \mathcal{H}(U)$ and each closed curve γ in U,*

$$\int_\gamma f\,dz = 0.$$

Proof. Apply the homotopy form of Cauchy's theorem 4.4 to a homotopy between γ and the constant closed curve $\gamma(0)$. □

We will now show that continuous mappings $f : Y \to X$, where Y is simply connected, admit liftings over covering maps $p : X' \to X$. In the proof we will use **concatenation of curves**: if $\gamma_i : [0,1] \to X$, $i = 1,2$, are curves such that $\gamma_1(1) = \gamma_2(0)$, then

$$\gamma_1 \cdot \gamma_2(t) := \begin{cases} \gamma_1(2t) & \text{if } t \in [0,1/2], \\ \gamma_2(2t-1) & \text{if } t \in [1/2,1]. \end{cases}$$

defines a curve $\gamma_1 \cdot \gamma_2 : [0,1] \to X$. (In homotopy theory this notation preferred in contrast to $\gamma_1 + \gamma_2$ used in homology theory.)

The property of being homotopic defines an equivalence relation on the set of all closed curves $\gamma : [0,1] \to X$ with fixed endpoint $\gamma(0) = \gamma(1) = a$. The concatenation of curves defines a binary operation on the set of equivalence classes which turns it into a group $\pi_1(X,a)$. This group is called the **first homotopy group** or **fundamental group** of X with base point a. One can show that the fundamental group is independent of the base point if X is pathwise connected; then one simply writes $\pi_1(X)$. Note that X is simply connected if and only if $\pi_1(X)$ is trivial.

Exercise 5. Show that concatenation of curves defines a binary operation on the set of all homotopy classes and turns it into a group $\pi_1(X,a)$.

Theorem 4.6 (existence of liftings)**.** *Let X, X' be manifolds and $p : X' \to X$ a covering map. Let Y be a connected simply connected manifold. Let $a' \in X'$ and $a = p(a')$. Suppose that $f : Y \to X$ is continuous and $f(y_0) = a$ for some $y_0 \in Y$. Then f has a lifting $f' : Y \to X'$ such that $f'(y_0) = a'$.*

Proof. Let $y_0, y \in Y$ and let $\gamma : [0,1] \to Y$ be a curve from y_0 to y. Then $\mu = f \circ \gamma$ is a curve in X starting at a which admits a lifting μ' to X' with $\mu'(0) = a'$, by the curve lifting property of coverings 1.2. We define

$$f'(y) := \mu'(1).$$

Let us prove that $f'(y)$ is independent of γ. Set $\gamma_0 = \gamma$ and let γ_1 be another curve in Y from y_0 to y. Since Y is simply connected there is a homotopy H between γ_0 and γ_1 fixing the endpoints. Then $f \circ H$ is a homotopy between μ and $\mu_1 := f \circ \gamma_1$ fixing the endpoints. If μ_1' is the lifting of μ_1 to X' with $\mu_1'(0) = a'$, then μ' and μ_1' have the same endpoints, by Corollary 4.2. Thus $f'(y)$ is independent of γ.

Clearly, f' satisfies $p \circ f' = f$. It remains to show that f' is continuous. Let $y_1 \in Y$ and let γ_1 be a curve in Y from y_0 to y_1. With the notation as above, let $x_1 = f(y_1)$, $x'_1 = f'(y_1) = \mu'_1(1)$ and let U', U, V be pathwise connected neighborhoods of x'_1, x_1, y_1, respectively, such that $p|_{U'} : U' \to U$ is a homeomorphism and $f(V) \subseteq U$. For each $y \in V$ choose a curve λ in V from y_1 to y. Then $\gamma_1 \cdot \lambda =: \gamma$ is a curve from y_0 to y in Y and $f \circ \gamma = \mu_1 \cdot (f \circ \lambda)$. The lifting of $f \circ \gamma$ starting at a' is $\mu'_1 \cdot (p|_{U'}^{-1} \circ f \circ \lambda)$, and $f'(y) = (\mu'_1 \cdot (p|_{U'}^{-1} \circ f \circ \lambda))(1) = p|_{U'}^{-1}(f(y)) \in U'$. Thus $f'(V) \subseteq U'$ so that f' is continuous. □

Corollary 4.7 (holomorphic liftings)**.** *Let $U, U' \subseteq \mathbb{C}$ be domains and let $p : U' \to U$ be a holomorphic covering map. Suppose that $V \subseteq \mathbb{C}$ is a simply connected region and let $f : V \to U$ be holomorphic. Then f has a holomorphic lifting over p.*

Proof. Theorem 4.6 implies that that f admits a lifting $f' : V \to U'$. We claim that f' is holomorphic. Let $w \in V$ and $z = f(w) \in U$. Let D be an open neighborhood of z in U and D' an open neighborhood of $z' = f'(w)$ in U' such that $p|_{D'} : D' \to D$ is a homeomorphism, and hence a biholomorphism. Then $(f')^{-1}(D') = f^{-1}(D) := B$ is an open neighborhood of w in V. Since $f'|_B = p|_{D'}^{-1} \circ f|_B$ is holomorphic, the assertion follows. □

Theorem 4.8 (branches of the logarithm)**.** *Let $U \subseteq \mathbb{C}$ be a simply connected domain. Let $n \geq 2$ be an integer. If $f \in \mathcal{H}(U)$ is nowhere-vanishing in U, then there exist $g, h \in \mathcal{H}(U)$ such that $e^g = f$ and $h^n = f$.*

Proof. This follows from Corollary 4.7 and Example 1.4. □

Theorem 4.9 (primitives)**.** *Let $U \subseteq \mathbb{C}$ be a simply connected domain. Any $f \in \mathcal{H}(U)$ has a primitive on U.*

Proof. We give two proofs. First, fix $z_0 \in U$ and define

$$F(z) := \int_{\gamma_z} f(\zeta)\, d\zeta, \quad z \in U,$$

where γ_z is any path in U from z_0 to z; this is well-defined since $\int_\gamma f\, dz = 0$ for every closed curve γ in U, by Corollary 4.5. Let $c \in U$ and $r > 0$ such that $D_r(c) \subseteq U$. Then

$$\begin{aligned} \frac{F(z) - F(c)}{z - c} &= \frac{1}{z - c}\Big(\int_{\gamma_c + [c,z]} f(\zeta)\, d\zeta - \int_{\gamma_c} f(\zeta)\, d\zeta \Big) \\ &= \frac{1}{z - c} \int_{[c,z]} f(\zeta)\, d\zeta \to f(c) \quad \text{as } z \to c, \end{aligned}$$

that is, $F'(c) = f(c)$.

Alternatively: Consider the mapping $\varphi : U \to \mathcal{O}$ which sends z to the germ of (U, f) at z. By Theorem 4.6, φ has a lifting $\Phi : U \to \mathcal{O}$ over the covering map $d : \mathcal{O} \to \mathcal{O}$ (cf. Proposition 3.1). Define $F : U \to \mathbb{C}$ by setting $F(z) := \Phi(z)(z)$. We must show that $F \in \mathcal{H}(U)$; then since $d \circ \Phi = \varphi$ we have $F' = f$ on U. Let $z_0 \in U$ and let (V, G) be a representative of $\Phi(z_0)$. That Φ is continuous means that, for z in a neighborhood W of z_0, $\Phi(z)$ is the germ at z of (V, G). Thus, $F|_W = G|_W$. This implies that $F \in \mathcal{H}(U)$. □

Remark 4.10. *Theorem 4.9 implies Theorem 4.8.* If every $f \in \mathcal{H}(U)$ has a primitive then for every non-vanishing $f \in \mathcal{H}(U)$ there are $g, h \in \mathcal{H}(U)$ such that $e^g = f$ and $h^n = f$. Indeed, suppose that f is non-vanishing in U. If the existence of g is established, then $h := e^{g/n}$ is as required. Since $f'/f \in \mathcal{H}(U)$, there is $g \in \mathcal{H}(U)$

with $g' = f'/f$. Adding a constant to g we may achieve that $e^{g(c)} = f(c)$ for some $c \in U$. Then $(fe^{-g})' = 0$ and thus $fe^{-g} = \text{const} = 1$. Consequently, $e^g = f$.

CHAPTER 2

Calculus of residues

5. The winding number

Usually, one defines the index or winding number of a closed path γ in $\mathbb{C}$ by the path integral

$$\mathrm{ind}_\gamma(z) := \frac{1}{2\pi i}\int_\gamma \frac{d\zeta}{\zeta - z}, \quad z \in \mathbb{C} \setminus |\gamma|. \tag{5.1}$$

Then ind_γ is an integer valued function $\mathrm{ind}_\gamma : \mathbb{C} \setminus |\gamma| \to \mathbb{Z}$ that is constant in each connected component of $\mathbb{C} \setminus |\gamma|$ and 0 in the unbounded component of $\mathbb{C} \setminus |\gamma|$; cf. [**12**, Theorem 12.2].

Here we give a different definition of the index which is more in the spirit of the previous chapter, and show then that it can be computed by the integral in (5.1).

Let $\gamma : [0,1] \to \mathbb{C}$ be a closed curve in $\mathbb{C}$, and let $z \in \mathbb{C} \setminus |\gamma|$. Recall that $\mathbb{C} \mapsto \mathbb{C}_z^* := \mathbb{C} \setminus \{z\}$, $\zeta \mapsto z + e^\zeta$, is a covering map; cf. Example 1.4. Let $\tilde\gamma$ be a lifting of γ over this covering map. We define the **index** or **winding number** of γ at z by

$$\mathrm{ind}_\gamma(z) := \frac{1}{2\pi i}(\tilde\gamma(1) - \tilde\gamma(0)). \tag{5.2}$$

It is easy to see that the index is independent of the choice of the lifting $\tilde\gamma$. Moreover, it is clear from (5.2) that $\mathrm{ind}_\gamma(z) \in \mathbb{Z}$.

Proposition 5.1. *The index is given by formula* (5.1).

Proof. Let $\eta : \mathbb{C}_z^* \to \mathcal{O}$ be the mapping which assigns to w the germ at w of the function $\zeta \mapsto 1/(\zeta - z)$. Consider $\Gamma := \eta \circ \gamma$ and let $\tilde\Gamma$ be a lifting of Γ over $d : \mathcal{O} \to \mathcal{O}$. Fix $w = \gamma(t)$, $t \in [0,1]$, and let F_w be the germ $\tilde\Gamma(t)$. Let (D, F) be a representative of F_w on some disk centered at w. Then $F'(\zeta) = 1/(\zeta - z)$ for $\zeta \in D$, and thus $d/d\zeta((\zeta - z)e^{-F(\zeta)}) = 0$ for $\zeta \in D$.

Let $\gamma_1(t)$ be the value at $\gamma(t)$ of the germ $\tilde\Gamma(t)$. For s sufficiently close to t,

$$(\gamma(s) - z)e^{-\gamma_1(s)} = (\gamma(s) - z)e^{-F(\gamma(s))}.$$

Since $\zeta \mapsto (\zeta - z)e^{-F(\zeta)}$ is constant on D, $s \mapsto (\gamma(s) - z)e^{-\gamma_1(s)}$ is locally constant, hence constant on $[0,1]$. Let $c \in \mathbb{C}$ be such that $e^c = (\gamma(t) - z)e^{-\gamma_1(t)}$ for all $t \in [0,1]$. Then $t \mapsto \tilde\gamma(t) := \gamma_1(t) + c$ is a lifting of γ over $\zeta \mapsto z + e^\zeta$. Then,

$$2\pi i\, \mathrm{ind}_\gamma(z) = \tilde\gamma(1) - \tilde\gamma(0) = \tilde\Gamma(1)(\gamma(1)) - \tilde\Gamma(0)(\gamma(0)) = \int_\gamma \frac{d\zeta}{\zeta - z},$$

by (3.2). □

Exercise 6. Use the homotopy form of Cauchy's theorem 4.4 to conclude that $\mathrm{ind}_{\gamma_1}(z) = \mathrm{ind}_{\gamma_2}(z)$, if γ_1, γ_2 are closed homotopic curves in $\mathbb{C}_z^*$.

6. The homology form of Cauchy's theorem

We consider formal sums $\gamma_1 + \cdots + \gamma_n$ of curves in $\mathbb{C}$ and define

$$\int_{\gamma_1+\cdots+\gamma_n} f\,dz := \int_{\gamma_1} f\,dz + \cdots + \int_{\gamma_n} f\,dz \tag{6.1}$$

Such formal sums of curves are called **chains**. (More formally, chains can be defined as formal sums $\gamma_1+\cdots+\gamma_n$ of linear functionals $\gamma_i(f) = \int_{\gamma_i} f\,dz$ for $f \in C(\bigcup_i |\gamma_i|)$, cf. [**13**, 10.34].)

Chains are considered identical if they yield the same path integral for all functions f. Thus two chains are identical if one is obtained from the other by permutation of curves, subdivision of curves, fusion of sub-curves, reparameterization of curves, cancellation of opposite curves. Chains can be added and (6.1) remains valid for arbitrary chains. If identical chains are added, we denote the sum as a multiple. By allowing $a(-\gamma) = -a\gamma$, every chain can be written as a finite linear combination

$$\gamma = a_1\gamma_1 + \cdots + a_n\gamma_n,$$

where $a_i \in \mathbb{Z}$, all γ_i are different, and no two γ_i are opposite. We allow zero coefficients, in particular, the zero chain 0. Clearly, a chain can be represented as a sum of paths in many ways.

For a formal sum $\gamma = \gamma_1+\cdots+\gamma_n$ of paths γ_i we set $|\gamma| = \bigcup_{i=1}^n |\gamma_i|$ and $|0| = \emptyset$. Note that $|\gamma|$ depends on the representation of γ (due to cancellation of opposite curves). We will consider chains contained in a given domain $U \subseteq \mathbb{C}$. This means that the chains have a representation by paths in U and only such representations are considered.

A chain is called a **cycle** if it can be represented as a sum of closed curves. For a cycle γ and a point $z \notin |\gamma|$ the **index** of z with respect to γ is defined by

$$\operatorname{ind}_\gamma(z) = \frac{1}{2\pi i}\int_\gamma \frac{d\zeta}{\zeta - z}. \tag{6.2}$$

Clearly,

$$\operatorname{ind}_{\gamma_1+\gamma_2}(z) = \operatorname{ind}_{\gamma_1}(z) + \operatorname{ind}_{\gamma_2}(z), \quad \operatorname{ind}_{-\gamma}(z) = -\operatorname{ind}_\gamma(z). \tag{6.3}$$

A cycle γ in a domain $U \subseteq \mathbb{C}$ is said to be **homologous to zero** with respect to U if $\operatorname{ind}_\gamma(z) = 0$ for all $z \in \mathbb{C}\setminus U$; we write $\gamma \sim_U 0$. Two cycles γ_1, γ_2 in U are **homologous** in U, in symbols $\gamma_1 \sim_U \gamma_2$, if $\gamma_1 - \gamma_2 \sim_U 0$. By (6.3),

$$\gamma_1 \sim_U \gamma_2 \;\Leftrightarrow\; \operatorname{ind}_{\gamma_1}(z) = \operatorname{ind}_{\gamma_2}(z) \text{ for all } z \notin U.$$

This defines an equivalence relation on the set of cycles in U. The set of equivalence classes, called **homology classes**, forms an additive group, the **homology group**. If $\gamma \sim_U 0$ then $\gamma \sim_{U'} 0$ for all $U' \supseteq U$.

Lemma 6.1. *If $f \in \mathcal{H}(U)$ then*

$$g : U\times U \to \mathbb{C}, \quad g(z,w) := \begin{cases} \dfrac{f(z)-f(w)}{z-w} & z \neq w \\ f'(z) & z = w \end{cases} \tag{6.4}$$

is continuous.

Proof. We need to check continuity at points on the diagonal $z = w$. Fix $a \in U$ and $\epsilon > 0$. Since f' is continuous, there is a disk $D_r(a) \subseteq U$ such that $|f'(\zeta) - f'(a)| < \epsilon$

if $\zeta \in D_r(a)$. If $z, w \in D_r(a)$, $z \neq w$, then $\zeta(t) := (1-t)z + tw \in D_r(a)$, $t \in [0,1]$, and

$$|g(z,w) - g(a,a)| = \Big|\frac{f(z)-f(w)}{z-w} - f'(a)\Big| = \Big|\int_0^1 (f'(\zeta(t)) - f'(a))\,dt\Big| \le \epsilon.$$

Thus g is continuous at (a,a). □

Theorem 6.2 (homology form of Cauchy's theorem)**.** *Let $U \subseteq \mathbb{C}$ be a domain and let $f \in \mathcal{H}(U)$.*

(1) *If γ is a cycle that is homologous to zero in U, then*

$$\int_\gamma f\,dz = 0, \tag{6.5}$$

$$\operatorname{ind}_\gamma(z) f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta, \quad z \in U \setminus |\gamma|. \tag{6.6}$$

(2) *If γ_1 and γ_2 are homologous cycles in U, then*

$$\int_{\gamma_1} f\,dz = \int_{\gamma_2} f\,dz. \tag{6.7}$$

Proof. (1) Consider the continuous function g in (6.4), and define

$$h(z) := \frac{1}{2\pi i}\int_\gamma g(z,w)\,dw, \quad z \in U.$$

For each $w \in U$ we have $g(\cdot, w) \in \mathcal{H}(U)$, since the singularity at $z = w$ is removable by Riemann's theorem on removable singularities. Thus $h \in \mathcal{H}(U)$.

Our goal is to show that $h(z) = 0$ for $z \in U \setminus |\gamma|$ which is equivalent to (6.6) (by (6.2)). Set $U_1 := \{z \in \mathbb{C} \setminus |\gamma| : \operatorname{ind}_\gamma(z) = 0\}$ and define

$$h_1(z) := \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw, \quad z \in U_1.$$

Since $h_1(z) = h(z)$ for $z \in U \cap U_1$, there exists a function $\varphi \in \mathcal{H}(U \cup U_1)$ such that $\varphi|_U = h$ and $\varphi|_{U_1} = h_1$. Since γ is homologous to zero in U, the set U_1 contains $\mathbb{C} \setminus U$, so $U \cup U_1 = \mathbb{C}$ and φ is entire. By definition U_1 also contains the unbounded connected component of the complement of $|\gamma|$ on which $\operatorname{ind}_\gamma$ vanishes. Thus

$$\lim_{|z|\to\infty} \varphi(z) = \lim_{|z|\to\infty} h_1(z) = 0.$$

By Liouville's theorem, $\varphi = 0$ and hence $h = 0$. We proved (6.6).

Let us deduce (6.5) from (6.6). Fix $a \in U \setminus |\gamma|$ and set $F(z) := (z-a)f(z)$. Then, as $F(a) = 0$,

$$\frac{1}{2\pi i}\int_\gamma f\,dz = \frac{1}{2\pi i}\int_\gamma \frac{F(z)}{z-a}\,dz = \operatorname{ind}_\gamma(a)F(a) = 0.$$

(2) Apply (6.5) to $\gamma = \gamma_1 - \gamma_2$. □

Remark 6.3. Note that if γ_1, γ_2 are homotopic closed curves in a domain U, then γ_1, γ_2 are homologous in U. The converse is false; see [**12**, 26.2].

Exercise 7. Let f be holomorphic in a neighborhood of the disk $D_R(a)$. Prove that for each $r \in (0,R)$ there is a constant $C > 0$ such that

$$\|f\|_{L^\infty(D_r(a))} \le C\|f\|_{L^2(D_R(a))},$$

where $\|f\|_{L^\infty(U)} = \sup_{z\in U} |f(z)|$ and $\|f\|_{L^2(U)} = (\int_U |f(z)|^2\, dx\, dy)^{1/2}$. Conclude that a sequence $(f_n) \subseteq \mathcal{H}(U)$ which is a Cauchy sequence with respect to the norm $\|\cdot\|_{L^2(U)}$ converges uniformly on compact subsets of U to a holomorphic function.

7. Laurent series

We loosely follow the presentation in [**8**].

By a **Laurent series** we mean a (formal) series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-c)^n. \tag{7.1}$$

To discuss convergence of Laurent series, we must first agree on the meaning of the convergence of a doubly infinite series $\sum_{n=-\infty}^{\infty} \alpha_n$, $\alpha_n \in \mathbb{C}$. We say that such a series **converges** if both $\sum_{n=0}^{\infty} \alpha_n$ and $\sum_{n=1}^{\infty} \alpha_{-n}$ converge. In this case we set

$$\sum_{n=-\infty}^{\infty} \alpha_n = \sum_{n=0}^{\infty} \alpha_n + \sum_{n=1}^{\infty} \alpha_{-n}.$$

Exercise 8. Prove that $\sum_{n=-\infty}^{\infty} \alpha_n$ converges to a complex number α if and only if for each $\epsilon > 0$ there is $N \in \mathbb{N}_{>0}$ such that $|\sum_{n=-k}^{\ell} \alpha_n - \alpha| < \epsilon$ if $k, \ell \geq N$.

A set of the form

$$A_{r_1,r_2}(c) := D_{r_2}(c) \setminus \overline{D}_{r_1}(c), \quad 0 \leq r_1 \leq r_2 \leq \infty, \tag{7.2}$$

is called an **annulus** centered at c. In particular, $A_{0,\infty}(c) = \mathbb{C} \setminus \{c\} = \mathbb{C}^*_c$. We denote by $\overline{A}_{r_1,r_2}(c)$ the closure of $A_{r_1,r_2}(c)$.

Lemma 7.1. *Suppose that $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-c)^n$ converges at $z_1 \neq c$ and at $z_2 \neq c$ with $|z_1 - c| < |z_2 - c|$. Then $f(z)$ converges normally on $\overline{A}_{s_1,s_2}(c)$ whenever $|z_1 - c| < s_1 \leq s_2 < |z_2 - c|$.*

Proof. If $f(z_2)$ converges then $\sum_{n=0}^{\infty} a_n(z_2-c)^n$ converges and hence $\sum_{n=0}^{\infty} a_n(z-c)^n$ converges normally on $\overline{D}_{s_2}(c)$ whenever $s_2 < |z_2 - c|$ (by Abel's lemma). If $f(z_1)$ converges then so does $\sum_{n=1}^{\infty} a_{-n}(z_1-c)^{-n}$. For $z \in \overline{A}_{s_1,s_2}(c)$, we have $0 < |z_1 - c| < s_1 \leq |z-c|$ and hence $|z-c|^{-1} \leq s_1^{-1} < |z_1-c|^{-1}$. Thus $\sum_{n=1}^{\infty} a_{-n}(z-c)^{-n}$ converges normally for $|z-c| \geq s_1$. □

Corollary 7.2 (annulus of convergence)**.** *Let $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-c)^n$. There are unique numbers $r_1, r_2 \in [0,\infty]$ such that $f(z)$ converges absolutely for all $z \in A_{r_1,r_2}(c)$ and diverges for $z \notin \overline{A}_{r_1,r_2}(c)$. If $r_1 < s_1 \leq s_2 < r_2$ then $f(z)$ converges normally on $\overline{A}_{s_1,s_2}(c)$.*

Proof. Follows from Lemma 7.1. □

The function defined by a Laurent series on its annulus of convergence is holomorphic, since it is the uniform limit on compact subsets of a sequence of holomorphic functions. We will now prove the converse: any holomorphic function on an annulus is given by a Laurent series that converges on that annulus.

We start by proving that there is at most one such expansion.

Lemma 7.3 (uniqueness of the Laurent expansion)**.** *Let $0 \leq r_1 < r_2 \leq \infty$. If the Laurent series $\sum_{n=-\infty}^{\infty} a_n(z-c)^n$ converges to a function $f(z)$ on $A_{r_1,r_2}(c)$, then* *for every $r \in (r_1, r_2)$ and each $n \in \mathbb{Z}$,*

$$a_n = \frac{1}{2\pi i} \int_{\partial D_r(c)} \frac{f(\zeta)}{(\zeta - c)^{n+1}}\, d\zeta.$$

In particular, the a_n are uniquely determined by f.

Proof. Since the series converges uniformly on $\partial D_r(c)$,

$$\begin{aligned}\int_{\partial D_r(c)} \frac{f(\zeta)}{(\zeta - c)^{k+1}}\, d\zeta &= \int_{\partial D_r(c)} \sum_{n=-\infty}^{\infty} a_n (\zeta - c)^{n-k-1}\, d\zeta \\ &= \sum_{n=-\infty}^{\infty} a_n \int_{\partial D_r(c)} (\zeta - c)^{n-k-1}\, d\zeta = 2\pi i\, a_k.\end{aligned}$$

□

Theorem 7.4 (existence of the Laurent expansion)**.** *Let $0 \le r_1 < r_2 \le \infty$. If $f \in \mathcal{H}(A_{r_1,r_2}(c))$ then f has a unique Laurent expansion $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-c)^n$ which converges absolutely, and normally on $\overline{A}_{s_1,s_2}(c)$ whenever $r_1 < s_1 < s_2 < r_2$.*

Proof. The cycle $\partial D_{s_2}(c) - \partial D_{s_1}(c)$ is homologous to 0 in $A_{r_1,r_2}(c)$. By Cauchy's formula (6.6),

$$f(z) = \frac{1}{2\pi i} \int_{\partial D_{s_2}(c)} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i} \int_{\partial D_{s_1}(c)} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

for $z \in A_{s_1,s_2}(c)$. For the second integral, observe that

$$\frac{1}{\zeta - z} = -\frac{1}{z - c} \frac{1}{1 - (\zeta - c)/(z - c)} = -\frac{1}{z - c} \sum_{n=0}^{\infty} \Big(\frac{\zeta - c}{z - c}\Big)^n$$

converges uniformly on $\partial D_{s_1}(c)$ because $|(\zeta - c)/(z - c)| = s_1/|z - c| < 1$. Therefore,

$$\begin{aligned}-\int_{\partial D_{s_1}(c)} \frac{f(\zeta)}{\zeta - z}\, d\zeta &= \int_{\partial D_{s_1}(c)} f(\zeta) \sum_{n=0}^{\infty} \frac{(\zeta - c)^n}{(z - c)^{n+1}}\, d\zeta \\ &= \sum_{m<0} \Big(\int_{\partial D_{s_1}(c)} \frac{f(\zeta)}{(\zeta - c)^{m+1}}\, d\zeta \Big) (z - c)^m.\end{aligned}$$

Similarly for the first integral, cf. [**12**, Theorem 12.1]. This implies the statement in view of Corollary 7.2. □

Corollary 7.5. *Let $0 \le r_1 < r_2 \le \infty$ and $f \in \mathcal{H}(A_{r_1,r_2}(c))$. There exists a holomorphic function f^+ on $D_{r_2}(c)$ and a holomorphic function f^- on $\mathbb{C} \setminus \overline{D}_{r_1}(c)$ such that*

$$f(z) = f^+(z) + f^-(z), \quad z \in A_{r_1,r_2}(c). \tag{7.3}$$

This decomposition is unique if we require that $f^-(z) \to 0$ as $|z| \to \infty$.

Proof. Let $f(z) = \sum_{n=-\infty}^{\infty} a_n(z - c)^n$ be the Laurent expansion of f and set

$$f^+(z) := \sum_{n \ge 0} a_n (z - c)^n \quad \text{and} \quad f^-(z) := \sum_{n<0} a_n (z - c)^n.$$

Then (7.3) holds, and $f^-(z) \to 0$ as $|z| \to \infty$. Let $f(z) = f_1(z) + f_2(z)$ be another such decomposition. Consider the function h defined by $h(z) = f^+(z) - f_1(z)$ if $|z - c| < r_2$ and $h(z) = f_2(z) - f^-(z)$ if $|z - c| > r_1$. Then $h \in \mathcal{H}(\mathbb{C})$ and $h(z) \to 0$ as $|z| \to \infty$. By Liouville's theorem, $h = 0$. □

Let U be a domain and $c \in U$. A function $f \in \mathcal{H}(U \setminus \{c\})$ is said to have an **isolated singularity** at c. There are precisely three alternatives:

(1) The singularity c of f is called **removable** if f has a holomorphic extension to c, i.e., there is a holomorphic function $\tilde{f} : U \to \mathbb{C}$ with $\tilde{f}|_{U\setminus\{c\}} = f$. By Riemann's theorem on removable singularities, this is the case if and only if f is bounded near c.
(2) The singularity c of f is called a **pole** of **order** m if there are complex numbers $a_1, \ldots, a_m$, where $m > 0$ and $a_m \neq 0$, such that

$$f(z) - \sum_{k=1}^{m} \frac{a_k}{(z-c)^k}$$

has a removable singularity at c. Note that f has a pole at c if and only if $|f(z)| \to \infty$ as $z \to c$.
(3) Singularities that are neither removable nor poles are called **essential** singularities. For instance, $f(z) = \exp(1/z)$ has an essential singularity at 0. The big Picard theorem 27.2 says that the f assumes all values in $\mathbb{C}$ except possibly one in any neighborhood of an essential singularity.

Theorem 7.6 (classification of singularities via Laurent series)**.** *If $f \in \mathcal{H}(D_r^*(c))$, where $D_r^*(c) := D_r(c) \setminus \{c\}$, then f has a unique Laurent series expansion $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-c)^n$ on $D_r^*(c)$. There are three alternatives:*

(1) *$a_n = 0$ for all $n < 0$.*
(2) *$a_n = 0$ for all $n < k < 0$ and $a_k \neq 0$.*
(3) *Neither* (1) *nor* (2) *applies.*

They correspond precisely to the following cases:

(1') *c is a removable singularity of f.*
(2') *c is a pole of order $-k$.*
(3') *c is an essential singularity.*

Proof. (1) $\Rightarrow$ (1') The power series $f(z) = \sum_{n=0}^{\infty} a_n(z-c)^n$ converges on $D_r(c)$ and represents a holomorphic function on $D_r(c)$.

(1') $\Rightarrow$ (1) If $\tilde{f}$ is the holomorphic extension of f to c then $\tilde{f}$ has a power series expansion $\tilde{f}(z) = \sum_{n=0}^{\infty} b_n(z-c)^n$ on $D_r(c)$. By the uniqueness of the Laurent expansion 7.3, $a_n = b_n$ for $n \geq 0$ and $a_n = 0$ for $n < 0$.

(2) $\Leftrightarrow$ (2') The statement is immediate since the equivalence of (1) and (1') is already established.

(3) $\Leftrightarrow$ (3') These are the only remaining possibilities. $\square$

Let c be an isolated singularity of f. Let $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-c)^n$ be the Laurent expansion of f at c. If $f \not\equiv 0$, we define the **order** of f at c by

$$\operatorname{ord}_c(f) := \inf\{n : a_n \neq 0\}.$$

We also set $\operatorname{ord}_c(0) := \infty$. We call $\sum_{n=-\infty}^{-1} a_n(z-c)^n$ the **principal part** of f at c; note that it defines a function which is holomorphic in $\mathbb{C}_c^*$, cf. Corollary 7.5.

If c is a pole of f of order k, then we have

$$a_n = \lim_{z\to c} \frac{1}{(k+n)!} \Big(\frac{d}{dz}\Big)^{k+n} (z-c)^k f(z), \quad n \geq -k.$$

Example 7.7. (1) The Laurent series expansion of $f(z) = z/(z-1)$ about 1 is given by

$$f(z) = \frac{z}{z-1} = \frac{1+(z-1)}{z-1} = \frac{1}{z-1} + 1.$$

(2) The function $f(z) = z^{-1}(z-1)^{-1}$ is holomorphic in $\mathbb{C} \setminus \{0, 1\}$. It has two Laurent series expansions about 0. Namely, for $0 < |z| < 1$,

$$\frac{1}{z(z-1)} = -\frac{1}{z} - \frac{1}{1-z} = -\frac{1}{z} - 1 - z - z^2 - \cdots$$

and for $|z| > 1$,

$$\frac{1}{z(z-1)} = -\frac{1}{z} + \frac{1}{z}\frac{1}{1-z^{-1}} = \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \cdots$$

Exercise 9. The function $f(z) = 6z^{-1}(z+1)^{-1}(z-2)^{-1}$ is holomorphic in $\mathbb{C} \setminus \{0, -1, 2\}$. It has three Laurent expansions about 0. Compute them.

Let $U \subseteq \mathbb{C}$ be a domain such that $U \supseteq \{z \in \mathbb{C} : |z| > R\}$ for some $R > 0$. For $f \in \mathcal{H}(U)$ let us consider $\tilde{f} : \{z \in \mathbb{C} : 0 < |z| < 1/R\} \to \mathbb{C}$ defined by $\tilde{f}(z) := f(1/z)$. We say that

(1) f has a **removable singularity at** ∞ if $\tilde{f}$ has a removable singularity at 0.
(2) f has a **pole of order** k **at** ∞ if $\tilde{f}$ has a pole of order k at 0.
(3) f has a **essential singularity at** ∞ if $\tilde{f}$ has a essential singularity at 0.

The Laurent expansion of $\tilde{f}$ about 0, $\tilde{f}(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, yields a series expansion which converges for $|z| > R$,

$$f(z) = \tilde{f}(1/z) = \sum_{n=-\infty}^{\infty} a_n z^{-n} = \sum_{n=-\infty}^{\infty} a_{-n} z^n.$$

It is called the **Laurent expansion of** f **about** ∞. By Theorem 7.6, f has removable singularity at ∞ if and only if its Laurent series has no positive powers of z with nonzero coefficients. Furthermore, f has a pole (resp. essential singularity) at ∞ if and only if the Laurent series has only a finite number of (resp. infinitely many) positive powers of z with nonzero coefficients.

Proposition 7.8. *An entire function f has a pole at infinity if and only if f is a non-constant polynomial. It has a removable singularity at ∞ if and only if it is constant.*

Proof. Since f is entire,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{7.4}$$

for all $z \in \mathbb{C}$. Hence $\tilde{f}(z) = \sum_{n=0}^{\infty} a_n z^{-n}$ for all $z \in \mathbb{C} \setminus \{0\}$. Since the Laurent expansion is unique, this is the only possible Laurent expansion of $\tilde{f}$ about 0, and so (7.4) is the Laurent expansion of f about ∞. The assertions follow from the observations made above. $\square$

Let $U \subseteq \mathbb{C}$ be a domain, and let A be a **discrete** subset of U: by this we mean that A is closed in U and has no accumulation point in U. Recall that a function $f \in \mathcal{H}(U \setminus A)$ is said to be **meromorphic** in U if f has either a removable singularity or a pole at each point of A.

Let f be meromorphic on a domain U such that $U \supseteq \{z \in \mathbb{C} : |z| > R\}$ for some $R > 0$. We say that f is **meromorphic at** ∞ if $\tilde{f}(z) = f(1/z)$ is meromorphic on $D_{1/R}(0)$, or equivalently, f has a removable singularity or a pole at ∞ and no poles in $\{z \in \mathbb{C} : |z| > R'\}$ for some $R' > R$. We will see in Proposition 31.2 that the meromorphic functions on the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ are precisely the rational functions. A **rational function** is a quotient of polynomials.

8. The residue theorem

Let $U \subseteq \mathbb{C}$ be a domain and let $A \subseteq U$ be a discrete subset. Let $f \in \mathcal{H}(U \setminus A)$ and let $a \in A$. Choose $r > 0$ such that $\overline{D}_r(a) \subseteq U$ and $\overline{D}_r(a) \cap A = \{a\}$. Then f has a Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n, \quad 0 < |z-a| < r.$$

The number c_{-1} is called the **residue** of f at a, we write $\operatorname{res}(f; a) := c_{-1}$. If a is a pole of order m of f, then

$$\operatorname{res}(f; a) = \lim_{z \to a} \frac{1}{(m-1)!} \Big(\frac{d}{dz}\Big)^{m-1} (z-a)^m f(z).$$

Theorem 8.1 (residue theorem). *Let $U \subseteq \mathbb{C}$ be a domain and let $A \subseteq U$ be a discrete subset. Let γ be a cycle in $U \setminus A$ that is homologous to zero in U. Then, for any $f \in \mathcal{H}(U \setminus A)$, the set $\{a \in A : \operatorname{ind}_\gamma(a) \neq 0\}$ is finite and*

$$\frac{1}{2\pi i} \int_\gamma f \, dz = \sum_{a \in A} \operatorname{res}(f; a) \operatorname{ind}_\gamma(a). \tag{8.1}$$

Proof. Set $B := \{a \in A : \operatorname{ind}_\gamma(a) \neq 0\}$. Let V be any connected component of $\mathbb{C} \setminus |\gamma|$. If V is unbounded or if $V \cap (\mathbb{C} \setminus U) \neq \emptyset$, then $\operatorname{ind}_\gamma$ vanishes on V, since γ is homologous to zero in U and since $\operatorname{ind}_\gamma$ is locally constant. Since A has no accumulation point in U, B must be finite.

Let $a_1, \ldots, a_n$ be the points of B and let $g_1, \ldots, g_n$ be the principal parts of f at $a_1, \ldots, a_n$. The function $f - \sum_{j=1}^n g_j$ has removable singularities at $a_1, \ldots, a_n$ and thus application of the homology form of Cauchy's theorem 6.2 on the domain $U \setminus (A \setminus B)$ gives

$$\int_\gamma f \, dz = \sum_{j=1}^n \int_\gamma g_j \, dz.$$

(Note that γ is homologous to zero with respect to $U \setminus (A \setminus B)$ since $\operatorname{ind}_\gamma(z) = 0$ for all z in $\mathbb{C} \setminus (U \setminus (A \setminus B)) = (\mathbb{C} \setminus U) \cup (A \setminus B)$ by assumption and by the definition of B.) We have $g_j(z) = \sum_{n=-\infty}^{-1} c_{j,n} (z - a_j)^n$ on $\mathbb{C}^*_{a_j}$, and the series converges uniformly on $|\gamma|$, whence

$$\int_\gamma g_j \, dz = \sum_{n=-\infty}^{-1} c_{j,n} \int_\gamma (z-a_j)^n \, dz = 2\pi i \, c_{j,-1} \operatorname{ind}_\gamma(a_j).$$

Here we use that only the summand for $n = -1$ is non-zero, by Corollary 3.3, since $(z - a_j)^n$ has a primitive $(z - a_j)^{n+1}/(n+1)$ on $\mathbb{C}^*_{a_j}$ if $n \neq -1$. This implies (8.1). □

Theorem 8.2 (argument principle). *Let f be meromorphic in U with zeros a_j and poles b_k, and let γ be a cycle which is homologous to zero in U and does not pass through any of the zeros or poles. Then*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'}{f} \, dz = \sum_j \operatorname{ind}_\gamma(a_j) - \sum_k \operatorname{ind}_\gamma(b_k), \tag{8.2}$$

where multiple zeros or poles are repeated according to their order.

Proof. Suppose that c is a zero of order m of f, i.e., $f(z) = (z-c)^m g(z)$, where g is holomorphic and nowhere vanishing in a neighborhood of c. Thus,

$$\frac{f'(z)}{f(z)} = \frac{m}{z-c} + \frac{g'(z)}{g(z)},$$

i.e., f'/f has a simple pole with residue m at c. The same arguments show that if f has a pole of order m at c, then f'/f has a simple pole with residue $-m$ at c. So (8.2) follows from (8.1). □

Exercise 10. Prove: Let f be meromorphic in U with zeros a_j and poles b_k, and let γ be a cycle which is homologous to zero in U and does not pass through any of the zeros or poles. Then

$$\frac{1}{2\pi i}\int_\gamma \frac{zf'(z)}{f(z)}\,dz = \sum_j \operatorname{ind}_\gamma(a_j)a_j - \sum_k \operatorname{ind}_\gamma(b_k)b_k,$$

where multiple zeros or poles are repeated according to their order.

Theorem 8.3 (Rouché's theorem)**.** *Let $U \subseteq \mathbb{C}$ be a domain and $f, g \in \mathcal{H}(U)$. Suppose that $\overline{D}_r(c) \subseteq U$ and*

$$|f(z) - g(z)| < |f(z)| + |g(z)|, \quad z \in \partial D_r(c). \tag{8.3}$$

Then

$$\#(\textit{zeros of } f \textit{ in } D_r(c)) = \#(\textit{zeros of } g \textit{ in } D_r(c)) \tag{8.4}$$

where the zeros are counted with their multiplicity.

Proof. The condition (8.3) implies that f and g cannot vanish on $\partial D_r(c)$. Moreover, $f(z)/g(z)$ cannot take a value in $(-\infty, 0]$ for $z \in \partial D_r(c)$; otherwise

$$\Big|\frac{f(z)}{g(z)} - 1\Big| = -\frac{f(z)}{g(z)} + 1 = \Big|\frac{f(z)}{g(z)}\Big| + 1$$

which contradicts (8.3). It follows that $tf(z) + (1-t)g(z) \neq 0$ for each $t \in [0,1]$ and $z \in \partial D_r(c)$.

Consider the curve of holomorphic functions $f_t(z) = tf(z) + (1-t)g(z)$, $t \in [0,1]$, and the path integral

$$I(t) := \frac{1}{2\pi i}\int_{\partial D_r(c)} \frac{f_t'(z)}{f_t(z)}dz, \quad t \in [0,1].$$

Then $I(t)$ is a continuous function of $t \in [0,1]$ and $I(t) = \#(\text{zeros of } f_t \text{ in } D_r(c))$ by the argument principle. This implies (8.4). □

Example 8.4. Let us determine the number of zeros of $f(z) = z^7 + 5z^3 - z - 2$ in $\mathbb{D}$. Set $g(z) = 5z^3$. Then for $|z| = 1$,

$$|f(z) - g(z)| = |z^7 - z - 2| \leq 4 < |f(z)| + |g(z)|.$$

Rouché's theorem implies that f and g have the same number of zeros in $\mathbb{D}$, namely three.

Exercise 11. Deduce the fundamental theorem of algebra from Rouché's theorem: any polynomial $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ has n roots counted with their multiplicities.

Theorem 8.5 (Hurwitz' theorem)**.** *Let $U \subseteq \mathbb{C}$ be a region and let f_k be a sequence of non-vanishing holomorphic functions on U. If f_k converges uniformly on compact subsets of U to a function f, then either f is non-vanishing or $f = 0$.*

Proof. Assume that $f \neq 0$ and $f(c) = 0$ for some $c \in U$. There is $r > 0$ such that $\overline{D}_r(c) \subseteq U$ and f is non-zero on $\overline{D}_r(c) \setminus \{c\}$. By the argument principle 8.2,

$$\frac{1}{2\pi i} \int_{\partial D_r(c)} \frac{f'(z)}{f(z)} dz = \operatorname{ord}_c(f) \neq 0, \tag{8.5}$$

and for all k,

$$\frac{1}{2\pi i} \int_{\partial D_r(c)} \frac{f_k'(z)}{f_k(z)} dz = 0. \tag{8.6}$$

This leads to a contradiction, since the integrals in (8.6) tend to the integral in (8.5) as $k \to \infty$, because $f_k \to f$ and $f_k' \to f'$ uniformly on $\partial D_r(c)$. □

9. Evaluation of integrals

The calculus of residues provides a method of computing a wide range of integrals. Let us describe three standard classes of integrals.

Example 9.1. Consider an integral of the form

$$I = \int_0^{2\pi} R(\cos t, \sin t)\, dt,$$

where $R(x, y)$ is a rational function without a pole on the circle $x^2 + y^2 = 1$. If we set $z = e^{it}$, then

$$\cos t = \frac{e^{it} + e^{-it}}{2} = \frac{1}{2}\Big(z + \frac{1}{z}\Big), \quad \sin t = \frac{e^{it} - e^{-it}}{2i} = \frac{1}{2i}\Big(z - \frac{1}{z}\Big),$$

and thus

$$\begin{aligned} I &= \int_{S^1} \frac{1}{iz} R\Big(\frac{1}{2}\Big(z + \frac{1}{z}\Big), \frac{1}{2i}\Big(z - \frac{1}{z}\Big)\Big)\, dz \\ &= 2\pi \sum \operatorname{res} \Big[\frac{1}{z} R\Big(\frac{1}{2}\Big(z + \frac{1}{z}\Big), \frac{1}{2i}\Big(z - \frac{1}{z}\Big)\Big)\Big], \end{aligned}$$

where the sum is over all poles in $\mathbb{D}$ of the function in the square brackets.

For instance, for $a > 1$,

$$\int_0^{2\pi} \frac{dt}{a + \sin t} = 2\pi \sum \operatorname{res} \frac{2i}{z^2 + 2aiz - 1}.$$

The function on the right-hand side has two simple poles $p_1 := -ia + i\sqrt{a^2 - 1}$ and $p_2 := -ia - i\sqrt{a^2 - 1}$, but only the first pole lies in $\mathbb{D}$. Its residue is

$$\lim_{z \to p_1} (z - p_1) \frac{2i}{z^2 + 2aiz - 1} = \lim_{z \to p_1} \frac{2i}{z - p_2} = \frac{1}{\sqrt{a^2 - 1}}.$$

Therefore,

$$\int_0^{2\pi} \frac{dt}{a + \sin t} = \frac{2\pi}{\sqrt{a^2 - 1}}.$$

Example 9.2. Let $R(x) = P(x)/Q(x)$, where P, Q are polynomials in one variable such that $\deg Q \geq \deg P + 2$ and Q does not vanish on $\mathbb{R}$. Let $\alpha \in \mathbb{R}_{\geq 0}$. We claim that

$$\int_{-\infty}^{\infty} R(x) e^{i\alpha x}\, dx = 2\pi i \sum_{\operatorname{Im} a > 0} \operatorname{res}(R(z) e^{i\alpha z}; a).$$

The integral $\int_{-\infty}^{\infty} R(x) e^{i\alpha x}\, dx$ converges absolutely, since $\deg Q \geq \deg P + 2$. Let $\gamma(t) = \rho e^{it}$, $t \in [0, \pi]$, where $\rho > 0$. By the residue theorem 8.1,

$$\int_{-\rho}^{\rho} R(x) e^{i\alpha x}\, dx + \int_\gamma R(z) e^{i\alpha z}\, dz = 2\pi i \sum_{\operatorname{Im} a > 0} \operatorname{res}(R(z) e^{i\alpha z}; a).$$

provided that ρ is large enough. If $z = \rho e^{it}$, then there is a constant $M > 0$ such that $|R(z)| \le M/\rho^2$ for large ρ, since $\deg Q \ge \deg P + 2$. Moreover, $|e^{i\alpha z}| = e^{-\alpha \operatorname{Im} z} \le 1$ for $z \in |\gamma|$. Thus $|\int_\gamma R(z)e^{i\alpha z}\,dz| \le M\pi/\rho \to 0$ as $\rho \to \infty$.

Example 9.3. Let $R(x) = P(x)/Q(x)$, where P, Q are polynomials in one variable such that $\deg Q \ge \deg P + 2$, Q does not vanish on $\mathbb{R}_{>0}$, and Q has a zero of order at most 1 at 0. Let $0 < \alpha < 1$. We want to compute the integral

$$\int_0^\infty x^\alpha R(x)\,dx.$$

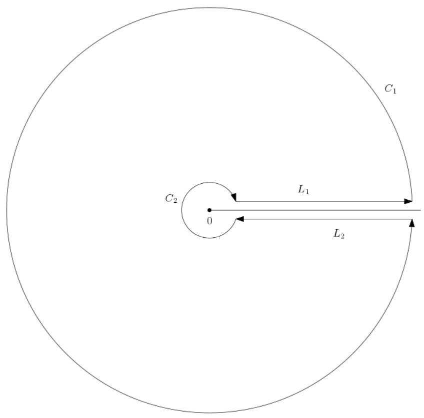
The set $U := \mathbb{C} \setminus \mathbb{R}_{\ge 0}$ is simply connected, so there is a unique branch of the logarithm g in U, i.e., $g \in \mathcal{H}(U)$ and $e^{g(z)} = z$ (cf. Theorem 4.8), such that

$$g(x+iy) \to \log x, \qquad \text{for } x > 0 \text{ as } y \to 0^+.$$

Consequently,

$$g(x-iy) \to \log x + 2\pi i, \qquad \text{for } x > 0 \text{ as } y \to 0^+.$$

Let $\gamma = L_1 + C_1 + L_2 + C_2$ be the path in the figure: L_1, L_2 are segments of the lines $\operatorname{Im} z = \epsilon$, $\operatorname{Im} z = -\epsilon$, and C_1, C_2 are segments of the circles $|z| = \rho$, $|z| = \delta$, where $\rho > \delta$, respectively.



If ρ is sufficiently large and δ sufficiently small, then

$$\int_\gamma e^{\alpha g(z)} R(z)\,dz = 2\pi i \sum_{a \in U} \operatorname{res}(e^{\alpha g(z)} R(z); a).$$

We have $|e^{\alpha g(z)}| = e^{\alpha \log|z|} = |z|^\alpha$. Since Q has a zero of order at most 1 at 0, there is a constant $M > 0$ such that $|R(z)| \le M/|z|$ near 0. Thus $|\int_{C_2} e^{\alpha g(z)} R(z)\,dz| \le 2M\pi\delta^\alpha \to 0$ as $\delta \to 0$. Since $\deg Q \ge \deg P + 2$, we have $|R(z)| \le N/|z|^2$ for large $|z|$ and some constant $N > 0$, and hence $|\int_{C_1} e^{\alpha g(z)} R(z)\,dz| \le 2N\pi\rho^{\alpha-1} \to 0$ as $\rho \to \infty$. For fixed δ and ρ,

$$\int_{L_1+L_2} e^{\alpha g(z)} R(z)\,dz \to (1 - e^{2\pi i \alpha}) \int_\delta^\rho e^{\alpha \log(x)} R(x)\,dx \qquad \text{as } \epsilon \to 0.$$

It follows that

$$\int_0^\infty x^\alpha R(x)\,dx = \frac{2\pi i}{1 - e^{2\pi i \alpha}} \sum_{a \in \mathbb{C} \setminus \mathbb{R}_{\ge 0}} \operatorname{res}(e^{\alpha g(z)} R(z); a).$$

Exercise 12. Show that
$$\int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}.$$
Hint: Integrate $1/(1+z^2)$ along the closed path formed by the segment $[0,R]$, the arc Re^{it}, $t \in [0,\pi]$, and the segment $[-R,0]$.

Exercise 13. Show that the function $z \mapsto \pi\cot(\pi z)$ is meromorphic in $\mathbb{C}$ with a simple pole with residue 1 at each integer n.

Exercise 14. Let $f(z) = P(z)/Q(z)$ be a rational function such that $\deg Q \geq \deg P + 2$. Let $a_1, \ldots, a_m$ be its poles, all of them of order 1, and $b_1, \ldots, b_m$ the respective residues, and assume that $a_i \notin \mathbb{Z}$ for all $i = 1, \ldots, m$. Let γ_n be the counter-clockwise oriented boundary of the square with vertices $(n+1/2)(\pm 1 \pm i)$, where n is a positive integer. Prove that there exist positive constants $C, K > 0$ independent of n such that $|\pi\cot(\pi z)| \leq C$ on $|\gamma_n|$ and $|f(z)| \leq K|z|^{-2}$ if $|z|$ is sufficiently large. Conclude that
$$\lim_{n\to\infty} \int_{\gamma_n} f(z)\pi\cot(\pi z)\,dz = 0,$$
and that
$$\lim_{n\to\infty} \sum_{k=-n}^{n} f(k) = -\sum_{i=1}^{m} b_i\pi\cot(\pi a_i).$$
Note that $\lim_{n,n'\to\infty} \sum_{k=-n}^{n'} f(k)$ exists, since $|f(z)| \leq K|z|^{-2}$ for large $|z|$, and hence the last identity is equivalent to
$$\sum_{k=-\infty}^{\infty} f(k) = -\sum_{i=1}^{m} b_i\pi\cot(\pi a_i).$$

Exercise 15. Use Exercise 14 to show that $\sum_{n=0}^{\infty} 1/(n^2+1) = (1+\pi\coth(\pi))/2$.

CHAPTER 3

Runge's theorem and its applications

10. The inhomogeneous Cauchy–Riemann equation

Holomorphic functions are characterized by the Cauchy–Riemann equation $\partial f/\partial\overline{z} = 0$. We shall now discuss the inhomogeneous equation $\partial f/\partial\overline{z} = g$.

We start with a Cauchy integral formula for C^1-functions. For such functions, Cauchy's theorem is a special case of Stokes' theorem. Let $U \subseteq \mathbb{C}$ be a bounded domain such that the boundary ∂U consists of a finite number of simple closed C^1-paths. If $g \in C^1(\overline{U})$, then by Stokes' theorem,

$$\int_{\partial U} g\,d\zeta = \iint_U dg \wedge d\zeta = \iint_U \left(g_\zeta\,d\zeta + g_{\overline{\zeta}}\,d\overline{\zeta}\right) \wedge d\zeta = \iint_U g_{\overline{\zeta}}\,d\overline{\zeta} \wedge d\zeta, \tag{10.1}$$

where ∂U is oriented such that U lies on the left of ∂U. So if g is also holomorphic in U, then $g_{\overline{\zeta}} = 0$ and hence $\int_{\partial U} g\,d\zeta = 0$.

Proposition 10.1 (inhomogeneous Cauchy integral formula)**.** *Let $U \subseteq \mathbb{C}$ be a bounded domain such that the boundary ∂U consists of a finite number of simple closed C^1-paths. If $f \in C^1(\overline{U})$, then*

$$f(z) = \frac{1}{2\pi i}\int_{\partial U} \frac{f(\zeta)}{\zeta - z}\,d\zeta + \frac{1}{2\pi i}\iint_U \frac{\partial f(\zeta)}{\partial\overline{\zeta}}\frac{1}{\zeta - z}\,d\zeta \wedge d\overline{\zeta}, \quad z \in U. \tag{10.2}$$

The boundary ∂U is oriented such that U lies on the left of ∂U.

Proof. For fixed z set $U_\epsilon := \{\zeta \in U : |z - \zeta| > \epsilon\}$, where $\epsilon > 0$ is smaller that the distance of z to the complement of U. We apply (10.1) to $g : U_\epsilon \to \mathbb{C}$, $g(\zeta) = f(\zeta)/(\zeta - z)$, and note that $U_\epsilon \ni \zeta \mapsto (\zeta - z)^{-1}$ is holomorphic,

$$\iint_{U_\epsilon} \frac{\partial f(\zeta)}{\partial\overline{\zeta}}\frac{1}{\zeta - z}\,d\overline{\zeta} \wedge d\zeta = \int_{\partial U} \frac{f(\zeta)}{\zeta - z}\,d\zeta - \int_0^{2\pi} f(z + \epsilon e^{it})i\,dt. \tag{10.3}$$

Now $\zeta \mapsto (\zeta - z)^{-1}$ is integrable over U, in fact, if $\zeta = \xi + i\eta = re^{i\varphi}$,

$$\iint_U |\zeta - z|^{-1}\,d(\xi,\eta) = \iint_{U-z} |\zeta|^{-1}\,d(\xi,\eta) \le \int_0^{2\pi}\int_0^R dr d\varphi < \infty,$$

since $U - z$ (being bounded) is contained in a large disk $D_R(0)$. Together with the fact that f and $\partial f/\partial\overline{\zeta}$ are continuous, it implies (10.2) by letting $\epsilon \to 0$ in (10.3). $\square$

In the following

$$d\xi d\eta = -\frac{1}{2i}d\zeta \wedge d\overline{\zeta}, \quad (\zeta = \xi + i\eta),$$

denotes the Lebesgue measure in the ζ-plane $\mathbb{C}$.

Theorem 10.2 (inhomogeneous CR-equation (I))**.** *Let $f \in C^k_c(\mathbb{C})$, $k = 1, 2, \dots, \infty$. Then the function*

$$u(z) = -\frac{1}{\pi}\iint_{\mathbb{C}} \frac{f(\zeta)}{\zeta - z}\,d\xi d\eta = \frac{1}{2\pi i}\iint_{\mathbb{C}} \frac{f(\zeta)}{\zeta - z}\,d\zeta \wedge d\overline{\zeta} \tag{10.4}$$

is in $C^k(\mathbb{C})$ *and satisfies*

$$\frac{\partial u}{\partial \overline{z}} = f. \tag{10.5}$$

Proof. We have

$$u(z) = -\frac{1}{\pi}\iint_{\mathbb{C}} \frac{f(\zeta)}{\zeta - z}\, d\xi d\eta = -\frac{1}{\pi}\iint_{\mathbb{C}} \frac{f(\zeta + z)}{\zeta}\, d\xi d\eta.$$

Since $\zeta \mapsto 1/\zeta$ is integrable on any compact subset of $\mathbb{C}$, u is continuous. If $h \in \mathbb{R}$, $h \neq 0$,

$$\frac{u(z+h) - u(z)}{h} = -\frac{1}{\pi}\iint_{\mathbb{C}} \frac{1}{\zeta}\frac{f(\zeta + z + h) - f(\zeta + z)}{h}\, d\xi d\eta$$

and letting $h \to 0$, we find

$$\frac{\partial u}{\partial x}(z) = -\frac{1}{\pi}\iint_{\mathbb{C}} \frac{1}{\zeta}\frac{\partial f(\zeta + z)}{\partial \xi}\, d\xi d\eta = -\frac{1}{\pi}\iint_{\mathbb{C}} \frac{1}{\zeta - z}\frac{\partial f(\zeta)}{\partial \xi}\, d\xi d\eta,$$

and $\partial u/\partial x$ is continuous. Similarly $\partial u/\partial y$ is continuous and

$$\frac{\partial u}{\partial y}(z) = -\frac{1}{\pi}\iint_{\mathbb{C}} \frac{1}{\zeta - z}\frac{\partial f(\zeta)}{\partial \eta}\, d\xi d\eta.$$

Iterating this procedure we find that $u \in C^k(\mathbb{C})$. The formulas for $\partial u/\partial x$ and $\partial u/\partial y$ give

$$\frac{\partial u}{\partial \overline{z}}(z) = -\frac{1}{\pi}\iint_{\mathbb{C}} \frac{1}{\zeta - z}\frac{\partial f(\zeta)}{\partial \overline{z}}\, d\xi d\eta$$

and by the inhomogeneous Cauchy integral formula (10.2) this equals f. □

The discussion of (10.5) will be continued in Theorem 12.2.

Exercise 16. Let $f \in C_c^k(\mathbb{C})$. Show that $u(z) = -1/\pi \iint_{\mathbb{C}} f(\zeta)/(\zeta - z)\, d\xi d\eta$ tends to 0 as $|z| \to \infty$. Prove that u is the only solution of $\partial u/\partial \overline{z} = f$ with this property. Hint: All other solutions are of the form $u + v$, where v is entire.

Exercise 17. Let $f \in C_c^k(\mathbb{C})$ and let u be a solution of $\partial u/\partial \overline{z} = f$ with compact support. Let D be a large disk which contains $\operatorname{supp} u$. Prove that

$$\iint_D f(z)\, dz \wedge d\overline{z} = 0.$$

Conclude that there are functions $f \in C_c^k(\mathbb{C})$ such that no solution u of $\partial u/\partial \overline{z} = f$ has compact support. Hint: Use Stokes' theorem.

Exercise 18. Suppose that $f \in C_c^\infty(\mathbb{C})$ satisfies $\iint_{\mathbb{C}} f(z) z^n\, dx\, dy = 0$ for every integer $n \geq 0$. Prove that the solution (10.4) of (10.5) has compact support. Hint: Expand the kernel $1/(\zeta - z)$ into a geometric series for ζ in some disk D containing $\operatorname{supp} f$ and $z \notin \overline{D}$.

Theorem 10.3 (variant of the Cauchy integral formula)**.** *Let $U \subseteq \mathbb{C}$ be a domain, $K \subseteq U$ compact. Let $\psi \in C_c^\infty(U)$ be 1 on a neighborhood of K. Then for every $f \in \mathcal{H}(U)$, we have*

$$f(z) = -\frac{1}{\pi}\iint_U \frac{\partial \psi(\zeta)}{\partial \overline{\zeta}}\frac{f(\zeta)}{\zeta - z}\, d\xi d\eta, \quad z \in K. \tag{10.6}$$

Proof. Define $\varphi \in C_c^\infty(\mathbb{C})$ by $\varphi(z) := \psi(z) f(z)$ for $z \in U$ and $\varphi(z) := 0$ if $z \notin U$. For $z \in K$,

$$f(z) = \varphi(z) = -\frac{1}{\pi}\iint_U \frac{\partial \varphi(\zeta)}{\partial \overline{\zeta}}\frac{1}{\zeta - z}\, d\xi d\eta = -\frac{1}{\pi}\iint_U \frac{\partial \psi(\zeta)}{\partial \overline{\zeta}}\frac{f(\zeta)}{\zeta - z}\, d\xi d\eta. \qquad \square$$

In comparison with the Cauchy integral formula (6.6) the integration along a path γ has be replaced with integration over $\mathbb{C}$ (or actually over $\operatorname{supp}(\psi) \setminus K$) and the winding number $\operatorname{ind}_\gamma(z)$ no longer appears.

11. Runge's theorem

Let $U \subseteq \mathbb{C}$ be a domain and let $K \subseteq U$ be compact. For continuous functions f on K, we use the notation

$$|f|_K = \sup_{z \in K} |f(z)|.$$

We define a topology on $\mathcal{H}(U)$ by taking as a fundamental system of neighborhoods of $f \in \mathcal{H}(U)$ the sets

$$\{g \in \mathcal{H}(U) : |f - g|_K < \epsilon\}, \quad K \subseteq U \text{ compact}, \epsilon > 0. \tag{11.1}$$

This topology is metrizable, namely it is defined by the metric

$$d(f,g) = \sum_{n=1}^{\infty} 2^{-n} \frac{|f-g|_{K_n}}{1 + |f-g|_{K_n}}, \quad f, g \in \mathcal{H}(U), \tag{11.2}$$

where K_n is a compact exhaustion of U (i.e. K_n are compact subsets of U with $K_{n-1} \subseteq \operatorname{int} K_n$ for all n and $U = \bigcup_{n \geq 1} K_n$). It makes $\mathcal{H}(U)$ a complete metric space. This topology is called the **topology of compact convergence**, since a sequence f_n of functions in $\mathcal{H}(U)$ converges in $\mathcal{H}(U)$ if and only if f_n converges uniformly on any compact set in U. Sometimes it is also called the **compact open topology**.

Exercise 19. Show that d defined by (11.2) is a metric on $\mathcal{H}(U)$ and that $(\mathcal{H}(U), d)$ is a complete metric space. Prove that a sequence in $\mathcal{H}(U)$ converges uniformly on every compact subset of U if and only if it converges for the metric d.

Exercise 20. Prove that the mapping $f \mapsto f'$ from $\mathcal{H}(U)$ to itself is continuous.

Let $K \subseteq \mathbb{C}$ be compact. Let $\mathcal{O}(K)$ denote the set of all $f \in C(K)$ such that there exists an open neighborhood U of K and $F \in \mathcal{H}(U)$ with $F|_K = f$. We consider $\mathcal{O}(K)$ as a subspace of the Banach space $C(K)$; in general it is not closed.

If K is a compact subset of a domain $U \subseteq \mathbb{C}$, then we denote by $\rho_K : \mathcal{H}(U) \to \mathcal{O}(K)$ the restriction mapping $\rho_K(f) = f|_K$.

Theorem 11.1 (Runge's theorem (I))**.** *Let $U \subseteq \mathbb{C}$ be a domain and let $K \subseteq U$ be compact. The following are equivalent:*

1. *Every function which is holomorphic in a neighborhood of K can be approximated uniformly on K by functions in $\mathcal{H}(U)$, i.e., $\rho_K(\mathcal{H}(U))$ is dense in $\mathcal{O}(K)$.*
2. *No connected component of $U \setminus K$ is relatively compact in U.*
3. *For each $a \in U \setminus K$ there exists $f \in \mathcal{H}(U)$ with $|f(a)| > |f|_K$.*

We begin with an easy observation which we will use several times. Occasionally, we will write $V \Subset U$ if we mean that V is an open relatively compact subset of a domain U.

Lemma 11.2. *Let $U \subseteq \mathbb{C}$ be a domain and let $K \subseteq U$ be compact. Suppose that V is a connected component of $U \setminus K$ which is relatively compact in U. Then $\partial V \subseteq K$.*

Proof. Suppose that $a \in \partial V$ and $a \notin K$. Since V is relatively compact in U, we have $a \in U \setminus K$. There is $r > 0$ such that $D = D_r(a) \subseteq U \setminus K$. Since $D \cap V \neq \emptyset$, $D \cup V$ is connected and $D \cup V \subseteq U \setminus K$. This implies $D \subseteq V$ and thus $a \notin \partial V$. $\square$

Proof of Runge's theorem (I). (1) $\Rightarrow$ (2) Suppose that $U \setminus K$ has a connected component V which is relatively compact in U. Let $z_0 \in V$ and $f(z) = 1/(z - z_0)$. Then f is holomorphic in a neighborhood of K. If there is a sequence $f_n \in \mathcal{H}(U)$ which converges to $f|_K$ uniformly on K, then by the maximum principle and since $\partial V \subseteq K$, by Lemma 11.2,

$$|f_n - f_m|_{\overline{V}} = |f_n - f_m|_{\partial V} \leq |f_n - f_m|_K.$$

So $f_n|_V$ converges to a function $g \in \mathcal{H}(V)$ uniformly on V. On the other hand, $(z - z_0)f_n(z) \to 1$ uniformly for $z \in \partial V \subseteq K$, and consequently, $(z - z_0)f_n(z) \to 1$ for $z \in V$, again by the maximum principle. Thus $(z - z_0)g(z) = 1$ for $z \in V$, a contradiction.

(2) $\Rightarrow$ (1) Let $E := \rho_K(\mathcal{H}(U))$. By the Hahn–Banach theorem, E is dense in $\mathcal{O}(K)$ if and only if every bounded linear functional on $C(K)$ which vanishes on E also vanishes on $\mathcal{O}(K)$; see e.g. [**13**, Theorem 5.19]. By the Riesz representation theorem (e.g. [**13**, Theorem 6.19]), it suffices to show that, if μ is a complex Borel measure on K such that $\int_K g\,d\mu = 0$ for all $g \in E$, then also $\int_K f\,d\mu = 0$ for all $f \in \mathcal{O}(K)$. (A proof which does not use the Riesz representation theorem can be found in [**11**].)

Consider the function $h \in \mathcal{H}(\mathbb{C} \setminus K)$ defined by

$$h(z) := \int_K \frac{d\mu(\zeta)}{\zeta - z}, \quad z \in \mathbb{C} \setminus K.$$

(Holomorphy of h can be proved along the lines of [**12**, Theorem 12.1]: if $z \in D = D_r(a) \subseteq \mathbb{C} \setminus K$ then $|z - a|/|\zeta - a| \leq |z - a|/r < 1$, and hence the geometric series $\sum_{n=0}^{\infty} (z - a)^n/(\zeta - a)^{n+1} = 1/(\zeta - z)$ converges uniformly on K, for fixed $z \in D$. By interchanging summation and integration, we see that h is representable by a power series in D.) If $z \in \mathbb{C} \setminus U$ then

$$h^{(k)}(z) = k! \int_K \frac{d\mu(\zeta)}{(\zeta - z)^{k+1}} = 0, \quad k \geq 0,$$

by the assumption on μ, because $\zeta \mapsto 1/(\zeta - z)^{k+1}$ belongs to $\mathcal{H}(U)$. Thus h vanishes in every connected component of $\mathbb{C} \setminus K$ which intersects $\mathbb{C} \setminus U$, by the identity theorem. (Note that we need that all derivatives of h vanish at $z \in \mathbb{C} \setminus U$, since the intersection of $\mathbb{C} \setminus K$ and $\mathbb{C} \setminus U$ need not be open.) Every bounded connected component V of $\mathbb{C} \setminus K$ is of this type. For, otherwise $V \subseteq U$ and $\partial V \subseteq K \subseteq U$ (by Lemma 11.2, since V is relatively compact in $\mathbb{C}$), hence $\overline{V} \subseteq U$. In this case V is a connected component of $U \setminus K$ which is relatively compact in U, a contradiction. That h vanishes also in the unbounded component of $\mathbb{C} \setminus K$ follows from the fact that, for fixed $|z| > \sup_{\zeta \in K} |\zeta|$,

$$-\sum_{n=0}^{N} \frac{\zeta^n}{z^{n+1}} \to \frac{1}{\zeta - z} \quad \text{as } N \to \infty,$$

uniformly for $\zeta \in K$. Summarizing, we showed that $h = 0$ on $\mathbb{C} \setminus K$.

Let $f \in \mathcal{O}(K)$. We must show that $\int_K f\,d\mu = 0$. Let W be an open neighborhood of K and $F \in \mathcal{H}(W)$ such that $F|_K = f$. Choose $\psi \in C_c^{\infty}(W)$ such that $\psi = 1$ on a neighborhood W_0 of K. By the variant of the Cauchy integral formula

10.3, for $z \in K$

$$f(z) = -\frac{1}{\pi} \iint_W \frac{\partial \psi(\zeta)}{\partial \bar{\zeta}} \frac{F(\zeta)}{\zeta - z} \, d\xi d\eta = -\frac{1}{\pi} \iint_{W \setminus W_0} \frac{\partial \psi(\zeta)}{\partial \bar{\zeta}} \frac{F(\zeta)}{\zeta - z} \, d\xi d\eta.$$

By Fubini's theorem,

$$\begin{aligned} \int_K f(z) \, d\mu(z) &= -\frac{1}{\pi} \int_K \iint_{W \setminus W_0} \frac{\partial \psi(\zeta)}{\partial \bar{\zeta}} \frac{F(\zeta)}{\zeta - z} \, d\xi d\eta \, d\mu(z) \\ &= \frac{1}{\pi} \iint_{W \setminus W_0} \frac{\partial \psi(\zeta)}{\partial \bar{\zeta}} F(\zeta) h(\zeta) \, d\xi d\eta = 0, \end{aligned}$$

because h vanishes on $\mathbb{C} \setminus K$.

(2) $\Rightarrow$ (3) Let $U \setminus K = \bigcup_\alpha V_\alpha$ be the decomposition of $U \setminus K$ into connected components. By assumption, none of the sets $\overline{V}_\alpha$ is compactly contained in U. Let $a \in U \setminus K$ and $a \in V_\beta$. Set $L := K \cup \{a\}$. Then $U \setminus L = \bigcup_{\alpha \neq \beta} V_\alpha \cup V_\beta \setminus \{a\}$ is the decomposition of $U \setminus L$ into connected components. No component of $U \setminus L$ is relatively compact in U. By the implication (2) $\Rightarrow$ (1), the set $\rho_L(\mathcal{H}(U))$ is dense in $\mathcal{O}(L)$. The function φ defined by $\varphi = 0$ on K and $\varphi(a) = 1$ belongs to $\mathcal{O}(L)$, since $a \notin K$. There is $f \in \mathcal{H}(U)$ such that $|f - \varphi|_L < 1/2$, and hence

$$|f(a)| > \frac{1}{2} > |f|_K.$$

(3) $\Rightarrow$ (2) If V is a connected component of $U \setminus K$ which is relatively compact in U, then $\partial V \subseteq K$, by Lemma 11.2. By the maximum principle, if $a \in V$, then

$$|f(a)| \le |f|_{\partial V} \le |f|_K$$

for all $f \in \mathcal{H}(U)$, contradicting (3). $\square$

Let $U \subseteq \mathbb{C}$ be a domain and K a compact subset of U. Let us define the $\mathcal{H}(U)$**-hull** of K by

$$\widehat{K} = \widehat{K}_U := \{z \in U : |f(z)| \le |f|_K \text{ for all } f \in \mathcal{H}(U)\}.$$

Lemma 11.3 (properties of $\widehat{K}$)**.** *We have:*

1. $\operatorname{dist}(K, \mathbb{C} \setminus U) = \operatorname{dist}(\widehat{K}, \mathbb{C} \setminus U)$.
2. $\widehat{K}$ *is contained in the convex hull of* K.
3. $\widehat{K}$ *is the union of* K *and the components of* $U \setminus K$ *which are relatively compact in* U.
4. $\widehat{K}$ *is compact and* $\widehat{\widehat{K}} = \widehat{K}$.
5. $\mathbb{C} \setminus \widehat{K}$ *has only finitely many connected components.*

Proof. (1) Clearly, $\operatorname{dist}(K, \mathbb{C} \setminus U) \ge \operatorname{dist}(\widehat{K}, \mathbb{C} \setminus U)$ since $K \subseteq \widehat{K}$. For $\zeta \notin U$ the function $z \mapsto 1/(z - \zeta)$ belongs to $\mathcal{H}(U)$. So if $z \in \widehat{K}$ then

$$\frac{1}{|z - \zeta|} \le \sup_{z \in K} \frac{1}{|z - \zeta|} = \frac{1}{\operatorname{dist}(K, \zeta)}$$

which implies $\operatorname{dist}(K, \mathbb{C} \setminus U) \le \operatorname{dist}(\widehat{K}, \mathbb{C} \setminus U)$.

(2) Let $a \in \mathbb{C}$. Then $f(z) = e^{az}$ belongs to $\mathcal{H}(U)$. If $z \in \widehat{K}$ then

$$|e^{az}| \le \sup_{w \in K} |e^{aw}|$$

or equivalently

$$\operatorname{Re} a \operatorname{Re} z - \operatorname{Im} a \operatorname{Im} z \le \sup_{w \in K} (\operatorname{Re} a \operatorname{Re} w - \operatorname{Im} a \operatorname{Im} w).$$

That means that z is contained in the closed half-plane normal to $(\operatorname{Im} a, \operatorname{Re} a)$ which contains K. Since $a \in \mathbb{C}$ is arbitrary, z lies in the intersection of all closed half-planes containing K which is precisely the convex hull of K, because K is compact.

(3) Let V be a connected component of $U \setminus K$ which is relatively compact in U. Then $\partial V \subseteq K$, by Lemma 11.2, and by the maximum principle

$$|f|_{\overline{V}} \leq |f|_K, \qquad \text{for all } f \in \mathcal{H}(U).$$

That is, $V \subseteq \widehat{K}$. This shows that the union of K and all such components is contained in $\widehat{K}$. Let us denote this union by $\widetilde{K}$.

For the opposite inclusion $\widehat{K} \subseteq \widetilde{K}$, note first that $U \setminus \widetilde{K}$ is open since it is a union of open components of $U \setminus K$. Thus $\widetilde{K}$ is closed. We claim that $\widetilde{K}$ is compact. To this end let $\Omega \Subset U$ with $K \subseteq \Omega$. We assert that there are only finitely many connected components V of $U \setminus K$ relatively compact in U and such that $V \not\subseteq \Omega$. This implies the claim, since $\widetilde{K}$ is contained in the finite union of Ω and such V, and is therefore relatively compact in U. There is a finite family $D_1, \ldots, D_k$ of open disks disjoint from K which cover $\partial\Omega$. It suffices to show that every component V of $U \setminus K$ which is relatively compact in U and satisfies $V \not\subseteq \Omega$ contains some disk D_j; no two components of $U \setminus K$ can contain the same disk. Since $\partial V \subseteq K$, by Lemma 11.2, we have $\Omega \cap V \neq \emptyset$. Moreover, $\partial\Omega \cap V \neq \emptyset$ since otherwise $V = (V \cap \Omega) \cup (V \cap (U \setminus \overline{\Omega}))$ would be a partition of V into disjoint nonempty open sets. Therefore $V \cap D_j \neq \emptyset$ for some j, so that $V \cup D_j$ is a connected set contained in $U \setminus K$. It follows that $D_j \subseteq V$. This shows that $\widetilde{K}$ is compact.

No connected component of $U \setminus \widetilde{K}$ is relatively compact in U by the definition of $\widetilde{K}$. Thus, Runge's theorem (I) 11.1 implies that for each $a \in U \setminus \widetilde{K}$ there exists $f \in \mathcal{H}(U)$ with $|f(a)| > |f|_{\widetilde{K}}$, i.e., $\widehat{K} \subseteq \widetilde{K}$.

(4) We saw in the proof of (3) that $\widehat{K}$ is compact. That $\widehat{\widehat{K}} = \widehat{K}$ is obvious.

(5) Let D be an open disk which contains $\widehat{K}$. Since $\mathbb{C} \setminus D$ is connected, there is precisely one connected component V_0 of $\mathbb{C} \setminus \widehat{K}$ containing $\mathbb{C} \setminus D$. Let $V_1, V_2, \ldots$ be the other connected components of $\mathbb{C} \setminus \widehat{K}$; they are all contained in D. We assert that $V_j \not\subseteq U$ for $j \geq 1$. We have $\partial V_j \subseteq \widehat{K}$, by Lemma 11.2. If $V_j \subseteq U$ then V_j is a connected component of $U \setminus \widehat{K}$ and $\overline{V}_j \subseteq U$. That means V_j is a connected component of $U \setminus \widehat{K}$ which is relatively compact in U, contradicting (3).

Suppose that the set $\{V_j\}_{j \geq 1}$ is infinite. By the claim we may choose $z_j \in V_j \setminus U$. Since $V_j \subseteq D$ there is a subsequence again denoted by z_j which converges to some point $z \in \mathbb{C} \setminus U$. Let B be an open disk centered at z and disjoint from $\widehat{K}$ (which is possible since $\widehat{K}$ is compact and contained in U). Then B is contained in some connected component of $\mathbb{C} \setminus \widehat{K}$. But the disk B meets infinitely many V_j, a contradiction. $\square$

So for every compact $K \subseteq U$ the $\mathcal{H}(U)$-hull $\widehat{K}$ is a compact subset of U containing K for which the hypotheses of Runge's theorem are satisfied. Consequently, one may choose an increasing sequence K_j of compact sets in U such that $K_j = \widehat{K}_j$ and every compact subset of U is contained in some K_j.

The next theorem is a version of Runge's theorem for two open sets. We need two topological lemmas.

Lemma 11.4. *Let X be a locally compact Hausdorff space, and let K be a connected component of X which is compact. Then K has a fundamental system of neighborhoods N in X which are both open and closed in X.*

Proof. [**11**, Ch. 5 §3 Proposition 1]. □

Lemma 11.5. *Let Y be a locally compact Hausdorff space, X a closed subset of Y, and K a compact connected component of X. Then there is a fundamental system of neighborhoods U of K in Y such that $\partial U \cap X = \emptyset$.*

Proof. Let Ω be an open relatively compact neighborhood of K in Y. There is a compact set $N \subseteq X$ which is open in X such that $K \subseteq N \subseteq \Omega$, by Lemma 11.4. Then $A := X \setminus N$ is closed in X and hence in Y. There exist disjoint open subsets U_1, U_2 of Y such that $N \subseteq U_1 \subseteq \Omega$ and $A \subseteq U_2$. (This is true since one of the closed sets we want to separate is compact. Since Y is locally compact and Hausdorff, we may consider its one-point compactification $\hat{Y}$ which is compact Hausdorff and hence normal. So there exist disjoint open subsets $\hat{U}_1, \hat{U}_2$ of $\hat{Y}$ such that $\hat{U}_1$ contains N and $\hat{U}_2$ contains the closure $\overline{A}$ of A. Note that N and $\overline{A}$ are disjoint, because $\overline{A}$ differs from A at most by the added point 'infinity' (which is not contained in any compact subset of Y). Since Y is open in $\hat{Y}$, the sets $U_i = Y \cap U_i$, $i = 1, 2$, are as required.) Then $\overline{U}_1 \cap A = \emptyset$ and $\partial U_1 \cap N = \emptyset$. Consequently, $\partial U_1 \cap X = \partial U_1 \cap (N \cup A) = \emptyset$. □

Theorem 11.6 (Runge's theorem (II))**.** *Let $U_1 \subseteq U_2$ be domains in $\mathbb{C}$. The following are equivalent:*

(1) *Every function in $\mathcal{H}(U_1)$ can be approximated by functions in $\mathcal{H}(U_2)$ uniformly on every compact subset of U_1, i.e., $\rho_{U_1}(\mathcal{H}(U_2))$ is dense in $\mathcal{H}(U_1)$.*
(2) *No connected component of $U_2 \setminus U_1$ is compact.*

Proof. (2) $\Rightarrow$ (1) Let K be a compact subset of U_1 and set $L := \widehat{K}_{U_1}$. We claim that $L = \widehat{L}_{U_2}$, i.e., $U_2 \setminus L$ has no relatively compact components in U_2. In fact, if V is a component of $U_2 \setminus L$ which is relatively compact in U_2 then $\partial V \subseteq L \subseteq U_1$, by Lemma 11.2, and thus $V \not\subseteq U_1$ (otherwise V would be a component of $U_1 \setminus L$ which is relatively compact in U_1). Let $a \in V \cap (U_2 \setminus U_1)$ and let C be the component of $U_2 \setminus U_1$ containing a. Then $V \cap C \neq \emptyset$ and so $V \cup C$ is connected, whence $C \subseteq V$. But C is closed in U_2 and V is relatively compact in U_2, and hence C is compact, contradicting (2). Thus we proved that $L = \widehat{L}_{U_2}$.

Let $f \in \mathcal{H}(U_1)$ and $\epsilon > 0$. Then $f|_L \in \mathcal{O}(L)$ and so by Runge's theorem (I) 11.1 there exists $F \in \mathcal{H}(U_2)$ such that $|f - F|_K \leq |f - F|_L < \epsilon$. This shows (1), since K and ϵ were arbitrary.

(1) $\Rightarrow$ (2) Suppose that $U_2 \setminus U_1$ has a compact connected component C. By Lemma 11.5, there is an open relatively compact neighborhood V of C in U_2 with $\partial V \cap (U_2 \setminus U_1) = \emptyset$, i.e., $\partial V \subseteq U_1$. If $a \in C$ then $f(z) = 1/(z-a)$ belongs to $\mathcal{H}(U_1)$. By (1), there is a sequence of functions $F_n \in \mathcal{H}(U_2)$ such that $F_n \to f$ uniformly on ∂V. By the maximum principle,

$$|F_n - F_m|_{\overline{V}} \leq |F_n - F_m|_{\partial V} \to 0 \quad \text{as } n, m \to \infty,$$

and so F_n converges uniformly on V to a function F. Again by the maximum principle, $1 = \lim_{n\to\infty}(z-a)F_n(z) = (z-a)F(z)$ for all $z \in V$, a contradiction. □

Theorem 11.7 (classical Runge theorem)**.** *Let $U \subseteq \mathbb{C}$ be a domain and let $\mathbb{C} \setminus U = \bigcup_{\alpha \in A} C_\alpha$ be the decomposition of $\mathbb{C} \setminus U$ into connected components C_α. Let $A' := \{\alpha \in A : C_\alpha \text{ compact}\}$ and for each $\alpha \in A'$ choose $c_\alpha \in C_\alpha$. Then each $f \in \mathcal{H}(U)$ can be approximated uniformly on compact subsets of U by rational functions all of whose poles are contained in the set $\{c_\alpha\}_{\alpha \in A'}$.*

Proof. Let K be a compact subset of U. By Lemma 11.3, $\mathbb{C} \setminus \widehat{K}$ has finitely many connected components $V_0, V_1, \ldots, V_k$; assume that V_0 is the unbounded one. We saw in the proof of Lemma 11.3(5) that $V_j \not\subseteq U$ for $j \geq 1$. So if C is the connected component of $\mathbb{C} \setminus U$ containing a point $z \in V_j \setminus U$, then $C \subseteq V_j$, and thus C is compact. Therefore, for each $j \geq 1$ there is $\alpha_j \in A'$ such that $C_{\alpha_j} \subseteq V_j$.

Consider $U_0 := \mathbb{C} \setminus \{c_{\alpha_1}, \ldots, c_{\alpha_k}\}$. Then $\widehat{K} \subseteq U_0$ and the connected components of $U_0 \setminus \widehat{K}$ are V_0, $V_1 \setminus \{c_{\alpha_1}\}, \ldots, V_k \setminus \{c_{\alpha_k}\}$, none of which is relatively compact in U_0. Thus, by Runge's theorem (I) 11.1, if $f \in \mathcal{H}(U)$ and $\epsilon > 0$, then $f|_{\widehat{K}} \in \mathcal{O}(\widehat{K})$ and there exists $F \in \mathcal{H}(U_0)$ with $|f - F|_{\widehat{K}} < \epsilon$. If $g_j(z) = \sum_{n=-\infty}^{-1} a_{j,n}(z - c_{\alpha_j})^n$ denotes the principal part of F at c_{α_j} then $F = h + g_1 + \ldots + g_k$ for $h \in \mathcal{H}(\mathbb{C})$. There is a polynomial p with $|p - h|_{\widehat{K}} < \epsilon$. Moreover, if $g_j^N(z) := \sum_{n=-N}^{-1} a_{j,n}(z - c_{\alpha_j})^n$ then $|g_j - g_j^N|_{\widehat{K}} < \epsilon$ for sufficiently large N. Thus, $G := p + g_1^N + \cdots + g_k^N$ is a rational function whose poles are among the points $c_{\alpha_1}, \ldots, c_{\alpha_k}$ and which satisfies

$$|G - f|_{\widehat{K}} \leq |G - F|_{\widehat{K}} + |F - f|_{\widehat{K}} < \epsilon(k + 1) + \epsilon.$$

The proof is complete. □

Corollary 11.8. *Let $U \subseteq \mathbb{C}$ be a domain. Then $\{p|_U : p$ polynomial$\}$ is dense in $\mathcal{H}(U)$ if and only if $\mathbb{C} \setminus U$ has no compact connected component.*

Proof. This follows from Theorem 11.6 and Theorem 11.7. □

Let us briefly discuss a result related to Runge's theorem. Let $K \subseteq \mathbb{C}$ be compact and let $f : K \to \mathbb{C}$ be a function. Under what conditions is f the uniform limit on K of rational functions with poles in $\widehat{\mathbb{C}} \setminus K$? There are two obvious necessary conditions: $f \in C(K)$ and $f \in \mathcal{H}(\mathring{K})$. The Weierstrass approximation theorem states that these conditions are also sufficient if K is an interval in $\mathbb{R}$. Runge's theorem (I) 11.1 asserts that at least functions in $\mathcal{O}(K)$ have this property.

We state without proof a striking result of Mergelyan which says that the mentioned necessary conditions are also sufficient provided that $\widehat{\mathbb{C}} \setminus K$ has finitely many connected components; for proofs see [**13**] or [**8**].

Theorem 11.9 (Mergelyan's theorem)**.** *Let $K \subseteq \mathbb{C}$ be compact and such that $\widehat{\mathbb{C}} \setminus K$ has only finitely many connected components. Let $f : K \to \mathbb{C}$ be continuous and holomorphic in the interior of K. For each $\epsilon > 0$ there is a rational function r with poles in $\widehat{\mathbb{C}} \setminus K$ such that $|f - r|_K < \epsilon$. In particular, if $\widehat{\mathbb{C}} \setminus K$ is connected then r can be taken to be a polynomial.*

Exercise 21. Let $K_1 = \overline{D}_1(4)$, $K_2 = \overline{D}_1(4i)$, $K_3 = \overline{D}_1(-4)$, and $K_4 = \overline{D}_1(-4i)$. Show that there exists a sequence of entire functions f_n such that $f_n \to j$ uniformly on K_j for $j = 1, 2, 3, 4$.

Exercise 22. Prove that there exists a sequence of polynomials p_n such that $p_n \to 1$ uniformly on compact subsets of $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$, $p_n \to -1$ uniformly on compact subsets of $\{z \in \mathbb{C} : \operatorname{Re} z < 0\}$, and $p_n \to 0$ uniformly on compact subsets of $i\mathbb{R}$.

Exercise 23. Prove that there exists a sequence of entire functions f_n such that $f_n \to 1$ uniformly on compact subsets of the open upper half-plane and (f_n) does not converge at any point of the open lower half-plane.

12. The Mittag-Leffler theorem

Recall that $\mathbb{C}_a^* := \mathbb{C} \setminus \{a\}$, for $a \in \mathbb{C}$.

Theorem 12.1 (Mittag-Leffler theorem)**.** *Let $U \subseteq \mathbb{C}$ be a domain and let $A \subseteq U$ be a discrete subset. Suppose that for each $a \in A$ a function $p_a \in \mathcal{H}(\mathbb{C}_a^*)$ is given. Then there exists $f \in \mathcal{H}(U \setminus A)$ such that $f - p_a$ has a removable singularity at a for all $a \in A$.*

In particular, there is $f \in \mathcal{H}(U \setminus A)$ with prescribed principal parts at the points of A.

Proof. If $K \subseteq U$ is compact, then $\widehat{K}$ is compact and $U \setminus \widehat{K}$ has no components which are relatively compact in U, see Lemma 11.3. There is a sequence of compact sets $K_j = \widehat{K}_j$ such that $K_j \subseteq \mathring{K}_{j+1}$ and $\bigcup_{j \geq 1} K_j = U$.

Set $g_j := \sum_{a \in A \cap K_j} p_a$; the sum is finite, since A is discrete. Then $g_{j+1} - g_j = \sum_{a \in A \cap (K_{j+1} \setminus K_j)} p_a \in \mathcal{O}(K_j)$. Since $K_j = \widehat{K}_j$ there exists $h_j \in \mathcal{H}(U)$ such that $|g_{j+1} - g_j - h_j|_{K_j} < 2^{-j}$, by Runge's theorem (I) 11.1. We define

$$f := g_j + \sum_{k \geq j} (g_{k+1} - g_k - h_k) - h_1 - \cdots - h_{j-1} \quad \text{ on } K_j \setminus A.$$

Then f is well-defined and holomorphic in $U \setminus A$, since

$$\begin{aligned} g_j &+ \sum_{k \geq j} (g_{k+1} - g_k - h_k) - h_1 - \cdots - h_{j-1} \\ &= g_{j+1} + \sum_{k \geq j+1} (g_{k+1} - g_k - h_k) - h_1 - \cdots - h_j. \end{aligned}$$

The series $\sum_{k \geq j} (g_{k+1} - g_k - h_k)$ converges uniformly on K_j and thus its sum belongs to $\mathcal{H}(\mathring{K}_j)$. Moreover, $g_j - p_a$ is holomorphic at a if $a \in A \cap K_j$. $\square$

Theorem 12.2 (inhomogeneous CR-equation (II))**.** *Let $U \subseteq \mathbb{C}$ be a domain and let $f \in C^\infty(U)$. Then there exists $u \in C^\infty(U)$ with*

$$\frac{\partial u}{\partial \overline{z}} = f. \tag{12.1}$$

Proof. For each compact $K \subseteq U$ there is $v \in C^\infty(U)$ with $\partial v / \partial \overline{z} = f$ on a neighborhood of K; apply Theorem 10.2 to ψf, where $\psi \in C_c^\infty(U)$ and $\psi = 1$ on some neighborhood of K.

Let K_j be a sequence of compact sets in U such that $K_j \subseteq \mathring{K}_{j+1}$, $K_j = \widehat{K}_j$ and $U = \bigcup_j K_j$. Let $v_j \in C^\infty(U)$ be such that $\partial v_j / \partial \overline{z} = f$ on some neighborhood of K_j. Then $v_{j+1} - v_j \in \mathcal{O}(K_j)$, since $\partial / \partial \overline{z}(v_{j+1} - v_j) = 0$. By Runge's theorem (I) 11.1, there exists $h_j \in \mathcal{H}(U)$ such that $|v_{j+1} - v_j - h_j|_{K_j} < 2^{-j}$. We define

$$u := v_j + \sum_{k \geq j} (v_{k+1} - v_k - h_k) - h_1 - \cdots - h_{j-1} \quad \text{ on } K_j.$$

As in the proof of the Mittag-Leffler theorem 12.1, u is well-defined on U. Since $v_{k+1} - v_k - h_k$ is holomorphic on $\mathring{K}_j$ for $k \geq j$ and the series $\sum_{k \geq j} (v_{k+1} - v_k - h_k)$ converges uniformly on K_j, we have $u - v_j \in \mathcal{H}(\mathring{K}_j)$, and hence $\partial u / \partial \overline{z} = \partial v_j / \partial \overline{z} = f$ on $\mathring{K}_j$. The result follows, since j was arbitrary. $\square$

Exercise 24. Let $U \subseteq \mathbb{C}$ be a domain and let $f \in C^\infty(U)$. Prove that the equation $\Delta u = f$ admits a solution $u \in C^\infty(U)$. Here $\Delta = \partial_x^2 + \partial_y^2 = 4 \partial_{\overline{z}} \partial_z$ is the Laplace operator. Conclude that if $u \in C^2(U)$ satisfies $\Delta u = 0$, then u is actually in $C^\infty(U)$. Hint: Check that $\partial_{\overline{z}} \overline{u} = \overline{\partial_z u}$ and use Theorem 12.2 twice.

We will now discuss the cohomological form of the Mittag-Leffler theorem 12.1 which provides a solution to the **first (additive) Cousin problem** for domains in $\mathbb{C}$. In the following we use the convention $C^\infty(\emptyset) = \{0\}$ and $\mathcal{H}(\emptyset) = \{0\}$.

Proposition 12.3. *Let U be an open set in $\mathbb{R}^n$. Let $\mathfrak{U} = \{U_i\}_{i\in I}$ be an open cover of U. Suppose that for any pair $(i,j) \in I \times I$ there is a function $f_{ij} \in C^\infty(U_i \cap U_j)$, and that for any triple $(i,j,k) \in I \times I \times I$ we have*

$$f_{ik} = f_{ij} + f_{jk} \quad \text{on } U_i \cap U_j \cap U_k. \tag{12.2}$$

Then there exists a family of functions $\{f_i\}_{i\in I}$ with $f_i \in C^\infty(U_i)$ such that

$$f_i - f_j = f_{ij} \quad \text{on } U_i \cap U_j \text{ for all } i,j \in I.$$

Proof. Let $\{\varphi_i\}_{i\in I}$ be a partition of unity relative to $\mathfrak{U}$. The function

$$\begin{cases} \varphi_j(x) f_{ij}(x) & \text{if } x \in U_i \cap U_j, \\ 0 & \text{if } x \in U_i \setminus (U_i \cap U_j), \end{cases}$$

is in $C^\infty(U_i)$; we denote this function simply by $\varphi_j f_{ij}$. Define

$$f_i := \sum_{j \in I\setminus\{i\}} \varphi_j f_{ij} \quad \text{on } U_i.$$

This sum contains only finitely many nonzero terms near any point of U_i, since the family $\{\operatorname{supp}\varphi_i\}$ is locally finite. Thus $f_i \in C^\infty(U_i)$. Taking $i = j = k$ in (12.2) we may conclude that $f_{ii} = 0$ on U_i, and taking $k = i$ we find $f_{ij} + f_{ji} = f_{ii} = 0$, i.e., $f_{ij} = -f_{ji}$ on $U_i \cap U_j$. Then, with (12.2),

$$\begin{aligned} f_k - f_\ell &= \sum_{j \in I\setminus\{k,\ell\}} \varphi_j (f_{kj} - f_{\ell j}) + \varphi_\ell f_{k\ell} - \varphi_k f_{\ell k} \\ &= \sum_{j \in I\setminus\{k,\ell\}} \varphi_j f_{k\ell} + \varphi_\ell f_{k\ell} + \varphi_k f_{k\ell} = \Big(\sum_{j\in I} \varphi_j\Big) f_{k\ell} = f_{k\ell}. \end{aligned}$$

□

Theorem 12.4 (additive Cousin problem)**.** *Let $U \subseteq \mathbb{C}$ be a domain. Let $\mathfrak{U} = \{U_i\}_{i\in I}$ be an open cover of U. Suppose that for any pair $(i,j) \in I \times I$ there is a function $f_{ij} \in \mathcal{H}(U_i \cap U_j)$, and that for any triple $(i,j,k) \in I \times I \times I$ we have*

$$f_{ik} = f_{ij} + f_{jk} \quad \text{on } U_i \cap U_j \cap U_k.$$

Then there exists a family of functions $\{f_i\}_{i\in I}$ with $f_i \in \mathcal{H}(U_i)$ such that

$$f_i - f_j = f_{ij} \quad \text{on } U_i \cap U_j \text{ for all } i,j \in I.$$

Proof. By Proposition 12.3, there is a family $\{\varphi_i\}_{i\in I}$ where $\varphi_i \in C^\infty(U_i)$ and $\varphi_i - \varphi_j = f_{ij}$ on $U_i \cap U_j$ for all $i,j \in I$. In particular, $\partial\varphi_i/\partial\overline{z} - \partial\varphi_i/\partial\overline{z} = 0$ on $U_i \cap U_j$. So there exists $\varphi \in C^\infty(U)$ such that $\varphi|_{U_i} = \partial\varphi_i/\partial\overline{z}$ for all $i \in I$. By Theorem 12.2, there is $u \in C^\infty(U)$ satisfying $\partial u/\partial\overline{z} = \varphi$ on U. Set $f_i := \varphi_i - u$ on U_i. Then $\partial f_i/\partial\overline{z} = 0$ on U_i, i.e., $f_i \in \mathcal{H}(U_i)$. If $i,j \in I$ then $f_i - f_j = \varphi_i - \varphi_j = f_{ij}$ on $U_i \cap U_j$. □

Theorem 12.4 implies the Mittag-Leffler theorem 12.1. Let $U \subseteq \mathbb{C}$ be a domain and let $A \subseteq U$ be discrete. Let U_a be a neighborhood of $a \in A$ in U not containing any other point of A, and let $p_a \in \mathcal{H}(U_a \setminus \{a\})$. Let $*$ be some symbol and set $I := A \cup \{*\}$, $U_* := U \setminus A$, $p_* := 0$. For $i,j \in I$, put $f_{ij} := p_i - p_j$ on $U_i \cap U_j$. Then $f_{ij} \in \mathcal{H}(U_i \cap U_j)$. By Theorem 12.4, there is a family $\{f_i\}_{i\in I}$ with $f_i \in \mathcal{H}(U_i)$ and $f_i - f_j = p_i - p_j$ on $U_i \cap U_j$. Then there is a function f on $U \setminus A$ with $f = p_i - f_i$ on $U_i \setminus A$. In particular, $f = p_* - f_* = -f_* \in \mathcal{H}(U_*) = \mathcal{H}(U \setminus A)$ and $f - p_a = -f_a \in \mathcal{H}(U_a)$ for all $a \in A$. □

Exercise 25. Let U_1, U_2 be domains in $\mathbb{C}$ and let $f \in \mathcal{H}(U_1 \cap U_2)$. Show that there are functions $f_1 \in \mathcal{H}(U_1)$ and $f_2 \in \mathcal{H}(U_2)$ such that $f = f_1 - f_2$ on $U_1 \cap U_2$. For $U_1 = \{z \in \mathbb{C} : \operatorname{Re} z < 1\}$, $U_2 = \{z \in \mathbb{C} : \operatorname{Re} z > -1\}$, and $f(z) = 1/(z^2 - 1)$, find explicit functions f_1, f_2 satisfying the above properties.

13. The cohomology form of Cauchy's theorem

Let $U \subseteq \mathbb{C}$ be a domain. Let $\mathfrak{U} = \{U_i\}_{i \in I}$ be an open cover of U. Let $J := \{(i,j) \in I \times I : U_i \cap U_j \neq \emptyset\}$. For every open subset $V \subseteq \mathbb{C}$ let us consider $\mathbb{C}(V) := \{f : V \to \mathbb{C} : f \text{ locally constant}\}$. Let

$$C^1(\mathfrak{U}, \mathbb{C}) := \prod_{(i,j) \in J} \mathbb{C}(U_i \cap U_j).$$

An element of $C^1(\mathfrak{U}, \mathbb{C})$ is called a **1-cochain** of the cover $\mathfrak{U}$ with values in $\mathbb{C}$. The 1-cochains $(c_{ij})_{(i,j) \in J} \in C^1(\mathfrak{U}, \mathbb{C})$ which satisfy

$$c_{ij} + c_{jk} + c_{ki} = 0 \quad \text{on } U_i \cap U_j \cap U_k \text{ if } U_i \cap U_j \cap U_k \neq \emptyset$$

are called **1-cocycles** of the cover $\mathfrak{U}$ with values in $\mathbb{C}$. Let $Z^1(\mathfrak{U}, \mathbb{C})$ be the set of all 1-cocycles. Let us consider the set of **0-cochains** $C^0(\mathfrak{U}, \mathbb{C}) := \prod_{i \in I} \mathbb{C}(U_i)$ and define a mapping

$$\delta : C^0(\mathfrak{U}, \mathbb{C}) \to Z^1(\mathfrak{U}, \mathbb{C})$$

by assigning $c = (c_i)_{i \in I} \in C^0(\mathfrak{U}, \mathbb{C})$ the element $\delta c \in Z^1(\mathfrak{U}, \mathbb{C})$ given by

$$(\delta c)_{ij} = c_i|_{U_i \cap U_j} - c_j|_{U_i \cap U_j} = c_i - c_j \quad \text{on } U_i \cap U_j \text{ for } (i,j) \in J.$$

Set $B^1(\mathfrak{U}, \mathbb{C}) := \delta C^0(\mathfrak{U}, \mathbb{C})$. Observe that $Z^1(\mathfrak{U}, \mathbb{C})$ and $B^1(\mathfrak{U}, \mathbb{C})$ are complex vector spaces and δ is $\mathbb{C}$-linear. The quotient vector space

$$H^1(\mathfrak{U}, \mathbb{C}) := Z^1(\mathfrak{U}, \mathbb{C}) / B^1(\mathfrak{U}, \mathbb{C})$$

is the **first cohomology group** of the cover $\mathfrak{U}$ with values in $\mathbb{C}$.

Let $U \subseteq \mathbb{C}$ be a domain. Let $\mathfrak{U} = \{U_i\}_{i \in I}$ be an open cover of U by connected, simply connected sets U_i. We define a homomorphism of $\mathbb{C}$-vector spaces

$$\delta_{\mathfrak{U}} : \mathcal{H}(U) \to H^1(\mathfrak{U}, \mathbb{C})$$

as follows. Let $f \in \mathcal{H}(U)$. Then, since U_i is simply connected, there is a primitive F_i of f on U_i, by Theorem 4.9. Then $F_i' - F_j' = f - f = 0$ on $U_i \cap U_j$ so that $c_{ij} := F_i - F_j$ is locally constant on $U_i \cap U_j$. If $U_i \cap U_j \cap U_k \neq \emptyset$ then

$$c_{ij} + c_{jk} + c_{ki} = F_i - F_j + F_j - F_k + F_k - F_i = 0,$$

so that $(c_{ij})_{(i,j) \in J} \in Z^1(\mathfrak{U}, \mathbb{C})$. We let $\delta_{\mathfrak{U}}(f)$ be the class in $H^1(\mathfrak{U}, \mathbb{C})$ of $(c_{ij})_{(i,j) \in J}$.

To show that this definition is meaningful we need to check that it does not depend on the choice of the primitives F_i. Let $\{G_i\}_{i \in I}$ be a different choice. Then $G_i' - F_i' = 0$ on U_i, and since U_i is connected, $c_i := G_i - F_i$ is a constant. If $g_{ij} := G_i - G_j$ on $U_i \cap U_j$, then $g_{ij} - c_{ij} = c_i - c_j$ on $U_i \cap U_j$, i.e., $(g_{ij} - c_{ij})_{(i,j) \in J} \in B^1(\mathfrak{U}, \mathbb{C})$. So $\delta_{\mathfrak{U}}$ is well-defined.

Let us denote by $d = d_U : \mathcal{H}(U) \to \mathcal{H}(U)$ the derivative $d(f) = f'$.

Theorem 13.1 (cohomological form of Cauchy's theorem)**.** *Let $U \subseteq \mathbb{C}$ be a region and let $\mathfrak{U} = \{U_i\}_{i \in I}$ be an open cover of U by connected, simply connected sets U_i. Then the following sequence is exact*

$$0 \longrightarrow \mathbb{C} \xrightarrow{i_U} \mathcal{H}(U) \xrightarrow{d_U} \mathcal{H}(U) \xrightarrow{\delta_{\mathfrak{U}}} H^1(\mathfrak{U}, \mathbb{C}) \longrightarrow 0$$

where i_U sends $c \in \mathbb{C}$ to the constant function $z \mapsto c$ on U.

Proof. Clearly, i_U is injective. We have $d_U(f) = f' = 0$ if and only if f is constant since U is connected, i.e., $\mathrm{im}(i_U) = \ker(d_U)$.

Next we show $\ker(\delta_{\mathfrak{U}}) = \mathrm{im}(d_U)$. If $f = d_U(F)$ then $\delta_{\mathfrak{U}}(f)$ is the class of $F|_{U_i} - F|_{U_j} = 0$ on $U_i \cap U_j$, thus, $\delta_{\mathfrak{U}}(f) = 0$. That means $\mathrm{im}(d_U) \subseteq \ker(\delta_{\mathfrak{U}})$. Conversely, let $f \in \ker(\delta_{\mathfrak{U}})$. Let F_i be a primitive of f on U_i and set $c_{ij} = F_i - F_j$ on $U_i \cap U_j$. Since $\delta_{\mathfrak{U}}(f) = 0$ there exists $(c_i)_{i\in I}$, where c_i is (locally) constant on U_i, such that $c_i - c_j = c_{ij}$ on $U_i \cap U_j$. Thus, $F_i - c_i = F_j - c_j$ on $U_i \cap U_j$, and consequently, there is a function F on U with $F|_{U_i} = F_i - c_i$. Obviously, $F \in \mathcal{H}(U)$ and $d_U(F)|_{U_i} = (F_i - c_i)' = f|_{U_i}$. Hence $f \in \mathrm{im}(d_U)$.

It remains to prove $\delta_{\mathfrak{U}}(\mathcal{H}(U)) = H^1(\mathfrak{U}, \mathbb{C})$. Let $(c_{ij}) \in Z^1(\mathfrak{U}, \mathbb{C})$. Then c_{ij} is locally constant, in particular, $c_{ij} \in \mathcal{H}(U_i \cap U_j)$. By Theorem 12.4, there is a family $(F_i)_{i\in I}$ with $F_i \in \mathcal{H}(U_i)$ and $F_i - F_j = c_{ij}$ on $U_i \cap U_j$. Since $dF_i - dF_j = dc_{ij} = 0$ on $U_i \cap U_j$, there exists $f \in \mathcal{H}(U)$ such that $f|_{U_i} = dF_i$. Then $\delta_{\mathfrak{U}}(f)$ is the class in $H^1(\mathfrak{U}, \mathbb{C})$ of $((F_i - F_j)|_{U_i \cap U_j}) = (c_{ij})$. This proves the theorem. □

Corollary 13.2 (cohomological characterization of integrability)**.** *Let $U \subseteq \mathbb{C}$ be a domain. Then every $f \in \mathcal{H}(U)$ has a primitive if and only if $H^1(\mathfrak{U}, \mathbb{C}) = 0$ for some open cover $\mathfrak{U}$ of U by connected, simply connected sets. If this holds for one such cover, then it holds for any such cover.*

Proof. Fix an open cover $\mathfrak{U}$ of U by connected, simply connected sets. Since $\delta_{\mathfrak{U}}$ is surjective, by Theorem 13.1, $H^1(\mathfrak{U}, \mathbb{C}) = 0$ if and only if $\ker(\delta_{\mathfrak{U}}) = \mathcal{H}(U)$ which is the case if and only if $\mathcal{H}(U) = \mathrm{im}(d_U)$. □

Corollary 13.3. *Let $U \subseteq \mathbb{C}$ be a simply connected region. Then $H^1(\mathfrak{U}, \mathbb{C}) = 0$ for any open cover $\mathfrak{U}$ of U by connected, simply connected sets.*

Proof. Theorem 4.9 and Corollary 13.2. □

We shall see in Theorem 21.3 that also the converse holds.

14. Infinite products

Before we continue with further applications of Runge's theorem we need some background on infinite products.

Let $a_n \in \mathbb{C}$. An **infinite product** $\prod_{n=1}^\infty (1 + a_n)$ is said to **converge** if

- $a_n \neq -1$ for almost all $n \in \mathbb{N}$,
- if $n_0 > 0$ is such that $a_n \neq -1$ for $n > n_0$, then $\lim_{N\to\infty} \prod_{n=n_0+1}^N (1+a_n)$ exists and is nonzero.

If $\prod_{n=1}^\infty (1 + a_n)$ converges then we define its value to be

$$\prod_{n=1}^\infty (1 + a_n) := \prod_{n=1}^{n_0} (1 + a_n) \cdot \lim_{N\to\infty} \prod_{n=n_0+1}^{N} (1 + a_n).$$

This is independent of n_0. If $\prod_{n=1}^\infty (1 + a_n)$ converges then $\lim_{N\to\infty} \prod_{n=1}^N (1 + a_n)$ exists and equals the value of $\prod_{n=1}^\infty (1 + a_n)$. The converse is not true; e.g., $a_n = -1/2$ for all n.

Exercise 26. Show that if $\prod_{n=1}^\infty (1+a_n)$ converges then $\lim_{M,N\to\infty} \prod_{n=M}^N (1+a_n)$ exists and equals 1. In addition show that this is not necessarily true if we allow $\lim_{N\to\infty} \prod_{n=n_0+1}^N (1+a_n) = 0$ in the definition of the convergence of $\prod_{n=1}^\infty (1+a_n)$.

Proposition 14.1. *The infinite product $\prod_{n=1}^\infty (1 + |a_n|)$ converges if and only if $\sum_{n=1}^\infty |a_n|$ converges.*

Proof. Suppose that $\sum_{n=1}^{\infty} |a_n| = s < \infty$. Then, since $1 + x \le e^x$ for $x \ge 0$,

$$1 \le p_N := \prod_{n=1}^{N} (1 + |a_n|) \le \prod_{n=1}^{N} e^{|a_n|} = \exp \sum_{n=1}^{N} |a_n| \le e^s. \tag{14.1}$$

The sequence of partial products p_N is increasing and hence converges to a nonzero limit.

For the other direction observe that $e^x \le 1 + 2x$ for $0 \le x \le 1$. So if $|a_n| \le 1$,

$$p_N \ge \prod_{n=1}^{N} e^{|a_n|/2} = \exp \frac{1}{2} \sum_{n=1}^{N} |a_n|.$$

Thus convergence of p_N implies convergence of $\sum_{n=1}^{\infty} |a_n|$. □

Proposition 14.2. *Convergence of* $\prod_{n=1}^{\infty}(1 + |a_n|)$ *implies convergence of* $\prod_{n=1}^{\infty}(1 + a_n)$.

Proof. Suppose $\prod_{n=1}^{\infty}(1 + |a_n|)$ converges. By Proposition 14.1, $|a_n| \to 0$, in particular, there is n_0 such that $a_n \ne -1$ for $n \ge n_0$.

For $N > n_0$, set $q_N := \prod_{n=n_0+1}^{N}(1 + a_n)$ and $\tilde{q}_N := \prod_{n=n_0+1}^{N}(1 + |a_n|)$. Then, for $N > M > n_0$,

$$|q_N - q_M| = |q_M| \Big| \prod_{n=M+1}^{N} (1 + a_n) - 1 \Big| \le |\tilde{q}_M| \Big| \prod_{n=M+1}^{N} (1 + |a_n|) - 1 \Big| = |\tilde{q}_N - \tilde{q}_M|;$$

note that $\prod_{n=M+1}^{N}(1 + a_n) - 1$ is a sum of monomials in the a_j and $\prod_{n=M+1}^{N}(1 + |a_n|) - 1$ is the same sum, where each a_j is replaced by its absolute value $|a_j|$. So the convergence of the sequence $\tilde{q}_N$ implies the convergence of the sequence q_N. We may choose $M > n_0 + 1$ such that $\prod_{n=M}^{N}(1 + |a_n|) - 1 < 1/2$ for all $N > M$. Then, for such N, $|\prod_{n=M}^{N}(1+a_n) - 1| < 1/2$ and therefore $|\prod_{n=M}^{N}(1+a_n)| > 1/2$. It follows that

$$|q_N| = \Big| \prod_{n=n_0+1}^{M-1} (1 + a_n) \Big| \Big| \prod_{n=M}^{N} (1 + a_n) \Big| \ge \frac{1}{2} \Big| \prod_{n=n_0+1}^{M-1} (1 + a_n) \Big| > 0,$$

and so $\prod_{n=1}^{\infty}(1 + a_n)$ converges. □

Let us now consider infinite products of holomorphic functions.

Theorem 14.3. *Let $U \subseteq \mathbb{C}$ be a domain, and $f_n \in \mathcal{H}(U)$. If $\sum_{n=1}^{\infty} |f_n|$ converges uniformly on compact sets, then the sequence of partial products*

$$p_N(z) = \prod_{n=1}^{N} (1 + f_n(z))$$

converges uniformly on compact sets to a holomorphic limit function $f \in \mathcal{H}(U)$. The function f vanishes at a point $z_0 \in U$ if and only if $f_n(z_0) = -1$ for some n, and

$$\operatorname{ord}_{z_0}(f) = \sum_{n} \operatorname{ord}_{z_0}(1 + f_n).$$

Proof. Fix a compact set $K \subseteq U$. Since $\sum_{n=1}^{\infty} |f_n|$ converges uniformly on K, there is a constant C such that $\sum_{n=1}^{N} |f_n| \le C$ for all N uniformly on K. Then $p_N := \prod_{n=1}^{N}(1 + |f_n|) \le e^C$ for all N uniformly on K, by (14.1).

Let $0 < \epsilon < 1$ and choose L such that for $N \geq M \geq L$, $\sum_{n=M}^{N} |f_n(z)| < \epsilon$ for all $z \in K$. Then, by (14.1),

$$\begin{aligned}
|p_N(z) - p_M(z)| &\leq |p_M(z)| \cdot \Big| \prod_{n=M+1}^{N} (1 + |f_n(z)|) - 1 \Big| \\
&\leq p_M(z) \Big(\exp\Big(\sum_{n=M+1}^{N} |f_n(z)| \Big) - 1 \Big) \\
&\leq e^C (e^\epsilon - 1) \to 0 \quad \text{ as } \epsilon \to 0.
\end{aligned}$$

So the sequence p_N is uniformly Cauchy on K. This implies that f_N is uniformly convergent on K, by Proposition 14.2. Since K was arbitrary, the limit function f is holomorphic on U.

Suppose that $f(z_0) = 0$ for some $z_0 \in U$. By the definition of the convergence of infinite products, there is n_0 such that $\lim_{N\to\infty} \prod_{n=n_0+1}^{N} (1 + f_n(z))$ does not vanish at z_0. This limit represents a holomorphic function and thus is non-vanishing in some neighborhood of z_0. Since

$$f(z) = \prod_{n=1}^{n_0} (1 + f_n(z)) \cdot \lim_{N\to\infty} \prod_{n=n_0+1}^{N} (1 + f_n(z))$$

the statements about the zeros follow. □

Exercise 27. Let (a_n) be a sequence (with repetitions) of points in $\mathbb{D} \setminus \{0\}$ satisfying $\sum_{n=1}^{\infty} (1 - |a_n|) < \infty$. Show that the so-called **Blaschke product**

$$f(z) = \prod_{n=1}^{\infty} \frac{-\overline{a}_n}{|a_n|} \frac{z - a_n}{1 - \overline{a}_n z}$$

converges uniformly on every disk $\overline{D}_r(0)$ with $r < 1$ and defines a holomorphic function on $\mathbb{D}$ with $|f(z)| \leq 1$. Prove that the zeros of f are precisely the a_n's (counted according to their multiplicities). Hint: Apply Theorem 14.3.

15. The Weierstrass theorem

We will use a variant of Runge's theorem (I) 11.1 in which only non-vanishing functions are allowed.

Lemma 15.1. *Let $U \subseteq \mathbb{C}$ be a domain, and let a, b, $a \neq b$, lie in the same connected component of $\mathbb{C} \setminus U$. Then there exists $f \in \mathcal{H}(U)$ such that*

$$e^{f(z)} = \frac{z - a}{z - b}, \quad z \in U.$$

Proof. Let $g(z) = (z - a)/(z - b)$. We will show that g'/g, which is holomorphic on U, has a primitive h on U. Then $(e^{-h} g)' = e^{-h} g' - e^{-h} h' g = 0$ and thus $g = ce^h$ for some $c \neq 0$. If C is such that $c = e^C$, then $f = h + C$ is the desired function.

To see that g'/g has a primitive on U, let γ be any closed curve in U. Then

$$\int_\gamma \frac{g'}{g}\, dz = \int_\gamma \Big(\frac{1}{z - a} - \frac{1}{z - b} \Big)\, dz = 2\pi i (\operatorname{ind}_\gamma(a) - \operatorname{ind}_\gamma(b)) = 0,$$

since a, b lie in the same connected component of $\mathbb{C} \setminus |\gamma|$. □

Proposition 15.2 (variant of Runge's theorem)**.** *Let $U \subseteq \mathbb{C}$ be a domain, and let K be a compact subset of U such that $K = \widehat{K}_U$. Let $f \in \mathcal{O}(K)$ such that $f(z) \neq 0$*

for all $z \in K$. Then for every $\epsilon > 0$ there exists $F \in \mathcal{H}(U)$ such that $F(z) \neq 0$ for all $z \in U$ and $|F - f|_K < \epsilon$.

Proof. By Lemma 11.3, $\mathbb{C} \setminus K$ has only finitely many connected components $V_0, V_1, \ldots, V_n$, where V_0 is the unbounded component. As seen in the proof of Lemma 11.3, $V_j \not\subseteq U$ for all $j \geq 1$. So there exist $a_j \in V_j \setminus U$ for all $j = 1, \ldots, n$. Let $R > 0$ be such that $K \subseteq D_R(0)$ and set $a_0 := R$.

By the classical Runge theorem 11.7, there is a rational function g which is holomorphic and non-vanishing on a neighborhood of K and $|f - g|_K < \epsilon$, so that

$$g(z) = c \prod_{i=1}^{d} (z - b_i)^{m_i},$$

where $c \neq 0$, $m_i \in \mathbb{Z} \setminus \{0\}$, and $b_i \in \mathbb{C} \setminus K$, $i = 1, \ldots, d$.

For $0 \leq j \leq n$, let $A_j := \{i : b_i \in V_j\}$. Then

$$g(z) = c\, H(z)(z - R)^{n_0} \prod_{j=0}^{n} \prod_{i \in A_j} \Big(\frac{z - b_i}{z - a_j}\Big)^{m_i},$$

where

$$H(z) := \prod_{j=1}^{n} (z - a_j)^{n_j}, \quad n_j := \sum_{i \in A_j} m_i.$$

If $i \in A_j$ then a_j and b_i both lie in V_j. By Lemma 15.1, there exists $\phi_{i,j} \in \mathcal{O}(K)$ such that $(z - b_i)/(z - a_j) = e^{\phi_{i,j}(z)}$ on a neighborhood of K. Moreover, there is $\phi_0 \in \mathcal{H}(D_R(0))$ such that $z - R = e^{\phi_0(z)}$ on $D_R(0)$, by Theorem 4.8. Thus, there exists $\ell \in \mathcal{O}(K)$ such that

$$g(z) = c\, H(z) e^{\ell(z)},$$

for z in a neighborhood of K. By Runge's theorem (I) 11.1, for every $\delta > 0$ there is $L \in \mathcal{H}(U)$ with $|L - \ell|_K < \delta$. Then

$$G := cHe^L$$

satisfies $|G - g|_K < \epsilon$, if δ is sufficiently small, and hence $|f - G|_K < 2\epsilon$. Since $a_j \notin U$, H and thus G, does not vanish in U. $\square$

Next we will see that, if $U \subseteq \mathbb{C}$ is a domain and $A \subseteq U$ is discrete, there is a holomorphic function $f \in \mathcal{H}(U)$ which has zeros at the points of A of prescribed orders and is nonzero elsewhere.

Theorem 15.3 (Weierstrass theorem)**.** *Let $U \subseteq \mathbb{C}$ be a domain and let $A \subseteq U$ be discrete. Suppose that for each $a \in A$ an integer m_a is given. Then there is a meromorphic function f on U such that $f|_{U \setminus A}$ is holomorphic and nowhere zero, and $(z - a)^{-m_a} f(z)$ is holomorphic and nonzero at a for all $a \in A$.*

Proof. Let K_j be a sequence of compact sets in U such that $K_j \subseteq \mathring{K}_{j+1}$, $U = \bigcup_j K_j$, and $K_j = \widehat{K}_j$. Set

$$F_j(z) = \prod_{a \in A \cap K_j} (z - a)^{m_a}.$$

Then F_{j+1}/F_j belongs to $\mathcal{O}(K_j)$ and has no zeros on K_j. By Proposition 15.2, there exists $h_j \in \mathcal{H}(U)$ which has no zeros in U and

$$\Big|\frac{F_{j+1}}{F_j} h_j - 1\Big|_{K_j} < \frac{1}{2^{j+1}}, \quad j \geq 1.$$

Indeed, we apply Proposition 15.2 to F_j/F_{j+1} to get $h_j \in \mathcal{H}(U)$ such that

$$\Big| h_j - \frac{F_j}{F_{j+1}} \Big|_{K_j} < \frac{1}{2^{j+1} C_j},$$

where $C_j := |F_{j+1}/F_j|_{K_j}$. Then $\prod_{k \geq j} h_k F_{k+1}/F_k$ is holomorphic and non-vanishing on K_j, by Theorem 14.3. We define f by setting

$$f = F_j \prod_{k \geq j} \Big(\frac{F_{k+1}}{F_k} h_k \Big) h_1 \cdots h_{j-1} \quad \text{on } K_j.$$

Then f is meromorphic on U and has the required properties. □

Remark 15.4. For $U = \mathbb{C}$ the theorem can be proved by writing down an infinite product with the required properties. Define $E_0(z) := 1 - z$ and

$$E_p(z) = (1 - z) \exp\Big(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p} \Big), \quad p = 1, 2, \ldots$$

These functions are called **elementary factors**. They are all entire and their only zero is 1. Let us enumerate the points in $A \setminus \{0\}$ by $a_1, a_2, a_3, \ldots$ and write $m_n := m_{a_n}$. One can show that for a suitable sequence $\{p_n\}$ of positive integers, e.g., $p_n = |m_n| n$, the infinite product

$$z^{m_0} \prod_{n=1}^{\infty} \Big(E_{p_n}\Big(\frac{z}{a_n} \Big) \Big)^{m_n}$$

has the required properties. The general theorem can be proved along similar lines; for details see e.g. [**9**].

In particular, let f be an entire function. Suppose that f vanishes to order m at 0, $m \geq 0$. Let (a_n) be the other zeros of f listed with multiplicities, i.e., $m_n = 1$ for all $n \geq 1$. Then there is an entire function g and a sequence p_n such that

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_{p_n}\Big(\frac{z}{a_n} \Big), \quad z \in \mathbb{C}.$$

This result is called the **Weierstrass factorization theorem**. In fact, the entire function $h(z) = z^m \prod_{n=1}^{\infty} E_{p_n}(z/a_n)$ has the same zeros (with multiplicities) as f. So f/h has only removable singularities, hence can be extended to an entire non-vanishing function. Since $\mathbb{C}$ is simply connected, there is $g \in \mathcal{H}(\mathbb{C})$ such that $f = he^g$, by Theorem 4.8.

Exercise 28. One can show that the **second (multiplicative) Cousin problem** is always solvable for domains in $\mathbb{C}$: Let $U \subseteq \mathbb{C}$ be a domain. Let $\mathfrak{U} = \{U_i\}_{i \in I}$ be an open cover of U. Suppose that for any pair $(i, j) \in I \times I$ there is a function $f_{ij} \in \mathcal{H}(U_i \cap U_j)$ vanishing nowhere in $U_i \cap U_j$, and that for any triple $(i, j, k) \in I \times I \times I$ we have

$$f_{ik} = f_{ij} f_{jk} \quad \text{on } U_i \cap U_j \cap U_k.$$

Then there exists a family of functions $\{f_i\}_{i \in I}$ with $f_i \in \mathcal{H}(U_i)$ nowhere vanishing on U_i such that

$$f_i / f_j = f_{ij} \quad \text{on } U_i \cap U_j \text{ for all } i, j \in I.$$

Prove that this implies the Weierstrass theorem 15.3. Hint: Set $\varphi_a(z) := (z - a)^{m_a}$ for $z \in U_a := U \setminus \{a\}$ and $a \in A$, and $f_{ab} := \varphi_b / \varphi_a$.

As a consequence we shall now prove that every region $U \subseteq \mathbb{C}$ is a **domain of holomorphy**, i.e., there is a function $f \in \mathcal{H}(U)$ which cannot be extended to a holomorphic function on a domain larger than U.

Let $U \subseteq \mathbb{C}$ be a region, and $f \in \mathcal{H}(U)$. Let $a \in \partial U$. We say that f is **singular** at a if, given any curve $\gamma : [0,1] \to \mathbb{C}$ such that $\gamma(t) \in U$ for $0 \le t < 1$ and $\gamma(1) = a$, the germ at $\gamma(0)$ of f cannot be analytically continued along γ. If $f \in \mathcal{H}(U)$ is singular at every point of ∂U, we say that ∂U is a **natural boundary** of f.

Theorem 15.5 (domain of holomorphy)**.** *Let $U \subseteq \mathbb{C}$ be a region. Then there exists $f \in \mathcal{H}(U)$ such that ∂U is a natural boundary of f.*

Proof. Let D_n, $n \ge 1$, be a sequence of open disks such that $\overline{D}_n \subseteq U$, $\{D_n\}$ is locally finite, $U = \bigcup_n D_n$, and the radius r_n of D_n tends to 0 as $n \to \infty$. (The existence of such a sequence can easily be established using a compact exhaustion of U.) Choose a sequence of points $a_n \in D_n$, $n \ge 1$, such that $a_n \ne a_m$ if $n \ne m$. By the Weierstrass theorem 15.3, there is a function $f \in \mathcal{H}(U)$ with zeros precisely in the set $\{a_n\}$. We will show that ∂U is a natural boundary of f.

Let $\gamma : [0,1] \to \mathbb{C}$ be a curve such that $\gamma(t) \in U$ for $0 \le t < 1$ and $\gamma(1) = a \in \partial U$. Suppose, for contradiction, that the germ at $\gamma(0)$ of f can be continued analytically along γ, and let F_a be the germ at $\gamma(1)$ so obtained. Let (D, F) be a representative of F_a, where $D = D_r(a)$ is a small disk centered at a and $F \in \mathcal{H}(D)$. There is $\epsilon > 0$ such that for $1 - \epsilon \le t < 1$, $\gamma(t) \in D$ and $F_{\gamma(t)} = f_{\gamma(t)}$. If V is the connected component of $D \cap U$ containing $\{\gamma(t) : 1 - \epsilon \le t < 1\}$, then $F = f$ on V.

Let $D' = D_{r/2}(a)$ be the open disk with center a and half the radius of D. Then $\gamma(t) \in D' \cap V$ for t sufficiently close to 1, and so $D' \cap V$ cannot be contained in the union of finitely many disks D_n (which is relatively compact in U). Thus, there exists a sequence n_k such that $D_{n_k} \cap (D' \cap V) \ne \emptyset$ and the radius of D_{n_k} is $< r/4$ for all k. It follows that $D_{n_k} \subseteq V$ for all k, since V is connected. Hence $F(a_{n_k}) = f(a_{n_k}) = 0$, but the sequence a_{n_k} is contained in $\in D_{3r/4}(a)$ and thus has an accumulation point in D. Consequently, $F \equiv 0$ and, since U is connected, $f \equiv 0$ on U, a contradiction. $\square$

Exercise 29. Consider the power series

$$f(z) = \sum_{n=0}^{\infty} z^{2^n}, \quad |z| < 1,$$

with radius of convergence 1. Prove that the natural boundary of f is $\partial \mathbb{D}$. Hint: Let $\varphi = 2\pi\ell/2^k$, where $k, \ell \in \mathbb{N}$, and show that $|f(re^{i\varphi})| \to \infty$ as $r \to 1^-$.

A further consequence is the following.

Theorem 15.6 (characterization of meromorphic functions)**.** *Every meromorphic function in a domain $U \subseteq \mathbb{C}$ is a quotient of two holomorphic functions in U.*

Proof. Suppose that f is meromorphic in U. Let A be the set of poles of f in U, and for each $a \in A$ denote by m_a the order of the pole of f at a. By the Weierstrass theorem 15.3, there exists $h \in \mathcal{H}(U)$ whose zero set is precisely A and m_a is the order of the zero of h at a for each $a \in A$. The singularities of the function $g := fh$ at the points of A are removable. So g can be extended to a function holomorphic in U. $\square$

Finally, let us combine the Mittag-Leffler theorem 12.1 and the Weierstrass theorem 15.3:

Theorem 15.7. *Let $U \subseteq \mathbb{C}$ be a domain, and $A \subseteq U$ a discrete subset. Suppose that we are given for each $a \in A$ a neighborhood U_a of a in U, a function $\varphi_a \in \mathcal{H}(U_a \setminus \{a\})$, and an integer $m_a > 0$. Then there exists $f \in \mathcal{H}(U \setminus A)$ such that $f - \varphi_a$ is holomorphic at a and $\operatorname{ord}_a(f - \varphi_a) > m_a$ for all $a \in A$.*

Proof. By the Weierstrass theorem 15.3, there is $g \in \mathcal{H}(U)$ such that g has no zeros outside A and $\mathrm{ord}_a(g) > m_a$ for all $a \in A$. By the Mittag-Leffler theorem 12.1, there is $h \in \mathcal{H}(U \setminus A)$ such that $h - \varphi_a/g$ is holomorphic in some neighborhood V_a of a. We claim that $f := gh$ has the required properties. Clearly, $f \in \mathcal{H}(U \setminus A)$. Moreover, $f - \varphi_a = g(h - \varphi_a/g) \in \mathcal{H}(V_a)$ and $\mathrm{ord}_a(f - \varphi_a) \geq \mathrm{ord}_a(g) > m_a$. □

Furthermore, if we exclude essential singularities:

Theorem 15.8. *Let $U \subseteq \mathbb{C}$ be a domain, and $A \subseteq U$ a discrete subset. Suppose that we are given for each $a \in A$ a meromorphic function φ_a in a neighborhood of a, and an integer $m_a > 0$. Then there exists a meromorphic function f on U which is holomorphic and non-vanishing on $U \setminus A$ and such that $\mathrm{ord}_a(f - \varphi_a) > m_a$ for all $a \in A$.*

Proof. Let $A_0 := \{a \in A : \varphi_a \neq 0\}$ and let $n_a := \mathrm{ord}_a(\varphi_a)$ for $a \in A_0$. By the Weierstrass theorem 15.3, there is a meromorphic function g on U which is holomorphic and non-vanishing on $U \setminus A$ and such that $\mathrm{ord}_a(g) = n_a$ if $a \in A_0$ and $\mathrm{ord}_a(g) > m_a$ if $a \in A \setminus A_0$. Set $\psi_a := \varphi_a/g$ for $a \in A_0$. Then ψ_a is holomorphic at a and $\psi_a(a) \neq 0$. So there is a small disk D_a centered at a and $h_a \in \mathcal{H}(D_a)$ such that $\psi_a = e^{h_a}$ on D_a, by Theorem 4.8.

By Theorem 15.7, there is $h \in \mathcal{H}(U)$ such that $\mathrm{ord}_a(h - h_a) > |n_a| + m_a$ for all $a \in A_0$. Define $f := ge^h$. Evidently, f is holomorphic and non-vanishing on $U \setminus A$. We have

$$f - \varphi_a = g(e^h - \psi_a) = ge^h(1 - e^{h_a - h}).$$

For $a \in A_0$, $\mathrm{ord}_a(ge^h) = n_a$ and $\mathrm{ord}_a(1 - e^{h_a - h}) \geq |n_a| + m_a + 1$ so that $\mathrm{ord}_a(f - \varphi_a) \geq n_a + |n_a| + m_a + 1 > m_a$. If $a \in A \setminus A_0$ then $\mathrm{ord}_a(f - \varphi_a) = \mathrm{ord}_a(f) = \mathrm{ord}_a(g) > m_a$. □

16. Ideals in $\mathcal{H}(U)$

Let us consider some consequences for ideals in $\mathcal{H}(U)$. We will show that every finitely generated ideal in $\mathcal{H}(U)$ is principal, and that a proper ideal in $\mathcal{H}(U)$ is finitely generated if and only if it is closed.

We denote by $(g_1, \ldots, g_n) := \{\sum_{k=1}^n f_k g_k : f_k \in \mathcal{H}(U)\}$ the ideal generated by $g_1, \ldots, g_n \in \mathcal{H}(U)$; note that $(1) = \mathcal{H}(U)$. An ideal $\mathcal{I}$ is called **principal** if there exists $g \in \mathcal{I}$ such that $\mathcal{I} = (g)$.

Lemma 16.1. *Let $U \subseteq \mathbb{C}$ be a region. If $g_1, \ldots, g_n \in \mathcal{H}(U)$, no g_k is identically 0, and no point of U is a zero of all g_k, then $(g_1, \ldots, g_n) = (1)$.*

Proof. We proceed by induction on n. The case $n = 1$ is trivial. Let $n > 1$ and let $g_1, \ldots, g_n \in \mathcal{H}(U)$ have no common zero. By the Weierstrass theorem 15.3, there exists $f \in \mathcal{H}(U)$ such that at every point $z \in U$ the order of vanishing of f is the minimal order of vanishing of the functions $g_1, \ldots, g_{n-1}$. Then $h_k = g_k/f$, $1 \leq k \leq n-1$, belong to $\mathcal{H}(U)$ and have no common zero. By induction hypothesis, $(h_1, \ldots, h_{n-1}) = (1)$ and so $(g_1, \ldots, g_n) = (f, g_n)$. Note that g_n does not vanish on the zero set of f. By Theorem 15.7, there exists $\varphi \in \mathcal{H}(U)$ such that at each point of U the order of vanishing of $1 - \varphi g_n$ is at least as large as the order of vanishing of f. Consequently, there is $\psi \in \mathcal{H}(U)$ such that $1 = \varphi g_n + \psi f$, which shows that $(g_1, \ldots, g_n) = (1)$. □

Theorem 16.2 (finitely generated ideals in $\mathcal{H}(U)$ (I))**.** *Every finitely generated ideal in $\mathcal{H}(U)$ is principal.*

Proof. By treating connected components separately we may assume that U is a region. Let $G_1, \ldots, G_n \in \mathcal{H}(U)$. We may assume without loss of generality that no G_k is identically 0. By the Weierstrass theorem 15.3, there exists $f \in \mathcal{H}(U)$ such that at every point $z \in U$ the order of vanishing of f is the minimal order of vanishing of the functions $G_1, \ldots, G_n$. Then the functions $g_k = G_k/f$ are holomorphic and have no common zeros. By Lemma 16.1, $(g_1, \ldots, g_n) = (1)$ and hence $(G_1, \ldots, G_n) = (f)$. □

Next we will show that a proper ideal in $\mathcal{H}(U)$ is finitely generated if and only if it is closed.

Lemma 16.3. *Let X be a $\mathbb{C}$-linear closed subspace of $\mathcal{H}(U)$. Suppose that for every $z \in U$ there exists $f \in X$ with $f(z) \neq 0$. Then there exist two functions $g, h \in X$ which have no common zeros in U.*

Proof. For $z \in U$ consider $X(z) := \{f \in X : f(z) \neq 0\}$. Then $X(z)$ is open in X. We claim that $X(z)$ is dense in X. Let $f \in X$ and $g \in X$ with $g(z) \neq 0$. Then $f + \lambda g \in X(z)$ if $\lambda \neq -f(z)/g(z)$. So if $\lambda \to 0$, $\lambda \neq -f(z)/g(z)$, then $f + \lambda g \to f$ in $\mathcal{H}(U)$. Therefore, $\overline{X(z)} = X$.

$\mathcal{H}(U)$ is a complete metric space, and since $X \subseteq \mathcal{H}(U)$ is closed, so is X. Let $\{z_n\}$ a countable set in U. By Baire's theorem, $\bigcap_{n=1}^{\infty} X(z_n)$ is dense in X.

Let $0 \not\equiv g \in X$ and let $\{z_n\}$ be the set of zeros of g. Let $h \in \bigcap_{n=1}^{\infty} X(z_n)$; then $h(z_n) \neq 0$ for all n. □

Theorem 16.4 (finitely generated ideals in $\mathcal{H}(U)$ (II))**.** *Let $U \subseteq \mathbb{C}$ be a region, and let $\mathcal{I}$ be a proper ideal in $\mathcal{H}(U)$. Then $\mathcal{I}$ is finitely generated if and only if $\mathcal{I}$ is closed in $\mathcal{H}(U)$.*

Proof. Suppose that $\mathcal{I}$ is finitely generated. By Theorem 16.2, we may assume that $\mathcal{I} = (g)$ for $0 \not\equiv g \in \mathcal{I}$. Let $f_n \in (g)$ and suppose that $f_n \to f$ uniformly on compact sets in U. Then $f_n = h_n g$ for $h_n \in \mathcal{H}(U)$. Let $A := \{z \in U : g(z) = 0\}$.

Let $w \in U$. Let D be a disk centered at w such that $\overline{D} \subseteq U$ and $\partial D \cap A = \emptyset$. Then, by the maximum principle,

$$|h_n - h_m|_D = |h_n - h_m|_{\partial D} \leq (\inf_{z \in \partial D} |g(z)|)^{-1} |f_n - f_m|_{\partial D} \to 0,$$

as $n, m \to \infty$. Thus (h_n) converges uniformly on compact sets in U to some $h \in \mathcal{H}(U)$ so that $f = \lim f_n = \lim h_n g = hg$. Hence $\mathcal{I} = (g)$ is closed.

For the converse, suppose that $\mathcal{I} \neq (0)$ is a closed ideal in $\mathcal{H}(U)$. For $z \in U$ set $m_z := \inf_{f \in \mathcal{I}} \operatorname{ord}_z(f)$. The set $\{z \in U : m_z > 0\}$ is discrete, since $\mathcal{I} \neq (0)$. By the Weierstrass theorem 15.3, there exists $g \in \mathcal{H}(U)$ such that $\operatorname{ord}_z(g) = m_z$ for all $z \in U$. If $f \in \mathcal{I}$ then $\operatorname{ord}_z(f) \geq \operatorname{ord}_z(g)$, for all $z \in U$, so that $f/g \in \mathcal{H}(U)$. Consider the closed ideal $\mathcal{J} := \{f/g : f \in \mathcal{I}\}$ in $\mathcal{H}(U)$; that $\mathcal{J}$ is closed follows from the same arguments that showed that principal ideals in $\mathcal{H}(U)$ are closed. Note that for every $z \in U$ there is $h \in \mathcal{J}$ such that $h(z) \neq 0$: if $f \in \mathcal{I}$ and $\operatorname{ord}_z(f) = m_z$, then $\operatorname{ord}_z(f/g) = 0$.

By Lemma 16.3, there exist $h_1, h_2 \in \mathcal{J}$ without common zero. By Lemma 16.1, there are $k_1, k_2 \in \mathcal{H}(U)$ with $k_1 h_1 + k_2 h_2 = 1$, and hence $k_1(h_1 g) + k_2(h_2 g) = g$, i.e., $g \in \mathcal{I}$. This implies $\mathcal{I} = (g)$, since, for each $f \in \mathcal{I}$, $f/g \in \mathcal{H}(U)$ as noted before. □

Remark 16.5. *$\mathcal{H}(U)$ is not a Noetherian ring.* In fact, let (z_n) be a sequence in U without accumulation point in U, and set $\mathcal{I}_n := \{f \in \mathcal{H}(U) : f(z_m) = 0 \text{ for } m \geq n\}$. Then $\mathcal{I}_n$ is an ideal in $\mathcal{H}(U)$, $\mathcal{I}_n \subseteq \mathcal{I}_{n+1}$, and $\mathcal{I}_n \neq \mathcal{I}_{n+1}$ (by the Weierstrass theorem 15.3). Also, the proper ideal $\bigcup_{n \geq 1} \mathcal{I}_n$ is not finitely generated.

CHAPTER 4

Harmonic functions

17. The Poisson integral formula

Let $U \subseteq \mathbb{C}$ be a domain. A function $u \in C^2(U)$ is said to be **harmonic** if

$$\Delta u = \Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big) u = 4 \frac{\partial^2}{\partial z \partial \overline{z}} u = 0.$$

Proposition 17.1 (harmonic conjugate)**.** *Let $U \subseteq \mathbb{C}$ be a simply connected region, and let $u : U \to \mathbb{R}$ be harmonic. Then there is a C^∞-function v such that $u + iv : U \to \mathbb{C}$ is holomorphic.*

Proof. Consider the C^1-function $h = u_x - iu_y$. Then $ih_x = iu_{xx} + u_{yx} = h_y$, since $\Delta u = 0$, so that h is holomorphic. By Theorem 4.9, h has a primitive $H = \tilde{u} + i\tilde{v}$. Then $H' = \tilde{u}_x - i\tilde{u}_y = u_x - iu_y$ so that $\tilde{u}_x = u_x$ and $\tilde{u}_y = u_y$, and hence $\tilde{u} = u + c$ for a constant c. Thus, $H - c = u + i\tilde{v}$ is holomorphic. □

The imaginary part v is unique up to an additive constant; for, if $u + iv_1$ and $u + iv_2$ are holomorphic then $i(v_1 - v_2)$ is holomorphic but not open. Any function v such that $u + iv$ is holomorphic is called a **harmonic conjugate** of u.

Proposition 17.2 (maximum principle for harmonic functions)**.** *If $u : U \to \mathbb{R}$ is harmonic on a region $U \subseteq \mathbb{C}$ and there is a point $z \in U$ such that $u(z) = \sup_{\zeta \in U} u(\zeta)$, then u is constant on U.*

Proof. Let $M := \{z \in U : u(z) = \sup_{\zeta \in U} u(\zeta)\}$. We show that M is open and closed in U, and hence $M = U$, in particular u is constant on U. That M is closed follows from the continuity of u. Let $z \in M$ and let $D := D_r(z) \subseteq U$. By Proposition 17.1, there is $h \in \mathcal{H}(D)$ with $\operatorname{Re} h = u$. Define $f := e^h$. Then $|f(z)| = \sup_{\zeta \in U} |f(\zeta)|$ and by the maximum principle for holomorphic functions f is constant on D. Then u is constant on D and so M is open. □

By applying the proposition to $-u$ we obtain the minimum principle for harmonic functions, where sup is replaced by inf in the statement.

Corollary 17.3. *Let $U \subseteq \mathbb{C}$ be a bounded region and let $u : \overline{U} \to \mathbb{R}$ be continuous and harmonic in U. Then $\max_{\overline{U}} u = \max_{\partial U} u$ and $\min_{\overline{U}} u = \min_{\partial U} u$.* □

Exercise 30. Prove Liouville's theorem for harmonic functions: If $u : \mathbb{C} \to \mathbb{R}$ is harmonic and bounded on $\mathbb{C}$, then u is constant.

Proposition 17.4 (mean value property)**.** *Let $u : U \to \mathbb{R}$ be harmonic on a domain $U \subseteq \mathbb{C}$, and let $\overline{D}_r(a) \subseteq U$. Then*

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{it})\, dt.$$

Proof. By Proposition 17.1, there is a holomorphic function h defined in a neighborhood of $\overline{D}_r(a)$ such that $h = u + iv$. Then

$$\begin{aligned} u(a) + iv(a) &= \frac{1}{2\pi i} \int_{\partial D_r(a)} \frac{h(z)}{z-a}\, dz \\ &= \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{it})\, dt + i \frac{1}{2\pi} \int_0^{2\pi} v(a + re^{it})\, dt \end{aligned}$$

which implies the assertion. □

Let $u_i : \overline{\mathbb{D}} \to \mathbb{R}$, $i = 1, 2$, be continuous and harmonic in $\mathbb{D}$. If $u_1 = u_2$ on $\partial\mathbb{D}$, then $u_1 = u_2$ on $\mathbb{D}$, by the maximum principle for harmonic functions 17.2. So a harmonic function u on $\mathbb{D}$ that extends continuously to $\overline{\mathbb{D}}$ is completely determined by its values on the boundary $\partial\mathbb{D}$. Proposition 17.4 makes this precise for the origin. We will now derive a formula for the values of u at every point in $\mathbb{D}$.

To this end observe that, for every $a \in \mathbb{D}$, the mapping

$$\varphi_a(z) := \frac{z-a}{1-\overline{a}z} \tag{17.1}$$

is an automorphism of $\mathbb{D}$ which extends to a holomorphic and invertible map on a neighborhood of $\overline{\mathbb{D}}$ satisfying $\varphi_a^{-1} = \varphi_{-a}$ and $\varphi_a(a) = 0$.

Exercise 31. Let $a \in \mathbb{D}$. Prove that $\varphi_a(z) = (z-a)/(1-\overline{a}z)$ is holomorphic and invertible on a neighborhood of $\overline{\mathbb{D}}$ with $\varphi_a^{-1} = \varphi_{-a}$. Show that $|\varphi_a(z)| = 1$ for $z \in \partial\mathbb{D}$.

Theorem 17.5 (Poisson integral formula)**.** *Let u be a harmonic function on a neighborhood of $\overline{\mathbb{D}}$. Then*

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{it}) \frac{1-|z|^2}{|z-e^{it}|^2}\, dt, \quad z \in \mathbb{D}. \tag{17.2}$$

Proof. By the mean value property 17.4, applied to the harmonic function $u \circ \varphi_{-z}$ we get

$$u(z) = (u \circ \varphi_{-z})(0) = \frac{1}{2\pi} \int_0^{2\pi} u(\varphi_{-z}(e^{it}))\, dt = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{u(\varphi_{-z}(\zeta))}{\zeta}\, d\zeta.$$

The mapping φ_z restricts to a C^1-diffeomorphism $\partial\mathbb{D} \to \partial\mathbb{D}$ with $\varphi_z'(w) = (1 - |z|^2)/(1-\overline{z}w)^2$. Thus,

$$\begin{aligned} u(z) &= \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{u(w)}{\varphi_z(w)}\, \varphi_z'(w)\, dw \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{u(e^{it})(1-\overline{z}e^{it})}{e^{it}-z}\, \frac{1-|z|^2}{(1-\overline{z}e^{it})^2} e^{it}\, dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} u(e^{it}) \frac{1-|z|^2}{|z-e^{it}|^2}\, dt. \end{aligned}$$ □

Exercise 32. Show that if $u : U \to \mathbb{R}$ is harmonic and $h : V \to U$ is holomorphic, then $u \circ h$ is harmonic.

The expression

$$\frac{1}{2\pi} \frac{1-|z|^2}{|z-e^{it}|^2} = \frac{1}{2\pi} \operatorname{Re} \frac{e^{it}+z}{e^{it}-z}$$

is called the **Poisson kernel** of the unit disk. In polar coordinates $z = re^{i\theta}$ it takes the form

$$P_r(\theta - t) := \frac{1}{2\pi}\frac{1-r^2}{1-2r\cos(\theta-t)+r^2}$$

and (17.2) reads

$$u(re^{i\theta}) = \int_0^{2\pi} u(e^{it})P_r(\theta - t)\, dt.$$

For $u \equiv 1$ we obtain $1 = \int_0^{2\pi} P_r(\theta - t)\, dt$. If $0 < \delta < \pi/2$ and $\delta \le \vartheta \le 2\pi - \delta$, then

$$0 < P_r(\vartheta) \le \frac{1}{2\pi}\frac{1-r^2}{1-\cos^2\delta} \tag{17.3}$$

for $0 \le r < 1$. Indeed, if $\pi/2 \le \vartheta \le 3\pi/2$ then $\cos\vartheta \le 0$ so that $1 - 2r\cos(\vartheta) + r^2 \ge 1$. If $\delta \le \vartheta \le \pi/2$ then $0 \le \cos\vartheta \le \cos\delta$ and hence $1-2r\cos(\vartheta)+r^2 \ge 1-2r\cos(\delta)+r^2 = 1-\cos^2\delta + (r-\cos\delta)^2 \ge 1-\cos^2\delta$. Similarly for $3\pi/2 \le \vartheta \le 2\pi - \delta$.

Exercise 33. Derive a formula analogous to the Poisson integral formula (17.2) for the upper half plane $\mathbb{H}$, by mapping $\mathbb{H}$ biholomorphically to $\mathbb{D}$: if u is harmonic on $\mathbb{H}$, and continuous and bounded on $\overline{\mathbb{H}}$, then

$$u(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} u(t)\frac{y}{(x-t)^2+y^2}\, dt, \quad z = x+iy \in \mathbb{H}.$$

Theorem 17.6 (solution of the Dirichlet problem for the disk)**.** *Let f be a continuous function on $\partial\mathbb{D}$. Then*

$$u(z) := \begin{cases} \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\frac{1-|z|^2}{|z-e^{it}|^2}\, dt & \text{if } z \in \mathbb{D}, \\ f(z) & \text{if } z \in \partial\mathbb{D}, \end{cases}$$

is continuous on $\overline{\mathbb{D}}$ and harmonic on $\mathbb{D}$.

Proof. Let us show first that u is harmonic in $\mathbb{D}$. To this end we observe that

$$\frac{1-|z|^2}{|z-e^{it}|^2} = \frac{e^{it}}{e^{it}-z} + \frac{e^{-it}}{e^{-it}-\overline{z}} - 1$$

and thus for $z \in \mathbb{D}$,

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\frac{e^{it}}{e^{it}-z}\, dt + \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\frac{e^{-it}}{e^{-it}-\overline{z}}\, dt - \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\, dt.$$

The first integral is holomorphic, the second antiholomorphic, and the third constant in z. Since $\Delta = 4\partial^2/\partial z\partial\overline{z}$, we find that $\Delta u = 0$ on $\mathbb{D}$.

Fix $t_0 \in \mathbb{R}$. If $z = re^{i\theta} \in \mathbb{D}$ then

$$u(z) - f(e^{it_0}) = \int_0^{2\pi} (f(e^{it}) - f(e^{it_0}))P_r(\theta - t)\, dt.$$

Let $\epsilon > 0$. By continuity of f, there is $\delta > 0$ such that $|f(e^{it}) - f(e^{it_0})| < \epsilon$ if $|e^{it} - e^{it_0}| < \delta$. On the other hand, if $|e^{it} - e^{it_0}| \ge \delta$ and $e^{i\theta}$ is sufficiently close to e^{it_0}, then $|e^{i(t-\theta)} - 1| \ge \delta/2$. Thus, by (17.3),

$$\begin{aligned} |u(z) - f(e^{it_0})| &\le \int_{\{t:|e^{it}-e^{it_0}|<\delta\}} |f(e^{it}) - f(e^{it_0})|P_r(\theta-t)\, dt \\ &\quad + \int_{\{t:|e^{it}-e^{it_0}|\ge\delta\}} |f(e^{it}) - f(e^{it_0})|P_r(\theta-t)\, dt \\ &\le \epsilon \int_{\{t:|e^{it}-e^{it_0}|<\delta\}} P_r(\theta-t)\, dt \end{aligned}$$

$$+ C(\delta)(1-r^2)\int_{\{t:|e^{it}-e^{it_0}|\geq\delta\}} |f(e^{it}) - f(e^{it_0})|\,dt$$
$$\leq \epsilon + C(\delta)(1-r^2)\Big(\int_0^{2\pi} |f(e^{it})|\,dt + 2\pi|f(e^{it_0})|\Big) \leq 2\epsilon,$$

if r is sufficiently close to 1. $\square$

Remark 17.7. The condition that f is continuous on $\partial\mathbb{D}$ can be replaced by $f \in L^1(\partial\mathbb{D})$ (where we identify functions on $\partial\mathbb{D}$ with 2π-periodic functions on $\mathbb{R}$ and use the Lebesgue measure). Then u is harmonic on $\mathbb{D}$ and if f is continuous at e^{it_0} then $u(z) \to f(e^{it_0})$ as $z \to e^{it_0}$; the proof is essentially the same.

By a change of variables, we can conclude the following. Let f be continuous on $\partial D_r(a)$ (or just integrable). Then the function defined by

$$u(z) := \begin{cases} P_{a,r}(f)(z) & \text{if } z \in D_r(a), \\ f(z) & \text{if } z \in \partial D_r(a), \end{cases}$$

is continuous on $\overline{D}_r(a)$ (at points of $\partial D_r(a)$, where f is continuous) and harmonic on $D_r(a)$, where

$$P_{a,r}(f)(z) := \int_0^{2\pi} f(a+re^{it})P_{a,r}(z,t)\,dt \tag{17.4}$$

and

$$P_{a,r}(z,t) := \frac{1}{2\pi}\operatorname{Re}\frac{re^{it}+(z-a)}{re^{it}-(z-a)}.$$

Next we shall prove that a continuous function u with the mean value property is harmonic. Actually, it suffices that for each $a \in U$ there is $r_a > 0$ such that $\overline{D}_{r_a}(a) \subseteq U$ and for every $0 < r < r_a$

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a+re^{it})\,dt.$$

Following [**8**] we say that u has the **small circle mean value (SCMV) property** if this holds.

Lemma 17.8. *Let $U \subseteq \mathbb{C}$ be a region, and let $u : U \to \mathbb{R}$ be continuous with the SCMV property. If $u(z) = \sup_{\zeta\in U} u(\zeta)$ for some $z \in U$, then u is constant.*

Proof. The set $M = \{z \in U : u(z) = \sup_{\zeta\in U} u(\zeta) =: s\}$ is clearly closed and nonempty. Let us prove that M is open. Let $a \in M$. By assumption, for $0 < r < r_a$,

$$s = u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a+re^{it})\,dt \leq \frac{1}{2\pi}\int_0^{2\pi} s\,dt = s$$

so that $u(a+re^{it}) = s$ for all $0 \leq t \leq 2\pi$ and all $0 < r < r_a$. So M is open. $\square$

Theorem 17.9. *Let $U \subseteq \mathbb{C}$ be a domain, and let $f : U \to \mathbb{R}$ be continuous with the SCMV property. Then f is harmonic.*

Proof. Let D be a disk such that $\overline{D} \subseteq U$. By Theorem 17.6, there is a harmonic function $u_D : D \to \mathbb{R}$ such that

$$\tilde{u}_D(z) := \begin{cases} u_D(z) & \text{if } z \in D, \\ f(z) & \text{if } z \in \partial D, \end{cases}$$

is a continuous function on $\overline{D}$. We claim that $f = u_D$ on D so that f is harmonic on D, and thus, since D was arbitrary, also on U.

The function $h := f - \tilde{u}_D$ is continuous on $\overline{D}$, vanishes on ∂D, and fulfills the assumptions of Lemma 17.8 on D. Thus $h \le 0$ and $h \ge 0$ (by applying the same reasoning to $-h$). Thus $h = 0$ and $f = u_D$ on D. □

Corollary 17.10. *If $u_n : U \to \mathbb{R}$ is a sequence of harmonic functions which converges uniformly on compact sets to $u : U \to \mathbb{R}$, then u is harmonic.*

Proof. If $\overline{D}_{r_a}(a) \subseteq U$ then, by the mean value property 17.4,

$$u_n(a) = \frac{1}{2\pi} \int_0^{2\pi} u_n(a + re^{it})\, dt.$$

Letting $n \to \infty$ the assertion follows from Theorem 17.9. □

Exercise 34. Prove **Jensen's formula**: Let f be holomorphic in a neighborhood of $\overline{D}_r(0)$ with $f(0) \neq 0$. Assume that f does not vanish on $\partial D_r(0)$ and let $a_1, \dots, a_k$ be the zeros of f in $D_r(0)$ counted according to their multiplicities. Then

$$\log |f(0)| + \sum_{j=1}^{k} \log \frac{r}{|a_j|} = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{it})|\, dt. \tag{17.5}$$

Hint: Use Exercise 31 to conclude that

$$g(z) = \frac{f(z)}{\prod_{j=1}^{k} \varphi_{a_j/r}(z/r)},$$

where $\varphi_{a_j/r}$ is defined by (17.1), is holomorphic in a neighborhood of $\overline{D}_r(0)$ and has no zeros in $\overline{D}_r(0)$. Apply the mean value property to $\log |g|$ which is harmonic in a neighborhood of $\overline{D}_r(0)$.

18. The Schwarz reflection principle

Lemma 18.1 (Schwarz reflection principle for harmonic functions)**.** *Let $V \subseteq \mathbb{C}$ be a region such that $V \cap \mathbb{R} = (a, b)$. Let $U := \{z \in V : \operatorname{Im} z > 0\}$ and let $u : U \to \mathbb{R}$ be harmonic such that for each $x \in (a, b)$*

$$\lim_{z \to x} u(z) = 0.$$

Then the function

$$\tilde{u}(z) := \begin{cases} u(z) & \text{if } z \in U, \\ 0 & \text{if } z \in (a, b), \\ -u(\overline{z}) & \text{if } z \in \tilde{U} := \{\overline{z} : z \in U\}, \end{cases}$$

is harmonic on $U \cup (a, b) \cup \tilde{U}$.

Proof. Obviously, $\tilde{u}$ is continuous on $W := U \cup (a, b) \cup \tilde{U}$. By Theorem 17.9, it suffices to check that $\tilde{u}$ has the SCMV property. This is clear for points in U, since u is harmonic, and for points in $\tilde{U}$, since $z \mapsto -u(\overline{z})$ is harmonic. Let $x \in (a, b)$. Let $r_x > 0$ be such that $D_{r_x}(x) \subseteq W$. Then, for $0 < r < r_x$,

$$\begin{aligned} \int_0^{2\pi} \tilde{u}(x + re^{it})\, dt &= \int_0^{\pi} \tilde{u}(x + re^{it})\, dt + \int_0^{\pi} \tilde{u}(x + re^{i(t+\pi)})\, dt \\ &= \int_0^{\pi} u(x + re^{it})\, dt - \int_0^{\pi} u(x + re^{-i(t+\pi)})\, dt = 0 = 2\pi \tilde{u}(x). \end{aligned}$$

The lemma follows. □

Theorem 18.2 (Schwarz reflection principle for holomorphic functions)**.** *Let $V \subseteq \mathbb{C}$ be a region such that $V \cap \mathbb{R} = (a,b)$. Let $U := \{z \in V : \operatorname{Im} z > 0\}$ and let $f : U \to \mathbb{C}$ be holomorphic such that for each $x \in (a,b)$*

$$\lim_{z \to x} \operatorname{Im} f(z) = 0.$$

Let $\tilde{U} := \{\overline{z} : z \in U\}$. Then there exists a holomorphic function F on $U \cup (a,b) \cup \tilde{U}$ such that $F|_U = f$. In particular, $F(z) = \overline{f(\overline{z})}$ for $z \in \tilde{U}$ and $F(x) = \lim_{U \ni z \to x} \operatorname{Re} f(z)$ for each $x \in (a,b)$.

Proof. It is easy to see that $z \mapsto \overline{f(\overline{z})}$ defines a holomorphic function on $\tilde{U}$. If there is a holomorphic extension F on $W := U \cup (a,b) \cup \tilde{U}$ of f, then $F(z) = \overline{f(\overline{z})}$ on $\tilde{U}$, because $z \mapsto \overline{F(\overline{z})}$ is holomorphic on W and agrees with F on (a,b).

Let $x \in (a,b)$. Let D be a small disk centered at x and contained in W. Let $v(z) := \operatorname{Im} f(z)$ for $z \in D \cap U$. Then v is harmonic in $D \cap U$ and $v(z) \to 0$ as $z \to x \in (a,b) \cap D$. By the Schwarz reflection principle for harmonic functions 18.1, v extends to a harmonic function $\tilde{v}$ on D. Choose $\tilde{u}$ such that $\tilde{u} + i\tilde{v}$ is holomorphic on D. Then, on $D \cap U$, $\operatorname{Im}(f - (\tilde{u} + i\tilde{v})) = 0$ and hence $f = (\tilde{u} + C) + i\tilde{v}$ for some real constant C.

Thus $F_0 := (\tilde{u} + C) + i\tilde{v}$ is a holomorphic extension of f to D. Moreover, $z \mapsto \overline{F_0(\overline{z})}$ is a holomorphic function on D which coincides with F_0 on $D \cap \mathbb{R}$ and thus on D. It follows that the function F defined by setting $F(z) = f(z)$ for $z \in U$, $F(z) = \overline{f(\overline{z})}$ for $z \in \tilde{U}$ and $F(x) = \lim_{U \ni z \to x} \operatorname{Re} f(z)$ for $x \in (a,b)$, is holomorphic on U. □

Corollary 18.3. *Let $f \in C(\overline{\mathbb{D}})$ such that f is holomorphic in $\mathbb{D}$. If f vanishes on an open arc I of $\partial\mathbb{D}$, then $f \equiv 0$ on $\mathbb{D}$.*

Proof. If $I = \partial\mathbb{D}$ we may invoke the maximum principle. Otherwise there is a point in $\partial\mathbb{D} \setminus I$, and after applying a rotation we may assume that this point is -1. Let $\varphi : \overline{\mathbb{D}} \setminus \{-1\} \to \overline{\mathbb{H}}$ be the inverse Cayley transform, $\varphi(z) = i(1-z)/(1+z)$. Then $g := f \circ \varphi^{-1}$ is holomorphic on $\mathbb{H}$, continuous on $\overline{\mathbb{H}}$, and vanishes on the interval $J = \varphi(I) \subseteq \mathbb{R}$. Let $U \subseteq \mathbb{H}$ be an open half disk with $\partial U \cap \mathbb{R} \subseteq J$. By the Schwarz reflection principle for holomorphic functions 18.2, g extends to a holomorphic function on $U \cup J \cup \tilde{U}$. By the identity theorem, $g \equiv 0$ on U and hence on $\mathbb{H}$, which implies the assertion. □

Exercise 35. Let f be continuous on $\overline{\mathbb{D}}$ and holomorphic in $\mathbb{D}$. Assume that f is nowhere zero on $\overline{\mathbb{D}}$ and $|f(z)| = 1$ on $\partial\mathbb{D}$. Prove that the function

$$F(z) := \begin{cases} f(z) & \text{if } |z| \le 1, \\ 1/\overline{f(1/\overline{z})} & \text{if } |z| > 1, \end{cases}$$

is entire, and conclude that f must be constant. Hint: Show first that F is continuous, then use Morera's theorem.

19. Harnack's principle

Proposition 19.1 (Harnack's inequality)**.** *Let $u \ge 0$ be a harmonic function on a neighborhood of $\overline{D}_R(a)$. Then, for $z \in D_R(a)$,*

$$\frac{R - |z-a|}{R + |z-a|} u(a) \le u(z) \le \frac{R + |z-a|}{R - |z-a|} u(a)$$

Proof. Without loss of generality $a = 0$. For $z \in D_R(0)$,

$$\frac{R-|z|}{R+|z|} \leq \frac{R^2-|z|^2}{|Re^{it}-z|^2} \leq \frac{R+|z|}{R-|z|},$$

since $R-|z| \leq |Re^{it}-z| \leq R+|z|$. By the Poisson integral formula 17.2,

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(Re^{it})\frac{R^2-|z|^2}{|Re^{it}-z|^2}\,dt, \quad z \in D_R(0),$$

we may conclude (in view of the mean value property 17.4)

$$\frac{R-|z|}{R+|z|}u(0) \leq u(z) \leq \frac{R+|z|}{R-|z|}u(0). \qquad \square$$

Theorem 19.2 (Harnack's principle)**.** *Let $u_1 \leq u_2 \leq \cdots$ be harmonic functions on a region $U \subseteq \mathbb{C}$. Then either $u_n \to \infty$ uniformly on compact sets or there is a harmonic function u on U and $u_n \to u$ uniformly on compact sets.*

So, if there is just one point $z \in U$ such that $\{u_n(z)\}_n$ is bounded, then u_n converges to a harmonic function u uniformly on compact sets.

Proof. If $z \in U$ and $u_n(z) \to \infty$, then there is n_0 such that $u_{n_0}(z) > 0$ and hence there is $R > 0$ such that $\overline{D}_R(z) \subseteq U$ and $u_{n_0} > 0$ on $\overline{D}_R(z)$. By Harnack's inequality 19.1, for $\zeta \in D_{R/2}(z)$ and $n \geq n_0$,

$$u_n(\zeta) \geq \frac{R-R/2}{R+R/2}u_n(z) = \frac{1}{3}u_n(z) \to \infty.$$

If $z \in U$ and $u_n(z)$ is bounded, then, provided $\overline{D}_R(z) \subseteq U$, by Harnack's inequality 19.1, for $\zeta \in D_{R/2}(z)$ and $n \geq m$,

$$u_n(\zeta) - u_m(\zeta) \leq \frac{R+R/2}{R-R/2}(u_n(z) - u_m(z)) = 3(u_n(z) - u_m(z)) \to 0.$$

Then u_n converges to a harmonic function uniformly on $D_{R/2}(z)$.

We have proved that the set of points on which $u_n \to \infty$ is open as well as the set on which u_n is bounded. Since U is connected one of these two sets is empty. Every compact subset $K \subseteq U$ is covered by finitely many disks $D_{R/2}(z)$ and the theorem follows. $\square$

CHAPTER 5

The Riemann mapping theorem

20. The Riemann mapping theorem

We will use the **Arzela–Ascoli theorem**. Let (X, d) be a metric space, and let $\mathcal{F}$ be a family of functions $f : X \to \mathbb{C}$. Then $\mathcal{F}$ is called **equicontinuous** if for every $\epsilon > 0$ there is $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for all $f \in \mathcal{F}$ and all $x, y \in X$ with $d(x, y) < \delta$. We say that $\mathcal{F}$ is **pointwise bounded** if for each $x \in X$ the set $\{f(x) : f \in \mathcal{F}\}$ is bounded.

Theorem 20.1 (Arzela–Ascoli theorem)**.** *Let X be a separable metric space, and let $\mathcal{F}$ be a equicontinuous pointwise bounded family of functions $f : X \to \mathbb{C}$. Then every sequence (f_n) in $\mathcal{F}$ has a subsequence which converges uniformly on compact subsets of X.*

Proof. Let $E := \{x_1, x_2, x_3, \ldots\}$ be a dense subset in X. Set $S_0 := \mathbb{N}_{>0}$. Suppose that $k \geq 1$ and an infinite set $S_{k-1} \subseteq S_0$ has been chosen. Then $\{f_n(x_k) : n \in S_{k-1}\}$ is a bounded set in $\mathbb{C}$, and thus has a convergent subsequence. Let $S_k \subseteq S_{k-1}$ be the set of indices of this subsequence. Inductively, we obtain infinite sets $S_0 \supseteq S_1 \supseteq S_2 \supseteq \cdots$ such that $\lim f_n(x_j)$ exists for $1 \leq j \leq k$ if $n \to \infty$ within S_k.

Let r_k be the kth term in S_k, and define $S := \{r_1, r_2, r_3, \ldots\}$. Then, for every k, there are at most $k - 1$ terms of S not contained in S_k. It follows that $\lim f_n(x)$ exists for every $x \in E$ as $n \to \infty$ within S.

Let $K \subseteq X$ be compact, and let $\epsilon > 0$. By equicontinuity, there is $\delta > 0$ such that

$$|f_n(x) - f_n(y)| < \epsilon$$

if $d(x, y) < \delta$. We may cover K be open balls $B_1, \ldots, B_m$ of radius $\delta/2$. Since E is dense in X, there exist $x_i \in B_i \cap E$ for $1 \leq i \leq m$. Thus $\lim f_n(x_i)$ exists for every $1 \leq i \leq m$ as $n \to \infty$ within S, whence

$$|f_n(x_i) - f_m(x_i)| < \epsilon$$

for $1 \leq i \leq m$ provided that $n, m \in S$ and $n, m > N$, for some integer N. If $x \in K$, then $x \in B_i$ for some i, and hence $d(x, x_i) < \delta$. Consequently,

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f_n(x_i)| + |f_n(x_i) - f_m(x_i)| + |f_m(x_i) - f_m(x)| \leq 3\epsilon$$

if $n, m \in S$ and $n, m > N$. $\square$

Let $U \subseteq \mathbb{C}$ be a region and Y a complete metric space. Then $C(U, Y)$ denotes the set of continuous mappings $f : U \to Y$. A subset $\mathcal{F} \subseteq C(U, Y)$ is called a **normal family** if every sequence of members of $\mathcal{F}$ has a subsequence which converges uniformly on compact subsets of U. (The limit function is not required to be in $\mathcal{F}$.) We are mostly interested in the case $Y = \mathbb{C}$. Later we shall also deal with the case $Y = \widehat{\mathbb{C}}$.

Theorem 20.2 (Montel's theorem)**.** *Let $\mathcal{F} \subseteq \mathcal{H}(U)$ be uniformly bounded on each compact subset of the region U. Then $\mathcal{F}$ is a normal family.*

Proof. By assumption, for each compact $K \subseteq U$ there is $M_K > 0$ such that $|f(z)| \leq M_K$ for all $f \in \mathcal{F}$ and all $z \in K$. Let K_n be a sequence of compact sets in U such that $K_n \subseteq \mathring{K}_{n+1}$, and $U = \bigcup_j K_j$. There exist $\delta_n > 0$ such that $D_{2\delta_n}(z) \subseteq K_{n+1}$ for all $z \in K_n$. Let $z, w \in K_n$ such that $|z - w| < \delta_n$. Then, by Cauchy's integral formula,

$$\begin{aligned} f(z) - f(w) &= \frac{1}{2\pi i} \int_{\partial D_{2\delta_n}(z)} f(\zeta) \Big(\frac{1}{\zeta - z} - \frac{1}{\zeta - w} \Big)\, d\zeta \\ &= \frac{z - w}{2\pi i} \int_{\partial D_{2\delta_n}(z)} \frac{f(\zeta)}{(\zeta - z)(\zeta - w)}\, d\zeta. \end{aligned}$$

Since $|\zeta - z| = 2\delta_n$ and $|\zeta - w| > \delta_n$ for $\zeta \in |\gamma|$, we may conclude that

$$|f(z) - f(w)| \leq \frac{M_{K_{n+1}}}{\delta_n} |z - w| \tag{20.1}$$

for all $f \in \mathcal{F}$ and all $z, w \in K_n$ with $|z - w| < \delta_n$. That means that, for each K_n, the restrictions of the members of $\mathcal{F}$ to K_n form an equicontinuous family.

Let $(f_k) \subseteq \mathcal{F}$ be any sequence. The Arzela–Ascoli theorem 20.1 implies that there is a subsequence which converges uniformly on K_1. Applying the same argument again we find a subsequence of this sequence that converges uniformly on K_2, etc. By a diagonal argument we find a sequence $g_j \in \mathcal{F}$ that is a subsequence of each of the sequences formed above. Thus g_j converges uniformly on each K_n, and therefore on every compact $K \subseteq U$. □

Remark 20.3. This implies that $\mathcal{H}(U)$ has the **Heine–Borel property**: every closed bounded subset is compact. Thus $\mathcal{H}(U)$ is a so-called **Montel space**. A Montel space is a Hausdorff locally convex space which is barrelled and has the Heine–Borel property. (The space $\mathcal{H}(U)$ is a Fréchet space and hence barrelled.)

Exercise 36. Let $\mathcal{F}$ be the family of all $f \in \mathcal{H}(\mathbb{D})$ such that $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ with $|a_n| \leq n$ for all n. Show that $\mathcal{F}$ is a normal family.

Exercise 37. Let $U \subseteq \mathbb{C}$ be a region such that $\mathbb{C} \setminus U$ has interior points. Let $z_0 \in U$. Prove that $\mathcal{F} = \{f \in \mathcal{H}(\mathbb{D}) : f(\mathbb{D}) \subseteq U \text{ and } f(0) = z_0\}$ is compact in $\mathcal{H}(\mathbb{D})$. Hint: If $a \in \mathbb{C} \setminus \overline{U}$, then $z \mapsto 1/(z - a)$ maps U biholomorphically on a subset of a disk with finite radius.

Exercise 38. Consider the family $\mathscr{S} = \{f \in \mathcal{H}(\mathbb{D}) : f \text{ injective}, f(0) = 0, f'(0) = 1\}$ of **schlicht functions**.

(1) Let $f \in \mathscr{S}$. Let r be the maximal radius such that $D_r(0) \subseteq f(\mathbb{D})$. Prove that $r \leq 1$.
(2) Choose $a \in \partial D_r(0)$ with $a \notin f(\mathbb{D})$ and set $g := f/a$. Then $\mathbb{D} \subseteq g(\mathbb{D})$ and $1 \notin g(\mathbb{D})$. Conclude that there is a holomorphic function $\varphi : g(\mathbb{D}) \to \mathbb{C}^*$ such that $\varphi(z)^2 = z - 1$ for all $z \in g(\mathbb{D})$.
(3) Set $h := \varphi \circ g$. Show that $w \in h(\mathbb{D})$ implies $-w \notin h(\mathbb{D})$.
(4) Let (f_n) be a sequence of functions in $\mathscr{S}$, and let a_n, g_n, h_n be as defined in (1), (2), (3) relative to f_n. Use Exercise 37 to conclude that (h_n) and (f_n) have convergent subsequences.
(5) Conclude that $\mathscr{S}$ is compact in $\mathcal{H}(\mathbb{D})$. Hint: To see that the limit function is injective use the argument principle 8.2.

Theorem 20.4 (Riemann mapping theorem)**.** *Every simply connected region $U \neq \mathbb{C}$ is biholomorphic to $\mathbb{D}$.*

The plane $U = \mathbb{C}$ has to be excluded, by Liouville's theorem.

Proof. Let $U \neq \mathbb{C}$ be a simply connected region and let $w_0 \notin U$. Define $\mathcal{F} := \{f \in \mathcal{H}(U) : f \text{ injective}, f(U) \subseteq \mathbb{D}\}$. It suffices to prove that some $f \in \mathcal{F}$ is surjective onto $\mathbb{D}$.

First we show that $\mathcal{F} \neq \emptyset$. Since U is simply connected there is $\psi \in \mathcal{H}(U)$ such that $\psi^2(z) = z - w_0$ for all $z \in U$, by Theorem 4.8. Clearly, ψ is injective and there are no points $z_1 \neq z_2$ in U such that $\psi(z_1) = -\psi(z_2)$. By the open mapping theorem, $\psi(U)$ contains a disk $D_r(c)$ where $0 < r < |c|$. Thus $D_r(-c) \cap \psi(U) = \emptyset$ and $f(z) := r/(\psi(z) + c)$ belongs to $\mathcal{F}$.

Next we claim: *If $f \in \mathcal{F}$ is such that $f(U) \neq \mathbb{D}$ and $z_0 \in U$, then there is $f_1 \in \mathcal{F}$ with $|f_1'(z_0)| > |f'(z_0)|$.* We will use the functions

$$\varphi_a(z) := \frac{z-a}{1-\overline{a}z}, \quad a \in \mathbb{D},$$

which are automorphisms of $\mathbb{D}$ with inverse φ_{-a}. Let $f \in \mathcal{F}$ and $a \in \mathbb{D} \setminus f(U)$. Then $\varphi_a \circ f \in \mathcal{F}$ and $\varphi_a \circ f$ does not vanish on U. So there exists $g \in \mathcal{H}(U)$ such that $g^2 = \varphi_a \circ f$, by Theorem 4.8. It follows that $g \in \mathcal{F}$. Moreover, if $f_1 := \varphi_\beta \circ g$ where $\beta = g(z_0)$, then also $f_1 \in \mathcal{F}$. Setting $s(z) = z^2$ we have

$$f = \varphi_{-a} \circ s \circ g = \varphi_{-a} \circ s \circ \varphi_{-\beta} \circ f_1.$$

Thus, for $F = \varphi_{-a} \circ s \circ \varphi_{-\beta}$, we obtain,

$$f'(z_0) = F'(0) f_1'(z_0).$$

Since $F(\mathbb{D}) \subseteq \mathbb{D}$ and F is not injective, the Schwarz lemma (see Exercise 39 below) implies that $|F'(0)| < 1$, and the claim follows. Indeed, application of the Schwarz lemma to $\varphi_{f(z_0)} \circ F$ gives $|\varphi'_{f(z_0)}(f(z_0))F'(0)| < 1$ and since $\varphi_a'(a) = (1 - |a|^2)^{-1}$, we have $|F'(0)| < 1 - |f(z_0)|^2 \le 1$.

Fix $z_0 \in U$ and set $\eta := \sup_{f \in \mathcal{F}} |f'(z_0)|$. By the claim, any $f \in \mathcal{F}$ with $\eta = |f'(z_0)|$ satisfies $f(U) = \mathbb{D}$. To finish the proof we must show the existence of such an f. The family $\mathcal{F}$ is uniformly bounded by 1 on U, and so it is a normal family, by Montel's theorem 20.2. There is a sequence $f_k \in \mathcal{F}$ such that $|f_k'(z_0)| \to \eta$ as $k \to \infty$. This sequence has a subsequence (again denoted by f_k) which converges uniformly on compact subsets of U to $f \in \mathcal{H}(U)$ and $|f'(z_0)| = \eta$. Since $\mathcal{F} \neq \emptyset$ we have $\eta > 0$ and so f is not constant. From $f_k(U) \subseteq \mathbb{D}$ for all k we may conclude $f(U) \subseteq \overline{\mathbb{D}}$, and by the open mapping theorem, $f(U) \subseteq \mathbb{D}$. To see that f is injective fix $c \in U$ and set $a = f(c)$ and $a_k = f_k(c)$. Then each function $f_k - a_k$ is nowhere-vanishing in $U \setminus \{c\}$, since f_k is injective. By Hurwitz' theorem 8.5, also the limit function $f - a$ is nowhere-vanishing in $U \setminus \{c\}$, i.e., f is injective. Hence $f \in \mathcal{F}$ and the proof is complete. □

Exercise 39. Prove the **Schwarz lemma**: Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic with $f(0) = 0$. Then $|f(z)| \le |z|$ for $z \in \mathbb{D}$ and $|f'(0)| \le 1$. If for some $c \in \mathbb{D}^*$ we have either $|f(c)| = |c|$ or $|f'(0)| = 1$, then f is a rotation, i.e., $f(z) = az$ for some a with $|a| = 1$. Hint: Use the maximum principle for the holomorphic function $z \mapsto f(z)/z$.

Exercise 40. Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic. Show that, if f has two fixed points, then $f(z) = z$ for all $z \in \mathbb{D}$. Give an example of a holomorphic function $f : \mathbb{D} \to \mathbb{D}$ without fixed point.

Exercise 41. The **pseudo-hyperbolic distance** between two points $z, w \in \mathbb{D}$ is defined by

$$\rho(z,w) := \left|\frac{z-w}{1-\overline{w}z}\right|.$$

Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic. Show that

$$\rho(f(z), f(w)) \le \rho(z,w), \quad z, w \in \mathbb{D},$$

and that equality holds if $f \in \operatorname{Aut}(\mathbb{D})$. Hint: Use the Schwarz lemma (Exercise 39).

Exercise 42. Prove the **Schwarz–Pick lemma**: Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic. Then

$$\frac{|f'(z)|}{1-|f(z)|^2} \le \frac{1}{1-|z|^2}, \quad z \in \mathbb{D}.$$

Hint: Use Exercise 41.

Exercise 43. For $w \in \mathbb{C}$ and $z \in \mathbb{D}$ we define the **hyperbolic length** of w at z by

$$\|w\|_z := \frac{|w|}{1-|z|^2}.$$

The **hyperbolic distance** of two points $z_1, z_2 \in \mathbb{D}$ is defined by

$$d(z_1, z_2) := \inf\Big\{ \int_0^1 \|\gamma'(t)\|_{\gamma(t)}\, dt : \gamma \in C^1([0,1], \mathbb{D}), \gamma(0) = z_1, \gamma(1) = z_2 \Big\}.$$

Use the Schwarz–Pick lemma to prove that, for holomorphic $f : \mathbb{D} \to \mathbb{D}$,

$$d(f(z_1), f(z_2)) \le d(z_1, z_2), \quad z_1, z_2 \in \mathbb{D}.$$

Show that equality holds if $f \in \operatorname{Aut}(\mathbb{D})$.

Exercise 44. Show that the hyperbolic distance of 0 and $s \in (0,1)$ is given by

$$d(0,s) = \frac{1}{2}\log\frac{1+s}{1-s}.$$

Derive a formula for the hyperbolic distance of two arbitrary points $z_1, z_2 \in \mathbb{D}$. Hint: Find an automorphism φ of $\mathbb{D}$ such that $\varphi(z_1) = 0$ and $\varphi(z_2) \in (0,1)$.

21. Characterization of simply connected regions

We need a preparatory results which is of independent interest.

Lemma 21.1. *Let $\Gamma = \{\gamma_1, \dots, \gamma_N\}$ be a finite collection of oriented intervals (line segments) $[a,b]$, $a, b \in \mathbb{C}$. Suppose that for all $z \in \mathbb{C}$,*

$$|\{\gamma \in \Gamma : \gamma \text{ starts at } z\}| = |\{\gamma \in \Gamma : \gamma \text{ ends at } z\}|. \tag{21.1}$$

Then $\gamma_1 + \gamma_2 + \cdots + \gamma_N$ is a cycle.

Proof. Choose $\beta_1 = [a_0, a_1] \in \Gamma$. Assume that distinct members $\beta_1, \dots, \beta_k$ of Γ have been chosen such that $\beta_i = [a_{i-1}, a_i]$, for $1 \le i \le k$. If $a_k = a_0$ we stop. Otherwise $a_k \ne a_0$ and if precisely r of the intervals $\beta_1, \dots, \beta_k$ end at a_k then only $r-1$ of them start at a_k. By (21.1), there exists an interval $\beta_{k+1} \in \Gamma$ that starts at a_k. Since Γ is finite, we must return to a_0 after finitely many, say n, steps. Then $\beta_1 + \beta_2 + \cdots + \beta_n$ forms a closed path. The remaining members of Γ form a collection Γ' that still satisfies (21.1). So the construction can be repeated for Γ'. It follows that the members of Γ can be numbered in such a way that they form finitely many closed paths. Their sum is a cycle. $\square$

Proposition 21.2. *Let $U \subseteq \mathbb{C}$ be a domain and $K \subseteq U$ a compact subset. There is a cycle γ in $U \setminus K$ such that the Cauchy formula*

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} \, d\zeta \tag{21.2}$$

holds for every $f \in \mathcal{H}(U)$ and every $z \in K$.

Proof. Let $\eta := \operatorname{dist}(K, U^c)/2 > 0$. Consider the (closed) squares of side length η formed by the lattice $\eta\mathbb{Z}^2$. Let $Q_1, \ldots, Q_m$ be those squares which intersect K; they are all contained in U. Let c_k denote the center of Q_k and let $c_k + d$ be one of its vertices. If we set

$$\gamma_{k,j} := [c_k + i^j d, c_k + i^{j+1} d]$$

then $\partial Q_k = \sum_{j=1}^4 \gamma_{k,j}$. Clearly, $\operatorname{ind}_{\partial Q_k}(z)$ is either 1 if $z \in \mathring{Q}_k$ or 0 if $z \notin Q_k$. Let $\tilde{\Gamma} := \{\gamma_{k,j} : 1 \le k \le m, 1 \le j \le 4\}$. Then $\tilde{\Gamma}$ satisfies (21.1). Let us remove all members of $\tilde{\Gamma}$ whose opposites also belong to $\tilde{\Gamma}$. The collection Γ of the remaining members still satisfies (21.1). Let γ be the cycle constructed from Γ by Lemma 21.1. By construction, γ is a cycle in $U \setminus K$. Indeed, if E is an edge of some Q_k that intersects K then the two squares in whose boundaries E lies intersect K. So $\tilde{\Gamma}$ contains two opposite intervals with range E, and hence these intervals do not occur in Γ.

By construction, $\operatorname{ind}_\gamma(z) = \sum_{k=1}^m \operatorname{ind}_{\partial Q_k}(z)$ if z is not in the boundary of any Q_k, and thus

$$\operatorname{ind}_\gamma(z) = \begin{cases} 1 & \text{if } z \in \mathring{Q}_k \text{ for some } 1 \le k \le m, \\ 0 & \text{if } z \text{ lies in no } Q_k. \end{cases}$$

If $z \in K$, then $z \notin |\gamma|$ and z is a limit point of the interior of some Q_k. Since $\operatorname{ind}_\gamma$ is constant in each component of $\mathbb{C} \setminus |\gamma|$, we may conclude

$$\operatorname{ind}_\gamma(z) = \begin{cases} 1 & \text{if } z \in K, \\ 0 & \text{if } z \notin U. \end{cases}$$

In particular, γ is homologous to zero in U and the statement follows from the homology form of Cauchy's theorem 6.2. □

Theorem 21.3 (characterization of simply connected regions)**.** *Let $U \subseteq \mathbb{C}$ be a region. The following are equivalent:*

(1) *U is homeomorphic to $\mathbb{D}$.*
(2) *U is simply connected.*
(3) *$\mathbb{C} \setminus U$ has no compact connected components.*
(4) *$\widehat{\mathbb{C}} \setminus U$ is connected.*
(5) *Any closed curve in U is homologous to 0 in U, i.e., $\operatorname{ind}_\gamma(z) = 0$ for all $z \in \mathbb{C} \setminus U$.*
(6) *Any $f \in \mathcal{H}(U)$ can be approximated by polynomials, uniformly on compact sets.*
(7) *For any open cover $\mathfrak{U}$ of U by connected, simply connected sets, we have $H^1(\mathfrak{U}, \mathbb{C}) = 0$.*
(8) *Any $f \in \mathcal{H}(U)$ has a primitive.*
(9) *If $f \in \mathcal{H}(U)$ is nowhere zero, then there exists $g \in \mathcal{H}(U)$ with $e^g = f$.*
(10) *If $f \in \mathcal{H}(U)$ is nowhere zero, then there exists $g \in \mathcal{H}(U)$ with $g^2 = f$.*

Proof. (1) ⇒ (2) Suppose that $\varphi : U \to \mathbb{D}$ is a homeomorphism, and let $\gamma : [0,1] \to U$ be a closed curve in U. Then $H(s,t) := \varphi^{-1}(s\varphi(\gamma(t)))$ defines a homotopy

$H : [0,1]^2 \to U$, where $H(0,t) = \varphi^{-1}(0)$, $H(1,t) := \gamma(t)$, and $H(s,0) = H(s,1)$ because $\gamma(0) = \gamma(1)$. Thus U is simply connected.

(2) $\Rightarrow$ (8) $\Rightarrow$ (9) $\Rightarrow$ (10) was shown in Theorem 4.8, Theorem 4.9, and Remark 4.10.

(10) $\Rightarrow$ (1) Clearly, $\mathbb{C}$ is homeomorphic to $\mathbb{D}$, for instance, via $z \mapsto z/(1+|z|)$. If $U \neq \mathbb{C}$, then the (proof of the) Riemann mapping theorem 20.4 gives even a biholomorphism between U and $\mathbb{D}$.

(7) $\Leftrightarrow$ (8) is the cohomological characterization of integrability 13.2.

(5) $\Leftrightarrow$ (8) That (5) $\Rightarrow$ (8) follows from the homology form of Cauchy's theorem 6.2 and the proof of Theorem 4.9. Conversely, if $c \in \mathbb{C} \setminus U$, then $1/(z-c) \in \mathcal{H}(U)$ and has a primitive, by (8). Then $2\pi i \operatorname{ind}_\gamma(c) = \int_\gamma 1/(z-c)\, dz = 0$ for every closed path γ in U.

(5) $\Rightarrow$ (4) If $\widehat{\mathbb{C}} \setminus U$ is not connected, then $\widehat{\mathbb{C}} \setminus U$ is the union of two nonempty disjoint closed sets H and K. If we assume that $\infty \in H$, then $\mathbb{C} \setminus H = U \cup K$ and K is compact. By Proposition 21.2, there is a cycle γ in $U = (\mathbb{C} \setminus H) \setminus K$ such that $\operatorname{ind}_\gamma(z) = 1$ for all $z \in K$, which contradicts (5).

(4) $\Rightarrow$ (3) Suppose that $\mathbb{C} \setminus U$ has a compact connected component C. By Lemma 11.4, there is a neighborhood N of C in $\mathbb{C} \setminus U$ which is open and closed in $\mathbb{C} \setminus U$, and relatively compact in $\mathbb{C}$. Since N is closed in $\mathbb{C} \setminus U$, and hence also in $\mathbb{C}$, N is compact. N is open in $\widehat{\mathbb{C}} \setminus U$, since it is open in $\mathbb{C} \setminus U$. N is also closed in $\widehat{\mathbb{C}} \setminus U$, since it is compact. Being both open and closed, N is the union of connected components of $\widehat{\mathbb{C}} \setminus U$, none of which can contain ∞. This contradicts (4).

(3) $\Leftrightarrow$ (6) Corollary 11.8.

(6) $\Rightarrow$ (8) Let $f \in \mathcal{H}(U)$ and let γ be a closed curve in U. There is a sequence of polynomials p_n which converges to f, uniformly on $|\gamma|$. Then $\int_\gamma f\, dz = \lim_{n\to\infty} \int_\gamma p_n\, dz = 0$. The proof of Theorem 4.9 implies (8). $\square$

22. Continuity at the boundary

A **Jordan curve** or **simple closed curve** is an injective continuous function $\gamma : S^1 \to \mathbb{C}$. The celebrated Jordan curve theorem asserts that, if γ is a Jordan curve, then $\mathbb{C} \setminus |\gamma|$ is the union for two disjoint open sets, one is unbounded and the other is homeomorphic to $\mathbb{D}$. We will take this result for granted.

A bounded region $U \subseteq \mathbb{C}$ whose boundary is a Jordan curve is called a **Jordan domain**. A Jordan domain is simply connected, cf. Theorem 21.3. We will prove in this section that a biholomorphic mapping $\varphi : U_1 \to U_2$ between Jordan domains extends to a homeomorphism $\tilde{\varphi} : \overline{U}_1 \to \overline{U}_2$.

Lemma 22.1. *Let U be a Jordan domain bounded by the Jordan curve γ. There is a function η defined for small $r > 0$ with $\eta(r) \to 0$ as $r \to 0$ such that if $a, b \in |\gamma|$ with $|a-b| \le r$ then there is a unique arc of γ having endpoints a, b and diameter $\le \eta(r)$.*

Proof. Since $\gamma : S^1 \to |\gamma|$ is an bijective continuous mapping between compact Hausdorff spaces, it has a continuous inverse. So there is $r_0 > 0$ such that $|\gamma(\zeta) - \gamma(\zeta')| \le r_0$ implies $|\zeta - \zeta'| < 2$. Let σ be the unique shorter arc of S^1 having endpoints ζ, ζ'. Let $\rho := \gamma \circ \sigma$. By continuity of γ^{-1}, we have $\operatorname{diam}(|\rho|) \to 0$ uniformly for $|\gamma(\zeta) - \gamma(\zeta')| \to 0$. For $0 < r < r_0$ we set

$$\eta(r) := \sup\{\operatorname{diam}(|\rho|) : |\gamma(\zeta) - \gamma(\zeta')| \le r\}.$$

Then $\eta(r) \to 0$ as $r \to 0$, and if $r_1 < r_0$ is such that $\eta(r_1) < \operatorname{diam}(|\gamma|)/2$ then the statement of the lemma holds for $r \le r_1$. $\square$

If $a, b \in |\gamma|$ with $|a-b|$ sufficiently small then we say that the unique arc of γ with endpoints a, b having diameter $\le \eta(|a-b|)$ is the **smaller** arc of γ joining a and b.

Theorem 22.2 (Carathéodory's theorem)**.** *Let $U_1, U_2 \subseteq \mathbb{C}$ be Jordan domains. If $\varphi : U_1 \to U_2$ is a biholomorphic mapping, then φ extends to a homeomorphism $\tilde{\varphi} : \overline{U}_1 \to \overline{U}_2$.*

Proof. Let us first assume that $U_1 = \mathbb{D}$. Fix $\zeta \in \partial\mathbb{D}$. We will construct a continuous extension of φ to ζ. Let γ_r denote the arc $\partial D_r(\zeta) \cap \mathbb{D}$ for $0 < r < 1$. The curve $\varphi \circ \gamma_r$ has length

$$L(r) = \int_{t_1(r)}^{t_2(r)} |\varphi'(\zeta + re^{it})| r\, dt,$$

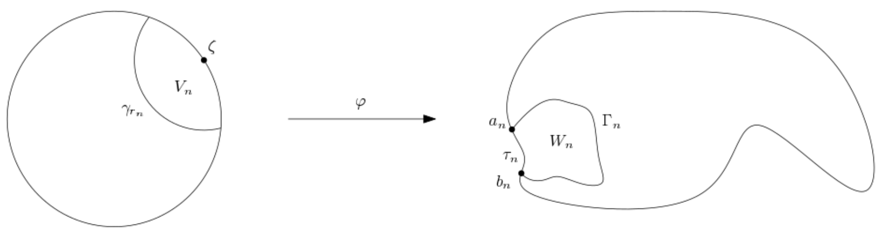
where $0 \le t_1(r) < t_2(r) < 2\pi$ are the solutions of $|\zeta + re^{it}| = 1$. Let M denote the area of $\varphi(D_{1/2}(\zeta) \cap \mathbb{D})$, which is finite since U_2 is bounded. Then, by the Cauchy–Schwarz inequality,

$$\begin{aligned}\int_0^{1/2} \frac{L(r)^2}{\pi r}\, dr &= \int_0^{1/2} \Big(\int_{t_1(r)}^{t_2(r)} |\varphi'(\zeta + re^{it})| r\, dt \Big)^2 \frac{1}{\pi r}\, dr \\ &\le \int_0^{1/2} \int_{t_1(r)}^{t_2(r)} |\varphi'(\zeta + re^{it})|^2 r\, dt \int_{t_1(r)}^{t_2(r)} r\, dt \frac{1}{\pi r}\, dr \le M < \infty.\end{aligned}$$

Since $1/r$ is not integrable at 0, there must exists a sequence $r_n \to 0$ such that $L(r_n) \to 0$.

Let a_n, b_n denote the endpoints of $\varphi \circ \gamma_{r_n}$; they exist since each $L(r_n) < \infty$. Since $\varphi : \mathbb{D} \to U_2$ is a homeomorphism, we have $a_n, b_n \in \partial U_2$.

Set $\Gamma_n := |\varphi \circ \gamma_{r_n}|$. If $a_n \ne b_n$ let τ_n be the smaller (in the sense specified after Lemma 22.1) of the two boundary arcs of U_2 connecting a_n and b_n. Then $\Gamma_n \cup \tau_n$ forms a Jordan curve. If $a_n = b_n$, then $\Gamma_n \cup \{a_n\}$ is a Jordan curve. In either case it surrounds a bounded region W_n, by the Jordan curve theorem. Let $V_n := D_{r_n}(\zeta) \cap \mathbb{D}$. Then either $\varphi(V_n) = W_n$ or $\varphi(V_n) = U_2 \setminus \overline{W}_n$. We claim that $\varphi(V_n) = W_n$ if n is sufficiently large.



In fact, let $T_n := \mathbb{D} \setminus \overline{V}_n$ and let n be fixed. If $w \in W_n$ then $w = \varphi(z)$ for some $z \in V_n \cup T_n$. If $z \in V_n$ then $\varphi(V_n) = W_n$, by connectivity. If $z \in T_n$ then $\varphi(T_n) \subseteq W_n$. Let us prove that this is impossible for large n. Observe that $\operatorname{area}(\varphi(V_n)) = \iint_{V_n} |\varphi'|^2\, dx\, dy \to 0$ as $n \to \infty$, and thus $\operatorname{area}(\varphi(T_n)) \to \operatorname{area}(U_2)$. We have $|a_n - b_n| \le L(r_n)$ so that $\operatorname{diam}(\tau_n) \le \eta(L(r_n)) \to 0$, by Lemma 22.1. It follows that the entire Jordan curve $\Gamma_n \cup \tau_n$, and thus also W_n, lies in the disk centered at a_n with radius $L(r_n) + \eta(L(r_n))$. Consequently, $\operatorname{area}(W_n) \to 0$ so that $\varphi(T_n) \not\subseteq W_n$.

So we have proved that $\varphi(V_n) = W_n$ if n is sufficiently large, as well as $\operatorname{diam}(W_n) \to 0$ and $\operatorname{area}(W_n) \to 0$ as $n \to \infty$.

Let $\delta_i : [0,1] \to \overline{\mathbb{D}}$, $i = 1, 2$, be any curves such that $\delta_i(t) \in \mathbb{D}$, for $t \in [0,1)$, and $\delta_i(1) = \zeta$. We claim that the limits $\lim_{t \to 1} \varphi(\delta_i(t))$, $i = 1, 2$, exist and coincide.

Let $\epsilon > 0$. Choose N sufficiently large that $\operatorname{diam}(W_N) < \epsilon$. If t is sufficiently close to 1, then $\varphi(\delta_i(t)) \in W_N$ and hence $|\varphi(\delta_1(t)) - \varphi(\delta_2(t))| < \epsilon$. Thus the limits $\lim_{t\to 1} \varphi(\delta_i(t))$ exist and coincide with the unique point in $\bigcap_n \overline{W}_n$.

This provides the continuous extension of φ to $\partial\mathbb{D}$: if $\zeta \in \partial\mathbb{D}$ choose a curve $\delta : [0,1] \to \overline{\mathbb{D}}$ such that $\delta(t) \in \mathbb{D}$, for $t \in [0,1)$, and $\delta(1) = \zeta$, and define $\tilde{\varphi}(\zeta) := \lim_{t\to 1} \varphi(\delta(t))$; the limit exists and is independent of δ by the previous paragraph.

Let us check injectivity of $\tilde{\varphi} : \overline{\mathbb{D}} \to \overline{U}_2$. It is enough to check that $\tilde{\varphi}$ is injective on $\partial\mathbb{D}$, since $\varphi : \mathbb{D} \to U_2$ is injective and $\tilde{\varphi}(\partial\mathbb{D}) \subseteq \partial U_2$. Let $\zeta, \zeta' \in \partial\mathbb{D}$ and $\tilde{\varphi}(\zeta) = \tilde{\varphi}(\zeta')$. Consider $R := \{r\zeta : 0 \le r \le 1\} \cup \{r\zeta' : 0 \le r \le 1\}$. By assumption $\varphi(R)$ is a Jordan curve, let W be its interior. If V_1, V_2 are the connected components of $\mathbb{D} \setminus R$, then either $\varphi(V_1) = W$ or $\varphi(V_2) = W$. Suppose without loss of generality that $\varphi(V_1) = W$ and let μ denote the segment on $\partial\mathbb{D}$ which bounds V_1. Then $\tilde{\varphi}(\mu) \subseteq \overline{W} \cap \partial U_2 = \{\tilde{\varphi}(\zeta)\}$, i.e., $\tilde{\varphi}$ is constant on μ. By the Schwarz reflection principle, see Corollary 18.3, φ is constant, a contradiction.

Thus we have shown that $\varphi : \mathbb{D} \to U_2$ extends to a bijective continuous mapping $\tilde{\varphi} : \overline{\mathbb{D}} \to \overline{U}_2$, thus, it is a homeomorphism. In the general case, when $\varphi : U_1 \to U_2$, let $\varphi_i : \mathbb{D} \to U_i$, $i = 1, 2$, be biholomorphic mappings, which exist by the Riemann mapping theorem 20.4. Then φ_i extends to a homeomorphism $\tilde{\varphi}_i : \overline{\mathbb{D}} \to \overline{U}_i$. Similarly, the mapping $\varphi_2^{-1} \circ \varphi \circ \varphi_1$ extends to a homeomorphism $\overline{\mathbb{D}} \to \overline{\mathbb{D}}$. This implies the theorem. □

Remark 22.3. The theorem extends without problems to Jordan domains in the extended plane $\widehat{\mathbb{C}}$.

Exercise 45. Let $U \subseteq \mathbb{C}$ be a bounded simply connected region with real analytic boundary, i.e., the boundary is locally the graph of a function given by a convergent power series. Let $f : \mathbb{D} \to U$ be biholomorphic. Prove that f has a holomorphic extension to some neighborhood of $\mathbb{D}$. Hint: The problem is purely local. Use a change of variables to reduce to the case that both boundaries are flat and apply the Schwarz reflection principle.

23. Biholomorphisms of annuli

By the Riemann mapping theorem 20.4, there are, up to biholomorphism, only two domains that are homeomorphic to the disk, namely, the disk and the plane. If we allow holes, then the situation becomes more involved. We demonstrate this by looking at annuli. If $c > 0$ and $r_1 < r_2$ then clearly the annuli $A_{r_1,r_2}(0)$ and $A_{cr_1,cr_2}(0)$ are biholomorphic under the mapping $z \mapsto cz$. Surprisingly, these are essentially the only circumstances under which two annuli are biholomorphic.

Theorem 23.1. *Let $A_i = \{z \in \mathbb{C} : 1 < |z| < R_i\}$, $i = 1, 2$, where $R_i > 1$. Then A_1, A_2 are biholomorphic if and only if $R_1 = R_2$. Moreover,* $\operatorname{Aut}(A_i) = \{e^{it}z, R_i e^{it}/z : t \in \mathbb{R}\}$.

Proof. Suppose that $f : A_1 \to A_2$ is a biholomorphism. Then also $g = R_2/f : A_1 \to A_2$ is a biholomorphism. We claim that

(1) $|f(z)|$ has a limit as $|z| \to 1$ which is either 1 or R_2,
(2) if $\lim_{|z|\to 1} |f(z)| = 1$ then $|f(z)| \to R_2$ as $|z| \to R_1$, and if $\lim_{|z|\to 1} |f(z)| = R_2$ then $|f(z)| \to 1$ as $|z| \to R_1$.

Set $F := f$ if $\lim_{|z|\to 1} |f(z)| = 1$ and $F := g$ if $\lim_{|z|\to 1} |f(z)| = R_2$. Then $|F(z)| \to 1$ as $|z| \to 1$ and $|F(z)| \to R_2$ as $|z| \to R_1$.

Consider the function $u(z) := \log|F(z)| - c\log|z|$ for $z \in A_1$, where $c = \log R_2/\log R_1 > 0$. Then u is harmonic on A_1 and extends continuously by 0 to ∂A_1. By the maximum principle for harmonic functions 17.2, $u \equiv 0$ on A_1. Thus, $|F(z)| = |z|^c$ for $z \in A_1$. Let $D = D_r(a) \subseteq A_1$. Let $h \in \mathcal{H}(D)$ be a branch of the logarithm, i.e., $z = e^{h(z)}$ on D, by Theorem 4.8. Then $|F(z)e^{-ch(z)}| = 1$ on D so that $F(z) = e^{ch(z)+i\theta}$ on D for some real constant θ. Analytic continuation of the germ F_a of F at a along the curve $\gamma(t) = ae^{it}$, $0 \le t \le 2\pi$, leads back to F_a, since F is holomorphic on A_1, while analytic continuation of the germ h_a of h at a along γ leads to $h_a + 2\pi i$. It follows that $e^{2\pi i c} = 1$ so that $c = n \in \mathbb{Z}$ and $z^{-n} \in \mathcal{H}(A_1)$. Consequently, $F(z) = e^{i\theta}z^n$. Since F is injective, only $n = \pm 1$ are possible, and since $|F(z)| \to R_2$ as $|z| \to R_1$ we have $n = 1$. Thus, either $f(z) = e^{i\theta}z$ or $R_2/f(z) = g(z) = e^{i\theta}z$. Since $|f(z)| \to R_2$ as $|z| \to R_1$ in the first case and $|f(z)| \to 1$ as $|z| \to R_1$ in the second, the theorem follows.

It remains to prove the claim. Since $f : A_1 \to A_2$ is biholomorphic, if a sequence z_n in A_1 converges to the boundary of A_1 (i.e., it has no interior accumulation point) then so does the sequence $f(z_n)$ in A_2. In particular, for small $\epsilon > 0$, the set $f(\{z : 1 < |z| < 1 + \epsilon\})$ does not intersect $\{z : |z| = (1 + R_2)/2\}$. There is n_0 such that, for $n \ge n_0$, $f(z_n)$ is contained in a fixed component of $\{z : |z| \ne (1 + R_2)/2\}$. This implies (1). Let us assume that $\lim_{|z|\to 1} |f(z)| = 1$. By the same reasoning as before, $|f(z)|$ has a limit as $|z| \to R_1$ which is either 1 or R_2. By the maximum principle, only the second possibility can occur. The claim is proved. $\square$

CHAPTER 6

Elliptic functions and Picard's theorem

24. Elliptic functions

Let f be meromorphic in $\mathbb{C}$. Let $\mathrm{per}(f)$ be the set of all periods of f (including 0); recall that w is a **period** of f if $f(z+w) = f(z)$ for all z. Clearly, $\mathrm{per}(f)$ is a module over $\mathbb{Z}$ (i.e., if $w_1, w_2 \in \mathrm{per}(f)$ and $n_1, n_2 \in \mathbb{Z}$ then $n_1 w_1 + n_2 w_2 \in \mathrm{per}(f)$), so we call $\mathrm{per}(f)$ the **period module** of f.

The identity theorem implies that $\mathrm{per}(f)$ is discrete unless f is constant.

Lemma 24.1. *For a discrete module $\Lambda \subseteq \mathbb{C}$ over $\mathbb{Z}$ we have three possibilities:*

(1) $\Lambda = \{0\}$,
(2) $\Lambda = \mathbb{Z}w$ *for some* $w \in \mathbb{C}^*$,
(3) $\Lambda = \mathbb{Z}w_1 + \mathbb{Z}w_2$ *for some* $w_1, w_2 \in \mathbb{C}^*$ *such that* $w_2/w_1 \notin \mathbb{R}$.

Proof. If $\Lambda \neq \{0\}$ then there exists $w_1 \in \Lambda$ with minimal absolute value, since Λ is discrete. Suppose that there is an element $w_2 \in \Lambda$ which is not an integer multiple of w_1, and we may assume that w_2 has minimal absolute value. If $w_2/w_1 \in \mathbb{R}$, then there is an integer n such that $n < w_2/w_1 < n+1$ and hence $0 < |nw_1 - w_2| < |w_1|$, which contradicts minimality of w_1.

Let us show that $\Lambda = \mathbb{Z}w_1 + \mathbb{Z}w_2$. Since $w_2/w_1 \notin \mathbb{R}$, we have $\mathbb{C} = \mathbb{R}w_1 + \mathbb{R}w_2$. Let $w = \lambda_1 w_1 + \lambda_2 w_2 \in \Lambda$. There exist integers m_1, m_2 such that $|\lambda_i - m_i| \le 1/2$, $i = 1, 2$. We have $w' := (\lambda_1 - m_1)w_1 + (\lambda_2 - m_2)w_2 \in \Lambda$ and $|w'| < |w_1|/2 + |w_2|/2 \le |w_2|$, where the first inequality is strict, because $w_2/w_1 \notin \mathbb{R}$. By the way w_2 was chosen, w' must be an integer multiple of w_1, and thus $w \in \mathbb{Z}w_1 + \mathbb{Z}w_2$. $\square$

We assume henceforth that the third alternative occurs: $\Lambda = \mathbb{Z}w_1 + \mathbb{Z}w_2$ for some $w_1, w_2 \in \mathbb{C}^*$ such that $w_2/w_1 \notin \mathbb{R}$.

A **basis** of the module Λ is any pair (w_1, w_2) such that every $w \in \Lambda$ has a unique representation $w = n_1 w_1 + n_2 w_2$. If (w_1, w_2) and (w_1', w_2') are two bases, then there exist integers a, b, c, d such that

$$\begin{pmatrix} w_1' \\ w_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

The same is true for the complex conjugates:

$$\begin{pmatrix} w_1' & \overline{w}_1' \\ w_2' & \overline{w}_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w_1 & \overline{w}_1 \\ w_2 & \overline{w}_2 \end{pmatrix}.$$

Since (w_1', w_2') also is a basis, there are integers a', b', c', d' such that

$$\begin{pmatrix} w_1 & \overline{w}_1 \\ w_2 & \overline{w}_2 \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} w_1' & \overline{w}_1' \\ w_2' & \overline{w}_2' \end{pmatrix}.$$

Consequently,

$$\begin{pmatrix} w_1 & \overline{w}_1 \\ w_2 & \overline{w}_2 \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w_1 & \overline{w}_1 \\ w_2 & \overline{w}_2 \end{pmatrix}$$

Now $w_1\overline{w}_2 - \overline{w}_1 w_2 \neq 0$, since otherwise any two elements in Λ would have a real ratio, and thus

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since the entries are integers, the determinant is ± 1. Linear transformations of this type with integer coefficients and determinant ± 1 are said to be **unimodular**. So any two bases of a module Λ are connected by a unimodular transformation.

The so-called **modular group** is the group of Möbius transformations

$$f(z) = \frac{az+b}{cz+d},$$

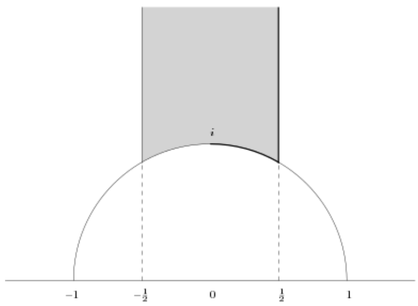
with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. It is a discrete subgroup of the automorphism group $\mathrm{Aut}(\mathbb{H}) = \{z \mapsto (az+b)/(cz+d) : a, b, c, d \in \mathbb{R},\ ad - bc = 1\}$ of the upper half plane, and is isomorphic to $\mathrm{PSL}(2, \mathbb{Z}) = \mathrm{SL}(2, \mathbb{Z})/\{\pm \mathrm{Id}\}$.

Proposition 24.2 (canonical basis)**.** *There is a basis (w_1, w_2) such that $\tau := w_2/w_1$ satisfies*

1. $\operatorname{Im} \tau > 0$,
2. $-1/2 < \operatorname{Re} \tau \leq 1/2$,
3. $|\tau| \geq 1$,
4. $\operatorname{Re} \tau \geq 0$ *if* $|\tau| = 1$.

The ratio τ is uniquely determined by these conditions, and there is a choice of 2, 4, *or* 6 *corresponding bases.*

The set of all τ satisfying (1)–(4) (depicted below) is called the **fundamental domain of the modular group**; strictly speaking it is not a domain since it is not open.



Proof. Select w_1, w_2 as in the proof of Lemma 24.1. Then $|w_1| \leq |w_2|$, $|w_2| \leq |w_1 + w_2|$, and $|w_2| \leq |w_1 - w_2|$, or equivalently, $|\tau| \geq 1$ and $|\operatorname{Re} \tau| \leq 1/2$. If $\operatorname{Im} \tau < 0$ replace (w_1, w_2) by $(-w_1, w_2)$, which makes $\operatorname{Im} \tau > 0$ without changing the condition on $\operatorname{Re} \tau$. If $\operatorname{Re} \tau = -1/2$ replace (w_1, w_2) by $(w_1, w_1 + w_2)$, and if $|\tau| = 1$, $\operatorname{Re} \tau < 0$ replace (w_1, w_2) by $(-w_2, w_1)$.

Next we show uniqueness of τ. We saw that two bases differ by a unimodular transformation. Hence if the new ratio is τ' then

$$\tau' = \frac{a\tau + b}{c\tau + d}, \quad ad - bc = \pm 1, \tag{24.1}$$

and hence

$$\operatorname{Im} \tau' = \frac{\pm \operatorname{Im} \tau}{|c\tau + d|^2},$$

where the sign is the same as that of $ad - bc$. Suppose that τ and τ' are in the fundamental domain. We must show that $\tau = \tau'$. If τ, τ' are in the fundamental domain, then $\operatorname{Im}\tau' = \operatorname{Im}\tau/|c\tau+d|^2$ and thus $ad-bc=1$. By symmetry, we may assume that $\operatorname{Im}\tau' \geq \operatorname{Im}\tau$ and hence $|c\tau+d| \leq 1$. Since c, d are integers, very few cases must be checked.

First let $c=0$ and $d=\pm 1$. Then we have $ad=1$ and hence $a=d=1$ or $a=d=-1$. By (24.1), $\tau' = \tau \pm b$. Since τ and τ' satisfy (2), $|b| = |\operatorname{Re}\tau' - \operatorname{Re}\tau| < 1$ which implies $b=0$. So $\tau'=\tau$.

Suppose that $d=0$. In this case $bc=-1$ and hence either $b=1$, $c=-1$ or $b=-1$, $c=1$. In any case $|c\tau+d| \leq 1$ becomes $|\tau| \leq 1$, and so $|\tau|=1$, by (3). By (24.1), $\tau' = \pm a - 1/\tau = \pm a - \overline{\tau}$. We have $|a| = |\operatorname{Re}\tau' + \operatorname{Re}\tau| \leq 1$, by (2). Consequently, either $\tau' = \tau = e^{i\pi/3}$ or $a=0$. In the latter case $\tau' = -\overline{\tau}$ which can only happen if $\tau'=\tau$, in view of (4).

Finally, let $c \neq 0$ and $d \neq 0$. Then $|cd| \geq 1$ and thus, by (2) and (3),

$$|c\tau+d|^2 = c^2|\tau|^2 + d^2 + 2cd\operatorname{Re}\tau \geq c^2 + d^2 - |cd| = (|c|-|d|)^2 + |cd| \geq 1.$$

Our assumption $|c\tau+d| \leq 1$ implies that equality holds everywhere in this computation. That means $|\tau|=1$ and $\operatorname{Re}\tau = 1/2$, or equivalently, $\tau = e^{i\pi/3}$. Since $|c\tau+d|=1$ we have $\operatorname{Im}\tau' = \operatorname{Im}\tau$, whence τ' (subject to (1)–(4)) must equal τ.

The canonical basis (w_1, w_2) can always be replaced by $(-w_1, -w_2)$. There are other bases with the same ratio τ if and only if τ is a fixed point of (24.1). This happens only for $\tau = i$ which is a fixed point of $\tau \mapsto -1/\tau$ and $\tau = e^{\pi i/3}$ which is a fixed point of $\tau \mapsto -(\tau+1)/\tau$ and of $\tau \mapsto -1/(\tau+1)$. So there is a choice of 2, 4, or 6 corresponding bases. □

Let f be a meromorphic function in $\mathbb{C}$. Let Λ be a module with basis (w_1, w_2), where $w_2/w_1 \notin \mathbb{R}$ (not necessarily canonical), and assume that $\Lambda \subseteq \operatorname{per}(f)$. Then f is called an **elliptic** or **doubly periodic** function.

Let $a \in \mathbb{C}$ and let P_a be the parallelogram with vertices a, $a+w_1$, $a+w_2$, $a+w_1+w_2$. Then P_a, where part of the boundary is included, represents the quotient space $\mathbb{C}/\Lambda$ of the equivalence relation $z_1 \sim z_2$ if and only if $z_1 - z_2 \in \Lambda$. So we may regard f as a function on P_a; cf. the figure on p. 85.

Theorem 24.3 (properties of elliptic functions)**.** *Let f be an elliptic function with period module Λ.*

1. *If f has no poles then f is constant.*
2. *The sum of residues of f is zero.*
3. *If f is non-constant, then the number of poles of f equals the number of zeros of f.*
4. *If $a_1, \dots, a_n$ are the zeros and $b_1, \dots, b_n$ the poles of f, then $\sum a_i - \sum b_i \in \Lambda$.*

Proof. (1) If f has no poles, then f is bounded on $\overline{P}_a$ and thus on $\mathbb{C}$. The assertion follows from Liouville's theorem.

(2) We may choose $a \in \mathbb{C}$ so that no poles of f lie on ∂P_a. By the residue theorem 8.1, the sum of the residues at the poles in P_a equals

$$\frac{1}{2\pi i}\int_{\partial P_a} f(z)\,dz$$

which vanishes, because by periodicity of f the integrals over opposite side of P_a cancel against each other.

(3) The function f'/f is elliptic. The poles and zeros of f are simple poles of f'/f, and the orders are the residues of f'/f counted positive for zeros and negative for poles. Thus (3) follows from (2).

(4) We may assume that there are no zeros and poles of f on P_a. Moreover, we may assume that all a_i, b_i lie in P_a, by taking the right representatives. By the residue theorem 8.1 (cf. Exercise 10) and periodicity of f,

$$\begin{aligned}\sum a_i - \sum b_i &= \frac{1}{2\pi i}\int_{\partial P_a} \frac{zf'(z)}{f(z)}\,dz\\ &= \frac{1}{2\pi i}\Big(\Big(\int_{[a,a+w_1]} - \int_{[a+w_2,a+w_1+w_2]}\Big)\\ &\qquad + \Big(\int_{[a+w_1,a+w_1+w_2]} - \int_{[a,a+w_2]}\Big)\Big)\frac{zf'(z)}{f(z)}\,dz\\ &= \frac{-w_2}{2\pi i}\int_{[a,a+w_1]}\frac{f'(z)}{f(z)}\,dz + \frac{w_1}{2\pi i}\int_{[a,a+w_2]}\frac{f'(z)}{f(z)}\,dz\\ &= -w_2 \operatorname{ind}_{f([a,a+w_1])}(0) + w_1 \operatorname{ind}_{f([a,a+w_2])}(0) \in \Lambda. \qquad \square\end{aligned}$$

Corollary 24.4. *If f is a non-constant elliptic function, then f and $f - c$ have the same number of zeros for every $c \in \mathbb{C}$.*

Proof. f and $f - c$ have the same number of poles. $\square$

25. The Weierstrass $\wp$-function

Let Λ be a module with basis (w_1, w_2), where $w_2/w_1 \notin \mathbb{R}$. The simplest elliptic functions are of order 2. They have either a double pole with residue zero, or two simple poles with opposite residues.

Suppose that f has a double pole at the origin with residue zero. By multiplication with a constant we may assume that the singular part of f is z^{-2}. If f is elliptic and has only this singularity (up to periodicity), then f must be even. Indeed, $f(z)-f(-z)$ has the same periods and no singularity, so $f(z)-f(-z) = \text{const} = 0$, by setting $z = w_1/2$. Thus the Laurent development of f at 0 has the form

$$z^{-2} + a_2 z^2 + a_4 z^4 + \cdots, \tag{25.1}$$

if we assume without loss of generality that $a_0 = 0$.

We will show the existence of an elliptic function with this Laurent development. Let $\Lambda^* := \Lambda \setminus \{0\}$ and define

$$\wp(z) := \frac{1}{z^2} + \sum_{w\in\Lambda^*}\Big(\frac{1}{(z-w)^2} - \frac{1}{w^2}\Big). \tag{25.2}$$

We claim that the series converges uniformly on compact subsets of $\mathbb{C}\setminus\Lambda$. We have

$$\Big|\frac{1}{(z-w)^2} - \frac{1}{w^2}\Big| = \Big|\frac{z(2w-z)}{w^2(z-w)^2}\Big| = O(|w|^{-3}) \quad \text{as } |w| \to \infty.$$

Since $w_2/w_1 \notin \mathbb{R}$, there is a constant $K > 0$ such that $|n_1w_1+n_2w_2| \ge K(|n_1|+|n_2|)$ for all integers n_1, n_2. There are $4n$ pairs (n_1, n_2) of integers with $|n_1| + |n_2| = n$, whence

$$\sum \frac{1}{|w|^3} \le \frac{4}{K^3}\sum_{n=1}^{\infty}\frac{1}{n^2} < \infty.$$

The claim is proved.

By termwise differentiation,

$$\wp'(z) = -\frac{2}{z^3} - \sum_{w\in\Lambda^*}\frac{2}{(z-w)^3} = -2\sum_{w\in\Lambda}\frac{1}{(z-w)^3} \tag{25.3}$$

which is obviously elliptic. This implies that $\wp(z+w_1)-\wp(z)$ and $\wp(z+w_2)-\wp(z)$ are constant. Since $\wp$ is even, by (25.2), choosing $z = -w_1/2$ and $z = -w_2/2$

implies that the constants are zero. That means that $\wp$ has the periods w_1, w_2. We may conclude that $\wp$ has the Laurent expansion (25.1) at the origin.

The function $\wp$ is called the **Weierstrass $\wp$-function**. Since $\wp$ is even and periodic, $\wp(w_i - z) = \wp(z)$ and hence $\wp'(w_i - z) = -\wp'(z)$, in particular, $\wp'(w_i/2) = 0$ for $i = 1, 2$. Similarly, $\wp'((w_1 + w_2)/2) = 0$. The **half-periods** $w_1/2$, $w_2/2$, and $(w_1 + w_2)/2$ are precisely the three simple zeros of $\wp'$ which is of order 3. Let us set

$$e_1 = \wp(w_1/2), \ e_2 = \wp(w_2/2), \ e_3 = \wp((w_1 + w_2)/2). \tag{25.4}$$

Then the equation $\wp(z) = e_1$ has a double root at $w_1/2$, and since $\wp$ has order 2 there are no other roots in the fundamental parallelogram. Similarly, $\wp(z) = e_2$ has only a double root at $w_2/2$, and $\wp(z) = e_3$ has only a double root at $(w_1 + w_2)/2$. We may conclude that e_1, e_2, e_3 are distinct, for otherwise $\wp$ would have at least four roots contradicting the fact that it assumes each value with multiplicity 2.

We claim that

$$(\wp'(z))^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3). \tag{25.5}$$

Indeed, $f(z) := (\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3)$ vanishes in the fundamental parallelogram precisely at the points $w_1/2$, $w_2/2$, and $(w_1 + w_2)/2$ of order 2, respectively. Also $(\wp')^2$ has double zeros at these points. Moreover, f as well as $(\wp')^2$ has poles of order 6 at the points in Λ. It follows that $(\wp')^2/f$ is holomorphic and elliptic, thus constant, by Theorem 24.3. That this constant equals 4 follows from (25.2) and (25.3). Thus (25.5) is proved.

We remark that (25.5) takes the form

$$(\wp'(z))^2 = 4\wp(z)^3 - g_2\wp(z) - g_3, \tag{25.6}$$

where $g_2 = 60G_2$, $g_3 = 140G_3$, and $G_k := \sum_{w \in \Lambda^*} w^{-2k}$; cf. [**1**, p. 275]. We shall see in Proposition 34.1 that $(x, y) = (\wp(z), \wp'(z))$ parameterizes the elliptic curve $y^2 = 4x^3 - g_2 x - g_3$.

The differential equation (25.6) for $w = \wp(z)$ can be solved explicitly:

$$z - z_0 = \int_\gamma \frac{dw}{\sqrt{4w^3 - g_2 w - g_3}},$$

where γ is the image under $\wp$ of a path from z_0 to z that avoid zeros and poles of $\wp'(z)$ and where the sign of the square root is chosen such that it equals $\wp'(z)$. Integrals of this type appear in the computation of the arc length of an ellipse and are called elliptic integrals.

Proposition 25.1 (universality of $\wp$)**.** *Every elliptic function with period module Λ is a rational function of $\wp$ and $\wp'$.*

Proof. The proposition will follow from the claim that every even elliptic function with period module Λ is a rational function of $\wp$. In fact, we may write

$$f(z) = f_{\text{even}}(z) + f_{\text{odd}}(z) := \frac{f(z) + f(-z)}{2} + \frac{f(z) - f(-z)}{2}$$

as a sum of an even and an odd part. Since $f_{\text{odd}}/\wp'$ is even, the statement of the proposition follows from the claim (applied to f_{even} and $f_{\text{odd}}/\wp'$).

Let us prove the claim. Suppose that f is an even elliptic function. If f has a zero or pole at 0 it must be of even order, since f is an even function. Thus, there is an integer m such that $f\wp^m$ has no zero or pole at the points in Λ. Hence we may assume without loss of generality that f has no zero or pole on Λ.

We saw before that $\wp(z) - \wp(a)$ has a zero of order 2 if a is a half-period, and two simple zeros at $\pm a$ otherwise. If a is a zero of f, then so is $-a$, since f is even. We have $a - (-a) = 2a \in \Lambda$ if and only if a is a half-period; in this case the zero is of even order. Consequently, if $a_1, -a_1, a_2, -a_2, \ldots, a_m, -a_m$ (counted with multiplicities) are the zeros of f, then $(\wp(z) - \wp(a_1)) \cdots (\wp(z) - \wp(a_m))$ has exactly the same zeros as f. A similar argument applies to the poles of f. We may conclude that

$$g(z) := \frac{(\wp(z) - \wp(a_1)) \cdots (\wp(z) - \wp(a_m))}{(\wp(z) - \wp(b_1)) \cdots (\wp(z) - \wp(b_m))}$$

is elliptic and has the same zeros and poles as f. So f/g is holomorphic and elliptic, therefore constant, by Theorem 24.3. The claim is proved. □

26. Modular functions and the little Picard theorem

Recall that the modular group G is the group of Möbius transformations $f(z) = (az+b)/(cz+d)$, where $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. This is a subgroup of the automorphism group of the upper half plane $\mathbb{H}$. A **modular function** is a function $f : \mathbb{H} \to \mathbb{C}$ which is invariant under the action of some nontrivial subgroup of G, i.e., there is a nontrivial subgroup K of G such that

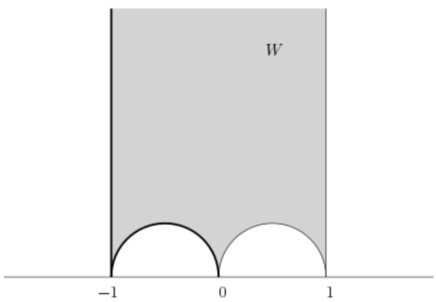
$$(f \circ \varphi)(z) = f(z), \qquad \text{for all } \varphi \in K,\ z \in \mathbb{H}.$$

We concentrate on the subgroup Γ of G generated by the two elements

$$\sigma(z) = \frac{z}{2z+1}, \quad \tau(z) = z + 2.$$

Consider the set

$$W := \{z \in \mathbb{H} : -1 \leq \operatorname{Re} z < 1,\ |2z+1| \geq 1,\ |2z-1| > 1\}.$$



Proposition 26.1. *W is a fundamental domain for the action of Γ on $\mathbb{H}$, that is:*

1. *If $f, g \in \Gamma$, $f \neq g$, then $f(W) \cap g(W) = \emptyset$.*
2. *$\mathbb{H} = \bigcup_{f \in \Gamma} f(W)$.*

Furthermore, $\Gamma = \{f(z) = (az+b)/(cz+d) \in G : a, d \text{ odd}, b, c \text{ even}\}$.

Proof. Let $\Gamma' := \{f(z) = (az+b)/(cz+d) \in G : a, d \text{ odd}, b, c \text{ even}\}$. It is easy to check that Γ' is a subgroup of G. Since $\sigma, \tau \in \Gamma'$ we have $\Gamma \subseteq \Gamma'$. Let (1′) be the statement obtained from (1) by replacing Γ with Γ'. Then $\Gamma = \Gamma'$ will follow from (1′) and (2).

(1′) Let $g, h \in \Gamma'$, $g \neq h$, and set $f := g^{-1} \circ h$. If $z \in g(W) \cap h(W)$, then $g^{-1}(z) \in W \cap f(W)$. So it suffices to prove $W \cap f(W) = \emptyset$ if $f \in \Gamma'$, $f \neq \mathrm{Id}$. If $f(z) = (az+b)/(cz+d)$, then

$$\operatorname{Im} f(z) = \frac{\operatorname{Im} z}{|cz+d|^2}. \tag{26.1}$$

We consider three cases.

If $c = 0$ then $ad = 1$ and hence $a = d = \pm 1$. Then $f(z) = z + 2n$ for some integer $n \neq 0$. Evidently, $W \cap f(W) = \emptyset$.

If $c = 2d$ then $c = \pm 2$ and $d = \pm 1$ (using $ad - bc = 1$). Then $f(z) = \sigma(z) + 2m$ for some $m \in \mathbb{Z}$. Observe that $\sigma(W) \subseteq \overline{D}_{1/2}(1/2)$ which implies the assertion.

If $c \neq 0$ and $c \neq 2d$, we claim that $|cz + d| > 1$ for all $z \in W$. Then, by (26.1), $\operatorname{Im} f(z) < \operatorname{Im} z$ for every $z \in W$. If $z \in W \cap f(W)$ then we could apply the same argument to f^{-1} and conclude that $\operatorname{Im} z = \operatorname{Im} f^{-1}(f(z)) < \operatorname{Im} f(z)$, a contradiction. So let us show that $|cz + d| > 1$ for all $z \in W$. Suppose, for some $z \in W$, $|cz + d| \leq 1$. Then $W \cap \overline{D}_{1/|c|}(-d/c) \neq \emptyset$. Since $-d/c \neq -1/2$, the open disk $D_{1/|c|}(-d/c)$ must contain one of the points $-1, 0, 1$ which is clear by a glance at the above picture. Hence $|cw + d| < 1$ for $w = -1$ or 0 or 1. But $cw + d$ is an odd integer, a contradiction.

(2) Let $U := \bigcup_{f \in \Gamma} f(W)$. Clearly, $U \subseteq \mathbb{H}$. Note that $\tau^n(W) \subseteq U$ for all $n \in \mathbb{Z}$, where $\tau^n(z) = z + 2n$. Since σ maps the circle $|2z + 1| = 1$ onto the circle $|2z - 1| = 1$, U contains all points $z \in \mathbb{H}$ with

$$|2z - (2m + 1)| \geq 1 \text{ for all } m \in \mathbb{Z}. \tag{26.2}$$

Fix $w \in \mathbb{H}$. Choose $f_0 \in \Gamma$ such that $|cw + d|$ is minimal; this is possible since as $\operatorname{Im} w > 0$ there are only finitely many c, d such that $|cw + d|$ lies below a given bound. By (26.1),

$$\operatorname{Im} f(w) \leq \operatorname{Im} f_0(w), \qquad \text{for all } f \in \Gamma.$$

Putting $z = f_0(w)$ this becomes

$$\operatorname{Im} f(z) \leq \operatorname{Im} z, \qquad \text{for all } f \in \Gamma.$$

In particular, for

$$(\sigma \circ \tau^{-n})(z) = \frac{z - 2n}{2z - 4n + 1} \qquad \text{and} \qquad (\sigma^{-1} \circ \tau^{-n})(z) = \frac{z - 2n}{-2z + 4n + 1}$$

we obtain, with (26.1),

$$|2z - 4n + 1| \geq 1 \qquad \text{and} \qquad |2z - 4n - 1| \geq 1, \qquad \text{for all } n \in \mathbb{Z}.$$

Thus z satisfies (26.2) and so $z \in U$. Therefore, $w = f_0^{-1}(z) \in U$. □

We will now construct a particular modular function, the so-called **elliptic modular function**.

Proposition 26.2 (elliptic modular function). *There exists $\lambda \in \mathcal{H}(\mathbb{H})$ with the following properties:*

1. *$\lambda \circ f = \lambda$ for every $f \in \Gamma$.*
2. *λ is injective on W.*
3. *$\lambda(\mathbb{H}) = \mathbb{C} \setminus \{0, 1\}$.*
4. *$\mathbb{R}$ is a natural boundary of λ.*

In particular, $\lambda : \mathbb{H} \to \mathbb{C} \setminus \{0, 1\}$ is a covering map.

Proof. Set $W^+ := \{z \in W : \operatorname{Re} z > 0\}$. By the Riemann mapping theorem 20.4, there is a biholomorphic mapping $g : W^+ \to \mathbb{H}$. By Carathéodory's theorem 22.2 and Remark 22.3, g extends to an bijective continuous mapping $g : \overline{W}^+ \to \overline{\mathbb{H}}$. Composing with a suitable automorphism we can assume $g(0) = 0$, $g(1) = 1$, and $g(\infty) = \infty$. By the Schwarz reflection principle for holomorphic functions 18.2, we may extend g across the y-axis by setting $g(-x + iy) = \overline{g(x + iy)}$. Then g is continuous on $\overline{W}$ and holomorphic in $\mathring{W}$ with $g(\mathring{W}) = \mathbb{C} \setminus \mathbb{R}_{\geq 0}$. Moreover, g is injective on W and $g(W) = \mathbb{C} \setminus \{0, 1\}$.

Since g is real on the boundary of W, we have

$$g(-1+iy) = \overline{g(1+iy)} = g(1+iy) = g(\tau(-1+iy)), \quad y > 0,$$

and

$$g(-\tfrac{1}{2}+\tfrac{1}{2}e^{it}) = \overline{g(\tfrac{1}{2}+\tfrac{1}{2}e^{i(\pi-t)})} = g(\tfrac{1}{2}+\tfrac{1}{2}e^{i(\pi-t)}) = g(\sigma(-\tfrac{1}{2}+\tfrac{1}{2}e^{it})), \quad 0 < t < \pi.$$

We define $\lambda : \mathbb{H} \to \mathbb{C}$ by setting

$$\lambda(z) := g(f^{-1}(z)), \qquad \text{for all } z \in f(W),\ f \in \Gamma;$$

by Proposition 26.1, each $z \in \mathbb{H}$ lies in $f(W)$ for precisely one $f \in \Gamma$. Then the properties (1)–(3) are obvious, and λ is holomorphic in the interior of each $f(W)$. It follows that λ is holomorphic in $\mathbb{H}$, by an application of Morera's theorem: it suffices to show that if φ is continuous in a region U and holomorphic in $U \setminus L = U_1 \cup U_2$, where L is a line segment or a circular arc and U_1, U_2 are regions, then φ is holomorphic in U. Up to a Möbius transformation, we may assume that L is a line segment. The integral of φ over every closed curve γ homologous to zero in U_1 or U_2 vanishes, and, by continuity of φ, this still holds if part of γ lies in L. If Δ is a triangle in U, then $\int_{\partial\Delta} \varphi\, dz$ is the sum of at most two such path integrals, and Morera's theorem implies the assertion.

Let us prove (4). Observe that the set $\{f(0) : f \in \Gamma\} = \{b/d : b, d \in \mathbb{Z},\ b \text{ even},\ d \text{ odd}\} = \lambda^{-1}(0)$ is dense in $\mathbb{R}$. So if λ could be extended to a neighborhood of $x \in \mathbb{R}$ then x would be an accumulation point of the zero-set of λ, hence $\lambda \equiv 0$, a contradiction. □

Theorem 26.3 (little Picard theorem)**.** *If f is an entire function such that the range of f omits two distinct complex numbers α, β, then f is constant.*

That the range of f can omit one point is shown by $f = \exp$.

Proof. We may assume that $\alpha = 0$ and $\beta = 1$, by replacing f by $(f-\alpha)/(\beta-\alpha)$. By Proposition 26.2, $\lambda : \mathbb{H} \to \mathbb{C} \setminus \{0,1\}$ is a covering map. By Corollary 4.7, there is a holomorphic function $g : \mathbb{C} \to \mathbb{H}$ such that $f = \lambda \circ g$. Then g is constant, by Liouville's theorem after composing with the Cayley mapping. Thus also f is constant. □

Exercise 46. Deduce from the little Picard theorem 26.3 that every periodic entire function has a fixed point.

Exercise 47. Let f and g be entire functions satisfying $e^f + e^g = 1$. Prove that f and g are both constant.

27. The big Picard theorem

The following theorem is a strengthening of Montel's theorem 20.2.

Theorem 27.1 (Montel–Carathéodory theorem)**.** *Let $U \subseteq \mathbb{C}$ be a domain and let $\mathcal{F} \subseteq \mathcal{H}(U)$ be such that $f(U) \subseteq \mathbb{C} \setminus \{0,1\}$ for all $f \in \mathcal{F}$. Then $\mathcal{F}$ is a normal family in $C(U, \widehat{\mathbb{C}})$. More precisely: if (f_n) is a sequence in $\mathcal{F}$, then there is a subsequence of (f_n) which either converges uniformly on compact sets to a holomorphic function $f : U \to \mathbb{C}$ or converges uniformly on compact sets to ∞.*

Proof. It suffices to show that $\mathcal{F}|_D$ is normal in $C(D, \widehat{\mathbb{C}})$ for each disk $D \subseteq U$. Let (f_n) be a sequence in $\mathcal{F}$. By Montel's theorem 20.2, it is enough to prove that there is either a subsequence that is uniformly bounded on compact subsets of D or a subsequence that converges to ∞ uniformly on compact subsets of D. Let c be the center of D. By passing to a subsequence we may assume that $f_n(c) \to \alpha \in \widehat{\mathbb{C}}$, since $\widehat{\mathbb{C}}$ is compact.

Assume first that $\alpha \neq 0, 1, \infty$. By Proposition 26.2, there is a holomorphic covering map $\mu : \mathbb{D} \to \mathbb{C} \setminus \{0, 1\}$ (compose λ with the Cayley mapping). Let V be a neighborhood of α and $W \subseteq \mathbb{D}$ such that $\mu|_W : W \to V$ is a biholomorphism. We may assume that $f_n(c) \in V$ for all n. By Corollary 4.7, for each n there is a holomorphic mapping $g_n : D \to \mathbb{D}$ such that $g_n(c) \in W$ and $\mu \circ g_n = f_n$ on D. Since (g_n) is uniformly bounded, there is a subsequence g_{n_k} which converges in $\mathcal{H}(D)$ to a holomorphic function g, by Montel's theorem 20.2. Clearly, $|g(z)| \le 1$ for all $z \in D$. We claim that actually $|g(z)| < 1$ for all $z \in D$. If there is $z \in D$ such that $|g(z)| = 1$ then g is constant, by the open mapping theorem, say $g = d$ with $|d| = 1$. But then $g_{n_k}(c) \to d$ and $g_{n_k}(c) = \mu|_W^{-1}(f_{n_k}(c)) \to \mu|_W^{-1}(\alpha)$, whence $\mu|_W^{-1}(\alpha) \notin \mathbb{D}$, a contradiction.

If $K \subset D$ is compact, then $|g|_K < r < 1$. Hence $|g_{n_k}|_K \le |g_{n_k} - g|_K + |g|_K \le r$ for k sufficiently large. Since μ is bounded on $D_r(0)$, we may conclude that (f_{n_k}) is uniformly bounded on K. The theorem follows.

Let us consider the remaining cases. Assume $\alpha = 1$. By Theorem 4.8, there are functions $h_n \in \mathcal{H}(D)$ such that $h_n^2 = f_n$. Since $f_n(c) \to 1$ we can choose the branch of the square roots so that $h_n(c) \to -1$. Clearly, $h_n(D) \subseteq \mathbb{C} \setminus \{0, 1\}$. Thus, by the above, there is a subsequence (h_{n_k}) which is uniformly bounded on compact subsets of D. Then also (f_{n_k}) has this property.

The case $\alpha = 0$ can be reduced to the previous case by setting $h_n = 1 - f_n$.

If $\alpha = \infty$ then set $h_n = 1/f_n$. The preceding case implies that there is a subsequence (h_{n_k}) which converges uniformly on compact subsets of D to a holomorphic function h. The functions h_{n_k} have no zeros while $h(c) = 0$, whence $h \equiv 0$, by Hurwitz' theorem 8.5. This means that $f_{n_k} \to \infty$ uniformly on compact subsets of D. □

Recall that the **Casorati–Weierstrass theorem** says that in a neighborhood of an essential singularity a holomorphic function assumes a dense set of values. Actually, by the **big Picard theorem**, all values except possibly one are assumed.

Theorem 27.2 (big Picard theorem)**.** *Let $D = D_R(a)$. Let $\alpha, \beta \in \mathbb{C}$, $\alpha \neq \beta$. If $f : D \setminus \{a\} \to \mathbb{C} \setminus \{\alpha, \beta\}$ is holomorphic, then the singularity at a is either removable or a pole.*

Proof. We may assume that $\alpha = 0$ and $\beta = 1$, by replacing f by $(f - \alpha)/(\beta - \alpha)$. Moreover, we may suppose that $a = 0$. Define $f_n : D \setminus \{0\} \to \mathbb{C} \setminus \{0, 1\}$ by $f_n(z) := f(z/2^n)$. By the Montel–Carathéodory theorem 27.1, $\{f_n\}$ is a normal family in $C(D \setminus \{0\}, \widehat{\mathbb{C}})$. So there is a subsequence (f_{n_k}) that converges uniformly on compact subsets of $D \setminus \{0\}$ to a holomorphic function $g : D \setminus \{0\} \to \widehat{\mathbb{C}}$.

Fix $0 < r < R$ and let $C := \{z : |z| = r\} \subseteq D \setminus \{0\}$. Then $f_{n_k} \to g$ uniformly on C. If $\infty \notin \operatorname{im} g$, then there is a constant M such that $|f_{n_k}(z)| \le M$ if $|z| = r$, or equivalently, $|f(z)| \le M$ if $|z| = r/2^{n_k}$. By the maximum principle, $|f(z)| \le M$ for z near 0, and so 0 is a removable singularity of f.

If $\infty \in \operatorname{im} g$, then $g \equiv \infty$, by the Montel–Carathéodory theorem 27.1. Hence $1/f_{n_k} \to 0$ uniformly on C, and thus for every $\epsilon > 0$, there is k_0 such that $|1/f_{n_k}(z)| < \epsilon$ for all $|z| = r$ and $k \ge k_0$. Then $|1/f(z)| < \epsilon$ for all $|z| = r/2^{n_k}$ and $k \ge k_0$. By the maximum principle, $|f(z)| \to \infty$ as $|z| \to 0$ so that 0 is a pole of f. □

The big Picard theorem implies the little Picard theorem. Let f be an entire function. If f is bounded near ∞ then it is bounded on $\mathbb{C}$ and Liouville's theorem implies that f is constant. So, if f is non-constant, it has either a pole or an essential singularity at ∞. If it has a pole at infinity, then f is a polynomial and the fundamental theorem of algebra implies that f assumes all complex values. If f has an essential singularity at infinity, then by the big Picard theorem f assumes all complex values except possibly one. $\square$

CHAPTER 7

Subharmonic functions and the Dirichlet problem

28. Subharmonic functions

Let $X = (X, d)$ be a metric space. A function $u : X \to \mathbb{R} \cup \{-\infty\}$ is said to be **upper semicontinuous (usc)** if for each $a \in X$,

$$\limsup_{x \to a} u(x) \le u(a),$$

or equivalently, for every $\beta \in \mathbb{R}$, the set $u^{-1}([-\infty, \beta))$ is open in X.

Let us collect a few facts on usc functions. Let $u : X \to \mathbb{R} \cup \{-\infty\}$ be usc. Then u is bounded above on every compact $K \subseteq X$. In fact, $K \subseteq \bigcup_{n \ge 1} \{x \in X : u(x) < n\}$ and hence K is contained in a finite union of the open sets $\{x \in X : u(x) < n\}$.

Assume that u is bounded above and let $M := \sup_X u(x)$. Then $\{x \in X : u(x) = M\}$ is closed in X. Indeed, $\{x \in X : u(x) = M\} = \{x \in X : u(x) \ge M\}$ is the complement of the open set $\{x \in X : u(x) < M\}$.

If X is compact then there is $x_0 \in X$ such that $u(x_0) = \sup_X u(x)$. Namely, if $M := \sup_X u(x)$, then there is a sequence $x_n \in X$ such that $u(x_n) > M - 1/n$. Since X is compact there is a subsequence x_{n_k} which converges to some $x_0 \in X$. Then $M \ge u(x_0) \ge \limsup_{k \to \infty} u(x_{n_k}) \ge \limsup_{k \to \infty} M - 1/n_k = M$.

If u_1, u_2 are usc, then so are $u_1 + u_2$, $\max\{u_1, u_2\}$, and λu_1, for each constant $\lambda \ge 0$; note that, for any $\beta \in \mathbb{R}$, the set $\{x : \max\{u_1, u_2\}(x) < \beta\} = \bigcap_{i=1,2} \{x : u_i(x) < \beta\}$ is open.

If $\{u_i\}_{i \in I}$ is an arbitrary family of usc functions on X, then $u := \inf_{i \in I} u_i$ is usc. In fact, for any $\beta \in \mathbb{R}$, the set $\{x : u(x) < \beta\} = \bigcup_{i \in I} \{x : u_i(x) < \beta\}$ is open. In particular, if (u_n) is a sequence of usc functions such that $u_n \ge u_{n+1}$, then the pointwise limit $u := \lim_{n \to \infty} u_n$ is usc; we shall write $u_n \downarrow u$.

Lemma 28.1 (approximation by continuous functions)**.** *Let $u : X \to \mathbb{R} \cup \{-\infty\}$, $u \not\equiv -\infty$, be usc. Then there exists a sequence $(u_n)_{n \ge 1}$ of continuous functions such that $u_n \downarrow u$.*

Proof. We may assume that u is bounded above. In fact, since u is locally bounded above there is a continuous function v on X such that $u \le v$. Then $u - v$ is usc and bounded above. If $v_n \downarrow u - v$ then $v_n + v \downarrow u$.

Let $M \in \mathbb{R}$ such that $u < M$. Define, for $n \ge 1$,

$$u_n(x) := \sup_{y \in X} (u(y) - nd(x, y)), \quad x \in X.$$

Since $u(y) > -\infty$ for some $y \in X$, we have $u_n(x) > -\infty$ for all $x \in X$. Clearly, $u_n \le M$. We have $u_n(x) \ge u(x) - nd(x, x) = u(x)$ and $u_n \ge u_{n+1}$ is immediate from the definition.

Let us prove that $u_n(x) \to u(x)$. Assume first that $u(x) > -\infty$. Since u is usc, for every $\epsilon > 0$ there is $\delta > 0$ such that $u(y) < u(x) + \epsilon$ if $d(x, y) < \delta$. If $d(x, y) \ge \delta$, then $u(y) - nd(x, y) \le M - n\delta < -n\delta/2$ if n is sufficiently large. Thus, if n is large,

$$u(x) \le u_n(x) \le \max\{u(x) + \epsilon, -n\delta/2\} = u(x) + \epsilon$$

and so $u_n(x) \to u(x)$. If $u(x) = -\infty$ then for any $N > 0$ there is $\delta > 0$ such that $u(y) < -N$ if $d(x,y) < \delta$. Thus, $u_n(x) \le \max\{M - n\delta, -N\} = -N$ for large n, and so $u_n(x) \to -\infty$.

It remains to show that u_n is continuous. Let $\epsilon > 0$. There is $y \in X$ such that $u_n(x) < u(y) - nd(x,y) + \epsilon$ and hence, for $x, x' \in X$,

$$u_n(x) - u_n(x') \le n(d(x',y) - d(x,y)) + \epsilon \le nd(x,x') + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, $u_n(x) - u_n(x') \le nd(x,x')$. Interchanging x,x' we may conclude that $|u_n(x) - u_n(x')| \le nd(x,x')$. $\square$

Exercise 48. Let u be usc and $u \ge 0$. Show that $v(x) := \log u(x)$ if $u(x) \ne 0$ and $v(x) := -\infty$ if $u(x) = 0$ is usc.

Let $U \subseteq \mathbb{C}$ be a domain and let $u : U \to \mathbb{R} \cup \{-\infty\}$ be usc. Then u is called **subharmonic** if:

(1) $u \not\equiv -\infty$ on any connected component of U.
(2) Let $V \Subset U$ and let $h : \overline{V} \to \mathbb{R}$ be continuous and harmonic on V. Then $u(z) \le h(z)$ for all $z \in \partial V$ implies $u(z) \le h(z)$ for all $z \in V$.

Subharmonicity is the complex-analytic analogue of convexity: on $\mathbb{R}$ the analogue of the Laplacian is d^2/dt^2 and the solutions of $d^2h/dt^2 = 0$ are the affine linear functions. The analogy with convex functions is apparent.

Exercise 49. Let u be a subharmonic function on $D_R(0)$ such that $u(z) = u(|z|)$ for $z \in D_R(0)$. Prove that $r \mapsto u(r)$, $r \in (0,R)$, is a convex function of $\log r$: if $\ell(r) := a \log r + b$, $r \in (0,R)$, and $r_1, r_2 \in (0,R)$ are such that $u(r_1) \le \ell(r_1)$ and $u(r_2) \le \ell(r_2)$, then $u(r) \le \ell(r)$ for all $r \in (r_1, r_2)$. Hint: $\ell(z) := \ell(|z|)$ is harmonic on $D_R(0) \setminus \{0\}$.

Theorem 28.2 (characterization of subharmonicity)**.** *Let $U \subseteq \mathbb{C}$ be a domain and let u be usc on U. Suppose that $u \not\equiv -\infty$ on any connected component of U. If u is subharmonic on U, $a \in U$ and $r > 0$ such that $\overline{D}_r(a) \subseteq U$, then*

$$u(z) \le P_{a,r}(u)(z), \quad z \in D_r(a),$$

cf. (17.4)*; in particular,*

$$u(a) \le \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{it})\, dt. \tag{28.1}$$

Conversely, if for every $a \in U$ there exists $r_a > 0$ such that (28.1) *holds for all $0 < r < r_a$, then u is subharmonic.*

Proof. Assume that u is subharmonic on U. By Lemma 28.1, there is a sequence of continuous functions $u_n \downarrow u$. By Theorem 17.6, $h_n := P_{a,r}(u_n)$ is continuous on $\overline{D}_r(a)$, harmonic on $D_r(a)$, and $h_n = u_n \ge u$ on $\partial D_r(a)$. Since u is subharmonic,

$$u(z) \le h_n(z) = \int_0^{2\pi} u_n(a + re^{it}) P_{a,r}(z,t)\, dt, \quad z \in D_r(a).$$

By the monotone convergence theorem,

$$u(z) \le \int_0^{2\pi} u(a + re^{it}) P_{a,r}(z,t)\, dt = P_{a,r}(u)(z),$$

in particular, (28.1).

For the converse, let $V \Subset U$ and let h be continuous on $\overline{V}$ and harmonic on V. Suppose that $u \le h$ on ∂V. For contradiction, assume that there is $z \in V$ such that $u(z) > h(z)$. Let $f := u - h$ on $\overline{V}$. If $M := \max_{\overline{V}} f$ then $K := \{z \in V : f(z) = M\}$

is a compact subset of V, in particular, K is a proper subset of V. Let $w \in \partial K$. For some small $r > 0$ there is a point on $\partial D_r(w)$ at which f is strictly less than M. Since f is usc, f is less than M on an open arc J of $\partial D_r(w)$. Thus,

$$\begin{aligned} f(w) = M &> \frac{1}{2\pi}\int_0^{2\pi} f(w + re^{it})\,dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} u(w+re^{it})\,dt - \frac{1}{2\pi}\int_0^{2\pi} h(w+re^{it})\,dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} u(w+re^{it})\,dt - h(w), \end{aligned}$$

since h is harmonic. This contradicts (28.1). □

This theorem implies that subharmonicity is a local condition: if u is usc on U and each $a \in U$ has a neighborhood V such that $u|_V$ is subharmonic, then u is subharmonic.

Corollary 28.3. *Let u_1, u_2 be subharmonic on U. Then $u_1 + u_2$ and $\max\{u_1, u_2\}$ are subharmonic. If $\lambda \geq 0$ then λu_1 is subharmonic.*

Proof. Follows from Theorem 28.2. □

Corollary 28.4. *If $f : U \to \mathbb{C}$ is harmonic, then $|f|$ is subharmonic.*

Proof. Use the mean value property 17.4 and Theorem 28.2. □

Corollary 28.5. *Let $(u_n)_{n\geq 1}$ be a sequence of subharmonic functions on U such that $u_n \downarrow u$ and $u \not\equiv -\infty$ on any connected component of U. Then u is subharmonic on U.*

Proof. Follows from Theorem 28.2 and the monotone convergence theorem. □

Corollary 28.6. *If u is subharmonic on U and $\varphi : \mathbb{R} \to \mathbb{R}$ is nondecreasing and convex and $\varphi(-\infty) := \lim_{t\to-\infty} \varphi(t)$, then $\varphi \circ u$ is subharmonic on U.*

Proof. Note that φ is continuous (since convex) and $\{x \in \mathbb{R} : \varphi(x) < \beta\}$ is either empty if $\varphi(-\infty) \geq \beta$, of the form $(-\infty, \alpha)$ if $\varphi(-\infty) < \beta \leq \varphi(\infty)$, or $\mathbb{R}$ if $\beta > \varphi(\infty)$, since φ is nondecreasing. Consequently, $\varphi \circ u$ is usc.

If $\overline{D}_r(a) \subseteq U$ and $\int_0^{2\pi} u(a + re^{it})\,dt > -\infty$, then

$$\varphi(u(a)) \leq \varphi\Big(\frac{1}{2\pi}\int_0^{2\pi} u(a+re^{it})\,dt\Big) \leq \frac{1}{2\pi}\int_0^{2\pi} \varphi(u(a+re^{it}))\,dt,$$

by Jensen's inequality (e.g. [**13**, p. 62]). If $\int_0^{2\pi} u(a + re^{it})\,dt = -\infty$, then $u(a) = -\infty$, since u is subharmonic. Since φ is nondecreasing, $\varphi(-\infty) = \varphi(u(a)) \leq \varphi(u(a + re^{it}))$ and hence $\varphi(u(a)) \leq \frac{1}{2\pi}\int_0^{2\pi} \varphi(u(a + re^{it}))\,dt$. □

For instance, e^f is subharmonic if f is, and f^2 is subharmonic provided that $f \geq 0$ is subharmonic.

Exercise 50. Let $u : U \to \mathbb{R}$ be harmonic and let $\varphi : \mathbb{R} \to \mathbb{R}$ be convex (not necessarily nondecreasing). Show that $\varphi \circ u$ is subharmonic. Give an example of a subharmonic u and a convex φ such that $\varphi \circ u$ is not subharmonic.

Exercise 51. Let f be holomorphic on some domain $U \subseteq \mathbb{C}$. Use Exercise 34 to show that $u = \log|f|$ is subharmonic on U.

Exercise 52. Let $\{u_i\}_{i\in I}$ be an arbitrary family of subharmonic functions on U. Suppose that $u(z) := \sup_{i\in I} u_i(z)$, $z \in U$, is usc and $u(z) < \infty$ for all $z \in U$. Prove that u is subharmonic.

Exercise 53. Deduce **Hadamard's three circles theorem**: Let f be holomorphic on $D_R(0)$. Let $0 < r_1 < r_2 < R$ and $M_i := \sup_{|z|=r_i} |f(z)|$, $i = 1, 2$. Then, if $r \in (r_1, r_2)$,

$$\sup_{|z|=r} |f(z)| \le M_1^{\lambda(r)} M_2^{1-\lambda(r)},$$

where

$$\lambda(r) = \frac{\log r_2 - \log r}{\log r_2 - \log r_1}.$$

Hint: Apply Exercise 49 to $u(z) = \sup_{t\in\mathbb{R}} \log |f(ze^{it})|$.

Theorem 28.7 (maximum principle for subharmonic functions)**.** *Let $U \subseteq \mathbb{C}$ be a bounded region. Let u be subharmonic on U. If*

$$M := \sup_{w\in\partial U} \limsup_{z\to w} u(z),$$

then $u(z) < M$ for all $z \in U$ unless u is constant.

Proof. We may assume that $M < +\infty$. Define

$$\varphi(w) := \begin{cases} u(w) & \text{if } w \in U, \\ \limsup_{z\to w} u(z) & \text{if } w \in \partial U. \end{cases}$$

We claim that φ is usc on $\overline{U}$. It suffices to show that $\limsup_{w\to a} \varphi(w) \le \varphi(a)$ if $a \in \partial U$. Let $\overline{U} \ni w_n \to a$. There exist $z_n \in U$ such that $|z_n - w_n| < 1/n$ and $\varphi(z_n) = u(z_n) > \varphi(w_n) - 1/n$. Hence

$$\varphi(a) = \limsup_{z\to a} u(z) \ge \limsup_{n\to\infty} u(z_n) \ge \limsup_{n\to\infty}(\varphi(w_n) - 1/n) = \limsup_{n\to\infty} \varphi(w_n).$$

Now suppose that $u(z) \ge M = \sup_{w\in\partial U} \varphi(w)$ for some $z \in U$. If $\tilde{M} := \sup_{w\in\overline{U}} \varphi(w)$ then the set $A := \{z \in U : u(z) = \tilde{M}\}$ is nonempty; in fact, the usc function φ attains its maximum on the compact set $\overline{U}$. Moreover, A is closed in U; see the remarks at the beginning of the section. Let us show that A is also open. Let $a \in A$ and let $\overline{D}_r(a) \subseteq U$. Then, by the characterization of subharmonicity 28.2,

$$\tilde{M} = u(a) \le \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{it})\, dt \le \tilde{M}$$

and hence $u(a + re^{it}) = \tilde{M}$ for $0 \le t \le 2\pi$ and small $r > 0$, i.e., A is open. Since U is connected, $A = U$, that is u is constant. $\square$

There is no minimum principle for subharmonic functions.

Proposition 28.8. *Let u be subharmonic on U. Then u cannot be $-\infty$ on any nonempty open subset of U.*

Proof. Let $A := \{z \in U : u(z) = -\infty\}$. Suppose that $V := \mathring{A} \ne \emptyset$. We will show that V is closed in U. This leads to a contradiction, since $u \not\equiv -\infty$ on any connected component of U, by the definition of subharmonicity.

Let $a \in \overline{V}$ and $\overline{D}_r(a) \subseteq U$ such that $\partial D_r(a) \cap V \ne \emptyset$. Since V is open, $a + re^{it} \in V$ for t is some open interval $I \subseteq [0, 2\pi]$, whence $u(a + re^{it}) = -\infty$ for

$t \in I$. Thus, since u is locally bounded above (being usc),

$$P_{a,r}(u)(z) = \int_0^{2\pi} u(a + re^{it}) P_{a,r}(z,t)\,dt = -\infty, \qquad \text{for } z \in D_r(a).$$

Since u is subharmonic, $u(z) \le P_{a,r}(u)(z)$ for $z \in D_r(a)$, and so $D_r(a) \subseteq V$. Thus V is closed in U. □

Let us prove that a subharmonic function is locally integrable.

Proposition 28.9. *Let u be subharmonic on U and let $\overline{D}_R(a) \subseteq U$. Then if $0 < \epsilon < R$ then there is $M = M(\epsilon, R, a, u)$ such that*

$$\int_0^{2\pi} |u(a + re^{it})|\,dt \le M, \qquad \textit{for } \epsilon \le r \le R.$$

Moreover, $u \in L^1_{\mathrm{loc}}(U)$, i.e., for every compact $K \subseteq U$,

$$\iint_K |u(z)|\,dx\,dy < \infty.$$

Proof. Let $u(a + re^{it}) = u^+(t) - u^-(t)$ be the decomposition into positive and negative part. Since u being usc is bounded above on $\overline{D}_R(a)$, there is a constant $C > 0$ such that $0 \le u^+(t) \le C$ for all t and $r \le R$. By Proposition 28.8, there is $z \in D_{\epsilon/2}(a)$ such that $u(z) > -\infty$. By the characterization of subharmonicity 28.2,

$$u(z) \le \int_0^{2\pi} (u^+(t) - u^-(t)) P_{a,r}(z,t)\,dt.$$

By Harnack's inequality 19.1,

$$\frac{r - |z-a|}{r + |z-a|} \le 2\pi P_{a,r}(z,t) \le \frac{r + |z-a|}{r - |z-a|}$$

which implies $1/3 \le 2\pi P_{a,r}(z,t) \le 3$ if $\epsilon \le r \le R$ and $|z - a| < \epsilon/2$. Then

$$\frac{1}{3} \int_0^{2\pi} u^-(t)\,dt \le 3 \int_0^{2\pi} u^+(t)\,dt - 2\pi u(z) < \infty$$

since $u^+(t) \le C$. This implies the first assertion.

Every $z \in U$ has a neighborhood of the form $A_{\epsilon,R}(a) = D_R(a) \setminus \overline{D}_\epsilon(a)$, where $\overline{D}_R(a) \subseteq U$. Since

$$\iint_{A_{\epsilon,R}(a)} |u(z)|\,dx\,dy = \int_\epsilon^R r \int_0^{2\pi} |u(a + re^{it})|\,dt\,dr < \infty,$$

the second statement follows. □

Next we show that subharmonic functions can be approximated by smooth subharmonic functions on relatively compact subsets. We need the following lemma.

Lemma 28.10. *Let u be subharmonic on $D_R(a)$. Then $\varphi(r) := \int_0^{2\pi} u(a + re^{it})\,dt$, for $0 \le r < R$, is increasing.*

Proof. We may assume that $a = 0$. We extend the definition of φ to complex values by putting $\varphi(z) := \int_0^{2\pi} u(ze^{it})\,dt$, for $z \in D_R(0)$; observe that $\varphi(z) = \varphi(|z|)$. We will show that φ is subharmonic on $D_R(0)$. Then the statement follows from the maximum principle for subharmonic functions 28.7: suppose that there are $0 \le r_1 < r_2 < R$ such that $\varphi(r_2) < \varphi(r_1)$. Then by the maximum principle φ is constant on $\overline{D}_{r_2}(0)$.

Let us first check that φ is usc on $D_R(0)$. If u is continuous so is φ. In general, there is a sequence of continuous functions u_n such that $u_n \downarrow u$, by Lemma 28.1.

Then $\varphi_n(z) := \int_0^{2\pi} u_n(ze^{it})\,dt$ is continuous and $\varphi_n \downarrow \varphi$, by the monotone convergence theorem. Consequently, φ is usc, by Corollary 28.5.

Next we prove that φ is subharmonic on $D_R(0)$. Let $z \in D_R(0)$. By the characterization of subharmonicity 28.2, we must check that, for small $r > 0$, $\varphi(z) \le \frac{1}{2\pi}\int_0^{2\pi} \varphi(z + re^{is})\,ds$, i.e.,

$$\int_0^{2\pi} u(ze^{it})\,dt \le \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} u((z+re^{is})e^{it})\,dt\,ds. \tag{28.2}$$

If $z = 0$ then, since u is subharmonic,

$$\int_0^{2\pi} u(0)\,dt = 2\pi u(0) \le \int_0^{2\pi} u(re^{is})\,ds,$$

and (28.2) holds. If $z \ne 0$ then, by Proposition 28.9, $\int_0^{2\pi}\int_0^{2\pi} |u((z+re^{is})e^{it})|\,dt\,ds < \infty$, and so (28.2) holds, by Fubini's theorem and Theorem 28.2,

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} u((z+re^{is})e^{it})\,dt\,ds &= \int_0^{2\pi} \frac{1}{2\pi}\int_0^{2\pi} u(ze^{it} + re^{i(s+t)})\,ds\,dt \\ &\ge \int_0^{2\pi} u(ze^{it})\,dt.\end{aligned}$$

□

Theorem 28.11 (approximation by smooth functions)**.** *Let $U \subseteq \mathbb{C}$ be a domain and let u be subharmonic on U. For any $V \Subset U$ there is a sequence of subharmonic functions $u_n \in C^\infty(V)$ such that $u_n \downarrow u$.*

Proof. Let $\varphi \in C_c^\infty(\mathbb{C})$ be a radially symmetric nonnegative function with $\operatorname{supp}\varphi \subseteq \mathbb{D}$ such that $\iint_{\mathbb{C}} \varphi\,dx\,dy = 1$. Then $\varphi_\epsilon(z) := \epsilon^{-2}\varphi(z/\epsilon)$, $\epsilon > 0$, is nonnegative, $\operatorname{supp}\varphi_\epsilon \subseteq D_\epsilon(0)$, and $\iint_{\mathbb{C}} \varphi_\epsilon\,dx\,dy = 1$.

Let $0 < \epsilon < \operatorname{dist}(\overline{V}, \mathbb{C}\backslash U)$. Set $u_\epsilon := u * \varphi_\epsilon$, i.e., $u_\epsilon(w) = \iint_{\mathbb{C}} u(w-z)\varphi_\epsilon(z)\,dx\,dy$, which is well-defined since u is locally integrable, by Proposition 28.9. Then $u_\epsilon \in C^\infty(U)$. Replacing z by $-\epsilon z$ and using radial symmetry of φ we get

$$u_\epsilon(w) = \iint_{\mathbb{D}} u(w+\epsilon z)\varphi(z)\,dx\,dy = \int_0^1 \rho\varphi(\rho)\int_0^{2\pi} u(w+\epsilon\rho e^{it})\,dt\,d\rho.$$

We claim that u_ϵ is subharmonic in V. Let $a \in V$ and $r > 0$ small. Then

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} u_\epsilon(a+re^{is})\,ds &= \frac{1}{2\pi}\int_0^{2\pi}\int_0^1 \rho\varphi(\rho)\int_0^{2\pi} u(a+re^{is}+\epsilon\rho e^{it})\,dt\,d\rho\,ds \\ &= \int_0^1 \rho\varphi(\rho)\int_0^{2\pi}\frac{1}{2\pi}\int_0^{2\pi} u(a+\epsilon\rho e^{it}+re^{is})\,ds\,dt\,d\rho \\ &\ge \int_0^1 \rho\varphi(\rho)\int_0^{2\pi} u(a+\epsilon\rho e^{it})\,dt\,d\rho = u_\epsilon(a),\end{aligned}$$

which implies the claim, by Theorem 28.2.

We have $u_\epsilon \le u_{\epsilon'}$ if $\epsilon \le \epsilon'$, because, for fixed w and ρ, $\epsilon \mapsto \int_0^{2\pi} u(w+\epsilon\rho e^{it})\,dt$ is increasing, by Lemma 28.10.

Finally, we show that, for $w \in V$, $u_\epsilon(w) \to u(w)$ as $\epsilon \to 0$. Since u is subharmonic,

$$u_\epsilon(w) = \int_0^1 \rho\varphi(\rho)\int_0^{2\pi} u(w+\epsilon\rho e^{it})\,dt\,d\rho \ge u(w)\,2\pi\int_0^1 \rho\varphi(\rho)d\rho = u(w).$$

Since u is usc, for $\delta > 0$ there is $\epsilon_0 > 0$ such that

$$u(w+\epsilon\rho e^{it}) \le u(w) + \delta \quad \text{if } 0 < \epsilon < \epsilon_0,\ 0 < \rho \le 1.$$

Thus

$$u_\epsilon(w) = \int_0^1 \rho\varphi(\rho) \int_0^{2\pi} u(w + \epsilon\rho e^{it})\, dt\, d\rho \le (u(w) + \delta)\, 2\pi \int_0^1 \rho\varphi(\rho) d\rho = u(w) + \delta$$

if $0 < \epsilon < \epsilon_0$. If $u(w) = -\infty$, replace $u(w) + \delta$ by $-1/\delta$. ☐

We end this section with a characterization of C^2 subharmonic functions. We need the following maximum principle.

Lemma 28.12 (maximum principle)**.** *Let $U \subseteq \mathbb{C}$ be a domain and let $u \in C^2(U)$ be real valued. Suppose that $\Delta u \ge 0$ on U. Then, for any open $V \Subset U$, we have $u(z) \le \sup_{w \in \partial V} u(w)$ for all $z \in V$.*

Proof. Let us first assume that $\Delta u > 0$ on U. Let $V \Subset U$ and let $z_0 \in \overline{V}$ be such that $u(z_0) = \sup_{w \in \overline{V}} u(w)$. Assume, for contradiction, that $u(z) \le \sup_{w \in \partial V} u(w)$ does not hold for all $z \in V$. Then $z_0 \in V$ and, consequently, $\Delta u(z_0) \le 0$, a contradiction.

In the general case, $\Delta u \ge 0$ on U, consider $u_\epsilon(z) := u(z) + \epsilon|z|^2$, for $\epsilon > 0$. Then $\Delta u_\epsilon = \Delta u + 4\epsilon > 0$ on U, and thus, for $V \Subset U$, we have $u_\epsilon(z) \le \sup_{w \in \partial V} u_\epsilon(w)$ for all $z \in V$. Letting $\epsilon \to 0$ implies the statement. ☐

Theorem 28.13 (characterization of C^2 subharmonic functions)**.** *Let $U \subseteq \mathbb{C}$ be a domain and let $u \in C^2(U)$ be real valued. Then u is subharmonic if and only if $\Delta u \ge 0$ on U.*

Proof. Assume that $u \in C^2(U)$ satisfies $\Delta u \ge 0$ on U. Let $V \Subset U$ and let h be continuous on $\overline{V}$, harmonic on V, and such that $u \le h$ on ∂V. Since $\Delta(u - h) = \Delta u \ge 0$ on V, for every $W \Subset V$, we have $u(z) - h(z) \le \sup_{w \in \partial W}(u(w) - h(w))$ for $z \in W$, by Lemma 28.12. We may infer $u(z) - h(z) \le \sup_{w \in \partial V}(u(w) - h(w)) \le 0$ for $z \in V$, by letting W run through an exhaustion of V by relatively compact sets. Thus u is subharmonic.

Let $u \in C^2(U)$ be subharmonic. Suppose that $\Delta u(z) < 0$ for some $z \in U$. Then there is a neighborhood V of z such that $\Delta u < 0$ on V so that $-u$ is subharmonic on V by the first part of the proof. By the characterization of subharmonicity 28.2, u is harmonic on V and hence $\Delta u = 0$ on V, a contradiction. ☐

We state without proof the following generalization of this result; see [**11**, p. 231] for a proof.

Theorem 28.14. *Let $U \subseteq \mathbb{C}$ be a domain and let $u \in L^1_{\text{loc}}(U)$ be real valued. Then there exists a subharmonic function $\tilde{u}$ on U such that $\tilde{u} = u$ a.e. if and only if $\Delta u \ge 0$ in the sense of distributions.*

Here Δu is the linear mapping $C_c^\infty(U) \to \mathbb{C}$ defined by

$$\langle \Delta u, \varphi \rangle := \iint_{\mathbb{C}} u(\Delta\varphi)\, dx\, dy,$$

and $\Delta u \ge 0$ *in the sense of distributions* means that $\langle \Delta u, \varphi \rangle \ge 0$ for all $\varphi \ge 0$, $\varphi \in C_c^\infty(U)$.

Exercise 54. Let U, V be regions in $\mathbb{C}$ and let $f : V \to U$ be a non-constant holomorphic mapping. Show that, if u is subharmonic on U, then $u \circ f$ is subharmonic on V. Hint: Use approximation by smooth functions 28.11 and the characterization of C^2 subharmonic functions 28.13.

29. The Dirichlet problem

The goal of this section is the solution of the **Dirichlet problem**: Let $U \subseteq \mathbb{C}$ be any bounded domain and let $f \in C(\partial U)$. Is there a continuous function u on $\overline{U}$ which is harmonic on U such that $u|_{\partial U} = f$?

We will say that the Dirichlet problem on U with boundary values f is solvable if this question has an affirmative answer. If the Dirichlet problem on U with boundary values f is solvable for all $f \in C(\partial U)$, then we say that the Dirichlet problem is solvable on U.

Exercise 55. Solve the Dirichlet problem on the strip $S = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$ for the boundary function f which is 0 on $\{z : \operatorname{Re} z = 0\}$ and 1 on $\{z : \operatorname{Re} z = 1\}$. Hint: Check that $z \mapsto \exp(i\pi z)$ is a biholomorphism between S and $\mathbb{H}$ which extends continuously to $\overline{S}$. Use Exercise 33.

The following example shows that the the Dirichlet problem is not always solvable.

Example 29.1. Let $U = \mathbb{D} \setminus \{0\}$ and let $f(z) = 1$ for $z \in \partial\mathbb{D}$ and $f(0) = 0$. Then f is continuous on $\partial U = \partial\mathbb{D} \cup \{0\}$. Suppose that the Dirichlet problem on U with boundary values f has a solution u. Note that $u(z) = u(e^{it}z)$ for any fixed $t \in \mathbb{R}$; this is because $u(e^{it}z)$ is a solution of the Dirichlet problem with boundary values f as well and since there is at most one solution.

The Laplace operator in polar coordinates reads

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial}{\partial r}\Big) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \tag{29.1}$$

and since u is independent of θ,

$$0 = \Delta u = \frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial u}{\partial r}\Big).$$

Thus $r\partial u/\partial r = c$ for some $c \in \mathbb{C}$ and therefore $u = c\log r + d$ for some $d \in \mathbb{C}$. But u cannot agree with f on ∂U.

Exercise 56. Prove that the Laplace operator in polar coordinates is given by the formula (29.1).

The domain in the example is typical for having no solution of the Dirichlet problem. We will show that the Dirichlet problem on U can be solved, if each connected component of ∂U contains more than one point.

Given a function u on U and $D := D_r(a)$ such that $\overline{D} \subseteq U$, let us define a function $P_D(u)$ on U by setting

$$P_D(u) := \begin{cases} u & \text{on } U \setminus D, \\ P_{a,r}(u) & \text{on } D. \end{cases}$$

Lemma 29.2. *If u is subharmonic on U then so is $P_D(u)$.*

Proof. Assume first that u is also continuous on U. By the characterization of subharmonicity 28.2, it suffices to check that for $b \in \partial D$ and small $\rho > 0$,

$$P_D(u)(b) \leq \frac{1}{2\pi}\int_0^{2\pi} P_D(u)(b + \rho e^{it})\, dt$$

We have $P_D(u)(z) = u(z)$ if $z \in \partial D$, and $u \le P_D(u)$ on D, since u is subharmonic. So $u \le P_D(u)$ on U and $u(b) = P_D(u)(b)$, whence

$$P_D(u)(b) = u(b) \le \frac{1}{2\pi}\int_0^{2\pi} u(b+\rho e^{it})\,dt \le \frac{1}{2\pi}\int_0^{2\pi} P_D(u)(b+\rho e^{it})\,dt,$$

as required.

In the general case, let $V \Subset U$ be an open neighborhood of $\overline{D}$, and let u_n be a sequence of C^∞ subharmonic functions on V such that $u_n \downarrow u$, which exists by Theorem 28.11. Then $P_D(u_n)$ is subharmonic on V and $P_D(u_n) \downarrow P_D(u)$, by the monotone convergence theorem. So $P_D(u)$ is subharmonic, by Corollary 28.5. $\square$

Let $U \subseteq \mathbb{C}$ be a bounded domain and let $f \in C(\partial U)$ be real valued. The family of functions

$$\mathfrak{P} = \mathfrak{P}_f := \Big\{ u \in C(U) : u \text{ is subharmonic, } \limsup_{z\to a} u(z) \le f(a) \text{ for all } a \in \partial U \Big\}$$

is called the **Perron family** of f. Note that ∂U is compact so that f is bounded below by some $m \in \mathbb{R}$. Thus $u(z) = m$ belongs to $\mathfrak{P}_f$.

Theorem 29.3 (Perron)**.** *Let $U \subseteq \mathbb{C}$ be a bounded domain and let $f \in C(\partial U)$ be real valued. Then the function*

$$h_f(z) := \sup_{u\in\mathfrak{P}_f} u(z), \quad z \in U,$$

*is harmonic on U. It is called the **Perron function** of f.*

Proof. Let $\mathcal{D} := \{D = D_r(a) : \overline{D} \subseteq U\}$. By Lemma 29.2, if $u \in \mathfrak{P}_f$ then also $P_D(u) \in \mathfrak{P}_f$ for all $D \in \mathcal{D}$.

Let $a \in U$ and let (u_n) be a sequence of functions in $\mathfrak{P}_f$ such that $u_n(a) \to h_f(a)$. Replacing u_n by $\max_{1\le i\le n} u_i$ we may assume that $u_1 \le u_2 \le \cdots$ on U.

Let $D = D_r(a) \in \mathcal{D}$ and set $v_n := P_D(u_n)$. Since $v_n \in \mathfrak{P}_f$ we have $v_n(a) \le h_f(a)$. Since $u_n \le v_n$ (cf. Lemma 29.2) and $u_n(a) \to h_f(a)$, also $v_n(a) \to h_f(a)$. Moreover, $u_n \le u_{n+1}$ implies $v_n \le v_{n+1}$. By Harnack's principle 19.2, v_n converges to a harmonic function h, uniformly on compact subsets of D. Clearly, $h \le h_f$ on D and $h(a) = h_f(a)$.

We claim that $h = h_f$ on D. Since D was arbitrary, this will show that h_f is harmonic on U. To prove the claim let $z \in D$ and let (w_n) be a sequence of functions in $\mathfrak{P}_f$ such that $w_n(z) \to h_f(z)$. Replacing w_n be $\max\{v_n, w_n\}$ we may assume that $v_n \le w_n$ on U. Setting $p_n := P_D(\max_{1\le i\le n} w_i)$ we have $p_n \ge v_n$, $p_{n+1} \ge p_n$, $p_n \in \mathfrak{P}_f$, $p_n|_D$ is harmonic, and $p_n(z) \to h_f(z)$ (similarly as before). By Harnack's principle 19.2, p_n converges to a harmonic function p, uniformly on compact subsets of D. Moreover, $h \le p \le h_f$ and $h(a) = p(a) = h_f(a)$. The maximum principle for harmonic functions 17.2 applied to $h - p$ implies that $h = p$ on D, and consequently $h(z) = p(z) = h_f(z)$. $\square$

If the Dirichlet problem on U is solvable with boundary values f, and H is the solution, then $H = h_f$. For, $H \le h_f$ because $H \in \mathfrak{P}_f$. On the other hand, if $u \in \mathfrak{P}_f$ then $\limsup_{z\to a}(u(z) - H(z)) \le f(a) - f(a) = 0$ for all $a \in \partial U$. Thus, $u \le H$ on U, by the maximum principle for subharmonic functions 28.7, and hence $h_f \le H$.

Thus, to solve the Dirichlet problem, we have only to look for conditions under which h_f converges to f at ∂U.

Let $U \subseteq \mathbb{C}$ be a bounded domain. A point $a \in \partial U$ is a **peak point** of U if there is an open set $a \in V \subseteq \mathbb{C}$ and a continuous subharmonic function p on $U \cap V$ such that:

(1) $p(z) \to 0$ as $z \to a$.
(2) $\limsup_{z\to b} p(z) < 0$ if $b \in (\partial U) \cap V$ and $b \neq a$.
(3) $p(z) < 0$ for $z \in U \cap V$.

Then p is called a **peaking function** or **barrier** at a.

Lemma 29.4. *If a is a peak point of U, then, given $\alpha, \beta \in \mathbb{R}$ and a small neighborhood W of a, there is a continuous subharmonic function u on U such that:*

(1) *$u(z) \to \alpha$ as $z \to a$.*
(2) *$\limsup_{z\to b} u(z) \leq \alpha$ for all $b \in (\partial U) \cap W$.*
(3) *$u(z) \leq \beta$ for $z \in U \setminus W$.*

Proof. We may suppose that $\beta < \alpha$. Let V and p be as in the definition of a peak point. Let W be a relatively compact neighborhood of a in V. Then there is $\delta > 0$ such that $p(z) \leq -\delta$ for $z \in (\partial W) \cap U$. For some $N > (\alpha - \beta)/\delta$ set

$$u(z) := \begin{cases} \beta & \text{if } z \in U \setminus W, \\ \alpha + \max\{\beta - \alpha, Np(z)\} & \text{if } z \in U \cap W. \end{cases}$$

Then u is subharmonic on $U \cap W$ and $u = \beta$ in a neighborhood of $(\partial W) \cap U$, thus u is subharmonic and continuous on U. It is then easy to check that u satisfies $u(z) \to \alpha$ as $z \to a$ and $\limsup_{z\to b} u(z) \leq \alpha$ for all $b \in (\partial U) \cap W$. $\square$

Proposition 29.5. *Let $U \subseteq \mathbb{C}$ be a bounded domain and let $a \in \partial U$ be a peak point of U. If $f \in C(\partial U)$, then $h_f(z) \to f(a)$ as $z \to a$.*

Proof. Let $M > 0$ be such that $|f|_{\partial U} \leq M$. Let $\epsilon > 0$. Let V be a neighborhood of a such that $|f(w) - f(a)| < \epsilon$ for all $w \in (\partial U) \cap V$.

By Lemma 29.4, there is a continuous subharmonic function u on U such that $u(z) \to f(a)$ as $z \to a$, $\limsup_{z\to b} u(z) \leq f(a)$ for all $b \in (\partial U) \cap V$, and $u(z) \leq -M$ for $z \in U \setminus V$. Then $v := u - \epsilon$ belongs to $\mathfrak{P}_f$. In fact, $\limsup_{z\to w} v(z) \leq -M - \epsilon < f(w)$ if $w \in (\partial U) \setminus V$ and $\limsup_{z\to w} v(z) \leq f(a) - \epsilon < f(w)$ if $w \in (\partial U) \cap V$. Consequently, $v \leq h_f$ and so

$$h_f(z) \geq v(z) = u(z) - \epsilon \to f(a) - \epsilon \quad \text{as } z \to a.$$

Since $\epsilon > 0$ is arbitrary, $\liminf_{z\to a} h_f(z) \geq f(a)$.

By Lemma 29.4, there is a continuous subharmonic function s on U such that $s(z) \to -f(a)$ as $z \to a$, $\limsup_{z\to b} s(z) \leq -f(a)$ for all $b \in (\partial U) \cap V$, and $s(z) \leq -M$ for $z \in U \setminus V$. Let $u \in \mathfrak{P}_f$. Then $\limsup_{z\to w}(u(z) + s(z)) \leq f(w) - f(a) < \epsilon$ if $w \in (\partial U) \cap V$ and $\limsup_{z\to w}(u(z) + s(z)) \leq f(w) - M \leq 0$ if $w \in (\partial U) \setminus V$. By the maximum principle for subharmonic functions 28.7, $u + s < \epsilon$ on U. Hence $h_f \leq \epsilon - s$ on U, and so

$$h_f(z) \leq \epsilon - s(z) \to \epsilon + f(a) \quad \text{as } z \to a.$$

Since $\epsilon > 0$ is arbitrary, $\limsup_{z\to a} h_f(z) \leq f(a)$. $\square$

Conversely, we have the following proposition.

Proposition 29.6. *Let $U \subseteq \mathbb{C}$ be a bounded domain and let $a \in \partial U$. Assume that, for every $f \in C(\partial U)$, we have $h_f(z) \to f(a)$ as $z \to a$. Then there is a harmonic function u on U such that $u(z) \to 0$ as $z \to a$ and $\limsup_{z\to b} u(z) < 0$ for all $b \in \partial U$, $b \neq a$. In particular, a is a peak point of U.*

Proof. Let $f(w) := |w - a|$, $w \in \partial U$. Then $f \in C(\partial U)$. The function $z \mapsto |z - a|$ belongs to $\mathfrak{P}_f$ (cf. Corollary 28.4). Thus, $|z - a| \leq h_f(z)$ for all $z \in U$, and so $\liminf_{z\to b} h_f(z) > 0$ for all $b \in \partial U$, $b \neq a$. By assumption, $h_f(z) \to f(a) = 0$

as $z \to a$. It follows that $u := -h_f$ is as required, since h_f is harmonic, by Theorem 29.3. $\square$

Thus, we have proved the following theorem.

Theorem 29.7 (solution of the Dirichlet problem (I))**.** *Let $U \subseteq \mathbb{C}$ be a bounded domain. The Dirichlet problem is solvable on U if and only if every boundary point of U is a peak point.*

In the rest of the section we will give geometric conditions which imply the existence of peaking functions.

Proposition 29.8 (Bouligand)**.** *Let $U \subseteq \mathbb{C}$ be a bounded domain and let $a \in \partial U$. Suppose that there is an open neighborhood V of a in $\mathbb{C}$ and a continuous subharmonic function p on $U \cap V$ such that $p(z) < 0$ for $z \in U \cap V$ and $p(z) \to 0$ as $z \to a$. Then there is a harmonic function h on U such that $h(z) \to 0$ as $z \to a$ and $\limsup_{z\to b} h(z) < 0$ for all $b \in \partial U$, $b \neq a$. In particular, a is a peak point of U.*

Note that, in the hypothesis of the proposition, condition (2) of the definition of a peaking function p is dropped.

Proof. Let $f(w) := |w - a|$, $w \in \partial U$. The proof of Proposition 29.6 shows that it suffices to prove that $h_f(z) \to 0$ as $z \to a$; then $h := -h_f$ is a function with the required properties.

By assumption, there is an open neighborhood V of a in $\mathbb{C}$ and a continuous subharmonic function p on $U \cap V$ such that $p(z) < 0$ for $z \in U \cap V$ and $p(z) \to 0$ as $z \to a$. Let $\overline{D}_r(a) \subseteq V$. Let $0 < \epsilon < r$ and $\rho > 0$. Set $I = I(\epsilon) := \partial D_\epsilon(a) \cap U$. Let C be a compact subset of I such that the measure of $I \setminus C$ is $< \rho$. Consider the function χ on $\partial D_\epsilon(a)$ defined by $\chi(\zeta) = M := \sup_{w\in\partial U} f(w)$ if $\zeta \in I \setminus C$ and $\chi(\zeta) = 0$ otherwise; then $\chi \in L^1(\partial D_\epsilon(a))$. The function

$$v(z) := \int_0^{2\pi} \chi(a + \epsilon e^{it}) P_{a,\epsilon}(z, t)\, dt, \quad z \in D_\epsilon(a),$$

is harmonic and > 0 on $D_\epsilon(a)$, and

$$v(a) = \frac{1}{2\pi} \int_0^{2\pi} \chi(a + \epsilon e^{it})\, dt < \frac{\rho M}{2\pi}.$$

Set $\delta := -\sup_{\zeta\in C} p(\zeta)$. Then $\delta > 0$, since C is a compact subset of $U \cap V$.

Let $u \in \mathfrak{P}_f$ (i.e., u is continuous subharmonic on U and $\limsup_{z\to w} u(z) \le f(w)$ for all $w \in \partial U$) and define a subharmonic function s on $D_\epsilon(a) \cap U$ by setting

$$s(z) := u(z) - \epsilon + \frac{M}{\delta} p(z) - v(z).$$

We claim that

$$\limsup_{z\to w} s(z) \le 0 \quad \text{for all } w \in \partial(D_\epsilon(a) \cap U). \tag{29.2}$$

We consider three cases which correspond to the decomposition $\partial(D_\epsilon(a) \cap U) = (\overline{D}_\epsilon(a) \cap \partial U) \cup (I \setminus C) \cup C$.

Let $w \in \overline{D}_\epsilon(a) \cap \partial U$. Since $u \in \mathfrak{P}_f$, $\limsup_{z\to w} u(z) \le f(w) = |w - a| \le \epsilon$. Moreover, $\limsup_{z\to w} p(z) \le 0$ and $\liminf_{z\to w} v(z) \ge 0$ which implies (29.2).

Let $w \in I \setminus C \subseteq U$. We have $v(z) \to M$ as $z \to w$, by Remark 17.7, and thus, since $p < 0$ and $u \le M$ on $U \cap V$ (by the maximum principle),

$$\limsup_{z\to w} s(z) \le u(w) - \epsilon - M \le 0.$$

Let $w \in C \subseteq U$. Then $p(z) \leq -\delta$ and thus, since $v > 0$,

$$\limsup_{z \to w} s(z) \leq u(w) - \epsilon + \frac{M}{\delta}(-\delta) \leq 0.$$

Thus the claim is proved.

Now, by (29.2) and the maximum principle for subharmonic functions 28.7, $s \leq 0$ on $D_\epsilon(a) \cap U$, i.e.,

$$u(z) + \frac{M}{\delta} p(z) \leq v(z) + \epsilon, \qquad \text{for } z \in D_\epsilon(a) \cap U.$$

Since this holds for all $u \in \mathfrak{P}_f$, we get

$$h_f(z) + \frac{M}{\delta} p(z) \leq v(z) + \epsilon, \qquad \text{for } z \in D_\epsilon(a) \cap U.$$

Letting $z \to a$ we find

$$\limsup_{z \to a} h_f(z) \leq v(a) + \epsilon < \frac{\rho M}{2\pi} + \epsilon.$$

Since $h_f \geq 0$ and ϵ, ρ were arbitrary, this implies $h_f(z) \to 0$ as $z \to a$, and the proposition follows. □

Theorem 29.9 (solution of the Dirichlet problem (II))**.** *Let $U \subseteq \mathbb{C}$ be a bounded domain such that no connected component of $\mathbb{C} \setminus U$ reduces to a point. Then the Dirichlet problem is solvable on U.*

Proof. Let $a \in \partial U$ and suppose that the connected component of $\mathbb{C} \setminus U$ containing a does not reduce to $\{a\}$. We will show that a is a peak point. This implies the theorem, by Theorem 29.7.

Let $b \neq a$ lie in the connected component of $\mathbb{C} \setminus U$ containing a. Then there exists $f \in \mathcal{H}(U)$ such that $e^{f(z)} = (z-a)/(z-b)$, by Lemma 15.1. Let $V := D_r(a)$. If $r > 0$ is sufficiently small then $|(z-a)/(z-b)| < 1$ for $z \in V$. If we set $p(z) := \operatorname{Re}(1/f(z)) = (\operatorname{Re} f(z))/|f(z)|^2$, $z \in U \cap V$, then $p(z) < 0$ and

$$|f(z)| \geq |\operatorname{Re} f(z)| = \left| \log \left| \frac{z-a}{z-b} \right| \right| \to \infty \qquad \text{as } z \to a,$$

so that $p(z) \to 0$ as $z \to a$. Proposition 29.8 implies that a is a peak point. □

CHAPTER 8

Introduction to Riemann surfaces

This chapter is intended as a short introduction to the basics on Riemann surfaces. The main goal is to convey the idea that Riemann surfaces are natural domains for holomorphic and meromorphic functions and to interpret some of the results in earlier chapters in this more general framework. The literature on Riemann surfaces is vast; we recommend [**2**], [**6**], and [**7**] for further reading.

30. Definitions, basic properties, and examples

Let X be a $2n$ dimensional manifold. A **complex structure** on X is an open cover $\{U_i\}_{i\in I}$ of X together with homeomorphisms $\varphi_i : U_i \to V_i$, V_i open in $\mathbb{C}^n$, such that the transition mappings

$$\varphi_i \circ \varphi_j^{-1}|_{\varphi_j(U_i\cap U_j)} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$$

are holomorphic for all $i, j \in I$. Two complex structures on X are said to be equivalent if their union is again a complex structure.

A **complex manifold** is a $2n$ dimensional manifold X equipped with an equivalence class of complex structures on X. Then n is the **complex dimension** of X. Given an open subset $U \subseteq X$ and a homeomorphism $\varphi : U \to V$ onto an open set $V \subseteq \mathbb{C}^n$, then (U, φ) is called a **chart** on X, if $(U, \varphi) \cup \{(U_i, \varphi_i)\}_{i\in I}$ is a complex structure, where $\{(U_i, \varphi_i)\}_{i\in I}$ is in the given equivalence class on X. The components $\varphi^1, \dots, \varphi^n$ of φ are called **local coordinates** on U.

Let X, Y be complex manifolds with a complex structures $\{(U_i, \varphi_i)\}_{i\in I}$, $\{(V_j, \psi_j)\}_{j\in J}$, respectively. A continuous mapping $f : X \to Y$ is said to be **holomorphic** if

$$\psi_j \circ f \circ \varphi_i^{-1} : \varphi_i(U_i \cap f^{-1}(V_j)) \to \psi_j(V_j)$$

is holomorphic for all $i \in I$, $j \in J$. A mapping $f : X \to Y$ is a **biholomorphism** if there is a holomorphic mapping $g : Y \to X$ such that $f \circ g = \mathrm{Id}_Y$ and $g \circ f = \mathrm{Id}_X$. The set of holomorphic mappings $f : X \to Y$ is denoted by $\mathcal{H}(X, Y)$. We set $\mathcal{H}(X) = \mathcal{H}(X, \mathbb{C})$, where $\mathbb{C}$ is equipped with the complex structure $(\mathbb{C}, \mathrm{Id})$.

A **Riemann surface** is a connected complex manifold X of complex dimension 1 having a countable base for its topology; the last condition is actually automatically satisfied by a theorem of Radó.

Note that a holomorphic mapping between Riemann surfaces is a biholomorphism if and only if it is a homeomorphism.

Many results for holomorphic functions defined in domains in $\mathbb{C}$ persist on Riemann surfaces:

Theorem 30.1. *Let X, Y be Riemann surfaces and let $f \in \mathcal{H}(X, Y)$. Then:*

(1) (Principle of analytic continuation). *If there is a nonempty open subset Ω of X such that $f|_\Omega = \mathrm{const} = y$, then $f \equiv y$ on X.*
(2) (Open mapping theorem). *If f is not constant, then f is an open mapping.*

(3) (Maximum principle). *If $f \in \mathcal{H}(X)$ and there is $a \in X$ such that $|f(x)| \leq |f(a)|$ for all $x \in X$, then f is constant.*
(4) *If $f_n \in \mathcal{H}(X)$ converges uniformly on compact subsets of X to f, then $f \in \mathcal{H}(X)$.*
(5) (Montel's theorem). *If $\mathcal{F} \subseteq \mathcal{H}(X)$ is bounded on compact subsets of X, then any sequence of functions in F has a subsequence which converges uniformly on compact subsets of X.*
(6) (Riemann's theorem on removable singularities). *If $f \in \mathcal{H}(X \setminus \{a\})$ is bounded in a neighborhood of a, then there is $F \in \mathcal{H}(X)$ such that $F|_{X\setminus\{a\}} = f$.*

Proof. (1) Let $A := \{x \in X : f \equiv y \text{ near } x\}$. Then A is open and $\Omega \subseteq A$. Since X is connected, it suffices to show that A is closed. Let $x \in \overline{A}$, and choose charts (U, φ), (V, ψ) such that $x \in U$ and $f(x) \in V$. Let W be the connected component of $\varphi(U \cap f^{-1}(V)) \subseteq \mathbb{C}$ containing $\varphi(x)$. Since A is open, any open set in X containing x intersects A in a nonempty open set. Thus $\varphi(A) \cap W$ is a nonempty open set, and $F := \psi \circ f \circ \varphi^{-1}$ is holomorphic on W and constant $\psi(y)$ on $\varphi(A) \cap W$. By the identity theorem, $F \equiv \psi(y)$ on W. Thus $x \in A$.

(2) Let $\Omega \subseteq X$ be open, $x \in \Omega$. Choose charts (U, φ), (V, ψ) such that $x \in U$ and $f(x) \in V$. Let Ω_0 be the connected component of $\Omega \cap U \cap f^{-1}(V)$ containing x. By (1), $f|_{\Omega_0}$ is not constant so that $F := \psi \circ f \circ \varphi^{-1}$ is not constant on $\varphi(\Omega_0)$. By the open mapping theorem in $\mathbb{C}$, $F(\varphi(\Omega_0))$ is open in $\mathbb{C}$ and thus $f(\Omega_0)$ is open in Y. So $f(\Omega)$ is a neighborhood of $f(x)$.

(3) The condition means that $f(X) \in D_{|f(a)|}(0)$. So $f(X)$ is not open, and the statement follows from (2).

(4), (5), (6) follow easily from their corresponding version in $\mathbb{C}$. □

Exercise 57. Prove the items (4), (5), and (6) of Theorem 30.1.

Corollary 30.2. *Any holomorphic function on a compact Riemann surface is constant.*

Proof. If $f \in \mathcal{H}(X)$ then $f(X)$ is compact in $\mathbb{C}$. So Theorem 30.1(2) implies that f is constant. □

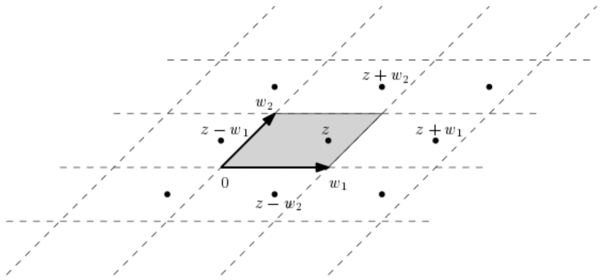
Let X be a Riemann surface defined by the complex structure $\{(U_i, \varphi_i)\}_{i\in I}$. If $U \subseteq X$ is open, then $\{(U \cap U_i, \varphi_i|_{U\cap U_i})\}_{i\in I}$ is a complex structure on U, called the **induced complex structure**. So each connected component of U is a Riemann surface.

Example 30.3 (connected components of $\mathcal{O}$)**.** Let $\mathcal{O}$ be the sheaf of germs of holomorphic functions on $\mathbb{C}$, and let $\pi : \mathcal{O} \to \mathbb{C}$ be the mapping given by $\pi(f_a) = a$. We saw in Lemma 2.3 that π is a local homeomorphism and $\mathcal{O}$ is a two dimensional manifold. Let $\{U_i\}_{i\in I}$ be an open cover of $\mathcal{O}$ such that $\pi_i := \pi|_{U_i}$ is a homeomorphism onto $\pi(U_i)$. Then $\{(U_i, \pi_i)\}_{i\in I}$ is a complex structure on $\mathcal{O}$, since the transition maps $\pi_i \circ \pi_j^{-1}|_{\pi_j(U_i\cap U_j)}$ are the identity maps. So any connected component of $\mathcal{O}$ is a Riemann surface in a natural way.

Example 30.4 (Riemann sphere)**.** The one point compactification $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is homeomorphic to $S^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$ via the stereographic projection. Let $U_1 := \mathbb{C}$ and $U_2 := \mathbb{C}^* \cup \{\infty\}$. Let $\varphi_1 := \mathrm{Id} : U_1 \to \mathbb{C}$ and let $\varphi_2 : U_2 \to \mathbb{C}$ be defined by $\varphi_2(z) = 1/z$ if $z \in \mathbb{C}^*$ and $\varphi_2(\infty) = 0$. Then φ_1, φ_2 are homeomorphisms, and the transition map $\varphi_1 \circ \varphi_2^{-1} : \varphi_2(U_1 \cap U_2) \to \varphi_1(U_1 \cap U_2)$ is the mapping $z \to 1/z$ from $\mathbb{C}^*$ to itself. This complex structure makes the **Riemann sphere** $\widehat{\mathbb{C}}$ to a compact Riemann surface, also called the **complex projective line** and denoted by $\mathbb{P}^1$.

Exercise 58. The complex projective line is the quotient space $\mathbb{P}^1 := \mathbb{C}^2 \setminus \{0\}/\sim$, where $(z_1, z_2) \sim (w_1, w_2)$ if and only if there exists $\lambda \in \mathbb{C}^*$ such that $(z_1, z_2) = \lambda(w_1, w_2)$. It is endowed with the quotient topology, i.e., the largest topology for which the quotient projection $\mathbb{C}^2 \setminus \{0\} \to \mathbb{P}^1$ is continuous. The equivalence class of (z_1, z_2) is denoted by $[z_1, z_2]$. Show that $\mathbb{P}^1$ is a complex manifold which is biholomorphic to the Riemann sphere. Hint: Show that $\varphi_1 : \mathbb{P}^1 \setminus \{[1,0]\} \to \mathbb{C}, [z,1] \mapsto z$, and $\varphi_2 : \mathbb{P}^1 \setminus \{[0,1]\} \to \mathbb{C}, [1,z] \mapsto z$, define two charts which cover $\mathbb{P}^1$. Compute the transition map $\varphi_1 \circ \varphi_2^{-1}$.

Example 30.5 (complex tori)**.** Let $w_1, w_2 \in \mathbb{C}^*$ be such that $\operatorname{Im} \tau > 0$, where $\tau = w_2/w_1$, and let $\Lambda = \mathbb{Z}w_1 + \mathbb{Z}w_2$. Then Λ is a subgroup of $\mathbb{C}$ and acts on $\mathbb{C}$ by $\lambda(z) = z + \lambda$, $\lambda \in \Lambda$, $z \in \mathbb{C}$. Consider the equivalence relation on $\mathbb{C}$ defined by $z \sim w$ if $z - w \in \Lambda$ and the corresponding quotient space $X = \mathbb{C}/\Lambda$. Then X is Hausdorff and the quotient map $\pi : \mathbb{C} \to X$ is a covering map. Let $\{V_i\}_i$ be an open cover of $\mathbb{C}$ by disks such that $\pi_i := \pi|_{V_i}$ is a homeomorphism onto $U_i := \pi(V_i)$. We claim that $\{(U_i, \pi_i^{-1})\}_i$ is a complex structure on X. Let $x \in U_i \cap U_j$ and $z_i = \pi_i^{-1}(x)$, $z_j = \pi_j^{-1}(x)$. Then $\lambda = z_i - z_j \in \Lambda$. The transition map $\pi_i^{-1} \circ \pi_j : \pi_j^{-1}(U_i \cap U_j) \to \pi_i^{-1}(U_i \cap U_j)$ is the mapping $z \mapsto z + \lambda$. The Riemann surface X defined by this complex structure is a **complex torus**. The mapping $\pi : \mathbb{C} \to X$ is holomorphic. A model of X is obtained by identifying opposite side of the parallelogram with vertices 0, w_1, w_2, and $w_1 + w_2$.



Let $X_1 = \mathbb{C}/\Lambda_1$, $X_2 = \mathbb{C}/\Lambda_2$ be two biholomorphic complex tori, where $\Lambda_1 = \mathbb{Z} + \mathbb{Z}\tau_1$ and $\Lambda_2 = \mathbb{Z} + \mathbb{Z}\tau_2$. What can be said about the relationship of τ_1,τ_2? If $f : X_1 \to X_2$ is a biholomorphism, then there is a biholomorphism $\tilde{f} : \mathbb{C} \to \mathbb{C}$ such that the following diagram commutes.

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "\tilde{f}"] \arrow[d, "\pi_1"'] & \mathbb{C} \arrow[d, "\pi_2"] \\
X_1 \arrow[r, "f"] & X_2
\end{tikzcd}$$

This follows from Corollary 4.7. Then $\tilde{f}$ induces a group isomorphism $\psi : \Lambda_1 \to \Lambda_2$ by $\lambda \mapsto \tilde{f} \circ \lambda \circ \tilde{f}^{-1}$. Since $\tilde{f} \in \operatorname{Aut}(\mathbb{C})$, $\tilde{f}(z) = az + b$ for some $a \in \mathbb{C}^*$, $b \in \mathbb{C}$. Therefore, there are $k, \ell, m, n \in \mathbb{Z}$ such that

$$a = \psi(1) = k + \ell\tau_2, \quad a\tau_1 = \psi(\tau_1) = m + n\tau_2$$

and since ψ is an isomorphism the matrix $\left(\begin{smallmatrix} k & \ell \\ m & n \end{smallmatrix}\right)$ is invertible, thus $kn - \ell m = \pm 1$. It follows that

$$\tau_1 = \frac{m + n\tau_2}{k + \ell\tau_2}$$

and since $\tau_1, \tau_2 \in \mathbb{H}$, we have $kn - \ell m = 1$. Conversely, if τ_1, τ_2 are related in this way then they correspond to biholomorphic complex tori. By Proposition 24.2, we have shown that the set of equivalence classes of complex tori up to biholomorphism (called **moduli space**) is in one-to-one correspondence with the fundamental domain of the modular group, depicted on p. 62.

Example 30.6 (orbit spaces $\mathbb{H}/\Gamma$)**.** Let Γ be a discrete fixed point free subgroup of $\mathrm{Aut}(\mathbb{H}) = \{z \mapsto (az+b)/(cz+d) : a, b, c, d \in \mathbb{R},\ ad - bc = 1\}$. The orbit space $X = \mathbb{H}/\Gamma$ is Hausdorff and the quotient map $\pi : \mathbb{H} \to X$ is a covering map. In analogy to Example 30.5, there is a natural complex structure on X (with $z \mapsto \gamma(z)$, $\gamma \in \Gamma$, as transition maps) which makes X to a Riemann surface and the projection $\pi : \mathbb{H} \to X$ holomorphic. A particular example was discussed in section 26.

Exercise 59. Prove: $f(z) = (az+b)/(cz+d) \in \mathrm{Aut}(\mathbb{H})$ is fixed point free in $\mathbb{H}$ if and only if $|a+d| \ge 2$.

Remark 30.7. The **uniformization theorem** states that any simply connected Riemann surface is biholomorphic to either $\mathbb{C}$, $\widehat{\mathbb{C}}$, or $\mathbb{H}$; note that this is a generalization of the Riemann mapping theorem 20.4. See e.g. [**6**].

Let X, Y be connected topological spaces and let $p : Y \to X$ be a covering map. Then $p : Y \to X$ is called a **universal covering** of X if it satisfies the following universal property: for every covering map $q : Z \to X$, where Z is connected, and all $y_0 \in Y$, $z_0 \in Z$ with $p(y_0) = q(z_0)$ there is a unique mapping $f : Y \to Z$ such that $p = q \circ f$ and $f(y_0) = z_0$. There is up to isomorphism at most one universal covering of X.

Let X be a connected manifold. If Y is a connected, simply connected manifold and $p : Y \to X$ is a covering map, then p is the universal covering of X. If X is a connected manifold, then there always exists a connected, simply connected manifold $\widetilde{X}$ and a covering map $p : \widetilde{X} \to X$; thus $p : \widetilde{X} \to X$ is the universal covering of X. If X is a Riemann surface, then $\widetilde{X}$ has a unique complex structure which makes it a Riemann surface and the mapping $p : \widetilde{X} \to X$ holomorphic.

Let X be a Riemann surface and let $p : \widetilde{X} \to X$ be its universal covering. Let G be the group of homeomorphisms $g : \widetilde{X} \to \widetilde{X}$ such that $p = p \circ g$, i.e., the group of **deck transformations**. Then G is isomorphic to the fundamental group $\pi_1(X)$ and it acts properly discontinuously and fixed point freely on $\widetilde{X}$. With respect to the complex structure on $\widetilde{X}$, G is actually a group of holomorphic automorphism of $\widetilde{X}$.

By the uniformization theorem we have only three candidates for $\widetilde{X}$, namely $\mathbb{C}$, $\widehat{\mathbb{C}}$, or $\mathbb{H}$. Each of these domains has the property that its automorphism group is a group of Möbius transformations. It follows that every Riemann surface X is biholomorphic to D/G, where D is either $\mathbb{C}$, $\widehat{\mathbb{C}}$, or $\mathbb{H}$ and G is a group of Möbius transformations isomorphic to $\pi_1(X)$ which acts properly discontinuously and fixed point freely on D.

31. Meromorphic functions

Let X be a Riemann surface, let $a \in X$, and let U be a neighborhood of a in X. Let $f \in \mathcal{H}(U \setminus \{a\})$. We say that a is a **removable singularity** of f if there exists $F \in \mathcal{H}(U)$ with $F|_{U \setminus \{a\}} = f$. We say that a is a **pole** of f if $|f(x)| \to \infty$ as $x \to a$. If a is neither a removable singularity nor a pole, then a is called an **essential singularity** of f.

Let A be a discrete subset of X and let $f \in \mathcal{H}(X \setminus A)$. We say that f is **meromorphic** on X if each point of A is either a removable singularity or a pole of f. Let $B \subseteq A$ be the set of poles of f. Then f defines a holomorphic mapping $f : X \setminus B \to \mathbb{C}$, and induces a mapping $F : X \to \widehat{\mathbb{C}}$ by

$$F(x) := \begin{cases} f(x) & \text{if } x \in X \setminus B, \\ \infty & \text{if } x \in B. \end{cases}$$

Proposition 31.1. *The mapping $F : X \to \widehat{\mathbb{C}}$ is holomorphic.*

Proof. F is continuous since $|f(x)| \to \infty$ as $x \to b$ for each $b \in B$. Let $D_1(\infty) := \{z \in \widehat{\mathbb{C}} : |z| > 1\}$. Then $U := F^{-1}(D_1(\infty))$ is open in X. It suffices to check that $F : U \to \widehat{\mathbb{C}}$ is holomorphic, that is, that $g := \varphi \circ F$ is holomorphic, where $\varphi : D_1(\infty) \to \mathbb{D}$ with $\varphi(z) = 1/z$ if $z \neq \infty$ and $\varphi(\infty) = 0$. This follows from Riemann's theorem on removable singularities 30.1(6), since $g(x) \to 0 = g(b)$ as $x \to b$ for all $b \in B$. □

If, conversely, $F : X \to \widehat{\mathbb{C}}$ is a non-constant holomorphic mapping and $B := F^{-1}(\infty)$, then $f = F|_{X \setminus B}$ is holomorphic on $X \setminus B$, meromorphic on X, and B is the set of its poles. Thus, non-constant meromorphic functions are just non-constant holomorphic functions into $\widehat{\mathbb{C}}$.

Meromorphic functions can be added and multiplied. If f is meromorphic on X and $f \not\equiv 0$ then $1/f$ is meromorphic on X. So the set $\mathcal{M}(X)$ of meromorphic functions on a Riemann surface X forms a field, the so-called **function field** of X.

Let $f \in \mathcal{M}(X)$ and $a \in X$. The **order** of f at a is defined by

$$\operatorname{ord}_a(f) := \operatorname{ord}_{\varphi(a)}(f \circ \varphi^{-1}),$$

where (U, φ) is any chart with $a \in U$. It is well-defined, since the order is invariant under biholomorphisms. If $\operatorname{ord}_a(f) = k > 0$ then a is a zero of order k, if $\operatorname{ord}_a(f) = -k < 0$ then a is a pole of order k.

Proposition 31.2 (function field of the Riemann sphere)**.** *The function field $\mathcal{M}(\widehat{\mathbb{C}})$ consists precisely of the rational functions.*

Proof. Clearly, every rational function is in $\mathcal{M}(\widehat{\mathbb{C}})$. Let $f \in \mathcal{M}(\widehat{\mathbb{C}})$, $f \neq 0$. Let $a_1, \ldots, a_n \in \mathbb{C}$ be the poles of f in $\mathbb{C}$ and let $-k_j = \operatorname{ord}_{a_j}(f)$; there are finitely many since $\widehat{\mathbb{C}}$ is compact. The function $g = f \prod_{j=1}^n (z - a_j)^{k_j}$ is meromorphic on $\widehat{\mathbb{C}}$ and has no poles in $\mathbb{C}$. Then $w \mapsto g(1/w)$ is meromorphic in a neighborhood of $w = 0$. Thus, there is $M > 0$, $\rho > 0$ such that

$$|w^N g(1/w)| \le M \quad \text{for } 0 < |w| < \rho,$$

where $N = -\operatorname{ord}_0(g(1/w))$. Hence

$$|g(z)| \le M|z|^N \quad \text{for } 1/\rho < |w| < \infty.$$

Since g is entire, it must be a polynomial (by the Cauchy inequalities). Thus f is a rational function. □

Proposition 31.3 (function field of complex tori)**.** *Let $\Lambda = \mathbb{Z}w_1 + \mathbb{Z}w_2$, where $w_1, w_2 \in \mathbb{C}^*$ with $\tau := w_2/w_1 \in \mathbb{H}$. Let $\mathbb{C}/\Lambda$ be the corresponding complex torus. The function field $\mathcal{M}(\mathbb{C}/\Lambda)$ is in one-to-one correspondence with the elliptic functions with period group Λ.*

Proof. Let $f \in \mathcal{M}(\mathbb{C}/\Lambda)$. We may assume that f is non-constant. Thus $f : \mathbb{C}/\Lambda \to \widehat{\mathbb{C}}$ is holomorphic and hence $\tilde{f} := f \circ \pi : \mathbb{C} \to \widehat{\mathbb{C}}$ is holomorphic, where $\pi : \mathbb{C} \to \mathbb{C}/\Lambda$

is the quotient map. Thus $\tilde{f}$ is meromorphic and Λ-invariant, i.e., $\Lambda \subseteq \operatorname{per}(f)$. Conversely, every elliptic function $\tilde{f}$ with period group Λ induces a meromorphic function on $\mathbb{C}/\Lambda$. □

32. Holomorphic mappings between Riemann surfaces

Let X, Y be Riemann surfaces and let $p \in \mathcal{H}(X, Y)$ be non-constant. Then p is open and every fiber $p^{-1}(y)$, $y \in Y$, is discrete, by Theorem 30.1. A function $f \in \mathcal{H}(X)$ (or $f \in \mathcal{M}(X)$) can be considered as a **multi-valued function** on Y: if $y \in Y$ and $p^{-1}(y) = \{x_i : i \in I\}$, then $f(x_i)$, $i \in I$, are the different values of f at the point y.

Example 32.1 (Riemann surface of the logarithm)**.** Consider $p = \exp : \mathbb{C} \to \mathbb{C}^*$. Then $f = \mathrm{Id} : \mathbb{C} \to \mathbb{C}$ corresponds to the multi-valued logarithm $\log : \mathbb{C}^* \to \mathbb{C}$. A geometric model for the Riemann surface of the logarithm is obtained as follows: the preimage $p^{-1}(\mathbb{C}^-)$ of the set $\mathbb{C}^- = \mathbb{C} \setminus \mathbb{R}_{\le 0}$ has infinitely many components S_n, $n \in \mathbb{Z}$, on which p is bijective onto $\mathbb{C}^-$. We may think of the components S_n to be all copies of $\mathbb{C}^-$ and stacked one above the other. Then the second quadrant edge of S_n is glued to the third quadrant edge of S_{n+1} for all n, which results in an infinite spiral, the **Riemann surface of the logarithm**. On this Riemann surface the logarithm is a single-valued function.

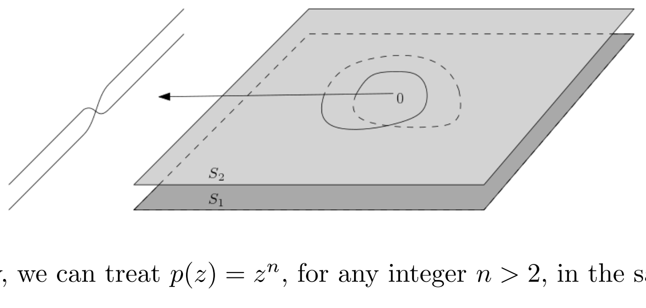
A point $x \in X$ is called a **branch point** or **ramification point** of p if there is no neighborhood U of x such that $p|_U$ is injective.

Lemma 32.2. *Let X, Y be Riemann surfaces and let $p \in \mathcal{H}(X, Y)$ be non-constant. Then p has no branch points if and only if p is a local homeomorphism.*

Proof. This follows easily from the fact that p is continuous and open. □

For instance, $\exp : \mathbb{C} \to \mathbb{C}^*$ has no branch points.

Example 32.3 (Riemann surface of the square root)**.** Let $p : \mathbb{C} \to \mathbb{C}$ be the mapping $z \mapsto z^2$. Then $0 \in \mathbb{C}$ is a branch point of p. Here $f = \mathrm{Id} : \mathbb{C} \to \mathbb{C}$ corresponds to the multi-valued square root. Let us look at the graph $X := \{(z, w) \in \mathbb{C}^2 : w = z^2\}$ of p. Then the projection $\mathrm{pr}_1 : X \to \mathbb{C}$, $(z, w) \mapsto z$, defines a biholomorphism between X and $\mathbb{C}$; it is the single-valued square root function $z = \sqrt{w}$. As in Example 32.1 we may construct a geometric model of the **Riemann surface of the square root** by gluing the components of the preimage $p^{-1}(\mathbb{C} \setminus \mathbb{R}_{<0})$ which consist of two sheets S_1 and S_2, both copies of $\mathbb{C} \setminus \mathbb{R}_{<0}$. Here the second quadrant edge of S_1 is is glued to the third quadrant edge of S_2 and the second quadrant edge of S_2 is is glued to the third quadrant edge of S_1.



Evidently, we can treat $p(z) = z^n$, for any integer $n > 2$, in the same way.

Proposition 32.4 (local form of holomorphic maps)**.** *Let X, Y be Riemann surfaces and let $f \in \mathcal{H}(X, Y)$ be non-constant. Let $a \in X$ and $b = f(a)$. Then there is* *an integer $n \geq 1$ and charts (U, φ), (V, ψ) on X, Y, respectively, such that $a \in U$, $\varphi(a) = 0$, $b \in V$, $\psi(b) = 0$, $f(U) \subseteq V$ and*

$$\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V) : z \mapsto z^n.$$

Proof. Let $F := \psi \circ f \circ \varphi^{-1}$. Then $F(0) = 0$ and so there is a positive integer n such that $F(z) = z^n g(z)$, where $g(0) \neq 0$. Thus, there is a neighborhood of 0 and a holomorphic function on this neighborhood such that $h^n = g$. The mapping $\alpha(z) := zh(z)$ is a biholomorphism from an open neighborhood of 0 onto an open neighborhood of 0. Replacing the chart φ by $\alpha \circ \varphi$ the statement follows. □

The number n is the **ramification number** or **multiplicity** of f at a, we write $m_a(f)$. The mapping $f : X \to Y$ is said to **take the value** $b \in Y$**,** m **times** (counting multiplicities) if $m = \sum_{x \in f^{-1}(b)} m_x(f)$.

A continuous mapping $f : X \to Y$ between manifolds is called **proper** if the preimage of every compact set is compact. Any proper mapping is closed, i.e., maps closed sets to closed sets.

Exercise 60. Prove that a proper mapping $f : X \to Y$ between manifolds is closed.

Let X, Y be Riemann surfaces. A proper non-constant holomorphic mapping $f : X \to Y$ is called a **branched covering**.

Theorem 32.5 (degree)**.** *Let X, Y be Riemann surfaces, and let $f : X \to Y$ be a branched covering. Then there is a positive integer n such that f takes every value $b \in Y$, n times. The number n is called the* ***degree*** *of f.*

Proof. The set of branch points A of f is closed and discrete by Proposition 32.4. Since f is proper, also $B = f(A)$ is closed and discrete. Set $Y' := Y \setminus B$ and $X' := X \setminus f^{-1}(B)$. Then $f|_{X'} : X' \to Y'$ is a holomorphic covering map with a finite number, say n, of sheets: by the uniqueness of liftings 1.1 of curves, any two fibers $f^{-1}(y)$ have the same cardinality, which is finite since f is proper. Let $b \in B$, $f^{-1}(b) = \{x_1, \ldots, x_k\}$ and $m_j = m_{x_j}(f)$. By Proposition 32.4, there exist disjoint neighborhoods U_j of x_j and V_j of b such that for each $c \in V_j \setminus \{b\}$ the set $f^{-1}(c) \cap U_j$ consists of exactly m_j points. We claim that there is a neighborhood $V \subseteq V_1 \cap \cdots \cap V_k$ of b such that $f^{-1}(V) \subseteq U_1 \cup \cdots \cup U_k$ (take $V := Y \setminus f(X \setminus (V_1 \cap \cdots \cap V_k))$ which is open since f is closed). Then, for every $c \in V \cap Y'$, the fiber $f^{-1}(c)$ consists of $m_1 + \cdots + m_k$ points. Thus $n = m_1 + \cdots + m_k$. □

Corollary 32.6. *Let X be a compact Riemann surface X and let $f \in \mathcal{M}(X)$ be non-constant. Then f has as many zeros as poles (counted according multiplicities).*

Proof. The mapping $f : X \to \widehat{\mathbb{C}}$ is proper, since X is compact. Apply Theorem 32.5. □

For the special case of complex tori we already proved this in Theorem 24.3; cf. Proposition 31.3.

Corollary 32.7 (fundamental theorem of algebra)**.** *Any polynomial $p(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n \in \mathbb{C}[z]$, $a_0 \neq 0$, has n roots (counted according multiplicities).*

Proof. $p \in \mathcal{M}(\widehat{\mathbb{C}})$ has a pole of order n at ∞. □

Let us state without proof that every covering map of $\mathbb{D}^* := \mathbb{D} \setminus \{0\}$ is either isomorphic to the covering given by the exponential mapping or else by the nth power; for a proof see [**7**, Theorem 5.10].

Theorem 32.8. *Let X be a Riemann surface and let $p : X \to \mathbb{D}^*$ be a covering map. Then one of the following occurs:*

(1) *If p has an infinite number of sheets, then there is a biholomorphism $\varphi : X \to \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$ such that $p = e^{\varphi}$.*
(2) *If p has n sheets, then there is a biholomorphism $\varphi : X \to \mathbb{D}^*$ such that $p = \varphi^n$.*

Corollary 32.9. *Let X be a Riemann surface and let $p : X \to \mathbb{D}$ be a branched covering such that $p : p^{-1}(\mathbb{D}^*) \to \mathbb{D}^*$ is a covering map. Then there is an integer $n \geq 1$ and a biholomorphism $\varphi : X \to \mathbb{D}$ such that $p = \varphi^n$.*

Proof. By the previous theorem and Theorem 32.5, there exists $n \geq 1$ and a biholomorphism $\varphi : p^{-1}(\mathbb{D}^*) \to \mathbb{D}^*$ such that $p = \varphi^n$. We claim that $p^{-1}(0)$ consists of only one point $a \in X$. Then, by setting $\varphi(a) := 0$, φ extends to a biholomorphism $\varphi : X \to \mathbb{D}$ such that $p = \varphi^n$, by Theorem 30.1(6).

Suppose that $p^{-1}(0)$ consists of $k \geq 2$ points $a_1, \ldots, a_k$. Then there are disjoint open neighborhoods U_i of a_i and $r > 0$ such that $p^{-1}(D_r(0)) \subseteq U_1 \cup \cdots \cup U_k$. Set $D_r^*(0) := D_r(0) \setminus \{0\}$. Then $p^{-1}(D_r^*(0))$ is homeomorphic to $D^*_{r^{1/n}}(0)$, and thus connected. Every a_i is an accumulation point of $p^{-1}(D_r^*(0))$, and hence also $p^{-1}(D_r(0))$ is connected, a contradiction. □

33. Construction of Riemann surfaces by analytic continuation

Let us now consider the construction of Riemann surfaces by analytic continuation of function germs.

Let X be a Riemann surface. In analogy to section 2 we define the **sheaf of germs of holomorphic functions** on X: we set $\mathcal{O}_X := \bigsqcup_{x \in X} \mathcal{O}_{X,x}$, where $\mathcal{O}_{X,x}$ is the set of **germs** at $x \in X$. A germ of a holomorphic function at $x \in X$ is an equivalence class with respect to the equivalence relation

$$(U, f) \sim (V, g) :\Leftrightarrow \exists W \text{ such that } x \in W \subseteq U \cap V \text{ and } f|W = g|W,$$

where $U, V, W \subseteq X$ are open neighborhoods of x and f, g are holomorphic. Endowing $\mathcal{O}_X$ with the topology generated by the fundamental system of neighborhoods (cf. (2.1))

$$N(U, f) := \{f_x \in \mathcal{O}_{X,x} : f_x \text{ is the germ at } x \in U \text{ defined by } (U, f)\}$$

makes $\mathcal{O}_X$ to a Hausdorff space and the projection $\pi : \mathcal{O}_X \to X$, $\pi(f_x) = x$, to a local homeomorphism; this can be seen as in Lemma 2.2 and Lemma 2.3.

Let $f_x \in \mathcal{O}_{X,x}$ and let $\gamma : [0, 1] \to X$ be a curve with $\gamma(0) = x$. An **analytic continuation of f_x along** γ is a lifting $\tilde{\gamma}$ of γ to $\mathcal{O}_X$ such that $\tilde{\gamma}(0) = f_x$. By uniqueness of liftings 1.1, the analytic continuation of a germ is unique if it exists. The general monodromy theorem 4.1 implies that if γ_0, γ_1 are homotopic curves in X from a to b and $f_a \in \mathcal{O}_{X,a}$ is a germ which admits an analytic continuation along every curve in a homotopy $\{\gamma_s\}_{s \in [0,1]}$ connecting γ_0 and γ_1, then the analytic continuations of f_a along γ_0 and γ_1 result in the same germ. In particular, if X is simply connected and $f_a \in \mathcal{O}_{X,a}$ admits an analytic continuation along every curve starting in a, then there exists a unique globally defined holomorphic function $f \in \mathcal{H}(X)$ such that f_a is the germ at a of f.

In general, if X is not simply connected, by considering all germs that arise by analytic continuation from a given germ we obtain a multi-valued function. Let us make this precise.

First we make the following observation. Suppose that X, Y are Riemann surfaces and $p : Y \to X$ is a holomorphic mapping which is a local homeomorphism.

Since p is locally biholomorphic, it induces an isomorphism $p^* : \mathcal{O}_{X,p(y)} \to \mathcal{O}_{Y,y}$ (where $p^*(f) = f \circ p$). Let $p_* : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,p(y)}$ denote the inverse of p^*.

Let X be a Riemann surface, $a \in X$, and $f_a \in \mathcal{O}_{X,a}$. By an **analytic continuation** (Y, p, f, b) of f_a we mean the following data: Y is a Riemann surface and $p : Y \to X$ is a holomorphic mapping which is a local homeomorphism, $b \in p^{-1}(a)$, and $f \in \mathcal{H}(Y)$ is such that $p_*(f_b) = f_a$. An analytic continuation (Y, p, f, b) of f_a is called **maximal** if it has the following universal property: if (Z, q, g, c) is another analytic continuation of f_a then there is a holomorphic mapping $\varphi : Z \to Y$ such that $\varphi(c) = b$, $\varphi^*(f) = g$, and $\varphi^*(p) = q$. By uniqueness of liftings 1.1, a maximal analytic continuation is unique up to isomorphism.

$$\begin{tikzcd} Y \arrow[r, "p"] \arrow[d, "f"'] & X \\ \mathbb{C} & Z \arrow[l, "g"] \arrow[u, "q"'] \arrow[ul, "\varphi"] \end{tikzcd}$$

We will show that there always exists a maximal analytic continuation.

Theorem 33.1 (maximal analytic continuation)**.** *Let X be a Riemann surface, $a \in X$, and $f_a \in \mathcal{O}_{X,a}$. Then there exists a maximal analytic continuation (Y, p, f, b) of f_a.*

Proof. Let Y be the connected component of $\mathcal{O}_X$ containing f_a. Then $p : Y \to X$, $f_x \mapsto x$, is a local homeomorphism. There is a natural complex structure on Y which makes it a Riemann surface and $p : Y \to X$ holomorphic; this follows from the arguments in Example 30.3. Let $f : Y \to \mathbb{C}$ be defined by $f(h) := \mathrm{ev}_{p(h)}(h)$, i.e., $h \in Y$ is a germ at $p(h)$ and $f(h)$ is its value. Then $f \in \mathcal{H}(Y)$ and $p_*(f_h) = h$ for every $h \in Y$, in particular, for $b := f_a$, we have $p_*(f_b) = f_a$. Thus (Y, p, f, b) is an analytic continuation of f_a.

Let us show maximality. Let (Z, q, g, c) be another analytic continuation of f_a. Let $z \in Z$. The germ $q_*(g_z) \in \mathcal{O}_{X,q(z)}$ arises by analytic continuation along a curve from a to $q(z)$, and hence there is precisely one $h \in Y$ such that $q_*(g_z) = h$. Define a mapping $\varphi : Z \to Y$ by setting $\varphi(z) := h$. Then $\varphi(c) = b$, $\varphi^*(f) = g$, and $\varphi^*(p) = q$. $\square$

34. Elliptic curves

Consider the elliptic curve

$$w^2 = 4(z - e_1)(z - e_2)(z - e_3) =: p(z), \tag{34.1}$$

where $e_1, e_2, e_3 \in \mathbb{C}$ are pairwise distinct. Then $p'(e_i) \neq 0$ for all $i = 1, 2, 3$. We will construct a compact Riemann surface on which the function $w = \sqrt{p(z)}$ is single-valued. Set

$$X := \{(z, w) \in \mathbb{C}^2 : w^2 = p(z)\}.$$

Let $(z_0, w_0) \in X$ be such that $w_0 \neq 0$. Then $z_0 \neq e_i$, $i = 1, 2, 3$, and we may take $(z, w) \mapsto z$ as a local coordinate in a neighborhood of (z_0, w_0). At a point $(z_0, 0) \in X$, $z_0 = e_i$ for some i and hence $p'(z_0) \neq 0$. By the implicit function theorem, there is a holomorphic function f defined in a neighborhood of 0 such that $z = f(w)$ near $(z_0, 0)$ and $z_0 = f(0)$. Thus we may take $(z, w) \mapsto w$ as a local coordinate in a neighborhood of $(z_0, 0)$. This defines a complex structure on X and the projection $p = \mathrm{pr}_1 : X \to \mathbb{C}$, $\mathrm{pr}_1(z, w) = z$ is holomorphic. The Riemann surface X has two sheets, since to a general value of z correspond two values of w. (Note that also $\mathrm{pr}_2 : X \to \mathbb{C}$ is holomorphic.)

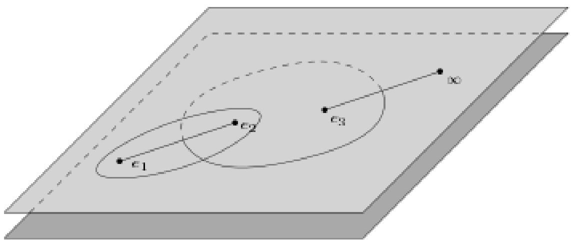
Let us complete X to a Riemann surface $\widehat{X}$ over $\widehat{\mathbb{C}}$. Let $\mathbb{P}^2$ be the **complex projective plane**, i.e., $\mathbb{P}^2 = \mathbb{C}^3 \setminus \{0\}/\sim$, where $(z_1, z_2, z_3) \sim (w_1, w_2, w_3)$ if there is $\lambda \in \mathbb{C}^*$ such that $(z_1, z_2, z_3) = \lambda(w_1, w_2, w_3)$. We denote by $[z_1, z_2, z_3]$ the equivalence class of (z_1, z_2, z_3) and say that $[z_1, z_2, z_3]$ are **homogeneous** coordinates. Define

$$\widehat{X} := \{[z, w, t] \in \mathbb{P}^2 : w^2 t = 4(z - e_1 t)(z - e_2 t)(z - e_3 t)\}.$$

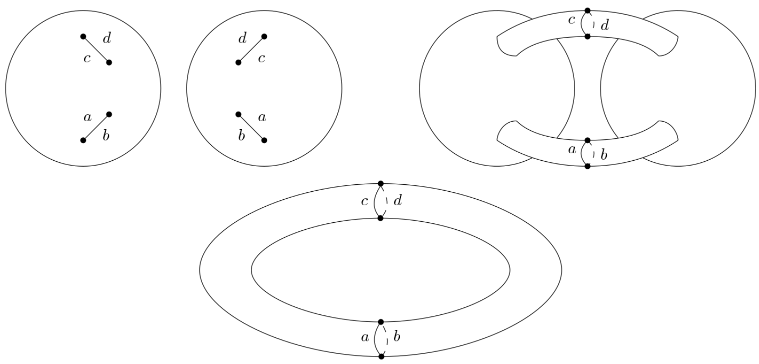
We identify X with $\{[z, w, t] \in \widehat{X} : t = 1\}$. The complement $\widehat{X} \setminus X$ consists of a single point, at infinity with the homogeneous coordinates $[0, 1, 0]$ (setting $t = 0$ gives $z^3 = 0$ thus $z = 0$). Let this point be denoted by ∞. In a neighborhood of ∞, we can take $\widehat{X} \ni [z, w, t] \to z/w \in \mathbb{C}$ as a local coordinate. In fact, replacing $[z, w, t]$ by $[z', w', t'] = [z/w, 1, t/w]$ gives

$$t' = 4(z' - e_1 t')(z' - e_2 t')(z' - e_3 t'),$$

and by the implicit function theorem, t' is a holomorphic function of z' in some neighborhood of $(z', t') = (0, 0)$. This defines a complex structure on $\widehat{X}$ and a holomorphic projection $\widehat{p} : \widehat{X} \to \widehat{\mathbb{C}}$ which coincides with p on X and sends $\infty \in \widehat{X}$ to $\infty \in \widehat{\mathbb{C}}$. The Riemann surface $\widehat{X}$ is the compactification of X and $\widehat{p} : \widehat{X} \to \widehat{\mathbb{C}}$ is a two-sheeted branched covering with branch points e_1, e_2, e_3, ∞. A geometric model of $\widehat{X}$ is obtained by slicing two copies of $\widehat{\mathbb{C}}$ along some path from e_1 to e_2 and some path from e_3 to ∞, say, and identifying the boundaries crosswise.



Topologically, the resulting surface is a torus, which is illustrated in the figure below.



Proposition 34.1. *Let $\Lambda = \mathbb{Z}w_1 + \mathbb{Z}w_2$, where $\operatorname{Im} w_2/w_1 > 0$, and assume that e_1, e_2, e_3 satisfy* (25.4) *for the associated Weierstrass $\wp$-function. Then the mapping $\varphi : \mathbb{C}/\Lambda \to \widehat{X}$,*

$$\varphi(z) := \begin{cases} [\wp(z), \wp'(z), 1] & \text{if } z \neq 0, \\ [\wp(z)/\wp'(z), 1, 1/\wp'(z)] & \text{if } z = 0, \end{cases}$$

is a biholomorphism.

Proof. Since $\wp$ satisfies the differential equation (25.5), φ maps $\mathbb{C}/\Lambda$ into $\widehat{X}$. We have the following commuting diagram.

$$\begin{tikzcd} \mathbb{C}/\Lambda \arrow[r, "\varphi"] \arrow[dr, "\wp"'] & \widehat{X} \arrow[d, "\widehat{p}"] \\ & \widehat{\mathbb{C}} \end{tikzcd}$$

Then φ is surjective, since $\wp$ is surjective, by Corollary 24.4. Let us show that φ is injective. Let $z, z' \in \mathbb{C}$ be such that $z' - z \notin \Lambda$ and $\wp(z) = \wp(z')$. Assume first that $2z \notin \Lambda$. We have $z' + z \in \Lambda$, since $\wp(-z) = \wp(z)$, because $\wp$ is even, but $\wp$ assumes every value exactly twice. Consequently, $\wp'(z) \neq \wp'(z')$ and hence $\varphi(z) \neq \varphi(z')$, since otherwise $\wp'(z) = \wp'(-z') = -\wp'(z') = -\wp'(z)$, as $\wp'$ is odd, and hence $\wp'(z) = 0$, which contradicts $2z \notin \Lambda$; recall that the zeros of $\wp'$ are $w_1/2$, $w_2/2$, and $(w_1 + w_2)/2$. In the case that $2z \in \Lambda$ we also have $2z' \in \Lambda$, because $\wp(-z') = \wp(z')$, since $\wp$ is even, and $\wp$ assumes every value exactly twice. Thus, $z, z' \in \{w_1/2, w_2/2, (w_1 + w_2)/2\}$ modulo Λ. But $e_1 = \wp(w_1/2)$, $e_2 = \wp(w_2/2)$, $e_3 = \wp((w_1 + w_2)/2)$ are pairwise distinct, which implies $z' - z \in \Lambda$. This implies that φ is injective.

Thus $\varphi : \mathbb{C}/\Lambda \to \widehat{X}$ is a holomorphic bijective mapping from a compact space to a Hausdorff space. It follows that φ is a homeomorphism, and hence a biholomorphism. $\square$

Also the converse is true: For every Riemann surface $\widehat{X}$ of an equation $w^2 = 4(z - e_1)(z - e_2)(z - e_3)$, where e_1, e_2, e_3 are distinct and satisfy $e_1 + e_2 + e_3 = 0$, there is a discrete subgroup $\Lambda = \mathbb{Z}w_1 + \mathbb{Z}w_2$, where $\operatorname{Im} w_2/w_1 > 0$, such that $\widehat{X}$ can be realized as in the proposition.

List of exercises

Exercise 1. Let n be a positive integer. Prove that $\mathbb{C}^* \to \mathbb{C}^*$, $z \mapsto z^n$, is a covering map. Determine the lifting $\tilde{\gamma}$ of $\gamma(t) = e^{2\pi i t}$, $t \in [0,1]$, with $\tilde{\gamma}(0) = 1$.

Exercise 2. Prove Lemma 2.1.

Exercise 3. Show that the mapping $\pi : \mathcal{O} \to \mathbb{C}$ does not have the curve lifting property and hence is not a covering map. Hint: Consider the germ φ at 1 of the function $z \mapsto 1/z$, and show that the curve $\gamma : [0,1] \to \mathbb{C}$, $\gamma(t) = 1 - t$, does not admit a lifting $\tilde{\gamma}$ to $\mathcal{O}$ with $\tilde{\gamma}(0) = \varphi$. Use Lemma 2.4.

Exercise 4. Let $f \in \mathcal{H}(\mathbb{C})$. Show that $N(\mathbb{C}, f)$ is the connected component in $\mathcal{O}$ of the germ f_0 at 0 of f. Hint: Use that an open subset X in the manifold $\mathcal{O}$ is connected if and only if X is pathwise connected.

Exercise 5. Show that concatenation of curves defines a binary operation on the set of all homotopy classes and turns it into a group $\pi_1(X, a)$.

Exercise 6. Use the homotopy form of Cauchy's theorem 4.4 to conclude that $\operatorname{ind}_{\gamma_1}(z) = \operatorname{ind}_{\gamma_2}(z)$, if γ_1, γ_2 are closed homotopic curves in $\mathbb{C}_z^*$.

Exercise 7. Let f be holomorphic in a neighborhood of the disk $D_R(a)$. Prove that for each $r \in (0, R)$ there is a constant $C > 0$ such that

$$\|f\|_{L^\infty(D_r(a))} \le C \|f\|_{L^2(D_R(a))},$$

where $\|f\|_{L^\infty(U)} = \sup_{z \in U} |f(z)|$ and $\|f\|_{L^2(U)} = (\int_U |f(z)|^2 \, dx \, dy)^{1/2}$. Conclude that a sequence $(f_n) \subseteq \mathcal{H}(U)$ which is a Cauchy sequence with respect to the norm $\| \cdot \|_{L^2(U)}$ converges uniformly on compact subsets of U to a holomorphic function.

Exercise 8. Prove that $\sum_{n=-\infty}^{\infty} \alpha_n$ converges to a complex number α if and only if for each $\epsilon > 0$ there is $N \in \mathbb{N}_{>0}$ such that $|\sum_{n=-k}^{\ell} \alpha_n - \alpha| < \epsilon$ if $k, \ell \ge N$.

Exercise 9. The function $f(z) = 6z^{-1}(z+1)^{-1}(z-2)^{-1}$ is holomorphic in $\mathbb{C} \setminus \{0, -1, 2\}$. It has three Laurent expansions about 0. Compute them.

Exercise 10. Prove: Let f be meromorphic in U with zeros a_j and poles b_k, and let γ be a cycle which is homologous to zero in U and does not pass through any of the zeros or poles. Then

$$\frac{1}{2\pi i} \int_\gamma \frac{z f'(z)}{f(z)} \, dz = \sum_j \operatorname{ind}_\gamma(a_j) a_j - \sum_k \operatorname{ind}_\gamma(b_k) b_k,$$

where multiple zeros or poles are repeated according to their order.

Exercise 11. Deduce the fundamental theorem of algebra from Rouché's theorem: any polynomial $P(z) = z^n + a_{n-1} z^{n-1} + \cdots + a_0$ has n roots counted with their multiplicities.

Exercise 12. Show that
$$\int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}.$$
Hint: Integrate $1/(1+z^2)$ along the closed path formed by the segment $[0,R]$, the arc Re^{it}, $t \in [0,\pi]$, and the segment $[-R,0]$.

Exercise 13. Show that the function $z \mapsto \pi \cot(\pi z)$ is meromorphic in $\mathbb{C}$ with a simple pole with residue 1 at each integer n.

Exercise 14. Let $f(z) = P(z)/Q(z)$ be a rational function such that $\deg Q \geq \deg P + 2$. Let $a_1, \dots, a_m$ be its poles, all of them of order 1, and $b_1, \dots, b_m$ the respective residues, and assume that $a_i \notin \mathbb{Z}$ for all $i = 1, \dots, m$. Let γ_n be the counter-clockwise oriented boundary of the square with vertices $(n+1/2)(\pm 1 \pm i)$, where n is a positive integer. Prove that there exist positive constants $C, K > 0$ independent of n such that $|\pi \cot(\pi z)| \leq C$ on $|\gamma_n|$ and $|f(z)| \leq K|z|^{-2}$ if $|z|$ is sufficiently large. Conclude that
$$\lim_{n\to\infty} \int_{\gamma_n} f(z)\pi\cot(\pi z)\,dz = 0,$$
and that
$$\lim_{n\to\infty} \sum_{k=-n}^{n} f(k) = -\sum_{i=1}^{m} b_i \pi \cot(\pi a_i).$$
Note that $\lim_{n,n'\to\infty} \sum_{k=-n}^{n'} f(k)$ exists, since $|f(z)| \leq K|z|^{-2}$ for large $|z|$, and hence the last identity is equivalent to
$$\sum_{k=-\infty}^{\infty} f(k) = -\sum_{i=1}^{m} b_i \pi \cot(\pi a_i).$$

Exercise 15. Use Exercise 14 to show that $\sum_{n=0}^{\infty} 1/(n^2+1) = (1+\pi\coth(\pi))/2$.

Exercise 16. Let $f \in C_c^k(\mathbb{C})$. Show that $u(z) = -1/\pi \iint_{\mathbb{C}} f(\zeta)/(\zeta - z)\, d\xi d\eta$ tends to 0 as $|z| \to \infty$. Prove that u is the only solution of $\partial u/\partial \overline{z} = f$ with this property. Hint: All other solutions are of the form $u + v$, where v is entire.

Exercise 17. Let $f \in C_c^k(\mathbb{C})$ and let u be a solution of $\partial u/\partial \overline{z} = f$ with compact support. Let D be a large disk which contains $\operatorname{supp} u$. Prove that
$$\iint_D f(z)\, dz \wedge d\overline{z} = 0.$$
Conclude that there are functions $f \in C_c^k(\mathbb{C})$ such that no solution u of $\partial u/\partial \overline{z} = f$ has compact support. Hint: Use Stokes' theorem.

Exercise 18. Suppose that $f \in C_c^\infty(\mathbb{C})$ satisfies $\iint_{\mathbb{C}} f(z) z^n \, dx\, dy = 0$ for every integer $n \geq 0$. Prove that the solution (10.4) of (10.5) has compact support. Hint: Expand the kernel $1/(\zeta - z)$ into a geometric series for ζ in some disk D containing $\operatorname{supp} f$ and $z \notin \overline{D}$.

Exercise 19. Show that d defined by (11.2) is a metric on $\mathcal{H}(U)$ and that $(\mathcal{H}(U), d)$ is a complete metric space. Prove that a sequence in $\mathcal{H}(U)$ converges uniformly on every compact subset of U if and only if it converges for the metric d.

Exercise 20. Prove that the mapping $f \mapsto f'$ from $\mathcal{H}(U)$ to itself is continuous.

Exercise 21. Let $K_1 = \overline{D}_1(4)$, $K_2 = \overline{D}_1(4i)$, $K_3 = \overline{D}_1(-4)$, and $K_4 = \overline{D}_1(-4i)$. Show that there exists a sequence of entire functions f_n such that $f_n \to j$ uniformly on K_j for $j = 1, 2, 3, 4$.

Exercise 22. Prove that there exists a sequence of polynomials p_n such that $p_n \to 1$ uniformly on compact subsets of $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$, $p_n \to -1$ uniformly on compact subsets of $\{z \in \mathbb{C} : \operatorname{Re} z < 0\}$, and $p_n \to 0$ uniformly on compact subsets of $i\mathbb{R}$.

Exercise 23. Prove that there exists a sequence of entire functions f_n such that $f_n \to 1$ uniformly on compact subsets of the open upper half-plane and (f_n) does not converge at any point of the open lower half-plane.

Exercise 24. Let $U \subseteq \mathbb{C}$ be a domain and let $f \in C^\infty(U)$. Prove that the equation $\Delta u = f$ admits a solution $u \in C^\infty(U)$. Here $\Delta = \partial_x^2 + \partial_y^2 = 4\partial_{\overline{z}}\partial_z$ is the Laplace operator. Conclude that if $u \in C^2(U)$ satisfies $\Delta u = 0$, then u is actually in $C^\infty(U)$. Hint: Check that $\partial_{\overline{z}}\overline{u} = \overline{\partial_z u}$ and use Theorem 12.2 twice.

Exercise 25. Let U_1, U_2 be domains in $\mathbb{C}$ and let $f \in \mathcal{H}(U_1 \cap U_2)$. Show that there are functions $f_1 \in \mathcal{H}(U_1)$ and $f_2 \in \mathcal{H}(U_2)$ such that $f = f_1 - f_2$ on $U_1 \cap U_2$. For $U_1 = \{z \in \mathbb{C} : \operatorname{Re} z < 1\}$, $U_2 = \{z \in \mathbb{C} : \operatorname{Re} z > -1\}$, and $f(z) = 1/(z^2 - 1)$, find explicit functions f_1, f_2 satisfying the above properties.

Exercise 26. Show that if $\prod_{n=1}^\infty (1+a_n)$ converges then $\lim_{M,N\to\infty} \prod_{n=M}^N (1+a_n)$ exists and equals 1. In addition show that this is not necessarily true if we allow $\lim_{N\to\infty} \prod_{n=n_0+1}^N (1+a_n) = 0$ in the definition of the convergence of $\prod_{n=1}^\infty (1+a_n)$.

Exercise 27. Let (a_n) be a sequence (with repetitions) of points in $\mathbb{D} \setminus \{0\}$ satisfying $\sum_{n=1}^\infty (1 - |a_n|) < \infty$. Show that the so-called **Blaschke product**

$$f(z) = \prod_{n=1}^\infty \frac{-\overline{a}_n}{|a_n|} \frac{z - a_n}{1 - \overline{a}_n z}$$

converges uniformly on every disk $\overline{D}_r(0)$ with $r < 1$ and defines a holomorphic function on $\mathbb{D}$ with $|f(z)| \le 1$. Prove that the zeros of f are precisely the a_n's (counted according to their multiplicities). Hint: Apply Theorem 14.3.

Exercise 28. One can show that the **second (multiplicative) Cousin problem** is always solvable for domains in $\mathbb{C}$: Let $U \subseteq \mathbb{C}$ be a domain. Let $\mathfrak{U} = \{U_i\}_{i \in I}$ be an open cover of U. Suppose that for any pair $(i,j) \in I \times I$ there is a function $f_{ij} \in \mathcal{H}(U_i \cap U_j)$ vanishing nowhere in $U_i \cap U_j$, and that for any triple $(i,j,k) \in I \times I \times I$ we have

$$f_{ik} = f_{ij} f_{jk} \quad \text{on } U_i \cap U_j \cap U_k.$$

Then there exists a family of functions $\{f_i\}_{i\in I}$ with $f_i \in \mathcal{H}(U_i)$ nowhere vanishing on U_i such that

$$f_i/f_j = f_{ij} \quad \text{on } U_i \cap U_j \text{ for all } i,j \in I.$$

Prove that this implies the Weierstrass theorem 15.3. Hint: Set $\varphi_a(z) := (z-a)^{m_a}$ for $z \in U_a := U \setminus \{a\}$ and $a \in A$, and $f_{ab} := \varphi_b/\varphi_a$.

Exercise 29. Consider the power series

$$f(z) = \sum_{n=0}^\infty z^{2^n}, \quad |z| < 1,$$

with radius of convergence 1. Prove that the natural boundary of f is $\partial\mathbb{D}$. Hint: Let $\varphi = 2\pi\ell/2^k$, where $k, \ell \in \mathbb{N}$, and show that $|f(re^{i\varphi})| \to \infty$ as $r \to 1^-$.

Exercise 30. Prove Liouville's theorem for harmonic functions: If $u : \mathbb{C} \to \mathbb{R}$ is harmonic and bounded on $\mathbb{C}$, then u is constant.

Exercise 31. Let $a \in \mathbb{D}$. Prove that $\varphi_a(z) = (z-a)/(1-\overline{a}z)$ is holomorphic and invertible on a neighborhood of $\overline{\mathbb{D}}$ with $\varphi_a^{-1} = \varphi_{-a}$. Show that $|\varphi_a(z)| = 1$ for $z \in \partial\mathbb{D}$.

Exercise 32. Show that if $u : U \to \mathbb{R}$ is harmonic and $h : V \to U$ is holomorphic, then $u \circ h$ is harmonic.

Exercise 33. Derive a formula analogous to the Poisson integral formula (17.2) for the upper half plane $\mathbb{H}$, by mapping $\mathbb{H}$ biholomorphically to $\mathbb{D}$: if u is harmonic on $\mathbb{H}$, and continuous and bounded on $\overline{\mathbb{H}}$, then

$$u(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} u(t)\frac{y}{(x-t)^2+y^2}\,dt, \quad z = x+iy \in \mathbb{H}.$$

Exercise 34. Prove **Jensen's formula**: Let f be holomorphic in a neighborhood of $\overline{D}_r(0)$ with $f(0) \neq 0$. Assume that f does not vanish on $\partial D_r(0)$ and let $a_1, \ldots, a_k$ be the zeros of f in $D_r(0)$ counted according to their multiplicities. Then

$$\log|f(0)| + \sum_{j=1}^{k} \log\frac{r}{|a_j|} = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{it})|\,dt. \tag{17.5}$$

Hint: Use Exercise 31 to conclude that

$$g(z) = \frac{f(z)}{\prod_{j=1}^{k}\varphi_{a_j/r}(z/r)},$$

where $\varphi_{a_j/r}$ is defined by (17.1), is holomorphic in a neighborhood of $\overline{D}_r(0)$ and has no zeros in $\overline{D}_r(0)$. Apply the mean value property to $\log|g|$ which is harmonic in a neighborhood of $\overline{D}_r(0)$.

Exercise 35. Let f be continuous on $\overline{\mathbb{D}}$ and holomorphic in $\mathbb{D}$. Assume that f is nowhere zero on $\overline{\mathbb{D}}$ and $|f(z)| = 1$ on $\partial\mathbb{D}$. Prove that the function

$$F(z) := \begin{cases} f(z) & \text{if } |z| \leq 1, \\ 1/\overline{f(1/\overline{z})} & \text{if } |z| > 1, \end{cases}$$

is entire, and conclude that f must be constant. Hint: Show first that F is continuous, then use Morera's theorem.

Exercise 36. Let $\mathcal{F}$ be the family of all $f \in \mathcal{H}(\mathbb{D})$ such that $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ with $|a_n| \leq n$ for all n. Show that $\mathcal{F}$ is a normal family.

Exercise 37. Let $U \subseteq \mathbb{C}$ be a region such that $\mathbb{C} \setminus U$ has interior points. Let $z_0 \in U$. Prove that $\mathcal{F} = \{f \in \mathcal{H}(\mathbb{D}) : f(\mathbb{D}) \subseteq U \text{ and } f(0) = z_0\}$ is compact in $\mathcal{H}(\mathbb{D})$. Hint: If $a \in \mathbb{C} \setminus \overline{U}$, then $z \mapsto 1/(z-a)$ maps U biholomorphically on a subset of a disk with finite radius.

Exercise 38. Consider the family $\mathscr{S} = \{f \in \mathcal{H}(\mathbb{D}) : f \text{ injective}, f(0) = 0, f'(0) = 1\}$ of **schlicht functions**.

(1) Let $f \in \mathscr{S}$. Let r be the maximal radius such that $D_r(0) \subseteq f(\mathbb{D})$. Prove that $r \leq 1$.
(2) Choose $a \in \partial D_r(0)$ with $a \notin f(\mathbb{D})$ and set $g := f/a$. Then $\mathbb{D} \subseteq g(\mathbb{D})$ and $1 \notin g(\mathbb{D})$. Conclude that there is a holomorphic function $\varphi : g(\mathbb{D}) \to \mathbb{C}^*$ such that $\varphi(z)^2 = z - 1$ for all $z \in g(\mathbb{D})$.
(3) Set $h := \varphi \circ g$. Show that $w \in h(\mathbb{D})$ implies $-w \notin h(\mathbb{D})$.
(4) Let (f_n) be a sequence of functions in $\mathscr{S}$, and let a_n, g_n, h_n be as defined in (1), (2), (3) relative to f_n. Use Exercise 37 to conclude that (h_n) and (f_n) have convergent subsequences.

(5) Conclude that $\mathscr{S}$ is compact in $\mathcal{H}(\mathbb{D})$. Hint: To see that the limit function is injective use the argument principle 8.2.

Exercise 39. Prove the **Schwarz lemma**: Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic with $f(0) = 0$. Then $|f(z)| \le |z|$ for $z \in \mathbb{D}$ and $|f'(0)| \le 1$. If for some $c \in \mathbb{D}^*$ we have either $|f(c)| = |c|$ or $|f'(0)| = 1$, then f is a rotation, i.e., $f(z) = az$ for some a with $|a| = 1$. Hint: Use the maximum principle for the holomorphic function $z \mapsto f(z)/z$.

Exercise 40. Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic. Show that, if f has two fixed points, then $f(z) = z$ for all $z \in \mathbb{D}$. Give an example of a holomorphic function $f : \mathbb{D} \to \mathbb{D}$ without fixed point.

Exercise 41. The **pseudo-hyperbolic distance** between two points $z, w \in \mathbb{D}$ is defined by

$$\rho(z, w) := \Big|\frac{z - w}{1 - \overline{w}z}\Big|.$$

Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic. Show that

$$\rho(f(z), f(w)) \le \rho(z, w), \quad z, w \in \mathbb{D},$$

and that equality holds if $f \in \operatorname{Aut}(\mathbb{D})$. Hint: Use the Schwarz lemma (Exercise 39).

Exercise 42. Prove the **Schwarz–Pick lemma**: Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic. Then

$$\frac{|f'(z)|}{1 - |f(z)|^2} \le \frac{1}{1 - |z|^2}, \quad z \in \mathbb{D}.$$

Hint: Use Exercise 41.

Exercise 43. For $w \in \mathbb{C}$ and $z \in \mathbb{D}$ we define the **hyperbolic length** of w at z by

$$\|w\|_z := \frac{|w|}{1 - |z|^2}.$$

The **hyperbolic distance** of two points $z_1, z_2 \in \mathbb{D}$ is defined by

$$d(z_1, z_2) := \inf\Big\{ \int_0^1 \|\gamma'(t)\|_{\gamma(t)}\, dt : \gamma \in C^1([0, 1], \mathbb{D}), \gamma(0) = z_1, \gamma(1) = z_2 \Big\}.$$

Use the Schwarz–Pick lemma to prove that, for holomorphic $f : \mathbb{D} \to \mathbb{D}$,

$$d(f(z_1), f(z_2)) \le d(z_1, z_2), \quad z_1, z_2 \in \mathbb{D}.$$

Show that equality holds if $f \in \operatorname{Aut}(\mathbb{D})$.

Exercise 44. Show that the hyperbolic distance of 0 and $s \in (0, 1)$ is given by

$$d(0, s) = \frac{1}{2} \log \frac{1 + s}{1 - s}.$$

Derive a formula for the hyperbolic distance of two arbitrary points $z_1, z_2 \in \mathbb{D}$. Hint: Find an automorphism φ of $\mathbb{D}$ such that $\varphi(z_1) = 0$ and $\varphi(z_2) \in (0, 1)$.

Exercise 45. Let $U \subseteq \mathbb{C}$ be a bounded simply connected region with real analytic boundary, i.e., the boundary is locally the graph of a function given by a convergent power series. Let $f : \mathbb{D} \to U$ be biholomorphic. Prove that f has a holomorphic extension to some neighborhood of $\mathbb{D}$. Hint: The problem is purely local. Use a change of variables to reduce to the case that both boundaries are flat and apply the Schwarz reflection principle.

Exercise 46. Deduce from the little Picard theorem 26.3 that every periodic entire function has a fixed point.

Exercise 47. Let f and g be entire functions satisfying $e^f + e^g = 1$. Prove that f and g are both constant.

Exercise 48. Let u be usc and $u \geq 0$. Show that $v(x) := \log u(x)$ if $u(x) \neq 0$ and $v(x) := -\infty$ if $u(x) = 0$ is usc.

Exercise 49. Let u be a subharmonic function on $D_R(0)$ such that $u(z) = u(|z|)$ for $z \in D_R(0)$. Prove that $r \mapsto u(r)$, $r \in (0, R)$, is a convex function of $\log r$: if $\ell(r) := a \log r + b$, $r \in (0, R)$, and $r_1, r_2 \in (0, R)$ are such that $u(r_1) \leq \ell(r_1)$ and $u(r_2) \leq \ell(r_2)$, then $u(r) \leq \ell(r)$ for all $r \in (r_1, r_2)$. Hint: $\ell(z) := \ell(|z|)$ is harmonic on $D_R(0) \setminus \{0\}$.

Exercise 50. Let $u : U \to \mathbb{R}$ be harmonic and let $\varphi : \mathbb{R} \to \mathbb{R}$ be convex (not necessarily nondecreasing). Show that $\varphi \circ u$ is subharmonic. Give an example of a subharmonic u and a convex φ such that $\varphi \circ u$ is not subharmonic.

Exercise 51. Let f be holomorphic on some domain $U \subseteq \mathbb{C}$. Use Exercise 34 to show that $u = \log |f|$ is subharmonic on U.

Exercise 52. Let $\{u_i\}_{i \in I}$ be an arbitrary family of subharmonic functions on U. Suppose that $u(z) := \sup_{i \in I} u_i(z)$, $z \in U$, is usc and $u(z) < \infty$ for all $z \in U$. Prove that u is subharmonic.

Exercise 53. Deduce **Hadamard's three circles theorem**: Let f be holomorphic on $D_R(0)$. Let $0 < r_1 < r_2 < R$ and $M_i := \sup_{|z|=r_i} |f(z)|$, $i = 1, 2$. Then, if $r \in (r_1, r_2)$,

$$\sup_{|z|=r} |f(z)| \leq M_1^{\lambda(r)} M_2^{1-\lambda(r)},$$

where

$$\lambda(r) = \frac{\log r_2 - \log r}{\log r_2 - \log r_1}.$$

Hint: Apply Exercise 49 to $u(z) = \sup_{t \in \mathbb{R}} \log |f(ze^{it})|$.

Exercise 54. Let U, V be regions in $\mathbb{C}$ and let $f : V \to U$ be a non-constant holomorphic mapping. Show that, if u is subharmonic on U, then $u \circ f$ is subharmonic on V. Hint: Use approximation by smooth functions 28.11 and the characterization of C^2 subharmonic functions 28.13.

Exercise 55. Solve the Dirichlet problem on the strip $S = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$ for the boundary function f which is 0 on $\{z : \operatorname{Re} z = 0\}$ and 1 on $\{z : \operatorname{Re} z = 1\}$. Hint: Check that $z \mapsto \exp(i\pi z)$ is a biholomorphism between S and $\mathbb{H}$ which extends continuously to $\overline{S}$. Use Exercise 33.

Exercise 56. Prove that the Laplace operator in polar coordinates is given by the formula (29.1).

Exercise 57. Prove the items (4), (5), and (6) of Theorem 30.1.

Exercise 58. The complex projective line is the quotient space $\mathbb{P}^1 := \mathbb{C}^2 \setminus \{0\}/\sim$, where $(z_1, z_2) \sim (w_1, w_2)$ if and only if there exists $\lambda \in \mathbb{C}^*$ such that $(z_1, z_2) = \lambda(w_1, w_2)$. It is endowed with the quotient topology, i.e., the largest topology for which the quotient projection $\mathbb{C}^2 \setminus \{0\} \to \mathbb{P}^1$ is continuous. The equivalence class of (z_1, z_2) is denoted by $[z_1, z_2]$. Show that $\mathbb{P}^1$ is a complex manifold which is biholomorphic to the Riemann sphere. Hint: Show that $\varphi_1 : \mathbb{P}^1 \setminus \{[1, 0]\} \to \mathbb{C}, [z, 1] \mapsto z$, and $\varphi_2 : \mathbb{P}^1 \setminus \{[0, 1]\} \to \mathbb{C}, [1, z] \mapsto z$, define two charts which cover $\mathbb{P}^1$. Compute the transition map $\varphi_1 \circ \varphi_2^{-1}$.

Exercise 59. Prove: $f(z) = (az + b)/(cz + d) \in \mathrm{Aut}(\mathbb{H})$ is fixed point free in $\mathbb{H}$ if and only if $|a + d| \geq 2$.

Exercise 60. Prove that a proper mapping $f : X \to Y$ between manifolds is closed.

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