Complex Analysis

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Preface

These are lecture notes for the course *Komplexe Analysis* held in Vienna in Spring 2015 (two semester hours). The presentation is based loosely on [**5**]. Other sources are [**1**], [**2**], [**4**], [**6**], and [**7**].

Most of the illustrations were made available by courtesy of Andreas Kriegl.

Contents

Preliminaries

1. Complex numbers

A complex number has the form $z = x+iy$, where $x, y \in \mathbb{R}$ and i is an imaginary number that satisfies $i^2 = -1$. We call $x = \operatorname{Re} z$ the **real part** and $y = \operatorname{Im} z$ the **imaginary part** of z. The complex numbers with zero imaginary part are precisely the real numbers, those with zero real part are called **purely imaginary**. The set of complex numbers is denoted by $\mathbb{C}$,

$$\boxed{\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}, \quad i^2 = -1.} \tag{1.1}$$

Assuming that the usual rules of arithmetic apply to complex numbers we observe that addition and multiplication preserve $\mathbb{C}$,

$$(x + iy) + (u + iv) = (x + u) + i(y + v),$$
$$(x + iy) \cdot (u + iv) = (xu - yv) + i(xv + yu).$$

If $x + iy = u + iv$ then $(x - u)^2 = -(y - v)^2$ and thus $x = u$ and $y = v$.

In the previous paragraph we have tacitly made the assumption that there is a field extension of $\mathbb{R}$ in which the equation $z^2 + 1 = 0$ is solvable. Let us now show that this assumption is justified.

1.1. Theorem. *The set $\mathbb{R}^2$ with the addition*

$$(x, y) + (u, v) = (x + u, y + v)$$

and the multiplication

$$(x, y) \cdot (u, v) = (xu - yv, xv + yu)$$

forms a field in which the equation $z^2 + 1 = 0$ has two solutions.

Proof. It is easy to check that $(\mathbb{R}^2, +, \cdot)$ is a field. The subfield $\{(x, 0) : x \in \mathbb{R}\}$ is isomorphic to $\mathbb{R}$. The element $i := (0, 1)$ satisfies $i^2 = (0, 1)^2 = (-1, 0) = -1$. Hence $\pm i$ are the solutions of the equation $z^2 + 1 = (z + i)(z - i) = 0$. $\square$

The field $(\mathbb{R}^2, +, \cdot)$ of Theorem 1.1 is by definition the field $\mathbb{C}$ of complex numbers. We arrive at the representation (1.1) if we set $i := (0, 1)$ and denote elements $(x, 0)$ simply by x,

$$(x, y) = (x, 0) + (0, 1) \cdot (y, 0) = x + iy.$$

The complex numbers can be visualized as the usual Euclidean plane by identifying $z = x + iy \in \mathbb{C}$ with the point $(x, y) \in \mathbb{R}^2$. Then the x-axis and the y-axis are called real and imaginary axis, respectively.

Addition of complex numbers corresponds to addition of the corresponding vectors in $\mathbb{R}^2$. We shall see below that multiplication corresponds to a rotation composed with a dilation.

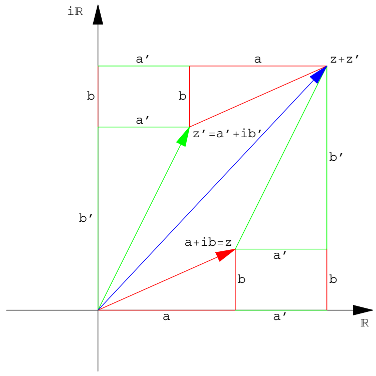

FIGURE 1. Addition of complex numbers.

The **complex conjugate** of $z = x + iy \in \mathbb{C}$ is defined by

$$\boxed{\overline{z} := x - iy.}$$

It is obtained by a reflection across the real axis. We have

$$\overline{z+w} = \overline{z} + \overline{w}, \quad \overline{zw} = \overline{z} \cdot \overline{w}, \quad \overline{\overline{z}} = z, \quad \operatorname{Re} z = \frac{z + \overline{z}}{2}, \quad \operatorname{Im} z = \frac{z - \overline{z}}{2i},$$

and $z \in \mathbb{R}$ if and only if $z = \overline{z}$. Thus conjugation $z \mapsto \overline{z}$ is a field automorphism $\mathbb{C} \to \mathbb{C}$ that is an involution and that fixes each point in $\mathbb{R}$.

The **absolute value** $|z|$ of a complex number $z = x + iy$ is defined as the distance of $(x, y) \in \mathbb{R}^2$ to the origin,

$$\boxed{|z| := \sqrt{x^2 + y^2}.}$$

Then it is easily seen that

$$|\overline{z}| = |z|, \quad |z|^2 = z\overline{z}, \quad |\operatorname{Re} z| \le |z|, \quad |\operatorname{Im} z| \le |z|,$$

and that $|\cdot| : \mathbb{C} \to \mathbb{R}$ is a **valuation** on $\mathbb{C}$, that is

- $|z| \ge 0$, and $|z| = 0$ if and only if $z = 0$,
- $|zw| = |z||w|$,
- $|z + w| \le |z| + |w|$.

So $\mathbb{C}$ (and $\mathbb{R}$) together with the absolute value $|\cdot|$ is a valued field.

1.2. Proposition. *The field $\mathbb{C}$ is isomorphic to*

(1) *the field of all matrices* $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$, $x, y \in \mathbb{R}$, *with matrix addition and multiplication,*
(2) *the quotient field* $\mathbb{R}[X]/(X^2 + 1)$.

Proof. In the first case the isomorphism is given by $x + iy \mapsto \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$.

The set $\mathbb{R}[X]$ of polynomials in the variable X with real coefficients is an integral domain (since $\mathbb{R}$ is a field).[1] The quotient ring $\mathbb{R}[X]/(X^2 + 1)$ is a field, since the polynomial $X^2 + 1$ is irreducible over $\mathbb{R}$. The field isomorphism is given by $a + ib \mapsto a + bX$. □

[1]An integral domain is a non-zero commutative ring in which the product of any two non-zero elements is non-zero.

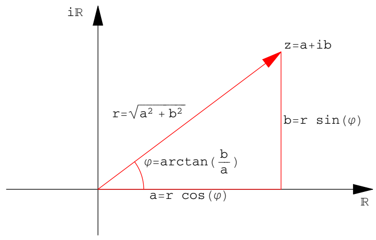

FIGURE 2. Polar coordinates.

We may infer that multiplication corresponds to a rotation composed with a dilation. In fact, if we represent $(a, b), (x, y) \in \mathbb{R}^2$ in polar coordinates

$$\begin{aligned} a = r\cos\varphi,\ b = r\sin\varphi, &\quad r > 0,\ \varphi \in \mathbb{R}, \\ x = s\cos\psi,\ y = s\sin\psi, &\quad s > 0,\ \psi \in \mathbb{R}, \end{aligned}$$

then the multiplication of $z = a + ib$ with $w = x + iy$ amounts to

$$\begin{aligned} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x & -y \\ y & x \end{pmatrix} &= rs \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix} \\ &= rs \begin{pmatrix} \cos(\varphi+\psi) & -\sin(\varphi+\psi) \\ \sin(\varphi+\psi) & \cos(\varphi+\psi) \end{pmatrix}. \end{aligned}$$

The *angles* φ and ψ which are determined uniquely up to addition of terms of the form $2\pi k$, $k \in \mathbb{Z}$, are called the **arguments** $\varphi = \arg z$ and $\psi = \arg w$ of z and w. Thus

$$\arg(zw) = \arg(z) + \arg(w).$$

Multiplying two complex numbers hence means adding the arguments and multiplying the absolute values.

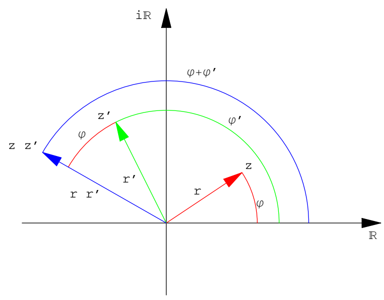

FIGURE 3. Multiplication of complex numbers.

We denote by

$$\boxed{\langle z, w\rangle := \mathrm{Re}(z\overline{w}) = xu + yv, \quad z = x + iy,\ w = u + iv,}$$

the **inner product** in the real vector space $\mathbb{C} \cong \mathbb{R}^2$ with respect to the basis $\{1, i\}$. We have, for all $z, w, a \in \mathbb{C}$,

$$\langle z, z\rangle = |z|^2, \quad \langle az, aw\rangle = |a|^2\langle z, w\rangle, \quad \langle z, w\rangle = \langle \overline{z}, \overline{w}\rangle.$$

It is easy to check that

$$\langle z, w\rangle^2 + \langle iz, w\rangle^2 = |z|^2|w|^2;$$

in particular, we get the **Cauchy–Schwarz inequality**

$$\boxed{|\langle z,w\rangle| \le |z||w|, \quad z,w\in\mathbb{C}.}$$

Moreover, we have

$$|z+w|^2 = |z|^2 + |w|^2 + 2\langle z,w\rangle.$$

Two vectors z and w are said to be **orthogonal** if $\langle z,w\rangle = 0$. Note that $\langle z, cz\rangle = \operatorname{Re}(cz\overline{z}) = |z|^2 \operatorname{Re} c$ and thus z and cz are orthogonal if and only if $c \in i\mathbb{R}$.

By the Cauchy–Schwarz inequality, $-1 \le \frac{\langle z,w\rangle}{|z||w|} \le 1$ if $z,w\in\mathbb{C}\setminus\{0\}$. There exists a unique real number $\varphi\in[0,\pi]$ such that

$$\cos\varphi = \frac{\langle z,w\rangle}{|z||w|}.$$

φ is called the **angle** between z and w; we write $\sphericalangle(z,w)=\varphi$.

2. Topological prerequisites

2.1. The metric space $\mathbb{C}$. The mapping $d:\mathbb{C}\times\mathbb{C}\to\mathbb{R}$, $d(z,w)=|z-w|$, is a **metric** on $\mathbb{C}$, that is

- $d(z,w)\ge 0$, and $d(z,w)=0$ if and only if $z=w$,
- $d(z,w)=d(w,z)$,
- $d(z,w)\le d(z,v)+d(v,w)$.

It is called the Euclidean metric on $\mathbb{C}$. The open balls in the metric space $(\mathbb{C},d)$ are the **open disks**

$$\boxed{D_r(a) := \{z\in\mathbb{C} : |z-a|<r\}, \quad r>0,\ a\in\mathbb{C},}$$

with center a and radius r. The **unit disk** will be denoted by $\mathbb{D} := D_1(0)$. A subset $U\subseteq\mathbb{C}$ is **open** (in $\mathbb{C}$) if for each $a\in U$ there exists $r>0$ such that $D_r(a)\subseteq U$. The empty set $\emptyset$ and $\mathbb{C}$ are open. Arbitrary unions and finite intersections of open sets are open. The **interior** $\mathring{S} := \bigcup\{U : U\subseteq S,\ U \text{ open}\}$ of an arbitrary set $S\subseteq\mathbb{C}$ is open. A subset $A\subseteq\mathbb{C}$ is **closed** (in $\mathbb{C}$) if its complement $\mathbb{C}\setminus A$ is open. The **closed disks**

$$\boxed{\overline{D}_r(a) := \{z\in\mathbb{C} : |z-a|\le r\}, \quad r>0,\ a\in\mathbb{C},}$$

are closed. Finite unions and arbitrary intersections of closed sets are closed. The **closure** $\overline{S} := \bigcap\{A : S\subseteq A,\ A \text{ closed}\}$ of an arbitrary set $S\subseteq\mathbb{C}$ is closed.

A set W is a **neighborhood** of a set $S\subseteq\mathbb{C}$ if there is an open subset $U\subseteq\mathbb{C}$ such that $S\subseteq U\subseteq W$. The **Hausdorff separation property** (which holds in every metric space) states that any two distinct points in $\mathbb{C}$ have disjoint neighborhoods, in particular, if $z\ne w\in\mathbb{C}$ then $D_r(z)\cap D_r(w)=\emptyset$ for $r:=|z-w|/2$.

A sequence $(z_n)_n$ in $\mathbb{C}$ is a **convergent** if there exists $z\in\mathbb{C}$ such that each neighborhood of z contains almost all (i.e. all except finitely many) z_n; then $z=\lim_{n\to\infty} z_n$ is called the limit of the sequence; we will also write $z_n\to z$. Equivalently,

$$z=\lim_{n\to\infty} z_n \ \Leftrightarrow\ \forall\epsilon>0\ \exists n_0\in\mathbb{N}\ \forall n\ge n_0 : |z_n-z|<\epsilon.$$

Non-convergent sequences are called **divergent**. For instance, $\lim_{n\to\infty} z^n = 0$ for every $z\in\mathbb{D}$, while the sequence z^n is divergent if $|z|>1$.

The Hausdorff separation property implies that the limit of a convergent sequence is unique. Note that a set $S\subseteq\mathbb{C}$ is closed if and only if the limit of each convergent sequence (z_n) with $z_n\in S$ belongs to S. A set $S\subseteq\mathbb{C}$ is said to be

dense in $\mathbb{C}$ if $\overline{S} = \mathbb{C}$, or equivalently, if for each $z \in \mathbb{C}$ there is a sequence $z_n \in S$ such that $z_n \to z$.

A point $z \in \mathbb{C}$ is called **accumulation point** of a set $S \subseteq \mathbb{C}$ if for each neighborhood U of z we have $U \cap (S \setminus \{z\}) \neq \emptyset$. If z is an accumulation point of S then there exists a sequence $z_n \in S \setminus \{z\}$ so that $\lim z_n = z$.

2.2. Convergent sequences and series of complex numbers. Every convergent sequence of complex numbers is **bounded**, i.e., there exists $M > 0$ so that $|z_n| < M$ for all n. If z_n and w_n are convergent sequences we have:

(1) For all $a, b \in \mathbb{C}$ the sequence $az_n + bw_n$ is convergent and
$$\lim(az_n + bw_n) = a \lim z_n + b \lim w_n.$$
(2) The sequence $z_n w_n$ is convergent and
$$\lim(z_n w_n) = \lim z_n \lim w_n.$$
(3) If $\lim w_n \neq 0$, then z_n / w_n (for sufficiently large n) is convergent and
$$\lim(z_n/w_n) = \lim z_n / \lim w_n.$$
(4) The sequence $|z_n|$ is convergent and $\lim |z_n| = |\lim z_n|$.
(5) The sequence $\overline{z_n}$ is convergent and $\lim \overline{z_n} = \overline{\lim z_n}$.

These rules can be proved in the same way as for real sequences (since the absolute value has the same properties on $\mathbb{C}$ and on $\mathbb{R}$). Moreover:

(6) A complex sequence z_n is convergent if and only if the real sequences $\operatorname{Re} z_n$ and $\operatorname{Im} z_n$ are convergent. In that case $\lim z_n = \lim \operatorname{Re} z_n + i \lim \operatorname{Im} z_n$.

This follows from (1) and (5), since $\operatorname{Re} z_n = \frac{1}{2}(z_n + \overline{z_n})$ and $\operatorname{Im} z_n = \frac{1}{2i}(z_n - \overline{z_n})$.

The field $\mathbb{C}$ is **complete**. That means that every Cauchy sequence z_n is convergent. A **Cauchy sequence** in $\mathbb{C}$ is a sequence z_n satisfying
$$\forall \epsilon > 0 \ \exists k \in \mathbb{N} \ \forall m, n \geq k : |z_m - z_n| < \epsilon.$$
Every convergent sequence is a Cauchy sequence (thanks to $|z_n - z_m| \leq |z_n - z| + |z - z_m|$ where $z = \lim z_n$). Conversely, completeness of $\mathbb{R}$ implies that every Cauchy sequence is convergent: the inequalities
$$|\operatorname{Re} z_m - \operatorname{Re} z_n| \leq |z_m - z_n|, \quad |\operatorname{Im} z_m - \operatorname{Im} z_n| \leq |z_m - z_n|$$
imply that $\operatorname{Re} z_n$ and $\operatorname{Im} z_n$ are Cauchy sequences if z_n is a Cauchy sequence. Since $\mathbb{R}$ is complete, they converge to real numbers a, b. By (1), the sequence z_n converges to $a + ib$ in $\mathbb{C}$.

Given a sequence $(a_k)_{k \geq j}$ of complex numbers, we call the sequence $(s_n)_{n \geq j}$ of **partial sums** $s_n := \sum_{k=j}^{n} a_k$ an (infinite) **series**, and write $\sum_{k=j}^{\infty} a_k$, $\sum_{k \geq j} a_k$ or just $\sum a_k$. A series $\sum a_k$ is **convergent** if the sequence of partial sums (s_n) converges, and we write $\sum a_k = \lim s_n$, otherwise it is called **divergent**. A series $\sum a_k$ is convergent if and only if
$$\forall \epsilon > 0 \ \exists \ell \in \mathbb{N} \ \forall m, n \geq \ell : \Big| \sum_{k=m+1}^{n} a_k \Big| < \epsilon; \tag{2.1}$$
this precisely means that s_n is a Cauchy sequence since $s_n - s_m = \sum_{k=m+1}^{n} a_k$.

The basic example is the **geometric series** $\sum_{k \geq 0} z^k$ with partial sums
$$\boxed{\sum_{k=0}^{n} z^k = \frac{1 - z^{n+1}}{1 - z}, \qquad \text{for } z \neq 1.}$$

Since $\lim z^{n+1} = 0$ if $|z| < 1$ we obtain

$$\boxed{\sum_{k=0}^{\infty} z^n = \frac{1}{1-z}, \qquad \text{for } |z| < 1.}$$

The terms a_k of every convergent series $\sum a_k$ must converge to 0, because $a_k = s_k - s_{k-1}$. The rules (1) and (5) imply: if $\sum_{k\geq j} a_k$ and $\sum_{k\geq j} b_k$ are convergent series and $a, b \in \mathbb{C}$ then $\sum_{k\geq j}(aa_k + bb_k)$ is convergent with

$$\sum_{k\geq j}(aa_k + bb_k) = a\sum_{k\geq j} a_k + b\sum_{k\geq j} b_k,$$

and $\sum_{k\geq j} \overline{a}_k$ is convergent with

$$\overline{\sum_{k\geq j} a_k} = \sum_{k\geq j} \overline{a}_k.$$

Consequently, $\sum a_k$ is convergent if and only if $\sum \operatorname{Re} a_k$ and $\sum \operatorname{Im} a_k$ are convergent; in that case

$$\sum_{k\geq j} a_k = \sum_{k\geq j} \operatorname{Re} a_k + i\sum_{k\geq j} \operatorname{Im} a_k.$$

A series $\sum a_k$ is called **absolutely convergent** if the series $\sum |a_k|$ is convergent. An absolutely convergent series $\sum a_k$ is convergent and satisfies $|\sum a_k| \leq \sum |a_k|$; this is a consequence of (2.1) and $|\sum_{k=m+1}^{n} a_k| \leq \sum_{k=m+1}^{n} |a_k|$.

Another simple consequence of (2.1) is the **majorant criterion** which states that $\sum a_k$ is absolutely convergent if $|a_k| \leq b_k$ (hence $b_k \in \mathbb{R}$ and $b_k \geq 0$) and $\sum b_k$ is convergent. It implies, in view of $\max\{|\operatorname{Re} a|, |\operatorname{Im} a|\} \leq |a| \leq |\operatorname{Re} a| + |\operatorname{Im} a|$, that a series $\sum a_k$ is absolutely convergent if and only if $\sum \operatorname{Re} a_k$ and $\sum \operatorname{Im} a_k$ are absolutely convergent.

The terms of absolutely convergent series can be arbitrarily reordered: a series $\sum_{k\geq 0} a_k$ is absolutely convergent if and only if it is **unconditionally convergent**, i.e., $\sum_{k\geq 0} a_{\sigma(k)} = \sum_{k\geq 0} a_k$ for every bijection $\sigma : \mathbb{N} \to \mathbb{N}$. For a proof we refer to [**3**, Satz 32.3].[2]

Given two series $\sum_{k\geq 0} a_k$ and $\sum_{k\geq 0} b_k$, we call every series $\sum_{k\geq 0} c_k$ such that each product $a_k b_\ell$ appears exactly once in the sequence $c_0, c_1, \ldots$ a **product series** of $\sum_{k\geq 0} a_k$ and $\sum_{k\geq 0} b_k$. The most important product series is the **Cauchy product** $\sum_{k\geq 0}\sum_{i+j=k} a_i b_j$. If $\sum_{k\geq 0} a_k$ and $\sum_{k\geq 0} b_k$ are absolutely convergent series, then every product series $\sum_{k\geq 0} c_k$ is absolutely convergent and satisfies

$$\Big(\sum_{k\geq 0} a_k\Big)\Big(\sum_{\ell\geq 0} b_\ell\Big) = \sum_{m\geq 0} c_m.$$

Let us prove this. For each ℓ there exists m such that $c_0, c_1, \ldots, c_\ell$ appear among the products $a_i b_j$, $0 \leq i, j \leq m$. Thus

$$\sum_{k=0}^{\ell} |c_k| \leq \Big(\sum_{i=0}^{m} |a_i|\Big)\Big(\sum_{j=0}^{m} |b_j|\Big) \leq \Big(\sum_{i=0}^{\infty} |a_i|\Big)\Big(\sum_{j=0}^{\infty} |b_j|\Big) < \infty.$$

Hence $\sum_{k=0}^{\infty} c_k$ is absolutely convergent, and therefore unconditionally convergent, whence

$$\sum_{k=0}^{\infty} c_k = \lim_{n\to\infty} \Big(\sum_{i=0}^{n} a_i\Big)\Big(\sum_{j=0}^{n} b_j\Big) = \Big(\sum_{i=0}^{\infty} a_i\Big)\Big(\sum_{j=0}^{\infty} b_j\Big).$$

[2]By reordering a real convergent, but not absolutely convergent, series one can obtain any real number. This is no longer true for complex series; see [**5**, 0.4.5].

2.3. Compact sets. A set $K \subseteq \mathbb{C}$ is **compact** if any of the following equivalent conditions holds:

- K is closed and bounded.
- Every sequence in K has a subsequence that converges to a point in K.
- Every open covering of K has a finite subcovering.

Every open set $U \subseteq \mathbb{C}$ is a countable union of compact subsets of U.

If $K_1 \supseteq K_2 \supseteq \cdots$ is a nested sequence of non-empty compact sets in $\mathbb{C}$ such that $\operatorname{diam} K_n := \sup_{z,w\in K_n} |z - w| \to 0$ as $n \to \infty$, then there is a unique point $c \in \mathbb{C}$ such that $c \in K_n$ for all n, i.e., $\bigcap_n K_n = \{c\}$. To see this choose a point z_n in K_n for all n. Then (z_n) forms a Cauchy sequence, since $\operatorname{diam} K_n \to 0$, and thus has a limit $c \in \mathbb{C}$. By compactness, c lies in each K_n. If there is a further point $c' \neq c$ with this property, then $0 < |c - c'| \leq \operatorname{diam} K_n$, a contradiction.

2.4. Continuous functions. Let X and Y be metric spaces. A mapping $f : X \to Y$ is **continuous** at a point $a \in X$ if the preimage $f^{-1}(V)$ of each neighborhood V of $f(a)$ in Y is a neighborhood of a in X. Equivalently,

$$\forall \epsilon > 0\ \exists \delta > 0\ \forall x \in X, d_X(x, a) < \delta : d_Y(f(x), f(a)) < \epsilon. \tag{2.2}$$

Moreover, $f : X \to Y$ is continuous at $a \in X$ if and only if for each sequence $x_n \to a$ in X we have $f(x_n) \to f(a)$ in Y. A mapping $f : X \to Y$ is called **continuous** if it is continuous at every point $x \in X$. Then f is continuous if and only if preimages of open sets are open, or equivalently, if preimages of closed sets are closed. The composite of continuous mappings is continuous. The image $f(K)$ of a compact set K under a continuous mapping f is compact. In particular, real valued continuous functions attain its maximum and minimum on every compact set. Every continuous mapping on a compact set is **uniformly continuous**, i.e., δ in (2.2) is independent of a.

Complex valued functions $f, g : X \to \mathbb{C}$ can be added and multiplied

$$(f + g)(x) := f(x) + g(x), \quad (fg)(x) := f(x)g(x).$$

Likewise we define

$$\overline{f}(x) := \overline{f(x)}, \quad (\operatorname{Re} f)(x) := \operatorname{Re}(f(x)), \quad (\operatorname{Im} f)(x) := \operatorname{Im}(f(x)).$$

For the real and imaginary part of f we shall consistently write

$$u := \operatorname{Re} f, \quad v := \operatorname{Im} f.$$

If f and g are continuous in $a \in X$ then so are $f + g$, fg, and $\overline{f}$. In particular, $f = u + iv$ is continuous in a if and only if u and v are continuous in a.

Let $C(X)$ (or $C^0(X)$) denote the set of all continuous functions $f : X \to \mathbb{C}$; it is a commutative $\mathbb{C}$-algebra with identity element. Since constant functions are continuous, we have a natural inclusion $\mathbb{C} \subseteq C(X)$. Conjugation defines an $\mathbb{R}$-linear automorphism that is also an involution. The functions $g \in C(X)$ with $g(x) \neq 0$ for all $x \in X$ are the units in the ring $C(X)$; in fact, $1/g \in C(X)$. Thus, also $f/g \in C(X)$ for all $f \in C(X)$.

2.5. Connected domains. Let X be a metric space. The following conditions are equivalent:

(1) Every locally constant[3] function $f : X \to \mathbb{C}$ is constant.
(2) If $A \subseteq X$ is non-empty open and closed then $A = X$.

[3]A function f is locally constant if every point has a neighborhood U such that $f|_U$ is constant.

(3) If $X = A \cup B$, where $A \cap B = \emptyset$ and A, B are open, then either A or B is the empty set.

To see that (1) implies (2) consider the characteristic function χ_A of the set A. It is locally constant, since A and $X \setminus A$ are both open. Thus it is constant, and since $A \neq \emptyset$, $\chi_A(x) = 1$ for all $x \in X$, i.e., $A = X$. Suppose that (2) holds and let $f : X \to \mathbb{C}$ be a locally constant function. For fixed $a \in X$ the fiber $A := f^{-1}(f(a))$ is non-empty open and closed in X. Then $A = X$ and so $f(x) = f(a)$ for all $x \in X$, i.e., (1). The equivalence of (2) and (3) is evident.

A metric space X satisfying the equivalent conditions (1)–(3) is called **connected**. A continuous mapping $f : X \to Y$ defined on a connected space X has a connected image $f(X)$. The connected subsets of $\mathbb{R}$ are precisely the intervals.

A continuous mapping $\gamma : \mathbb{R} \supseteq [a, b] \to X$ is called a **path** (or a **curve**) in X. The path γ is said to be **closed** if its endpoints coincide, $\gamma(a) = \gamma(b)$. We say that γ is a **simple** path it has no self-intersections, i.e., $\gamma(t) \neq \gamma(s)$ unless $t = s$, except at the endpoints if γ is closed. We shall denote by $|\gamma|$ the unparameterized path $|\gamma| := \gamma([a, b]) \subseteq X$.

If $\gamma_i : [a_i, b_i] \to X$, $i = 1, 2$, are paths such that $\gamma_1(b_1) = \gamma_2(a_2)$, we define the path $\gamma_1 + \gamma_2 : [a_1, b_1 + b_2 - a_2] \to X$ by setting

$$(\gamma_1 + \gamma_2)(t) := \begin{cases} \gamma_1(t) & t \in [a_1, b_1], \\ \gamma_2(t + a_2 - b_1) & t \in [b_1, b_1 + b_2 - a_2]; \end{cases}$$

it is the concatenation of the paths γ_1 and γ_2. Analogously, one defines $\gamma_1 + \gamma_2 + \cdots + \gamma_n$. This symbolic addition of paths is associative but not commutative.

A space X is called **path-connected** if any two points $x, y \in X$ can be connected by a path, i.e., there exists a path $\gamma : [a, b] \to X$ such that $\gamma(a) = x$ and $\gamma(b) = y$. A path-connected metric space is connected.[4] For, let A be an open and closed subset of X that contains x. Let $y \in X$ and choose a path $\gamma : [a, b] \to X$ such that $\gamma(a) = x$ and $\gamma(b) = y$. Then $\gamma^{-1}(A)$ is non-empty open and closed in $[a, b]$. Since $[a, b]$ is connected, $\gamma^{-1}(A) = [a, b]$ and hence $y = \gamma(b) \in A$. It follows that $A = X$.

Let us now consider the complex plane $\mathbb{C}$. The path $\gamma : [0, 1] \to \mathbb{C}$, $\gamma(t) = (1 - t)z_0 + tz_1$ is the **line segment** from z_0 to z_1; we will denote it by $[z_0, z_1]$. A **polygon** is a finite sum $[z_0, z_1] + [z_1, z_2] + \cdots + [z_{n-1}, z_n]$.

A non-empty open subset $U \subseteq \mathbb{C}$ is called a **domain**. For any domain $U \subseteq \mathbb{C}$ the following are equivalent:

(1) U is connected.
(2) Any two points $z, w \in U$ can be joined by an polygon P in U such that each line segment in P is parallel to the axes.
(3) U is path-connected.

It remains to show that (1) implies (2). Fix $z \in U$. Let $f : U \to \mathbb{C}$ be defined as follows: $f(w) := 1$ if w can be joined to z by a polygon in U with line segments parallel to the axes, otherwise $f(w) := 0$. Any two points in a disk D can be joined in D by a polygon in D with line segments parallel to the axes. Thus $f|_D$ is either 1 or 0. That means that f is locally constant and hence constant, since U is connected. So $f = 1$ (as $f(z) = 1$) which implies the assertion.[5]

A connected domain is called a **region**.

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[4]The converse is not true: the subspace $i[-1, 1] \cup \{x + i \sin(1/x) : x > 0\}$ of $\mathbb{C}$ is connected but not path-connected.

[5]The same argument gives: every connected locally path-connected space is path-connected.

We may say that two points in a domain U are equivalent if they can be joined by a path in U. This defines an equivalence relation on U. The equivalence classes are called the **connected components** of U. Every connected component is a region. A domain has at most countably many connected components. Indeed, every domain $U \subseteq \mathbb{C}$ has a countable dense subset (e.g. $U \cap \mathbb{Q}^2$) and thus is a countable union of open disks D_i. Every disk D_i is contained in precisely one connected component of U, and each connected component of U contains at least one D_i.

The **boundary** of a domain $U \subseteq \mathbb{C}$ is the set $\partial U := \overline{U} \setminus \mathring{U} = \overline{U} \setminus U$ which is always closed in $\mathbb{C}$. For disks we have $\partial D_r(a) = \{z \in \mathbb{C} : |z-a| = r\}$. For points $a \in U$ we define the **distance to the boundary**

$$d_a(U) := \inf\{|z-a| : z \in \partial U\} > 0, \quad d_a(\mathbb{C}) := \infty.$$

The number $d_a(U)$ is the maximal radius r such that $D_r(a) \subseteq U$.

3. Review of 1-forms

Let E and F be finite dimensional vector spaces, and let $U \subseteq E$ be open. An F-valued **1-form** is a mapping $\omega : U \to L(E,F)$. A continuous 1-form ω is called **exact** if there is a C^1 mapping $f : U \to F$ such that $df = \omega$.

Let $E = \mathbb{R}^n$ and let x_i denote the coordinate projection $x_i : \mathbb{R}^n \to \mathbb{R}$, $x = (x_1, \dots, x_n) \mapsto x_i$. The differential $dx_i : \mathbb{R}^n \to L(\mathbb{R}^n, \mathbb{R}) = (\mathbb{R}^n)^*$ is the constant 1-form $v = (v_1, \dots, v_n) \mapsto v_i$. An arbitrary 1-form $\omega : \mathbb{R}^n \supseteq U \to L(\mathbb{R}^n, F)$ is of the form

$$\omega = \sum_{i=1}^n \omega_i \, dx_i,$$

where the mappings $\omega_i : U \to F$ are given by $\omega_i(x) = \omega(x)(e_i)$ and where $(e_i)_{i=1}^n$ denote the standard unit vectors of $\mathbb{R}^n$. Indeed,

$$\omega(x)(v) = \omega(x)\Big(\sum_{i=1}^n v_i e_i\Big) = \sum_{i=1}^n \omega(x)(e_i) v_i = \sum_{i=1}^n \omega_i(x) \, dx_i(x)(v).$$

Thus $dx_1, \dots, dx_n$ form a basis for the module of real valued 1-forms over the ring of real valued functions.

If $\omega = df$ is exact, then its components $\omega_i(x) = \omega(x)(e_i) = df(x)(e_i) = \partial_i f(x)$ are the partial derivatives of f,

$$df = \sum_{i=1}^n \partial_i f \, dx_i.$$

Since for a C^2 mapping f the partial derivatives of second order commute, $\partial_i \partial_j f = \partial_j \partial_i f$, $i \ne j$, a necessary condition for a C^1 1-form $\omega = \sum_{i=1}^n \omega_i \, dx_i$ to be exact is that the integrability conditions $\partial_i \omega_j = \partial_j \omega_i$, $i \ne j$, are satisfied. In that case ω is said to be **closed**. A closed 1-form is locally exact.

Let $\omega : E \supseteq U \to L(E,F)$ be a continuous 1-form and let $\gamma : [a,b] \to U$ be a C^1-curve in U. The **integral of ω along** γ is defined by

$$\int_\gamma \omega := \int_a^b \omega(\gamma(t))\gamma'(t) \, dt.$$

For exact 1-forms $\omega = df$ the integral computes the primitive f. In fact, if $f : U \to F$ is C^1 then by the fundamental theorem of calculus

$$\int_\gamma df = \int_a^b df(\gamma(t))\gamma'(t) \, dt = \int_a^b (f \circ \gamma)'(t) \, dt = f(\gamma(b)) - f(\gamma(a)),$$

for all C^1-curves γ.

Complex differentiation

4. Holomorphic functions

4.1. $\mathbb{R}$-linear and $\mathbb{C}$-linear mappings. Since $\mathbb{C}$ is a vector space over $\mathbb{R}$ as well as over $\mathbb{C}$, one must distinguish $\mathbb{R}$-linear from $\mathbb{C}$-linear mappings $\mathbb{C} \to \mathbb{C}$.

Lemma. *A mapping $f : \mathbb{C} \to \mathbb{C}$ is $\mathbb{R}$-linear if and only if*

$$f(z) = f(1)x + f(i)y = \frac{f(1) - if(i)}{2} z + \frac{f(1) + if(i)}{2} \overline{z}, \quad z = x + iy.$$

An $\mathbb{R}$-linear mapping $f : \mathbb{C} \to \mathbb{C}$ is $\mathbb{C}$-linear if and only if $f(i) = if(1)$; in that case $f(z) = f(1)z$.

Proof. If $f : \mathbb{C} \to \mathbb{C}$ is $\mathbb{R}$-linear then $f(z) = f(x+iy) = f(1)x + f(i)y$; the converse is obvious. The second identity follows from $x = \frac{1}{2}(z + \overline{z})$ and $y = \frac{1}{2i}(z - \overline{z})$. The rest follows easily. $\square$

For instance $z \mapsto \overline{z}$ is $\mathbb{R}$-linear, but not $\mathbb{C}$-linear.

Let us identify $\mathbb{C} \cong \mathbb{R}^2$ by $x + iy \mapsto \begin{pmatrix} x \\ y \end{pmatrix}$. Every $\mathbb{R}$-linear mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ is of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}, \quad a, b, c, d \in \mathbb{R}.$$

Via the above identification we get

$$f(1) = a + ic, \quad f(i) = b + id.$$

Then, $f(i) = if(1)$ if and only if $c = -b$ and $d = a$. Thus, cf. Proposition 1.2:

Proposition. *The real matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ induces a $\mathbb{C}$-linear mapping $f : \mathbb{C} \to \mathbb{C}$ if and only if $c = -b$ and $d = a$.*

4.2. Complex differentiable functions. Let $U \subseteq \mathbb{C}$ be a domain. A function $f : U \to \mathbb{C}$ is called **complex differentiable** (or **$\mathbb{C}$-differentiable**) at $a \in U$, if there exists a function $f_1 : U \to \mathbb{C}$ that is continuous in a and

$$f(z) = f(a) + f_1(z)(z - a) \quad \text{for all } z \in U. \tag{4.1}$$

The function f_1 is uniquely determined by f: we have $f_1(z) = \frac{f(z) - f(a)}{z - a}$ if $z \neq a$ and by continuity of f_1,

$$f_1(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$

The number $f_1(a) \in \mathbb{C}$ is the **complex derivative** of f at a; we write

$$\boxed{f'(a) = \frac{df}{dz}(a) := f_1(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.}$$

A function $f : U \to \mathbb{C}$ is called **holomorphic** in U if it is $\mathbb{C}$-differentiable at every point $a \in U$. The set of all holomorphic functions in U will be denoted by $\mathcal{H}(U)$. The following lemma implies $\mathbb{C} \subseteq \mathcal{H}(U) \subseteq C(U)$.

Lemma. *If f is $\mathbb{C}$-differentiable at a then f is continuous at a.*

Proof. This follows immediately from (4.1). □

Example.

(1) Every power function $z \mapsto z^n$ is holomorphic in $\mathbb{C}$:
$$z^n = a^n + (z-a)(z^{n-1} + az^{n-2} + \cdots + a^{n-2}z + a^{n-1}),$$
$$(z^n)' = nz^{n-1} \quad \text{for all } z \in \mathbb{C}.$$

(2) The conjugation mapping $z \mapsto \overline{z}$ is nowhere $\mathbb{C}$-differentiable:
$$\frac{\overline{z+h} - \overline{z}}{h} = \frac{\overline{h}}{h}, \quad h \neq 0,$$
is 1 for $h \in \mathbb{R}$ and -1 for $h \in i\mathbb{R}$, thus does not converge as $h \to 0$. Analogously, $\operatorname{Re} z$, $\operatorname{Im} z$, and $|z|$ are nowhere $\mathbb{C}$-differentiable.

(3) The inversion $z \mapsto 1/z$ is holomorphic in $\mathbb{C} \setminus \{0\}$:
$$\frac{1}{z} = \frac{1}{a} + (z-a)\Big(-\frac{1}{za}\Big),$$
$$\Big(\frac{1}{z}\Big)' = -\frac{1}{z^2} \quad \text{for all } z \in \mathbb{C} \setminus \{0\}.$$

4.3. Complex and real differentiability. A mapping $f : \mathbb{R}^m \supseteq U \to \mathbb{R}^n$ is **real differentiable** (or **$\mathbb{R}$-differentiable**) at $a \in U$, if there is an $\mathbb{R}$-linear mapping $T : \mathbb{R}^m \to \mathbb{R}^n$ such that
$$\lim_{h \to 0} \frac{|f(a+h) - f(a) - T(h)|}{|h|} = 0.$$
Then T is unique and is called the **differential** $df(a)$. Clearly, $\mathbb{R}$-differentiability at a implies continuity at a. If we introduce coordinates and write $f = (f_1, \ldots, f_n)$ with $f_i = f_i(x_1, \ldots, x_m)$, then provided that f is $\mathbb{R}$-differentiable at a the partial derivatives $\partial_j f_i(a)$, $1 \le i \le n$, $1 \le j \le m$, exist and we have
$$df(a)(v) = J_f(a).v, \quad J_f(a) := \big(\partial_j f_i(a)\big)_{1 \le i \le n, 1 \le j \le m}.$$
The matrix $J_f(a)$ is called the **Jacobian matrix**.

Let $U \subseteq \mathbb{C}$ be a domain. A function $f : U \to \mathbb{C}$ may be considered as a mapping $f : \mathbb{R}^2 \supseteq U \to \mathbb{R}^2$ (namely, $z = x + iy$ and $f = u + iy$) and we can compare complex and real differentiability.

Theorem. *For a function $f : U \to \mathbb{C}$ and $a \in U$ the following are equivalent:*

(1) *f is $\mathbb{C}$-differentiable in a.*
(2) *f is $\mathbb{R}$-differentiable in a and the differential $df(a) : \mathbb{C} \to \mathbb{C}$ is $\mathbb{C}$-linear.*
(3) *f is $\mathbb{R}$-differentiable in a and the* ***Cauchy–Riemann equations*** *hold:*
$$\boxed{u_x(a) = v_y(a), \quad u_y(a) = -v_x(a).} \tag{4.2}$$

In this case we have $df(a)(z) = f'(a)z$ and $f'(a) = u_x(a) + iv_x(a) = v_y(a) - iu_y(a)$.

Proof. (1) ⇒ (2) By assumption the derivative $f'(a)$ exists. Define the $\mathbb{C}$-linear mapping $T(z) := f'(a)z$. Then
$$\frac{|f(a+h) - f(a) - T(h)|}{|h|} = \Big|\frac{f(a+h) - f(a)}{h} - f'(a)\Big| \to 0$$

as $h \to 0$. That is f is $\mathbb{R}$-differentiable and $df(a) = T$.

(2) $\Rightarrow$ (1) If f is $\mathbb{R}$-differentiable at a with $\mathbb{C}$-linear derivative $df(a)$ then

$$\left|\frac{f(a+h)-f(a)}{h} - df(a)(1)\right| = \frac{|f(a+h)-f(a)-df(a)(h)|}{|h|} \to 0$$

as $h \to 0$, since $df(a)(h) = h \cdot df(a)(1)$. So $f'(a)$ exists and equals $df(a)(1)$.

The equivalence of (2) and (3) follows from Proposition 4.1: the differential $df(a)$ is given by the Jacobian matrix

$$J_f(a) = \begin{pmatrix} u_x(a) & u_y(a) \\ v_x(a) & v_y(a) \end{pmatrix}$$

which is $\mathbb{C}$-linear if and only if (4.2).

Then $f'(a) = df(a)(1) = u_x(a) + iv_x(a) = v_y(a) - iu_y(a)$. □

A function $f = u + iv : \mathbb{C} \supseteq U \to \mathbb{C}$ is $\mathbb{R}$-differentiable in $a \in U$ if and only if $u : U \to \mathbb{R}$ and $v : U \to \mathbb{R}$ are $\mathbb{R}$-differentiable in a. A sufficient condition for $\mathbb{R}$-differentiability of a function $u : U \to \mathbb{R}$ is existence and continuity of the partial derivatives u_x and u_y; we say that u is **continuously differentiable** and write $u \in C^1(U)$. Thus follows:

Corollary. *If u and v are real continuously differentiable functions in $U \subseteq \mathbb{C}$, satisfying the Cauchy–Riemann equations*

$$u_x = v_y, \quad u_y = -v_x. \tag{4.3}$$

in U, then $f = u + iv$ is holomorphic in U.[6]

Suppose that $f = u + iv$ is $\mathbb{C}$-differentiable. Then

$$\boxed{|f'|^2 = \det\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = u_x^2 + v_x^2 = u_y^2 + v_y^2,} \tag{4.4}$$

that is, $|f'|^2$ is the **Jacobian determinant** of the mapping $(x, y) \mapsto (u, v)$. It is non-negative, and zero precisely at points where f' vanishes.

Proposition. *If f is holomorphic in a region and $f' = 0$, then f is constant.*

Proof. This follows from (4.4) and the mean value theorem. □

A different proof will be given in Section 13.3.

4.4. Example.

(1) The function

$$\tilde{e}(z) := e^x(\cos y + i \sin y), \quad z = x + iy,\ x, y \in \mathbb{R},$$

is $\mathbb{R}$-differentiable and the Cauchy–Riemann equations are satisfied. Thus $\tilde{e}$ is holomorphic in $\mathbb{C}$ and we have

$$\tilde{e}'(z) = \tilde{e}(z).$$

(2) The function

$$\tilde{\ell}(z) := \frac{1}{2}\log(x^2 + y^2) + i \arctan\frac{y}{x}, \quad z = x + iy,\ x, y \in \mathbb{R},\ x \neq 0,$$

[6] Actually, by the Looman–Menchoff theorem, a continuous function $f : \mathbb{C} \supseteq U \to \mathbb{C}$ whose partial derivatives exist (everywhere but a countable set) is holomorphic if and only if it satisfies the Cauchy–Riemann equations.

is $\mathbb{R}$-differentiable and the Cauchy–Riemann equations are satisfied. Thus $\tilde{\ell}$ is holomorphic in $\mathbb{C} \setminus \{z \in \mathbb{C} : \operatorname{Re} z = 0\}$ and

$$\tilde{\ell}'(z) = \frac{1}{z}.$$

4.5. Elementary properties of holomorphic functions. Let U be a domain in $\mathbb{C}$. Recall that a function f is holomorphic in U if it is $\mathbb{C}$-differentiable at every point $a \in U$. We say that a function is **holomorphic at** a if there is an open neighborhood V of a in U such that f is holomorphic in V, i.e., $f|_V \in \mathcal{H}(V)$.

The set of points, where a function is holomorphic, is always open in $\mathbb{C}$. If f is holomorphic at a then it is clearly $\mathbb{C}$-differentiable at a. Then converse is not true: The function $f(z) = x^3y^2 + ix^2y^3$, $z = x + iy$, $x, y \in \mathbb{R}$, is $\mathbb{C}$-differentiable on the coordinate axes, but nowhere else, and it is nowhere holomorphic.

Proposition. *Let U be a domain in $\mathbb{C}$ and let $f, g \in \mathcal{H}(U)$.*

(1) *For all $a, b \in \mathbb{C}$, $af + bg \in \mathcal{H}(U)$ and*
$$(af + bg)' = af' + bg'.$$
(2) *$fg \in \mathcal{H}(U)$ and*
$$(fg)' = f'g + fg'.$$
(3) *If $g(z) \neq 0$ for all $z \in U$ then $f/g \in \mathcal{H}(U)$ and*
$$\Big(\frac{f}{g}\Big)' = \frac{f'g - fg'}{g^2}.$$
(4) *If V is a domain in $\mathbb{C}$ such that $g(U) \subseteq V$, and $h \in \mathcal{H}(V)$, then $h \circ g \in \mathcal{H}(U)$ and*
$$(h \circ g)' = (h' \circ g) \cdot g'.$$

Proof. There are $f_1, g_1 : U \to \mathbb{C}$ that are continuous in $c \in U$ and such that

$$f(z) = f(c) + (z - c)f_1(z), \quad g(z) = g(c) + (z - c)g_1(z), \quad z \in U.$$

Let us show (2) ((1) is similar):

$$(fg)(z) = (fg)(c) + (z - c)\big[f_1(z)g(c) + f(c)g_1(z) + (z - c)f_1(z)g_1(z)\big].$$

The function in the square bracket is continuous in c, and so

$$(fg)'(c) = f_1(c)g(c) + f(c)g_1(c) = f'(c)g(c) + f(c)g'(c).$$

In order to show (3) consider

$$\frac{1}{g(z)} = \frac{1}{g(c)} + (z - c)\Big[-\frac{g_1(z)}{g(c)(g(c) + (z - c)g_1(z))}\Big]$$

which implies that $1/g \in \mathcal{H}(U)$ and $(1/g)' = -g'/g^2$; the general case follows from (2). Let us prove (4). By assumption

$$h(w) = h(g(c)) + (w - g(c))h_1(w), \quad w \in V,$$

where $h_1 : V \to \mathbb{C}$ is continuous in $g(c)$. Then

$$h(g(z)) = h(g(c)) + (g(z) - g(c))h_1(g(z)) = h(g(c)) + (z - c)g_1(z)h_1(g(z)), \quad z \in U,$$

and $z \mapsto g_1(z)h_1(g(z))$ is continuous in c. Thus $h \circ g$ is $\mathbb{C}$-differentiable in c and

$$(h \circ g)'(c) = h_1(g(c))g_1(c) = h'(g(c))g'(c). \qquad \square$$

As a consequence we obtain that every polynomial $p(z) = a_0 + a_1z + \cdots + a_nz^n \in \mathbb{C}[z]$ is holomorphic in $\mathbb{C}$, and $p'(z) = a_1 + 2a_2z + \cdots + na_nz^{n-1} \in \mathbb{C}[z]$. Moreover, every rational function is holomorphic in the complement of the zero set of the denominator.

4.6. Partial derivatives f_x, f_y, f_z, and $f_{\overline{z}}$. Let $f : \mathbb{C} \supseteq U \to \mathbb{C}$ be $\mathbb{R}$-differentiable at $a \in U$. Then

$$f(a+z) - f(a) = f_x(a)x + f_y(a)y + o(z), \qquad \text{as } z = x + iy \to 0,$$

and since $x = \frac{1}{2}(z + \overline{z})$ and $y = \frac{1}{2i}(z - \overline{z})$, we find

$$f(a+z) - f(a) = \frac{f_x(a) - if_y(a)}{2} z + \frac{f_x(a) + if_y(a)}{2} \overline{z} + o(z). \tag{4.5}$$

This suggests the introduction of the differential operators

$$\boxed{\frac{\partial}{\partial z} := \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big), \qquad \frac{\partial}{\partial \overline{z}} := \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big).}$$

We will denote by f_z and $f_{\overline{z}}$ the respective partial derivatives,

$$\boxed{f_z := \tfrac{\partial f}{\partial z} = \tfrac{1}{2}(f_x - if_y), \quad f_{\overline{z}} := \tfrac{\partial f}{\partial \overline{z}} = \tfrac{1}{2}(f_x + if_y).}$$

Then (4.5) reads

$$f(a+z) - f(a) = f_z(a)z + f_{\overline{z}}(a)\overline{z} + o(z).$$

The Cauchy–Riemann equations (4.3) are equivalent to

$$\boxed{if_x = f_y \;\Leftrightarrow\; f_{\overline{z}} = 0.}$$

Thus, if f is $\mathbb{C}$-differentiable at a then

$$f'(a) = f_x(a) = -if_y(a) = f_z(a).$$

The differential of f is the 1-form

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy,$$

where $dx : \mathbb{R}^2 \to \mathbb{R}$, $h \mapsto h_1$, and $dy : \mathbb{R}^2 \to \mathbb{R}$, $h \mapsto h_2$. Since

$$dz = dx + i\,dy, \quad d\overline{z} = dx - i\,dy,$$

we find

$$df = \frac{\partial f}{\partial z}\,dz + \frac{\partial f}{\partial \overline{z}}\,d\overline{z} = f_z\,dz + f_{\overline{z}}\,d\overline{z},$$

and $df = f_z\,dz = f'\,dz$ if f is holomorphic.

4.7. Holomorphic functions are harmonic. Let $f = u+iv$ be a complex valued function defined in a domain $U \subseteq \mathbb{C}$. If f is C^2 (i.e. the partial derivatives of second order exist and are continuous), we may consider the **Laplacian** of f,

$$\boxed{\Delta f := f_{xx} + f_{yy} = 4f_{\overline{z}z}.}$$

A function $f \in C^2(U)$ is called **harmonic** if $\Delta f = 0$ on U. If $f \in \mathcal{H}(U) \cap C^2(U)$ then $f_{\overline{z}} = 0$ and thus $\Delta f = 0$ on U. Since $\Delta f = \Delta(u + iv) = \Delta u + i\Delta v$, f is harmonic if and only u and v are harmonic. So we proved:

Lemma. *If $f \in \mathcal{H}(U) \cap C^2(U)$ then the real and the imaginary part of f are harmonic.*[7]

[7]We will see in Theorem 16.1 that $\mathcal{H}(U) \subseteq C^2(U)$.

5. Conformal mappings

5.1. Linear angle-preserving mappings. An $\mathbb{R}$-linear bijective mapping $f : \mathbb{C} \to \mathbb{C}$ is said to be **angle-preserving** if

$$\frac{\langle f(z), f(w)\rangle}{|f(z)||f(w)|} = \frac{\langle z, w\rangle}{|z||w|}, \quad z, w \in \mathbb{C} \setminus \{0\},$$

i.e., $\sphericalangle(f(z), f(w)) = \sphericalangle(z, w)$.

Lemma. *The following are equivalent:*

(1) $f : \mathbb{C} \to \mathbb{C}$ *is angle-preserving.*
(2) *There exists* $a \in \mathbb{C} \setminus \{0\}$ *such that either* $f(z) = az$ *or* $f(z) = a\overline{z}$ *for all* $z \in \mathbb{C}$.
(3) *There exists* $r > 0$ *such that* $\langle f(z), f(w)\rangle = r\langle z, w\rangle$ *for all* z, w.

Proof. (1) $\Rightarrow$ (2) Set $a := f(1)$. Since $\langle f(z), f(w)\rangle = 0$ if $\langle z, w\rangle = 0$, we have

$$0 = \langle f(1), f(i)\rangle = \operatorname{Re}(\overline{a}f(i)) = \operatorname{Re}(a\overline{a}a^{-1}f(i)) = |a|^2 \operatorname{Re}(a^{-1}f(i)).$$

Thus $a^{-1}f(i)$ is purely imaginary and so $f(i) = ira$ for some $r \in \mathbb{R}$. Since $\langle 1 + i, 1 - i\rangle = 0$,

$$\begin{aligned} 0 &= \langle f(1 + i), f(1 - i)\rangle = \langle f(1) + f(i), f(1) - f(i)\rangle = \langle a(1 + ir), a(1 - ir)\rangle \\ &= |a|^2 \operatorname{Re}((1 + ir)^2) = |a|^2(1 - r^2), \end{aligned}$$

i.e., $r = \pm 1$. This implies (2), in fact, $f(x + iy) = xf(1) + yf(i) = xa \pm iya$ and so either $f(z) = az$ or $f(z) = a\overline{z}$.

(2) $\Rightarrow$ (3) This follows from $\langle az, aw\rangle = |a|^2\langle z, w\rangle$ and $\langle z, w\rangle = \langle \overline{z}, \overline{w}\rangle$.

(3) $\Rightarrow$ (1) By assumption, $|f(z)| = \sqrt{r}|z|$ and so f is injective and hence bijective. Moreover,

$$|z||w|\langle f(z), f(w)\rangle = |z||w|r\langle z, w\rangle = |f(z)||f(w)|\langle z, w\rangle$$

which implies (1). $\square$

5.2. Angle-preserving and conformal mappings. Let $U \subseteq \mathbb{C}$ be a domain. An $\mathbb{R}$-differentiable mapping $f : U \to \mathbb{C}$ is called **angle-preserving** at $c \in U$ if the differential $df(c) : \mathbb{C} \to \mathbb{C}$ is angle-preserving. We say that f is angle-preserving in U if it is so at every point.

We say that $f : U \to \mathbb{C}$ is **antiholomorphic** if $\overline{f} : U \to \mathbb{C}$ is holomorphic; this is the case if and only if $f_z = 0$ in U.

Theorem. *Let* $U \subseteq \mathbb{C}$ *be a region and let* $f \in C^1(U)$. *The following are equivalent:*

(1) f *is holomorphic or antiholomorphic in* U *and* $f' \neq 0$ *or* $\overline{f}' \neq 0$ *in* U.
(2) f *is angle-preserving in* U.

Proof. By Lemma 5.1 the differential $df(c)h = f_z(c)h + f_{\overline{z}}(c)\overline{h}$ is angle-preserving if and only if either $f_{\overline{z}}(c) = 0$ and $f_z(c) \neq 0$ or $f_z(c) = 0$ and $f_{\overline{z}}(c) \neq 0$. This implies (1) $\Rightarrow$ (2) since $f_z = f'$ and $f_{\overline{z}} = \overline{\overline{f}'}$. For the direction (2) $\Rightarrow$ (1) we note that the function

$$U \ni c \mapsto \frac{f_z(c) - f_{\overline{z}}(c)}{f_z(c) + f_{\overline{z}}(c)}$$

is well-defined, continuous, and takes values in $\{-1, 1\}$. Since U is connected it must be constant. This means that either $f_{\overline{z}} = 0$ and $f_z \neq 0$ or $f_z = 0$ and $f_{\overline{z}} \neq 0$ in U. $\square$

An $\mathbb{R}$-differentiable mapping $f = u + iv : U \to \mathbb{C}$ is called **orientation-preserving** at $c \in U$ if

$$\det \begin{pmatrix} u_x(c) & u_y(c) \\ v_x(c) & v_y(c) \end{pmatrix} > 0.$$

We say that f is orientation-preserving in U if it is so at every point.

An angle- and orientation preserving mapping $f : U \to \mathbb{C}$ is called **conformal**.

Corollary. *Let $f \in C^1(U)$. Then f is holomorphic in U and $f' \neq 0$ in U if and only if f is conformal.*

Proof. This follows from the theorem and (4.4). □

Example. The mapping $f : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$, $f(z) = z^2$, is conformal. For $z = x + iy$ and $f = u + iv$,

$$u = x^2 - y^2, \quad v = 2xy.$$

So the lines $x = a$, $y = b$ parallel to the axes are mapped to the parabolas $v^2 = 4a^2(a^2 - u)$ and $v^2 = 4b^2(b^2 + u)$. The parabolas of the first family are open to the left, those of the second family to the right. Any two parabolas intersect orthogonally. The levels $u = c$ and $v = d$ are hyperbolas in the (x, y)-plane; any two of them intersect orthogonally.

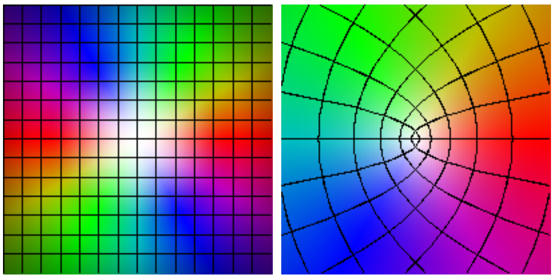


FIGURE 4. The function $z \mapsto z^2$. On the left we have the (x, y)-plane, on the right the (u, v)-plane. Brightness corresponds to $|z|$, and the color to $\arg z$.

Power series

6. Convergent sequences and series of functions

6.1. Uniformly convergent sequences and series of functions. Let X be a set. A sequence of functions $f_n : X \to \mathbb{C}$ is **uniformly convergent** to $f : X \to \mathbb{C}$ if

$$\forall \epsilon > 0 \ \exists k \in \mathbb{N} \ \forall n \geq k \ \forall x \in X : |f_n(x) - f(x)| < \epsilon,$$

or equivalently, if $\lim_{n\to\infty} \|f_n - f\| = 0$, where

$$\boxed{\|f\| = \|f\|_X := \sup_{x\in X} |f(x)|}$$

is the **sup norm**. Uniform convergence implies **pointwise convergence**, that is $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in X$; we simply write $f = \lim f_n$, the limit is unique.

If f_n and g_n are uniformly convergent sequences of functions $X \to \mathbb{C}$ then:

(1) For all $a, b \in \mathbb{C}$ the sequence $af_n + bg_n$ is uniformly convergent and

$$\lim(af_n + bg_n) = a \lim f_n + b \lim g_n.$$

(2) If $\|\lim f_n\| < \infty$ and $\|\lim g_n\| < \infty$, then the product sequence $f_n g_n$ is uniformly convergent and

$$\lim(f_n g_n) = \lim f_n \lim g_n.$$

Let X be a metric space. A sequence of functions $f_n : X \to \mathbb{C}$ is **locally uniformly convergent** to $f : X \to \mathbb{C}$ if each point $x \in X$ has a neighborhood U in X such that $f_n|_U$ is uniformly convergent.

Theorem. *The limit of a locally uniformly convergent sequence of continuous functions $f_n \in C(X)$ is continuous, $f = \lim f_n \in C(X)$.*

Proof. Fix $a \in X$ and let $\epsilon > 0$. There is a neighborhood U of a in X and $m \in \mathbb{N}$ such that $\|f - f_m\|_U < \epsilon$. Since f_m is continuous in a, there is a neighborhood $V \subseteq U$ of a such that $|f_m(x) - f(a)| < \epsilon$ for all $x \in V$. Thus

$$|f(x) - f(a)| \leq |f(x) - f_m(x)| + |f_m(x) - f_m(a)| + |f_m(a) - f(a)| < 3\epsilon$$

for all $x \in V$. □

A locally uniformly convergent sequence of functions converges uniformly on compact sets. If X is locally compact (i.e. each point in X has a compact neighborhood), then also the converse is trivially true. Note that domains in $\mathbb{C}$ are locally compact.

A series $\sum f_n$ of functions $f_n : X \to \mathbb{C}$ is **(locally) uniformly convergent** if the sequence of partial sums is (locally) uniformly convergent. Thus (1) remains true for series, and the limit of a locally uniformly convergent series of continuous functions in continuous.

6.2. Cauchy's convergence criterion. A sequence $f_n : X \to \mathbb{C}$ is called a **Cauchy sequence** if

$$\forall \epsilon > 0 \; \exists k \in \mathbb{N} \; \forall m, n \geq k : \|f_m - f_m\| < \epsilon.$$

Theorem. *A sequence $f_n : X \to \mathbb{C}$ is uniformly convergent if and only if it is a Cauchy sequence.*

Proof. Suppose that f_n is a Cauchy sequence. Then f_n is pointwise a Cauchy sequence, since $|f_m(x) - f_n(x)| \leq \|f_m - f_m\|$, and thus it is pointwise convergent; set $f := \lim f_n$. Let $\epsilon > 0$. There is $k \in \mathbb{N}$ such that $|f_m(x) - f_n(x)| < \epsilon$ for all $m, n \geq k$ and all $x \in X$. For $x \in X$ choose $m = m(x) \geq k$ such that $|f_m(x) - f(x)| < \epsilon$, then

$$|f_n(x) - f(x)| \leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)| < 2\epsilon$$

for all $x \in X$ and all $n \geq k$. □

Corollary. *A series $\sum f_n$ of functions $f_n : X \to \mathbb{C}$ is uniformly convergent if and only if for all $\epsilon > 0$ there is $k \in \mathbb{N}$ such that $\|f_{m+1} + \cdots + f_n\| < \epsilon$ for all $n > m \geq k$.*

6.3. Weierstrass majorant criterion.

Theorem. *Let $f_n : X \to \mathbb{C}$ be a sequence of functions, and let $M_n \geq 0$ be a sequence of real numbers such that $\|f_n\| \leq M_n$ and $\sum M_n < \infty$. Then $\sum f_n$ is uniformly convergent.*

Proof. This follows from Cauchy's convergence criterion, since

$$\Big\| \sum_{k=m+1}^{n} f_k \Big\| \leq \sum_{k=m+1}^{n} \|f_k\| \leq \sum_{k=m+1}^{n} M_k. \qquad \square$$

6.4. Normally convergent series. A series $\sum f_n$ of functions $f_n : X \to \mathbb{C}$ is called **normally convergent** if $\sum \|f_n\| < \infty$. The Weierstrass majorant criterion implies that a normally convergent series is uniformly convergent. Clearly, each subseries of a normally convergent series is normally convergent. Moreover, normally convergent series can be reordered arbitrarily:

Proposition. *If $\sum_{n=0}^{\infty} f_n$ is normally convergent to f, then $\sum_{n=0}^{\infty} f_{\sigma(n)}$ is normally convergent to f for each bijection $\sigma : \mathbb{N} \to \mathbb{N}$.*

Proof. Let $\sigma : \mathbb{N} \to \mathbb{N}$ be any bijection. Since $\sum_{n=0}^{\infty} \|f_n\| < \infty$, we also have $\sum_{n=0}^{\infty} \|f_{\sigma(n)}\| < \infty$, because absolutely convergent series of complex numbers can be arbitrarily reordered; cf. Section 2.2. Moreover, $f(x) = \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} f_{\sigma(n)}(x)$ for each $x \in X$, since $\sum_{n=0}^{\infty} f_n(x)$ is absolutely convergent for each $x \in X$. □

Lemma. *Let $\sum f_n$ and $\sum g_n$ be normally convergent series.*

1. *For all $a, b \in \mathbb{C}$ the series $\sum (af_n + bg_n)$ is normally convergent.*
2. *Every product series $\sum h_n$, where $h_0, h_1, \ldots$ runs through all products $f_k g_\ell$ exactly once, converges normally to $\sum f_n \sum g_n$.*

Proof. (1) follows immediately from the majorant criterion.

(2) is a consequence of the corresponding result for absolutely convergent series; cf. Section 2.2. Note that $\|f_k g_\ell\| \leq \|f_k\| \|g_\ell\|$. □

7. Convergent power series

We begin by discussing power series centered at the origin. Everything holds up to obvious changes for power series centered at arbitrary points; see Section 8.2.

7.1. Formal power series. A **formal power series** is a series $\sum_{n=0}^{\infty} a_n z^n$ with $a_n \in \mathbb{C}$. The numbers a_n are called the **coefficients** of the formal power series.

The set of all formal power series, denoted by $\mathbb{C}[[z]]$, forms a $\mathbb{C}$-algebra. In fact, we have the inclusion $\mathbb{C} \subseteq \mathbb{C}[[z]]$ by identifying $\mathbb{C} \ni a \mapsto a + \sum_{n=1}^{\infty} 0 \cdot z^n$ and define the sum and the product of two formal power series $f = \sum_{n=0}^{\infty} a_n z^n$ and $g = \sum_{n=0}^{\infty} b_n z^n$ by setting

$$f + g := \sum_{n=0}^{\infty} (a_n + b_n) z^n, \quad fg := \sum_{n=0}^{\infty} \Big(\sum_{k+\ell=n} a_k b_\ell \Big) z^n.$$

Note that this is the Cauchy product; cf. Section 2.2.

7.2. Abel's lemma. A (formal) power series $\sum_{n=0}^{\infty} a_n z^n$ is called **convergent** if there is a point $z_0 \in \mathbb{C} \setminus \{0\}$ such that the series $\sum_{n=0}^{\infty} a_n z_0^n$ is convergent; for $z_0 = 0$ this is trivially true.

Lemma. *If there exist $s, M > 0$ such that $|a_n| s^n \le M$ for all $n \in \mathbb{N}$, then the power series $\sum_{n=0}^{\infty} a_n z^n$ is normally convergent on each disk $\overline{D}_r(0)$ with $r < s$.*

Proof. We have $\|a_n z^n\|_{\overline{D}_r(0)} = |a_n| r^n = |a_n| s^n (r/s)^n \le M (r/s)^n$ and $0 < r/s < 1$. The majorant criterion implies the statement. ☐

Corollary. *If $\sum_{n=0}^{\infty} a_n z^n$ converges at $z_0 \ne 0$, then $\sum_{n=0}^{\infty} a_n z^n$ is normally convergent on each disk $\overline{D}_r(0)$ with $r < |z_0|$.*

Proof. The sequence $a_n z_0^n$ converges to 0, and thus is bounded. ☐

7.3. Radius of convergence. The **radius of convergence** of a power series $\sum_{n=0}^{\infty} a_n z^n$ is by definition

$$\boxed{\rho := \sup \big\{ s \ge 0 : |a_n| s^n \text{ is bounded} \big\} \in [0, \infty].}$$

The set $D_\rho(0)$ is called the **disk of convergence** of $\sum_{n=0}^{\infty} a_n z^n$.

Theorem. *Let ρ be the radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$. Then:*

(1) *For each $r < \rho$, the series $\sum_{n=0}^{\infty} a_n z^n$ converges normally for $|z| \le r$, and it converges absolutely for each z with $|z| < \rho$.*
(2) *The series $\sum_{n=0}^{\infty} a_n z^n$ diverges for $|z| > \rho$.*

*The radius of convergence is given by **Hadamard's formula**:*

$$\boxed{1/\rho = \limsup_{n \to \infty} |a_n|^{\frac{1}{n}},} \tag{7.1}$$

with the convention that $1/0 = \infty$ and $1/\infty = 0$.

Proof. Let us prove (1) and (2). There is nothing to prove if $\rho = 0$. Assume $\rho > 0$. For each $s \in (0, \rho)$ the sequence $|a_n| s^n$ is bounded. Abel's lemma implies (1). If $|z| > \rho$ then $a_n z^n$ is unbounded, and thus $\sum_{n=0}^{\infty} a_n z^n$ is divergent; this shows (2).

Let us prove formula (7.1). Set $L := (\limsup_{n \to \infty} |a_n|^{\frac{1}{n}})^{-1}$. In order to show $L \le \rho$ we prove that $0 < r < L$ implies $r \le \rho$. If $0 < r < L$ then $1/r >$

$\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$, and thus there exists $n_0 \in \mathbb{N}$ such that $1/r > |a_n|^{\frac{1}{n}}$ for all $n \geq n_0$.[8] That means that $|a_n| r^n$ is bounded, i.e., $r \leq \rho$.

In order to show $\rho \leq L$ we prove that $L < s < \infty$ implies $s \geq \rho$. If $L < s < \infty$ then $1/s < \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$. So there exists an infinite subset $M \subseteq \mathbb{N}$ such that $1/s < |a_m|^{\frac{1}{m}}$ for all $m \in M$. It follows that $a_n s^n$ cannot converge to zero. Hence $s \geq \rho$ (otherwise $\sum |a_n| s^n < \infty$ by (1), a contradiction). □

We may conclude that the sum of the power series $\sum_{n=0}^\infty a_n z^n$ is a function that is continuous in the disk of convergence $D_\rho(0)$; see Theorem 6.1.

7.4. Ratio test. Sometimes the radius of convergence is easier computed by the **ratio test**:

Proposition. *Let ρ be the radius of convergence of $\sum_{n=0}^\infty a_n z^n$, and assume that $a_n \neq 0$ for all n. Then*

$$\boxed{\liminf_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| \leq \rho \leq \limsup_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|},$$

in particular, $\rho = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ if the limit exists.

Proof. Let us set $L := \liminf_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ and $R := \limsup_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$. It suffices to show that $0 < r < L$ implies $r \leq \rho$ and that $R < s < \infty$ implies $s \geq \rho$.

If $0 < r < L$ then there exists $n_0 \in \mathbb{N}$ such that $\left|\frac{a_n}{a_{n+1}}\right| > r$ for all $n \geq n_0$. Then

$$|a_{n_0+k}| r^{n_0+k} = \left|\frac{a_{n_0+k}}{a_{n_0+k-1}}\right| \cdots \left|\frac{a_{n_0+1}}{a_{n_0}}\right| |a_{n_0}| r^{n_0+k} \leq |a_{n_0}| r^{n_0}$$

for all $k \geq 0$. Thus $|a_n| r^n$ is bounded, i.e., $r \leq \rho$.

If $R < s < \infty$ then there exists $n_0 \in \mathbb{N}$ such that $\left|\frac{a_n}{a_{n+1}}\right| < s$ for all $n \geq n_0$. Analogously, we obtain $|a_{n_0+k}| s^{n_0+k} \geq |a_{n_0}| s^{n_0} > 0$ for all $k \geq 0$. So $a_n s^n$ cannot converge to zero, and hence $s \geq \rho$. □

8. Analytic functions

8.1. Formal differentiation and integration of power series. For a formal power series $\sum_{n=0}^\infty a_n z^n$ we may consider the formal power series that arise by term-wise differentiation and integration:

$$\sum_{n=0}^\infty n a_n z^{n-1}, \quad \sum_{n=0}^\infty \frac{a_n}{n+1} z^{n+1}.$$

Lemma. *The radius of convergence remains unchanged by term-wise differentiation and integration.*

Proof. Let ρ be the radius of convergence of $\sum_{n=0}^\infty a_n z^n$, and ρ' the one of $\sum_{n=0}^\infty n a_n z^{n-1}$. Since boundedness of $n|a_n| s^{n-1}$ implies boundedness of $|a_n| s^n$, we have $\rho' \leq \rho$. In order to show $\rho \leq \rho'$ it suffices to prove that $r < \rho$ implies $r \leq \rho'$. Choose $s \in (r, \rho)$; then $|a_n| s^n$ is bounded. Thus $n|a_n| r^{n-1} = r^{-1} |a_n| s^n n (r/s)^n$ converges to zero, since $r < s$. Hence $r \leq \rho'$.

If $\tilde{\rho}$ denotes the radius of convergence of $\sum_{n=0}^\infty \frac{a_n}{n+1} z^{n+1}$, then $\tilde{\rho} = \rho$ by the first part. □

[8] Recall that $\limsup_{n\to\infty} b_n := \inf_{n\geq 0} \sup_{k\geq n} b_k$ and $\liminf_{n\to\infty} b_n := \sup_{n\geq 0} \inf_{k\geq n} b_k$.

We say that a function $f : U \to \mathbb{C}$ is **k-times $\mathbb{C}$-differentiable** in U if the higher order complex derivatives f, f' through $f^{(k-1)}$ are holomorphic in U, where we define $f^{(0)} := f$ and $f^{(j)} := (f^{(j-1)})'$ for $j \geq 1$. We say that f is **indefinitely $\mathbb{C}$-differentiable** if f is k-times $\mathbb{C}$-differentiable for all $k \in \mathbb{N}$.

Theorem. *Let $\sum_{n=0}^{\infty} a_n z^n$ have positive radius of convergence ρ. Then the function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is indefinitely $\mathbb{C}$-differentiable in $D_\rho(0)$. We have*

$$\boxed{f^{(k)}(z) = \sum_{n \geq k} k! \binom{n}{k} a_n z^{n-k}, \quad z \in D_\rho(0),\ k \in \mathbb{N}.} \tag{8.1}$$

In particular, we get the ***Taylor coefficients***

$$\boxed{a_n = \frac{f^{(n)}(0)}{n!}, \quad n \in \mathbb{N}.}$$

Proof. It suffices to show that f is holomorphic in $D_\rho(0)$ and that (8.1) holds for $k = 1$; the general assertion follows by iteration. The lemma implies that $g(z) := \sum_{n=1}^{\infty} n a_n z^{n-1}$ defines a (continuous) function $g : D_\rho(0) \to \mathbb{C}$. We shall show that $f' = g$. Fix $b \in D_\rho(0)$ and set

$$h_n(z) := z^{n-1} + z^{n-2}b + \cdots + zb^{n-2} + b^{n-1}, \quad z \in \mathbb{C},\ n \geq 1,$$

and $f_1(z) := \sum_{n=1}^{\infty} a_n h_n(z)$. Since $z^n - b^n = (z-b)h_n(z)$, we find

$$f(z) - f(b) = \sum_{n=1}^{\infty} a_n(z^n - b^n) = (z-b)\sum_{n=1}^{\infty} a_n h_n(z) = (z-b)f_1(z), \quad z \in D_\rho(0).$$

Since $f_1(b) = g(b)$, it remains to show that f_1 is convergent in $D_\rho(0)$ and continuous in b. This follows from the fact that $\sum_{n=1}^{\infty} a_n h_n(z)$ is normally convergent on $D_r(0)$ for every $r \in (|b|, \rho)$. In fact, for such r, $\|a_n h_n\|_{D_r(0)} \leq |a_n| n r^{n-1}$ and $\sum |a_n| n r^{n-1} < \infty$, by the lemma. $\square$

8.2. Power series centered at arbitrary points. We have so far dealt only with power series centered at the origin. More generally, a **power series centered at** $c \in \mathbb{C}$ is an expression of the form $\sum_{n=0}^{\infty} a_n(z-c)^n$. Everything we have said so far for power series centered at 0 holds for power series centered at arbitrary points by virtue of a simple translation.

(1) The radius of convergence ρ is still given by Hadamard's formula (7.1).
(2) We still have local normal convergence in the disk of convergence $D_\rho(c)$ which is now centered at c (and divergence outside).
(3) In the disk of convergence, $f(z) = \sum_{n=0}^{\infty} a_n(z-c)^n$ is indefinitely $\mathbb{C}$-differentiable and the iterated derivatives are given by term-wise differentiation (this follows, for instance, from the chain rule):

$$\boxed{f^{(k)}(z) = \sum_{n \geq k} k! \binom{n}{k} a_n (z-c)^{n-k}, \quad z \in D_\rho(c),\ k \in \mathbb{N}.}$$

The Taylor coefficients are

$$\boxed{a_n = \frac{f^{(n)}(c)}{n!}, \quad n \in \mathbb{N}.} \tag{8.2}$$

8.3. Power series expansions. A function $f : U \to \mathbb{C}$ defined in a domain $U \subseteq \mathbb{C}$ is said to have a **power series expansion** at $c \in U$ if there exists a power series $\sum_{n=0}^{\infty} a_n(z-c)^n$ centered at c with positive radius of convergence such that

$$f(z) = \sum_{n=0}^{\infty} a_n(z-c)^n \qquad \text{for all } z \text{ in a neighborhood of } c. \tag{8.3}$$

The coefficients a_n are unique by (8.2). The series (8.3) is called the **Taylor series** of f at c.

A function $f : U \to \mathbb{C}$ that has a power series expansion at every point in U is said to be **analytic** in U. An analytic function in U is indefinitely $\mathbb{C}$-differentiable on U, by Theorem 8.1, and thus holomorphic in U.

We shall see in Theorem 16.1 that also the converse is true: every holomorphic function is analytic.

Remark. Theorem 16.1 will show that a function is analytic if and only if it is holomorphic. In particular, it will imply that the sum of a convergent power series is analytic.

Let $\mathbb{C}\{z\}$ denote the set of all power series $\sum_{n=0}^{\infty} a_n z^n$ with positive radius of convergence. It follows that elements $f, g \in \mathbb{C}\{z\}$ can be

- added, multiplied, $f + g \in \mathbb{C}\{z\}$, $fg \in \mathbb{C}\{z\}$,
- divided, $f/g \in \mathbb{C}\{z\}$, if g has non-zero constant term,
- composed, $f \circ g \in \mathbb{C}\{z\}$, if g has vanishing constant term,
- inverted, $g^{-1} \in \mathbb{C}\{z\}$, if the constant term of g vanishes and the linear term does not vanish;

clearly the radii of convergence may change under these operations.

Elementary transcendental functions

A function is said to be transcendental if it does not satisfy a polynomial equation whose coefficients are themselves roots of polynomials.

9. Exponential, trigonometric, and hyperbolic functions

9.1. The exponential function. The **exponential function** is defined by the formula

$$\boxed{\exp z := \sum_{n=0}^{\infty} \frac{z^n}{n!}.} \tag{9.1}$$

The power series (9.1) has infinite radius of convergence by the ratio test 7.4. Thus (9.1) is normally convergent on every bounded subset of $\mathbb{C}$, and $\exp : \mathbb{C} \to \mathbb{C}$ is a holomorphic function. By Theorem 8.1,

$$\boxed{\exp' z = \exp z, \quad z \in \mathbb{C}.} \tag{9.2}$$

By Lemma 6.4,

$$\Big(\sum_{n=0}^{\infty} \frac{z^n}{n!}\Big)\Big(\sum_{n=0}^{\infty} \frac{w^n}{n!}\Big) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{z^k w^{n-k}}{k!(n-k)!} = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!},$$

thus, we have the **addition formula**

$$\boxed{(\exp z)(\exp w) = \exp(z+w), \quad z, w \in \mathbb{C}.} \tag{9.3}$$

If we define

$$\boxed{e^z := \exp z, \quad \text{where} \quad e := \exp 1,}$$

then (9.3) takes the form $e^z e^w = e^{z+w}$. As a special case we get $e^z e^{-z} = e^0 = 1$, and see that $(e^z)^{-1} = e^{-z}$ and that $e^z \neq 0$ for all $z \in \mathbb{C}$.

A further consequence of (9.3) is

$$e^z = e^x e^{iy}, \quad z = x + iy \in \mathbb{C}.$$

So it suffices to study the functions $\mathbb{R} \ni x \mapsto e^x$ and $\mathbb{R} \ni y \mapsto e^{iy}$.

Lemma. *The restriction of* $\exp$ *to* $\mathbb{R}$ *is a strictly increasing positive function satisfying* $\lim_{x\to\infty} e^x = \infty$ *and* $\lim_{x\to-\infty} e^x = 0$.

Proof. By the definition (9.1), $e^x > 1 + x$ for $x > 0$ and hence, by (9.2), exp is strictly increasing and positive on the positive real axis with $\lim_{x\to\infty} e^x = \infty$. The remaining properties follow from $e^{-x} = (e^x)^{-1}$. □

As a consequence we obtain

$$\boxed{|\exp z| = \exp(\operatorname{Re} z), \quad z \in \mathbb{C}.}$$

Indeed,

$$|\exp z|^2 = \exp z \cdot \overline{\exp z} = \exp z \cdot \exp \overline{z} = \exp(z + \overline{z}) = \exp(2 \operatorname{Re} z) = (\exp(\operatorname{Re} z))^2.$$

In particular,

$$\boxed{|\exp z| = 1 \quad \text{if and only if} \quad z \in i\mathbb{R}.} \tag{9.4}$$

9.2. The logarithmic series. The power series

$$\boxed{\lambda(z) := \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n}} \tag{9.5}$$

is called the **logarithmic series**. Its radius of convergence is 1 by the ratio test 7.4. Hence λ is holomorphic in the unit disk $\mathbb{D}$ with

$$\boxed{\lambda'(z) = \sum_{n=1}^{\infty} (-1)^{n-1} z^{n-1} = \frac{1}{1+z}.} \tag{9.6}$$

Lemma. *We have* $\exp \lambda(z) = 1 + z$ *for all* $z \in \mathbb{D}$.

Proof. The function $f(z) := (1+z)\exp(-\lambda(z))$ is holomorphic in $\mathbb{D}$ and satisfies $f' = 0$, by (9.2) and (9.6). Thus f is constant, by Proposition 4.3, and so $f = 1$. □

9.3. Theorem. *The exponential function* $\exp : \mathbb{C} \to \mathbb{C} \setminus \{0\}$ *is a surjective group homomorphism.*

Proof. The addition formula (9.3) means that exp is a group homomorphism from the additive group $\mathbb{C}$ to the multiplicative group $\mathbb{C} \setminus \{0\}$.

In order to show that it is surjective we first prove that $\exp(\mathbb{C})$ is open in $\mathbb{C} \setminus \{0\}$.[9] By Lemma 9.2, $D_1(1) \subseteq \exp(\mathbb{C})$. Let $c \in \exp(\mathbb{C})$. Then

$$D_{|c|}(c) = cD_1(1) \subseteq c\exp(\mathbb{C}) \overset{(9.3)}{=} \exp(\mathbb{C}),$$

and so $\exp(\mathbb{C})$ is open in $\mathbb{C} \setminus \{0\}$.

Since, by (9.3), $A := \exp(\mathbb{C})$ is a subgroup of $\mathbb{C} \setminus \{0\}$, we have a disjoint union of cosets

$$\mathbb{C} \setminus \{0\} = A \cup \bigcup_{b \in B} bA, \quad B := (\mathbb{C} \setminus \{0\})/A.$$

By the previous paragraph $\bigcup_{b\in B} bA$ is open. Since $\mathbb{C} \setminus \{0\}$ is connected and since $1 = \exp 0 \in A$, we get $A = \mathbb{C} \setminus \{0\}$. □

9.4. Periodicity of $\exp$**.** A function $f : \mathbb{C} \to \mathbb{C}$ is called **periodic** if there exists $w \in \mathbb{C} \setminus \{0\}$ such that $f(z + w) = f(z)$ for all $z \in \mathbb{C}$; w is called a **period** of f. The set of all periods together with 0,

$$\boxed{\operatorname{per}(f) := \{w \in \mathbb{C} : w \text{ is a period of } f\} \cup \{0\},}$$

is a commutative subgroup of $\mathbb{C}$.

Theorem. *The exponential function is periodic. We have*

$$\boxed{\operatorname{per}(\exp) = \ker(\exp) = 2\pi i\mathbb{Z},}$$

where π *is the unique positive real number such that the second identity holds.*

Proof. For $w \in \mathbb{C}$ we have $\exp(z+w) = \exp z \exp w = \exp z$ if and only if $\exp w = 1$. This gives the first identity

$$\operatorname{per}(\exp) = \ker(\exp) = \{w \in \mathbb{C} : \exp w = 1\}.$$

[9] This will be proved in greater generality in the open mapping theorem 19.1.

By Theorem 9.3 there exists $a \in \mathbb{C}$ such that $\exp a = -1$, and clearly $a \neq 0$. Then $\exp(2a) = (\exp a)^2 = 1$ shows that the subgroup $\ker(\exp)$ of $\mathbb{C}$ is not trivial. Since $|\exp w| = 1$ if and only if $w \in i\mathbb{R}$, by (9.4), we have $\ker(\exp) \subseteq i\mathbb{R}$.

Next we show that there is a neighborhood U of 0 in $\mathbb{C}$ such that $U \cap \ker(\exp) = \{0\}$. Otherwise there would exist a sequence $0 \neq z_n \to 0$ with $\exp z_n = 1$ which leads to a contradiction,

$$1 = \exp 0 = \exp' 0 = \lim_{n\to\infty} \frac{\exp z_n - \exp 0}{z_n} = 0.$$

Since exp is continuous and thus $\ker(\exp)$ is closed, there exists a smallest positive real number π such that $2\pi i \in \ker(\exp)$ (note that $\exp w = 1$ if and only if $\exp(-w) = 1$). By (9.3) we may conclude that $2\pi i\mathbb{Z} \subseteq \ker(\exp)$. Conversely, let $ir \in \ker(\exp)$, $r \in \mathbb{R}$. Since $\pi \neq 0$ there exists $n \in \mathbb{Z}$ such that $2n\pi \leq r < 2(n+1)\pi$. Since $i(r - 2n\pi) \in \ker(\exp)$ and $0 \leq r - 2n\pi < 2\pi$, we may conclude that $r = 2n\pi$ by the minimality of π. This finishes the proof. $\square$

We infer that

$$\boxed{e^{i\pi} = -1,} \tag{9.7}$$

since $1 = e^{2\pi i} = (e^{i\pi})^2$, thus $e^{i\pi} = \pm 1$, but $e^{i\pi} = 1$ is impossible by the minimality of π.

If we decompose the z-plane in infinitely many horizontal strips

$$S_n := \{z \in \mathbb{C} : 2n\pi \leq \operatorname{Im} z < 2(n+1)\pi\}, \quad n \in \mathbb{Z},$$

then the exponential function maps each strip S_n bijectively onto $\mathbb{C} \setminus \{0\}$ in the w-plane. Since $w = e^z = e^x e^{iy}$, the orthogonal cartesian coordinates are mapped to orthogonal polar coordinates.

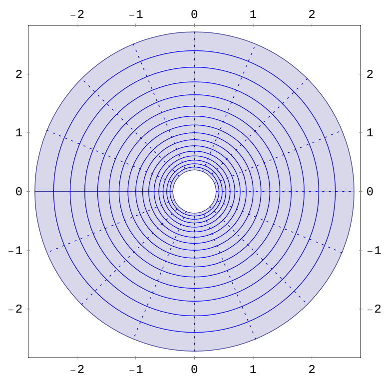


FIGURE 5. The image under exp of the rectangle $[-1, 1] \times [-\pi, \pi]$ in the (x, y)-plane.

9.5. Sine and cosine. The **sine** and the **cosine** functions are defined by

$$\boxed{\sin z := \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, \quad \cos z := \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}.} \tag{9.8}$$

These power series have infinite radius of convergence, since $\sum \frac{|z|^{2n+1}}{(2n+1)!}$ and $\sum \frac{|z|^{2n}}{(2n)!}$ are subseries of $\sum \frac{|z|^n}{n!}$. We obtain **Euler's formula**

$$\boxed{\exp(iz) = \cos z + i \sin z, \quad z \in \mathbb{C},}$$

by letting $N \to \infty$ in

$$\sum_{n=0}^{2N+1} \frac{(iz)^n}{n!} = \sum_{k=0}^{N} (-1)^k \frac{z^{2k}}{(2k)!} + i \sum_{k=0}^{N} (-1)^k \frac{z^{2k+1}}{(2k+1)!}.$$

The definitions in (9.8) imply

$$\boxed{\cos(-z) = \cos z, \quad \sin(-z) = -\sin z, \quad z \in \mathbb{C},}$$

thus $\exp(-iz) = \cos z - i \sin z$, and consequently,

$$\boxed{\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.} \tag{9.9}$$

Sine and cosine are holomorphic functions on $\mathbb{C}$ with

$$\boxed{\cos' z = -\sin z, \quad \sin' z = \cos z, \quad z \in \mathbb{C}.}$$

Moreover,

$$\begin{aligned} e^{i(z+w)} = e^{iz} e^{iw} &= (\cos z + i \sin z)(\cos w + i \sin w) \\ &= \cos z \cos w - \sin z \sin w + i(\sin z \cos w + \cos z \sin w) \end{aligned}$$

and similarly

$$e^{-i(z+w)} = \cos z \cos w - \sin z \sin w - i(\sin z \cos w + \cos z \sin w)$$

whence we obtain the addition formulas

$$\boxed{\begin{aligned} \cos(z+w) &= \cos z \cos w - \sin z \sin w, \\ \sin(z+w) &= \sin z \cos w + \cos z \sin w. \end{aligned}}$$

These formulas imply a multitude of further identities; we just mention a few:

$$\cos^2 z + \sin^2 z = 1, \quad \cos(2z) = \cos^2 z - \sin^2 z, \quad \sin(2z) = 2 \sin z \cos z,$$

$$\cos z - \cos w = -2 \sin(\tfrac{z+w}{2}) \sin(\tfrac{z-w}{2}), \tag{9.10}$$

$$\sin z - \sin w = 2 \cos(\tfrac{z+w}{2}) \sin(\tfrac{z-w}{2}). \tag{9.11}$$

9.6. Range, zeros, and periodicity of sine and cosine.

Proposition.

(1) *The functions* $\sin : \mathbb{C} \to \mathbb{C}$ *and* $\cos : \mathbb{C} \to \mathbb{C}$ *are surjective.*
(2) $\sin^{-1}(0) = \pi\mathbb{Z}$ *and* $\cos^{-1}(0) = \pi\mathbb{Z} + \pi/2$.
(3) $\mathrm{per}(\sin) = \mathrm{per}(\cos) = 2\pi\mathbb{Z}$.

Proof. (1) Let $c \in \mathbb{C}$. The equation $c = \cos z = \frac{1}{2}(e^{iz} + e^{-iz})$ is equivalent to $e^{iz} = c \pm \sqrt{c^2 - 1} \neq 0$. Since $\exp(\mathbb{C}) = \mathbb{C} \setminus \{0\}$ and $\ker(\exp) = 2\pi i\mathbb{Z}$ there exist countably many solutions z. The arguments for sin are similar.

(2) By (9.9),

$$2i \sin z = e^{-iz}(e^{2iz} - 1) = 0 \;\Leftrightarrow\; 2iz \in \ker(\exp) = 2\pi i\mathbb{Z} \;\Leftrightarrow\; z \in \pi\mathbb{Z}$$

$$2 \cos z = e^{i(\pi - z)}(e^{2i(z-\pi/2)} - 1) = 0 \;\Leftrightarrow\; 2i(z - \pi/2) \in 2\pi i\mathbb{Z} \;\Leftrightarrow\; z \in \pi\mathbb{Z} + \pi/2$$

since $e^{i\pi} = -1$ by (9.7).

(3) By (9.10) we have

$$\cos(z+w) - \cos z = -2 \sin(\tfrac{2z+w}{2}) \sin(\tfrac{w}{2})$$

and hence $w \in \mathrm{per}(\cos)$ if and only if $\sin(w/2) = 0$, i.e., $w \in 2\pi\mathbb{Z}$ by (2). The statement for the sine follows in the same way from (9.11). $\square$

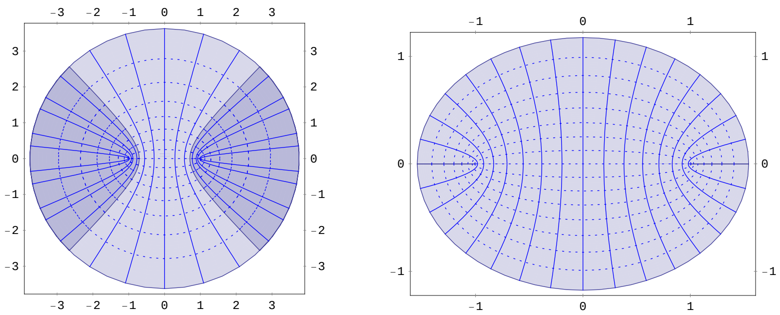


FIGURE 6. On the left we have the image under sin of the rectangle $[-\frac{3\pi}{4}, \frac{3\pi}{4}] \times [-2, 2]$ in the (x, y)-plane; on the right the image under cos of the rectangle $[0, \pi] \times [-1, 1]$

9.7. Polar coordinates. The **unit circle** $S^1 := \partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$ is a group with respect to multiplication. Theorems 9.3 and 9.4 and (9.4) imply the following lemma.

Lemma. *The mapping $\mathbb{R} \ni t \mapsto e^{it} \in S^1$ is a surjective group homomorphism with kernel $2\pi\mathbb{Z}$.*

Thus every $z \in \mathbb{C} \setminus \{0\}$ has a unique representation

$$\boxed{z = |z|e^{i\varphi} = |z|(\cos\varphi + i\sin\varphi), \quad \varphi \in [0, 2\pi).}$$

Indeed, since $z|z|^{-1} \in S^1$ there is $\varphi \in \mathbb{R}$ with $z = |z|e^{i\varphi}$. We may assume that $\varphi \in [0, 2\pi)$ since the kernel of the mapping $t \mapsto e^{it}$ is $2\pi\mathbb{Z}$. Suppose there is $\psi \in [0, 2\pi)$ such that $|z|e^{i\varphi} = |z|e^{i\psi}$. Without loss of generality assume $\psi \ge \varphi$. Then $e^{i(\psi-\varphi)} = 1$, thus $\psi - \varphi \in 2\pi\mathbb{Z}$ and hence $\psi = \varphi$ because $0 \le \psi - \varphi < 2\pi$.

The real numbers $|z|, \varphi$ are the **polar coordinates** of z, and φ is the **argument** of z, $\varphi = \arg z$. The restriction of φ to the interval $[0, 2\pi)$ is arbitrary. Any half-open interval of length 2π will do, e.g., $(-\pi, \pi]$.

Multiplying complex numbers given in polar coordinates is very easy, $|z|e^{i\varphi} \cdot |w|e^{i\psi} = |z||w|e^{i(\varphi+\psi)}$. In particular, we obtain **Moivre's formula**

$$\boxed{(e^{i\varphi})^n = e^{in\varphi} = \cos(n\varphi) + i\sin(n\varphi), \quad n \in \mathbb{Z}.}$$

9.8. Roots of unity. A number $z \in \mathbb{C}$ is called an nth **root of unity** if $z^n = 1$.

Lemma. *For every integer $n \ge 1$,*

$$\{z \in \mathbb{C} : z^n = 1\} = \{(e^{2\pi i/n})^k : k = 0, \dots, n-1\} =: G_n.$$

G_n is a cyclic subgroup of S^1 of order n.

Proof. By Moivre's formula each element of G_n is an nth root of unity. The n elements are pairwise distinct since $\ker(\exp) = 2\pi i\mathbb{Z}$. The statement follows since the polynomial $z^n - 1$ has at most n distinct roots. □

9.9. Tangent and cotangent. We define **tangent** and **cotangent** by

$$\boxed{\begin{aligned} \tan z &:= \frac{\sin z}{\cos z}, \quad z \in \mathbb{C} \setminus (\pi\mathbb{Z} + \tfrac{\pi}{2}), \\ \cot z &:= \frac{1}{\tan z} = \frac{\cos z}{\sin z}, \quad z \in \mathbb{C} \setminus \pi\mathbb{Z}. \end{aligned}}$$

Both functions are holomorphic in their domain of definition with

$$\boxed{\tan' z = \frac{1}{\cos^2 z} = 1 + \tan^2 z, \quad \cot' z = -\frac{1}{\sin^2 z} = -(1 + \cot^2 z).}$$

By (9.9),

$$\tan z = -i\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}} = i\frac{1 - e^{2iz}}{1 + e^{2iz}} = i\Big(1 - \frac{2}{1 + e^{-2iz}}\Big),$$
$$\cot z = i\frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}} = i\frac{e^{2iz} + 1}{e^{2iz} - 1} = i\Big(1 - \frac{2}{1 - e^{2iz}}\Big),$$

and thus, by Theorem 9.4, we obtain:

Lemma.

(1) $\mathrm{im}(\tan) = \mathrm{im}(\cot) = \mathbb{C} \setminus \{\pm i\}$.
(2) $\tan^{-1}(0) = \pi\mathbb{Z}$ *and* $\cot^{-1}(0) = \pi\mathbb{Z} + \frac{\pi}{2}$.
(3) $\mathrm{per}(\tan) = \mathrm{per}(\cot) = \pi\mathbb{Z}$.

We have the addition formulas

$$\boxed{\tan(z + w) = \frac{\tan z + \tan w}{1 - \tan z \tan w}, \quad \cot(z + w) = \frac{\cot z \cot w - 1}{\cot z + \cot w},}$$

in particular, $\cot(z + \frac{\pi}{2}) = -\tan z$ and $\tan(z + \frac{\pi}{2}) = -\cot z$ (in fact, $\tan(z + \frac{\pi}{2}) = -\cot(z + \pi) = -\cot z$).

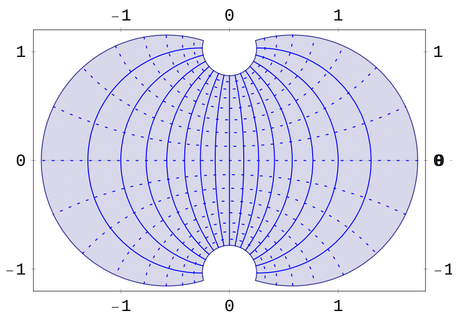


FIGURE 7. The image under tan of the square $[-\frac{\pi}{3}, \frac{\pi}{3}]^2$ in the (x, y)-plane.

9.10. Hyperbolic functions. The **hyperbolic sine** and the **hyperbolic cosine** functions are defined by

$$\boxed{\begin{aligned} \sinh z &:= \frac{e^z - e^{-z}}{2} = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}, \\ \cosh z &:= \frac{e^z + e^{-z}}{2} = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}. \end{aligned}}$$

These functions are holomorphic in $\mathbb{C}$ with

$$\boxed{\cosh' z = \sinh z, \quad \sinh' z = \cosh z, \quad z \in \mathbb{C}.}$$

We have

$$\cosh z = \cos(iz), \quad \sinh z = -i\sin(iz), \quad \cosh^2 z - \sinh^2 z = 1,$$
$$\cosh(z + w) = \cosh z \cosh w + \sinh z \sinh w,$$

$$\sinh(z+w) = \sinh z \cosh w + \cosh z \sinh w,$$
$$\cos(x+iy) = \cos x \cosh y - i \sin x \sinh y,$$
$$\sin(x+iy) = \sin x \cosh y + i \cos x \sinh y.$$

The **hyperbolic tangent** and the **hyperbolic cotangent** are defined by

$$\boxed{\begin{aligned} \tanh z &:= \frac{\sinh z}{\cosh z}, \quad z \in \mathbb{C} \setminus (\pi\mathbb{Z} + \tfrac{\pi}{2})i, \\ \coth z &:= \frac{1}{\tanh z} = \frac{\cosh z}{\sinh z}, \quad z \in \mathbb{C} \setminus \pi i\mathbb{Z}. \end{aligned}}$$

Both functions are holomorphic in their domain of definition with

$$\boxed{\tanh' z = \frac{1}{\cosh^2 z} = 1 - \tanh^2 z, \quad \coth' z = -\frac{1}{\sinh^2 z} = 1 - \coth^2 z.}$$

Lemma.

(1) $\mathrm{im}(\sinh) = \mathrm{im}(\cosh) = \mathbb{C}$ *and* $\mathrm{im}(\tanh) = \mathrm{im}(\coth) = \mathbb{C} \setminus \{\pm i\}$.
(2) $\sinh^{-1}(0) = \tanh^{-1}(0) = \pi i\mathbb{Z}$ *and* $\cosh^{-1}(0) = \coth^{-1}(0) = (\pi\mathbb{Z} + \frac{\pi}{2})i$.
(3) $\mathrm{per}(\sinh) = \mathrm{per}(\cosh) = 2\pi i\mathbb{Z}$ *and* $\mathrm{per}(\tanh) = \mathrm{per}(\coth) = \pi i\mathbb{Z}$.

Proof. This follows from Proposition 9.6 and Lemma 9.9 in view of $\cosh z = \cos(iz)$, $\sinh z = -i\sin(iz)$, $\tanh z = -i\tan(iz)$, and $\coth z = i\cot(iz)$. □

10. The complex logarithm

A complex number $b \in \mathbb{C}$ is said to be a logarithm of $a \in \mathbb{C}$ if $e^b = a$; we write $b = \log a$. The properties of the exponential function imply:

- 0 has no logarithm.
- Every positive real number $r > 0$ has precisely one real logarithm $\log r$.
- Every $z = |z|e^{i\varphi} \in \mathbb{C} \setminus \{0\}$ has countably many logarithms,

$$\log|z| + i\varphi + 2\pi i\mathbb{Z}.$$

We see that the logarithm is a multi-valued function. In order to make sense of the logarithm as a single-valued function we make the following definition.

10.1. Branches of the logarithm. We say that a holomorphic function $\ell \in \mathcal{H}(U)$ in a region $U \subseteq \mathbb{C}$ is a **branch of the logarithm** if $\exp(\ell(z)) = z$ for all $z \in U$.

Lemma. *Let U be a region and $\ell \in \mathcal{H}(U)$. Then ℓ is a branch of the logarithm if and only if $\ell'(z) = 1/z$ in U and $\exp(\ell(c)) = c$ for some $c \in U$.*

Proof. Let us prove the non-trivial direction. If we set $f(z) := z\exp(-\ell(z))$, $z \in U$, then

$$f'(z) := \exp(-\ell(z)) - z\ell'(z)\exp(-\ell(z)) = 0.$$

By Proposition 4.3 and the condition $\exp(\ell(c)) = c$, we get $f = 1$. □

If ℓ is a branch of the logarithm in a region U then $\{\ell + 2\pi in : n \in \mathbb{Z}\}$ is the set of all branches of the logarithm in U; in fact if $\tilde{\ell}$ is another branch then $\exp(\ell(z) - \tilde{\ell}(z)) = 1$.

Example. The function $\log z := \sum_{n=1}^\infty (-1)^{n-1}\frac{(z-1)^n}{n}$ is a branch of the logarithm in $D_1(1)$. Note that $\log z = \lambda(z-1)$, where λ is the logarithmic series from (9.5).

Remark. The "inverse" of a non-injective function $f : \mathbb{C} \supseteq U \to \mathbb{C}$ is not single-valued. One can analyze the multivaluedness by considering the mapping $\tilde{f} : U \to \Gamma(f) \subseteq \mathbb{C} \times \mathbb{C}$ given by $\tilde{f}(z) := (z, f(z))$, where $\Gamma(f) := \{(z, f(z)) \in \mathbb{C} \times \mathbb{C} : z \in U\}$ denotes the graph of f. Then $\tilde{f}$ is invertible with inverse $\mathrm{pr}_1 |_{\Gamma(f)} : \Gamma(f) \to U$, and $f = \mathrm{pr}_2 \circ \tilde{f}$. So, instead of f, we may investigate $\mathrm{pr}_2 |_{\Gamma(f)} : \Gamma(f) \to \mathbb{C}$. The graph $\Gamma(f) = \{(z, w) \in \mathbb{C} \times \mathbb{C} : f(z) = w\}$ is a complex submanifold of $\mathbb{C} \times \mathbb{C}$ of complex dimension one, a so-called **Riemann surface**.

For instance, consider $\exp : \mathbb{C} \to \mathbb{C} \setminus \{0\}$. The exponential function maps each of the horizontal strips $S_n = \{z \in \mathbb{C} : 2\pi n - \pi < \operatorname{Im} z \le 2\pi n + \pi\}$, $n \in \mathbb{Z}$, bijectively onto $\mathbb{C} \setminus \{0\}$. We imagine each of the copies $T_n = \exp(S_n)$ of $\mathbb{C} \setminus \{0\}$ cut along the negative axis $(-\infty, 0)$; the cut regions T_n are called the sheets of the Riemann surface. The graph $\Gamma(\exp)$ is obtained by gluing the sheets T_n, $n \in \mathbb{Z}$, along their cuts: the upper boundary of T_n is glued with the lower boundary of T_{n+1}. The result is the **Riemann surface of the logarithm**; it resembles an infinite spiral staircase. If we set

$$\mathrm{Log} : T_n \to \mathbb{C},\ z \mapsto \log|z| + i(\varphi + 2\pi n), \quad n \in \mathbb{Z},$$

where $z = |z|e^{i\varphi}$, $\varphi \in (-\pi, \pi]$, then Log is single-valued on the Riemann surface of the logarithm.

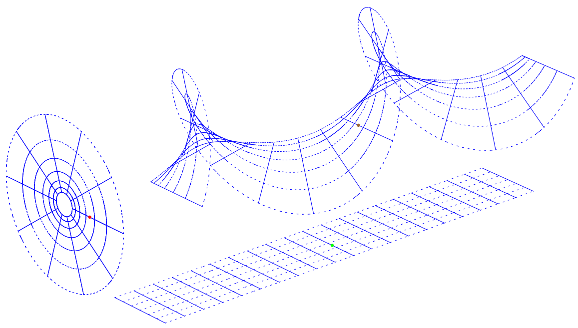


FIGURE 8. The image of the graph $\Gamma(\exp)$ under the injective projection $(x, y, e^x \cos y, e^x \sin y) \mapsto (y, e^x \cos y, e^x \sin y)$.

10.2. The principal branch of the logarithm. Let $\mathbb{C}^- := \mathbb{C} \setminus (-\infty, 0]$ be the **slit plane**. In $\mathbb{C}^-$ we define the **principal branch** of the logarithm by

$$\boxed{\log z := \log|z| + i\varphi, \quad z = |z|e^{i\varphi},\ \varphi \in (-\pi, \pi).} \tag{10.1}$$

Proposition. *The function* (10.1) *is a branch of the logarithm on the slit plane* $\mathbb{C}^-$. *On the disk* $D_1(1)$ *it coincides with the power series* $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(z-1)^n}{n}$.

Proof. Since $e^{\log r} = r$ if $r > 0$, we have

$$\exp(\log z) = \exp(\log|z| + i\varphi) = |z|e^{i\varphi} = z, \quad z \in \mathbb{C}^-.$$

The function $\log : \mathbb{C}^- \to \mathbb{C}$ is continuous, since $z \mapsto \log|z|$ and $z \mapsto i\varphi$ are continuous in $\mathbb{C}^-$.

Let us check that $\log \in \mathcal{H}(\mathbb{C}^-)$. Since exp is holomorphic, for $c = \log b$,

$$\exp(\log z) = \exp(\log b) + (\log z - \log b) f(\log z)$$

where f is continuous with $f(c) = \exp'(c) = \exp c$, i.e.,

$$\log z - \log b = (z - b)\frac{1}{f(\log z)};$$

f is non-zero in a neighborhood of c and $f \circ \log \in C(\mathbb{C}^-)$.

Thus log is a branch of the logarithm on $\mathbb{C}^-$. That it coincides with the logarithmic series follows from Lemma 10.1. □

Example. The function

$$\tilde{\ell}(z) := \frac{1}{2}\log(x^2 + y^2) + i \arctan\frac{y}{x}, \quad z = x + iy \in \mathbb{C} \setminus i\mathbb{R},$$

from Example 4.4 coincides with the principal value of the logarithm on the right half-plane $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. In the left half-plane $\tilde{\ell}$ is not a branch of the logarithm since $\exp(\tilde{\ell}(z)) = -z$ if $\operatorname{Re} z < 0$.

From now on log denotes the principal branch of the logarithm.

10.3. Properties of log**.** In contrast to the real logarithm, in general $\log(zw) \neq \log z + \log w$. If $z, w, zw \in \mathbb{C}^-$ then $z = |z|e^{i\varphi}$, $w = |w|e^{i\psi}$, and $zw = |zw|e^{i\theta}$, where $\varphi, \psi, \theta \in (-\pi, \pi)$ and $\theta = \varphi + \psi + \tau$ with $\tau \in \{-2\pi, 0, 2\pi\}$. Thus

$$\log(zw) = \log|zw| + i\theta = (\log|z| + i\varphi) + (\log|w| + i\psi) + i\tau = \log z + \log w + i\tau.$$

It follows that

$$\boxed{\log(zw) = \log z + \log w \;\Leftrightarrow\; \arg z + \arg w \in (-\pi, \pi).} \tag{10.2}$$

In particular, (10.2) holds if $\operatorname{Re} z > 0$ and $\operatorname{Re} w > 0$.

Since $\exp : \mathbb{C} \to \mathbb{C} \setminus \{0\}$ is not injective, log is not the inverse function of exp and the identity $\log(\exp z) = z$ does not hold in general. Let us investigate the composite function $\log \circ \exp$. It is not defined for $z = x + iy$ with $\exp z = e^x \cos y + ie^x \sin y \in (-\infty, 0]$, i.e., if $\sin y = 0$ and $\cos x \leq 0$, i.e., $y = (2n+1)\pi$, $n \in \mathbb{Z}$. Therefore $\log \circ \exp$ is well-defined in the union of the horizontal open strips

$$S_n := \{z \in \mathbb{C} : (2n-1)\pi < \operatorname{Im} z < (2n+1)\pi\}$$

of width 2π. If $z = x + iy \in S_n$ then $e^z = e^x e^{i(y-2n\pi)}$ where $y - 2n\pi \in (-\pi, \pi)$. Thus

$$\log(\exp z) = \log e^x + i(y - 2n\pi) = z - i2n\pi, \quad z \in S_n.$$

This implies the following proposition.

Proposition. *The restriction* $\exp : S_0 \to \mathbb{C}^-$ *is biholomophic*[10] *with inverse function* $\log : \mathbb{C}^- \to S_0$.

10.4. Power functions. If $\ell \in \mathcal{H}(U)$ is a branch of the logarithm, we can introduce the **power function** with exponent $\alpha \in \mathbb{C}$ with respect to ℓ,

$$\boxed{p_\alpha(z) := \exp(\alpha\ell(z)), \quad z \in U.}$$

Then $p_\alpha \in \mathcal{H}(U)$ and $p'_\alpha = \alpha p_{\alpha-1}$. For all $\alpha, \beta \in \mathbb{C}$ we have $p_\alpha p_\beta = p_{\alpha+\beta}$. If $n \in \mathbb{Z}$ then $p_n(z) = z^n$ for $z \in U$.

The power function with respect to the principal branch of the logarithm is defined by

$$\boxed{z^\alpha := e^{\alpha \log z}, \quad z \in \mathbb{C}^-,\ \alpha \in \mathbb{C}.}$$

[10] Cf. Section 21.1.

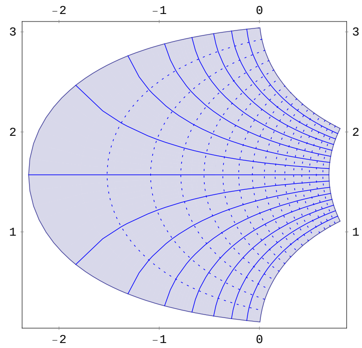


FIGURE 9. The image under log of the rectangle $[-1, 1] \times [\frac{1}{10}, 2]$ in the (x, y)-plane.

Then we have

$$\boxed{(z^\alpha)' = \alpha z^{\alpha-1}, \quad z^\alpha z^\beta = z^{\alpha+\beta}, \quad z \in \mathbb{C}^-.}$$

For instance, $1^\alpha = 1$, $i^i = e^{i \log i} = e^{i^2 \frac{\pi}{2}} = e^{-\frac{\pi}{2}}$, and $(z^{\frac{1}{n}})^n = (e^{\frac{1}{n} \log z})^n = e^{\log z} = z$. In particular, we use the symbol $\sqrt{z}$ for the **principal branch of the square root**,

$$\boxed{\sqrt{z} := e^{\frac{\log z}{2}}, \quad z \in \mathbb{C}^-.}$$

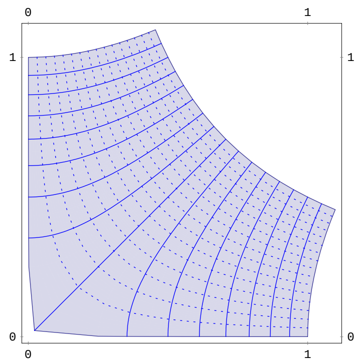


FIGURE 10. The image under the square root of the rectangle $[-1, 1] \times [0.001, 1]$ in the (x, y)-plane.

For $\alpha \in \mathbb{C}$ define the **binomial coefficients** by

$$\binom{\alpha}{0} := 1, \quad \binom{\alpha}{n} := \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}, \quad n = 1, 2, \ldots,$$

and consider the **binomial series**

$$\boxed{b_\alpha(z) := \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n.}$$

If $\alpha \in \mathbb{N}$ then b_α is a finite sum and the binomial formula results. For each $\alpha \in \mathbb{C} \setminus \mathbb{N}$ the binomial series has radius of convergence 1 which follows from the ratio test 7.4. Thus $b_\alpha \in \mathcal{H}(\mathbb{D})$ with

$$b'_\alpha(z) = \sum_{n=1}^{\infty} n \binom{\alpha}{n} z^{n-1} = \alpha \sum_{n=1}^{\infty} \binom{\alpha-1}{n-1} z^{n-1} = \alpha b_{\alpha-1}(z) = \frac{\alpha}{1+z} b_\alpha(z),$$

since $(1+z)b_{\alpha-1}(z) = \sum\left(\binom{\alpha-1}{n} + \binom{\alpha-1}{n-1}\right)z^n = \sum\binom{\alpha}{n}z^n$. The function $f(z) := b_\alpha(z)\exp(-\alpha\log(1+z))$ is holomorphic in $\mathbb{D}$ with $f' = 0$. By Proposition 4.3 and since $f(0) = 1$, $f = 1$. So we proved:

$$\boxed{(1+z)^\alpha = \sum_{n=0}^{\infty}\binom{\alpha}{n}z^n, \quad \alpha \in \mathbb{C},\ z \in \mathbb{D}.}$$

10.5. Inverse trigonometric functions. By Proposition 9.6 the sine and cosine function are surjective onto $\mathbb{C}$. Let us determine domains of injectivity. By (9.10), (9.11), and Proposition 9.6,

$$0 = \cos z - \cos w = -2\sin(\tfrac{z+w}{2})\sin(\tfrac{z-w}{2}) \ \Leftrightarrow\ z - w \in 2\pi\mathbb{Z} \text{ or } z + w \in 2\pi\mathbb{Z}$$
$$0 = \sin z - \sin w = 2\cos(\tfrac{z+w}{2})\sin(\tfrac{z-w}{2}) \ \Leftrightarrow\ z - w \in 2\pi\mathbb{Z} \text{ or } z + w \in 2\pi\mathbb{Z} + \pi$$

Thus sin is injective on each of the vertical strips

$$T_n := \{z \in \mathbb{C} : -\tfrac{\pi}{2} + n\pi < \operatorname{Re} z < \tfrac{\pi}{2} + n\pi\}, \quad n \in \mathbb{Z},$$

and cos is injective on each of the vertical strips

$$U_n := \{z \in \mathbb{C} : n\pi < \operatorname{Re} z < (n+1)\pi\}, \quad n \in \mathbb{Z}.$$

Since $V := \mathbb{C} \setminus \{x \in \mathbb{R} : |x| \ge 1\} = \sin(T_0) = \cos(U_0)$ the restrictions

$$\sin : T_0 \to V, \quad \cos : U_0 \to V,$$

are biholomorphic.[11]

Let us derive a formula for the inverse function $\sin^{-1} = \arcsin$:

$$w = \sin z = \frac{e^{iz} - e^{-iz}}{2i} \ \Leftrightarrow\ (e^{iz})^2 - 2iwe^{iz} - 1 = 0.$$

Then $e^{iz} = iw \pm \sqrt{1-w^2}$, where $\sqrt{\cdot}$ is the principal branch of the square root; if $w \in V$ then $1 - w^2 \in \mathbb{C}^-$. The image of T_0 under $z \mapsto e^{iz} = e^{i\operatorname{Re} z}e^{-\operatorname{Im} z}$ is the right half-plane $\{\zeta \in \mathbb{C} : \operatorname{Re}\zeta > 0\}$. So we have $e^{iz} = iw + \sqrt{1-w^2}$, in fact, the image of $\mathbb{C} \setminus \{ix \in i\mathbb{R} : |x| \ge 1\} \ni a \mapsto a + \sqrt{1+a^2}$ is the closed right-half plane:

$$\operatorname{Re}(a + \sqrt{1+a^2}) = \operatorname{Re} a + \operatorname{Re}\sqrt{1+a^2} \ge \operatorname{Re} a + |\operatorname{Re} a| \ge 0$$

since $\operatorname{Re}\sqrt{1+a^2} \ge |\operatorname{Re} a|$ which (as the image of $\sqrt{\cdot}$ is the right half-plane) is equivalent to

$$\left(\sqrt{1+a^2} + \overline{\sqrt{1+a^2}}\right)^2 \ge |a + \overline{a}|^2 \ \Leftrightarrow\ 2 + a^2 + \overline{a}^2 + 2|1+a^2| \ge a^2 + \overline{a}^2 + 2|a|^2$$
$$\Leftrightarrow\ 1 + |1+a^2| \ge |a|^2.$$

Hence

$$\boxed{\arcsin w = -i\log(iw + \sqrt{1-w^2}), \quad w \in \mathbb{C} \setminus \{x \in \mathbb{R} : |x| \ge 1\}.}$$

Similarly,

$$\boxed{\arccos w = -i\log(w + i\sqrt{1-w^2}), \quad w \in \mathbb{C} \setminus \{x \in \mathbb{R} : |x| \ge 1\},}$$

and $\arcsin w + \arccos w = \pi/2$. We also mention

$$\boxed{\arctan w = \frac{1}{2i}\log\frac{1+iw}{1-iw}, \quad w \in \mathbb{C} \setminus \{ix \in i\mathbb{R} : |x| \ge 1\}.}$$

Similarly, one may derive formulas for the inverse hyperbolic functions.

[11] See Theorem 21.1.

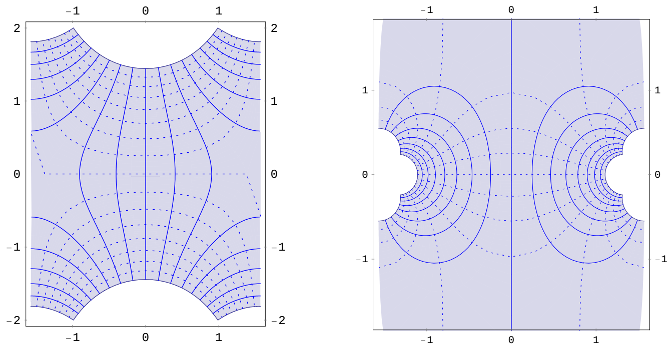


FIGURE 11. On the left we have the image under arcsin of the rectangle $[-\pi, \pi] \times [-2, 2]$ in the (x, y)-plane, on the right the image under arctan of the square $[-2, 2]^2$.

Complex integration

11. Integration along paths

11.1. Path integrals. A path $\gamma : [a, b] \to C$ is called **piecewise continuously differentiable (C^1)** if there are C^1-paths $\gamma_1, \dots, \gamma_n$ such that $\gamma = \gamma_1 + \cdots + \gamma_n$. Every polygon is a piecewise continuously differentiable path.

From now on we assume that any path is piecewise C^1.

Let $\gamma : [a, b] \to \mathbb{C}$ be a C^1-path and let $f \in C(|\gamma|)$ be a continuous function defined on the image $|\gamma| = \gamma([a, b])$; note that $|\gamma|$ is compact. The **path integral of f along γ** is defined by

$$\boxed{\int_\gamma f\,dz = \int_\gamma f(z)\,dz := \int_a^b f(\gamma(t))\gamma'(t)\,dt;}$$

the integral is well-defined, since $(f \circ \gamma) \cdot \gamma'$ is continuous on $[a, b]$.

For a path $\gamma = \gamma_1 + \cdots + \gamma_n$ with C^1-subpaths γ_i and $f \in C(|\gamma|)$ we define the path integral of f along γ by setting

$$\boxed{\int_\gamma f\,dz := \sum_{i=1}^n \int_{\gamma_i} f\,dz.}$$

It is easy to check that the integral is independent of the decomposition of γ into C^1-pieces.

Remark. The integral of f along γ is by definition the path integral of the 1-form $f\,dz = f(z)\,dz$ along γ. If we decompose $f = u + iv$ into real and imaginary part, then

$$f\,dz = (u + iv)(\,dx + i\,dy) = (u\,dx - v\,dy) + i(v\,dx + u\,dy)$$

and thus

$$\int_\gamma f\,dz = \int_\gamma (u\,dx - v\,dy) + i\int_\gamma (v\,dx + u\,dy).$$

11.2. Invariance under reparameterization. Two C^1-paths $\gamma_i : [a_i, b_i] \to \mathbb{C}$, $i = 1, 2$, are said to be **equivalent** if there is a bijective C^1-function $\varphi : [a_2, b_2] \to [a_1, b_1]$ with $\varphi' > 0$ and such that $\gamma_2 = \gamma_1 \circ \varphi$. The function φ is called a **reparameterization**. The requirement $\varphi' > 0$ implies that also the inverse φ^{-1} is C^1 and that φ preserves the orientation of the path γ.

Lemma. *Let γ_1 and γ_2 be equivalent C^1-paths and let f be any continuous function on $|\gamma_1| = |\gamma_2|$. Then*

$$\int_{\gamma_1} f\,dz = \int_{\gamma_2} f\,dz.$$

Proof. Let $\varphi : [a_2, b_2] \to [a_1, b_1]$ be a reparameterization with $\gamma_2 = \gamma_1 \circ \varphi$. Then

$$\begin{aligned}\int_{\gamma_2} f\,dz = \int_{a_2}^{b_2} f(\gamma_2(t))\gamma_2'(t)\,dt &= \int_{a_2}^{b_2} f(\gamma_1(\varphi(t)))\gamma_1'(\varphi(t))\varphi'(t)\,dt \\ &= \int_{a_1}^{b_1} f(\gamma_1(s))\gamma_1'(s)\,ds = \int_{\gamma_1} f\,dz. \qquad \square\end{aligned}$$

11.3. Properties of the path integral. Let us denote by $-\gamma$ the **inverse path** of γ given by $-\gamma(t) := \gamma(a + b - t)$. Note that

$$-(\gamma_1 + \cdots + \gamma_n) = (-\gamma_n) + (-\gamma_{n-1}) + \cdots + (-\gamma_1). \tag{11.1}$$

The **length** of a C^1-path $\gamma : [a, b] \to \mathbb{C}$ is defined by

$$\boxed{L(\gamma) := \int_a^b |\gamma'(t)|\,dt.}$$

Equivalent C^1-paths have the same length. If $\gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n$ is a path with C^1-pieces γ_i, we set

$$\boxed{L(\gamma) := L(\gamma_1) + L(\gamma_2) + \cdots + L(\gamma_n).}$$

Lemma. *Let γ be a path. Then:*

(1) *For all $f, g \in C(|\gamma|)$, $a, b \in \mathbb{C}$,*
$$\int_\gamma (af + bg)\,dz = a\int_\gamma f\,dz + b\int_\gamma g\,dz.$$

(2) *If $\tilde\gamma$ is a path that starts at the endpoint of γ, then for all $f \in C(|\gamma + \tilde\gamma|)$,*
$$\int_{\gamma+\tilde\gamma} f\,dz = \int_\gamma f\,dz + \int_{\tilde\gamma} f\,dz.$$

(3) *For all $f \in C(|\gamma|)$,*
$$\int_{-\gamma} f\,dz = -\int_\gamma f\,dz.$$

(4) *If $g \in \mathcal{H}(U)$ with continuous*[12] *derivative g' and γ is a path in U, then for all $f \in C(|g \circ \gamma|)$,*
$$\int_{g\circ\gamma} f(z)\,dz = \int_\gamma f(g(z))g'(z)\,dz.$$

(5) *For all $f \in C(|\gamma|)$,*
$$\Big|\int_\gamma f\,dz\Big| \le L(\gamma)\|f\|_\gamma. \tag{11.2}$$
where $\|f\|_\gamma := \|f\|_{|\gamma|} = \max_{z\in|\gamma|} |f(z)|$.

Proof. (1) and (2) are easy exercises. Let $\gamma : [a, b] \to \mathbb{C}$ be a C^1-path. Then

$$\begin{aligned}\int_{-\gamma} f\,dz &= \int_a^b f(\gamma(a + b - t))\gamma'(a + b - t)(-1)\,dt \\ &= \int_b^a f(\gamma(s))\gamma'(s)\,ds = -\int_\gamma f\,dz.\end{aligned}$$

[12] By Theorem 16.1, a holomorphic function has continuous complex derivatives of all orders.

Together with (11.1) it implies (3). Moreover, if $|\gamma| \subseteq U$ and $g \in \mathcal{H}(U)$ with continuous derivative g', then by the chain rule

$$\int_{g\circ\gamma} f\,dz = \int_a^b f(g(\gamma(t)))g'(\gamma(t))\gamma'(t)\,dt = \int_\gamma f(g(z))g'(z)\,dz,$$

that is (4). For (5) consider

$$\Big|\int_\gamma f\,dz\Big| \le \int_a^b |f(\gamma(t))||\gamma'(t)|\,dt \le \max_{z\in|\gamma|}|f(z)|\int_a^b |\gamma'(t)|\,dt = \|f\|_\gamma L(\gamma).$$

If $\gamma = \gamma_1+\gamma_2+\cdots+\gamma_n$ is a decomposition into C^1-pieces, then

$$\Big|\int_\gamma f\,dz\Big| \le \sum_{i=1}^n \Big|\int_{\gamma_i} f\,dz\Big| \le \sum_{i=1}^n L(\gamma_i)\|f\|_{\gamma_i} \le L(\gamma)\|f\|_\gamma,$$

since $|\gamma_i| \subseteq |\gamma|$. □

11.4. Integration and limits of functions. The estimate (11.2) implies that integration and limits of uniformly convergent continuous functions can be interchanged.

Proposition. *Let γ be a path and $f_n \in C(|\gamma|)$, $n \in \mathbb{N}$.*

(1) *If the sequence f_n converges uniformly to f on $|\gamma|$, then*

$$\lim \int_\gamma f_n\,dz = \int_\gamma \lim f_n\,dz = \int_\gamma f\,dz.$$

(2) *If the series $\sum f_n$ converges uniformly to f on $|\gamma|$, then*

$$\sum \int_\gamma f_n\,dz = \int_\gamma \sum f_n\,dz = \int_\gamma f\,dz.$$

Proof. (1) The integral $\int_\gamma f\,dz$ exists, since $f \in C(|\gamma|)$, by Theorem 6.1. By (11.2),

$$\Big|\int_\gamma f_n\,dz - \int_\gamma f\,dz\Big| = \Big|\int_\gamma (f_n - f)\,dz\Big| \le L(\gamma)\|f_n - f\|_\gamma \to 0.$$

(2) follows from (1). □

12. Index

We shall now see that special path integrals lead to analytic functions. As an application we introduce the index of a point with respect to a closed path.

12.1. Analytic functions defined by path integrals. [13]

Theorem. *Let γ be a path in $\mathbb{C}$, let $f \in C(|\gamma|)$, and set $U := \mathbb{C}\setminus|\gamma|$. Then the function*

$$F(z) := \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta, \quad z \in U,$$

is analytic on U. If $D_r(a)$ is any disk contained in U, then F has the power series expansion

$$F(z) = \sum_{n=0}^\infty \Big(\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - a)^{n+1}}\,d\zeta\Big)(z-a)^n, \quad z \in D_r(a), \tag{12.1}$$

[13] Essentially the same proof yields a much more general version of this theorem: the function $F(z) = \int_X \frac{d\mu(\zeta)}{\varphi(\zeta)-z}$, where μ is a complex measure on a measurable space X and $\varphi : X \to \mathbb{C}$ is a measurable function, is analytic in any domain $U \subseteq \mathbb{C}$ disjoint from $\varphi(X)$; see [**6**, 10.7].

and the derivatives

$$F^{(n)}(z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta, \quad z \in U,\ n \in \mathbb{N}. \tag{12.2}$$

Proof. Let $D = D_r(a)$ be any disk contained in U. For fixed $z \in D$, we find δ such that $|\frac{z-a}{\zeta-a}| < \delta < 1$ for all $\zeta \in |\gamma|$, and thus the geometric series

$$\frac{1}{\zeta - a}\sum_{n=0}^{\infty}\Big(\frac{z-a}{\zeta-a}\Big)^n = \frac{1}{\zeta - z}$$

converges uniformly on $|\gamma|$. Thus, by Proposition 11.4,

$$\begin{aligned} F(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta &= \frac{1}{2\pi i}\int_\gamma f(\zeta)\sum_{n=0}^{\infty}\frac{(z-a)^n}{(\zeta-a)^{n+1}}\, d\zeta \\ &= \sum_{n=0}^{\infty}\Big(\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-a)^{n+1}}\, d\zeta\Big)(z-a)^n, \end{aligned}$$

for $z \in D$. This shows (12.1), and hence F is analytic in U. The formulas in (12.2) follow from (12.1) and (8.2). □

12.2. The index of a point with respect to a closed path.

Theorem. *Let γ be a closed path in $\mathbb{C}$ and let $U := \mathbb{C} \setminus |\gamma|$. Then*

$$\boxed{\mathrm{ind}_\gamma(z) := \frac{1}{2\pi i}\int_\gamma \frac{d\zeta}{\zeta - z}, \quad z \in U,}$$

defines an integer valued function $\mathrm{ind}_\gamma : U \to \mathbb{Z}$ that is constant in each connected component of U and 0 in the unbounded component of U.

Since $|\gamma|$ is compact, it lies in some disk D. The complement of D is connected and lies in some connected component of U. So U has precisely one unbounded connected component.

Proof. Let $[0,1]$ be the parameter interval of γ. By definition

$$\mathrm{ind}_\gamma(z) := \frac{1}{2\pi i}\int_0^1 \frac{\gamma'(t)}{\gamma(t) - z}\, dt. \tag{12.3}$$

Let us set

$$\varphi(t) := \exp\int_0^t \frac{\gamma'(s)}{\gamma(s) - z}\, ds, \quad t \in [0,1], \tag{12.4}$$

and show that $\varphi(1) = 1$. This will prove that $\mathrm{ind}_\gamma(z)$ is an integer, since $\ker(\exp) = 2\pi i\mathbb{Z}$, by Theorem 9.4. By differentiating (12.4), we find

$$\varphi'(t) = \frac{\varphi(t)\gamma'(t)}{\gamma(t) - z}$$

except on a finite set of points where γ is not differentiable. It follows that the function $[0,1] \ni t \mapsto \frac{\varphi(t)}{\gamma(t)-z}$ is continuous and has vanishing derivative on all but finitely many points t, thus it is constant;

$$\partial_t\Big(\frac{\varphi(t)}{\gamma(t) - z}\Big) = \frac{1}{\gamma(t) - z}\Big(\varphi'(t) - \frac{\varphi(t)\gamma'(t)}{\gamma(t) - z}\Big).$$

Since $\varphi(0) = 1$ we have

$$\varphi(t) = \frac{\gamma(t) - z}{\gamma(0) - z},$$

and as the path γ is closed, we may conclude that $\varphi(1) = 1$ as required.

The function $\operatorname{ind}_\gamma$ is analytic in U, by Theorem 12.1, and hence maps connected sets to connected sets. Since $\operatorname{ind}_\gamma$ is integer valued it must be constant on each connected component of U.

We can infer from (11.2), or directly from (12.3), that $|\operatorname{ind}_\gamma(z)| < 1$ if $|z|$ is sufficiently large. So $\operatorname{ind}_\gamma(z) = 0$ on the unbounded component of U. □

The integer $\operatorname{ind}_\gamma(z)$ is called the **index** or **winding number** of z with respect to γ. It counts the number of times the path γ winds counter-clockwise around z. To see this write $\zeta - z = re^{i\varphi}$ in polar coordinates. Then $d\zeta = e^{i\varphi}\,dr + ire^{i\varphi}\,d\varphi$, thus

$$\frac{d\zeta}{\zeta - z} = \frac{dr}{r} + i\,d\varphi = d(\log r) + i\,d\varphi,$$

and consequently, if γ is parameterized in polar form $\gamma(t) - z = r(t)e^{i\varphi(t)}$ by C^1-functions $r(t)$ and $\varphi(t)$,

$$\operatorname{ind}_\gamma(z) = \frac{1}{2\pi i}\int_\gamma \frac{d\zeta}{\zeta - z} = \frac{1}{2\pi i}\int_\gamma d(\log r) + \frac{1}{2\pi}\int_\gamma d\varphi = \frac{\varphi(1) - \varphi(0)}{2\pi}, \qquad (12.5)$$

since γ is closed and thus $r(0) = r(1)$. That means that $2\pi \operatorname{ind}_\gamma(z)$ is the total change of the argument of $\gamma(t) - z$ as t runs from 0 to 1.

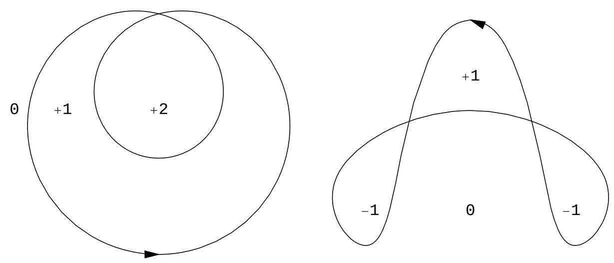


FIGURE 12. Examples for the index function; cf. (24.8).

Corollary. *Let D be a disk. Then*

$$\boxed{\operatorname{ind}_{\partial D}(z) = \begin{cases} 1 & z \in D \\ 0 & z \notin D \end{cases}} \qquad (12.6)$$

with the understanding that the circle ∂D is positively oriented.

Proof. Let $D = D_r(a)$ and $\gamma : [0, 2\pi] \to \mathbb{C}$, $\gamma(t) := a + re^{it}$. By Theorem 12.2 it suffices to compute $\operatorname{ind}_\gamma(a)$,

$$\operatorname{ind}_\gamma(a) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{rie^{it}}{a + re^{it} - a}\,dt = 1. \qquad □$$

13. Primitives and integrability

Next we investigate the existence of primitives of functions. We will see that a function f has a primitive in some domain U if and only if $\int_\gamma f\,dz = 0$ for all closed paths γ in U.

13.1. Primitives. Let $U \subseteq \mathbb{C}$ be a domain and let $f \in C(U)$. A **primitive** or **antiderivative** of f in U is a holomorphic function $F : U \to \mathbb{C}$ such that $F' = f$.

Proposition. *Let $f \in C(U)$. A function $F : U \to \mathbb{C}$ is a primitive of f if and only if for every path $\gamma : [a, b] \to U$ in U,*

$$\int_\gamma f\,dz = F(\gamma(b)) - F(\gamma(a)). \tag{13.1}$$

Proof. Assume $F' = f$. If γ is C^1 then, by the fundamental theorem of calculus,

$$\int_\gamma f\,dz = \int_a^b F'(\gamma(t))\gamma'(t)\,dt = \int_a^b (F \circ \gamma)'(t)\,dt = F(\gamma(b)) - F(\gamma(a)).$$

If $\gamma = \gamma_1 + \cdots + \gamma_n$ with C^1-pieces γ_i, we have

$$\int_\gamma f\,dz = \sum_{i=1}^n F(z_\Omega(\gamma_i)) - F(z_A(\gamma_i)) = F(\gamma(b)) - F(\gamma(a)),$$

where $z_A(\gamma_i)$ is the initial and $z_\Omega(\gamma_i)$ the endpoint of γ_i.

Conversely, assume that (13.1) holds for every path γ in U. We will show $F'(c) = f(c)$ for each $c \in U$. Let D be a small disk centered at c and contained in U. By (13.1),

$$F(z) = F(c) + \int_{[c,z]} f\,d\zeta \quad \text{for all } z \in D.$$

We define

$$F_1(z) := \begin{cases} \dfrac{1}{z-c}\displaystyle\int_{[c,z]} f\,d\zeta & z \in D \setminus \{c\} \\ f(c) & z = c \end{cases}$$

and show that F_1 is continuous in c; this will imply the assertion. For $z \in D \setminus \{c\}$,

$$|F_1(z) - F_1(c)| = \Big|\frac{1}{z-c}\int_{[c,z]} f(\zeta) - f(c)\,d\zeta\Big| \le \|f - f(c)\|_{[c,z]},$$

and thus continuity of f in c implies continuity of F_1 in c. $\square$

The proposition states that $f \in C(U)$ has a primitive if and only if the integral $\int_\gamma f\,dz$ depends only on the endpoints of the path γ and is **independent of the path** γ itself:

$$\int_{\gamma_1} f\,dz = \int_{\gamma_2} f\,dz \quad \text{for all paths } \gamma_i \text{ in } U \text{ with fixed endpoints.}$$

Taking $\gamma = \gamma_1 + (-\gamma_2)$ we get the following corollary.

Corollary. *If $f \in C(U)$ has a primitive then for every closed path γ in U,*

$$\int_\gamma f(z)\,dz = 0.$$

We will see shortly that also the converse holds.

13.2. Example.

(1) Let γ be a closed path in the complex plane and let $n \in \mathbb{Z}$. Then

$$\boxed{\int_\gamma z^n\,dz = \begin{cases} 0 & n \ge 0, \\ 2\pi i \operatorname{ind}_\gamma(0) & n = -1,\ 0 \notin |\gamma|, \\ 0 & n \le -2,\ 0 \notin |\gamma|. \end{cases}}$$

This follows from Theorem 12.2 and from the fact that for all integers $n \neq -1$, z^n has a primitive $z^{n+1}/(n+1)$. We may conclude that, given $c \in \mathbb{C}$, there is no neighborhood U of c such that $(z-c)^{-1} \in \mathcal{H}(\mathbb{C} \setminus \{c\})$ has a primitive in $U \setminus \{c\}$.

(2) If $f(z) = \sum a_n (z-c)^n$ is a convergent power series with disk of convergence $D_\rho(c)$ then $F(z) = \sum \frac{a_n}{n+1}(z-c)^{n+1}$ is a primitive of f on $D_\rho(c)$; this follows from Theorem 8.1.

13.3. Holomorphic functions with vanishing derivative are locally constant. We shall give now a complex proof of Proposition 4.3.

Corollary. *If $f \in \mathcal{H}(U)$ satisfies $f' = 0$, then f is locally constant.*

Proof. Assume without loss of generality that U is connected. Fix $c \in U$. For each $z \in U$ there is a path γ which joins c to z; see Section 2.5. Then

$$0 = \int_\gamma f' \, dz = f(z) - f(c),$$

that is $f(z) = f(c)$ for all $z \in U$. □

Hence the difference of two primitives of $f \in C(U)$ is locally constant in U.

13.4. Integrability. We will say that a function $f \in C(U)$ is **integrable** in U if it has a primitive $F \in \mathcal{H}(U)$.

Proposition. *A function $f \in C(U)$ is integrable if and only if $\int_\gamma f \, dz = 0$ for every closed path γ in U.*

Proof. One direction was observed in Corollary 13.1. For the other direction, suppose that $\int_\gamma f \, dz = 0$ for every closed path in U. We need to find a primitive F of f on U. We may assume that U is connected. Fix some point $c \in U$ and for every $z \in U$ choose a path γ_z in U that joins c to z. Define

$$F(z) := \int_{\gamma_z} f \, d\zeta, \quad z \in U.$$

Let $w \in U$ and let γ be a path from w to z. Then $\gamma_w + \gamma - \gamma_z$ is a closed path in U, and so

$$0 = \int_{\gamma_w + \gamma - \gamma_z} f \, d\zeta = \int_{\gamma_w} f \, d\zeta + \int_\gamma f \, d\zeta - \int_{\gamma_z} f \, d\zeta = F(w) + \int_\gamma f \, d\zeta - F(z).$$

This implies that F is a primitive of f by Proposition 13.1. □

13.5. Integrability on star-shaped domains. It is practically impossible to check that $\int_\gamma f \, dz = 0$ for *all* closed paths in a domain. For special domains the integrability criterion in Proposition 13.4 can be weakened considerably.

A subset $A \subseteq \mathbb{C}$ is called **star-shaped** if there is a point $c \in A$ such that for all $z \in A$ the segment $[c, z]$ lies in A. The point c is called the **center** of A. Clearly, every star-shaped domain in $\mathbb{C}$ is connected.

A subset $A \subseteq \mathbb{C}$ is **convex** if for any two points $z, w \in A$ also $[z, w] \subseteq A$. A convex set is star-shaped and every point is a center. The **convex hull** of a set $A \subseteq \mathbb{C}$ is the intersection of all convex sets that contain A; it is obviously convex.

Example. Every disk is convex. The slit plane $\mathbb{C}^-$ is star-shaped but not convex; every point in $(0, \infty)$ is a center. The punctured plane $\mathbb{C} \setminus \{0\}$ is not star-shaped.

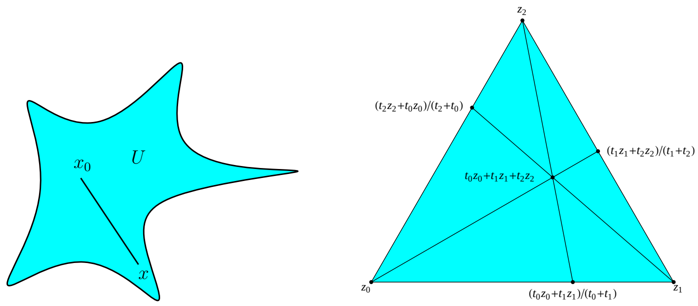


FIGURE 13. On the left we have a star-shaped domain U with center x_0. On the right we see the convex hull of the three points z_0, z_1, z_2, i.e., the triangle $\Delta(z_0, z_1, z_2) = \{t_0z_0+t_1z_0+t_2z_2 : t_i \geq 0,\ t_0+t_1+t_2 = 1\}$.

Let (a, b, c) be an ordered triple of complex numbers. The convex hull of $\{a, b, c\}$ defines a (closed) **triangle**

$$\boxed{\Delta = \Delta(a, b, c) = \{ra + sb + tc : r, s, t \geq 0,\ r + s + t = 1\}}$$

with oriented boundary

$$\boxed{\partial\Delta = [a, b] + [b, c] + [c, a]}$$

and vertices a, b, c.

Proposition. *Let $U \subseteq \mathbb{C}$ be a star-shaped domain with center c. Let $f \in C(U)$ satisfy $\int_{\partial\Delta} f\,dz = 0$ for every triangle $\Delta \subseteq U$ one vertex of which is c. Then f is integrable in U and*

$$F(z) := \int_{[c,z]} f\,d\zeta, \quad z \in U,$$

is a primitive of f.

Proof. The function F is well-defined since U is star-shaped. Let $w \in U$. Since U is open, there is a small open disk D centered at w and contained in U. For $z \in D$ the triangle $\Delta = \Delta(c, w, z)$ lies in U, because U is star-shaped. By assumption,

$$F(z) - F(w) = \int_{[c,z]} f\,d\zeta - \int_{[c,w]} f\,d\zeta = \int_{[w,z]} f\,d\zeta, \quad z \in D.$$

The arguments at the end of the proof of Proposition 13.1 show that $F'(w) = f(w)$. Thus F is a primitive of f in U. $\square$

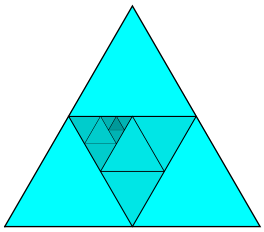
14. Cauchy's theorem for star-shaped domains

As we will prove in this section, holomorphic functions are always integrable on star-shaped domains. This is a local version of Cauchy's theorem (for a global version see Sections 24 and 26).

14.1. Goursat's lemma.

Lemma. *Let $f \in \mathcal{H}(U)$ and let Δ be a triangle contained in U. Then*

$$\int_{\partial\Delta} f\,dz = 0.$$



Proof. Subdivide the triangle Δ into four triangles Δ^i, $i = 1, 2, 3, 4$, by joining the midpoints of the three edges of Δ by line segments. Note that $L(\partial\Delta^i) = 2^{-1}L(\partial\Delta)$, $i = 1, 2, 3, 4$.

Let us set $I(\Delta) := \int_{\partial\Delta} f\,dz$. Then, by Lemma 11.3,

$$I(\Delta) = \int_{\partial\Delta} f\,dz = \sum_{i=1}^{4} \int_{\partial\Delta^i} f\,dz = \sum_{i=1}^{4} I(\Delta^i).$$

(The line segments connecting the midpoints are traversed exactly twice in opposite directions.) It follows that there is at least one i such that $|I(\Delta^i)| \ge 4^{-1}|I(\Delta)|$. Denote this triangle by Δ_1. Repeat the argument with Δ_1 in place of Δ, etc. We obtain a sequence of triangles $\Delta \supseteq \Delta_1 \supseteq \Delta_2 \supseteq \cdots$ such that

$$|I(\Delta)| \le 4^k|I(\Delta_k)|, \quad L(\partial\Delta) = 2^k L(\partial\Delta_k), \quad k \ge 1. \tag{14.1}$$

The intersection of all these triangles Δ_k, $k \ge 1$, contains precisely one point c; see Section 2.3.

Since f is $\mathbb{C}$-differentiable at c, there exists a function $h \in C(U)$[14] with $h(c) = 0$ and

$$f(z) = f(c) + (z - c)(f'(c) + h(z)), \quad z \in U.$$

By Example 13.2,

$$I(\Delta_k) = \int_{\partial\Delta_k} f(c) + (z-c)f'(c) + (z-c)h(z)\,dz = \int_{\partial\Delta_k} (z-c)h(z)\,dz$$

and thus, by (14.1) and (11.2),

$$\begin{aligned}|I(\Delta)| \le 4^k|I(\Delta_k)| &\le 4^k L(\partial\Delta_k) \sup_{z\in\partial\Delta_k} |(z-c)h(z)| \\ &\le 4^k L(\partial\Delta_k)^2 \|h\|_{\partial\Delta_k} \le L(\partial\Delta)^2\|h\|_{\partial\Delta_k}, \quad k \ge 1,\end{aligned}$$

since evidently the diameter of a triangle is bounded by the length of its boundary.

Let $\epsilon > 0$. By continuity of h at c and since $h(c) = 0$, there is a $\delta > 0$ such that $\|h\|_{D_\delta(c)} < \epsilon$. There exists k_0 so that $\Delta_k \subseteq D_\delta(c)$ for all $k \ge k_0$. Therefore, $\|h\|_{\partial\Delta_k} < \epsilon$ if $k \ge k_0$ and hence $|I(\Delta)| \le L(\partial\Delta)^2\epsilon$. Since $\epsilon > 0$ was arbitrary, we may conclude that $I(\Delta) = 0$. $\square$

14.2. Cauchy's theorem for star-shaped domains.

Theorem. *Let $U \subseteq \mathbb{C}$ be a star-shaped domain with center c. Each $f \in \mathcal{H}(U)$ is integrable. The function $F(z) := \int_{[c,z]} f\,d\zeta$ is a primitive of f on U. In particular,*

$$\int_\gamma f\,dz = 0$$

for every closed path γ in U.

---

[14]The definition of $\mathbb{C}$-differentiability gives continuity at c; at points $z \neq c$ continuity is inherited from f, in fact, $h(z) = \frac{f(z)-f(c)}{z-c} - f'(c)$.

Proof. This follows from Goursat's lemma 14.1 and the integrability criterion in Proposition 13.5. □

14.3. Evaluation of integrals. Cauchy's theorem can be used as a tool to evaluate certain integrals; in particular, real integrals. We will illustrate this by means of the following example.

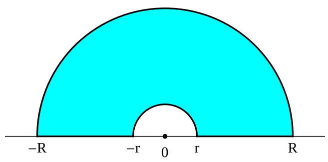
Example. We will show that

$$\boxed{\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.} \tag{14.2}$$

The function $f(z) := \frac{e^{iz}}{z}$ is holomorphic in $\mathbb{C} \setminus \{0\}$. By Cauchy's theorem 14.2,

$$\int_{\gamma_{R,r}} f\,dz = 0 \tag{14.3}$$

where $\gamma_{R,r}$ is the positively oriented boundary of $\{z \in \mathbb{C} : r \le |z| \le R,\ \operatorname{Im} z \ge 0\}$; the set $\mathbb{C} \setminus i(-\infty, 0]$ is star-shaped.



If $\gamma_\rho : [0,\pi] \to \mathbb{C}$, $\gamma_\rho(t) := \rho e^{it}$ parameterizes the upper half-circle of radius $\rho > 0$ centered at 0, then (14.3) reads

$$\int_{-R}^{-r} \frac{e^{ix}}{x}\,dx + \int_{-\gamma_r} \frac{e^{iz}}{z}\,dz + \int_r^R \frac{e^{ix}}{x}\,dx + \int_{\gamma_R} \frac{e^{iz}}{z}\,dz = 0.$$

By the substitution formula,

$$\int_{-R}^{-r} \frac{e^{ix}}{x}\,dx = -\int_r^R \frac{e^{-ix}}{x}\,dx.$$

Thus

$$2i \int_r^R \frac{\sin x}{x}\,dx = \int_r^R \frac{e^{ix} - e^{-ix}}{x}\,dx = \int_{\gamma_r} \frac{e^{iz}}{z}\,dz - \int_{\gamma_R} \frac{e^{iz}}{z}\,dz. \tag{14.4}$$

We have

$$\int_{\gamma_\rho} \frac{e^{iz}}{z}\,dz = i \int_0^\pi e^{i\rho(\cos t + i \sin t)}\,dt.$$

Since $e^{ir(\cos t + i \sin t)} \to 1$ as $r \to 0$ uniformly for $t \in [0,\pi]$ we find

$$\int_{\gamma_r} \frac{e^{iz}}{z}\,dz \to i \int_0^\pi dt = i\pi \quad \text{as } r \to 0.$$

On the other hand, for $0 < \epsilon < \pi$,

$$\Big|\int_0^\epsilon e^{iR(\cos t + i\sin t)}\,dt\Big| \le \int_0^\epsilon e^{-R\sin t}\,dt \le \int_0^\epsilon dt = \epsilon,$$
$$\Big|\int_{\pi-\epsilon}^\pi e^{iR(\cos t + i\sin t)}\,dt\Big| \le \int_{\pi-\epsilon}^\pi e^{-R\sin t}\,dt \le \int_{\pi-\epsilon}^\pi dt = \epsilon,$$
$$\Big|\int_\epsilon^{\pi-\epsilon} e^{iR(\cos t + i\sin t)}\,dt\Big| \le \int_\epsilon^{\pi-\epsilon} e^{-R\sin t}\,dt \le e^{-R\sin\epsilon} \int_\epsilon^{\pi-\epsilon} dt = (\pi - 2\epsilon)e^{-R\sin\epsilon},$$

and thus

$$\int_{\gamma_R} \frac{e^{iz}}{z}\, dz \to 0 \quad \text{ as } R \to \infty.$$

This implies (14.2) in view of (14.4).

15. Cauchy's integral formula

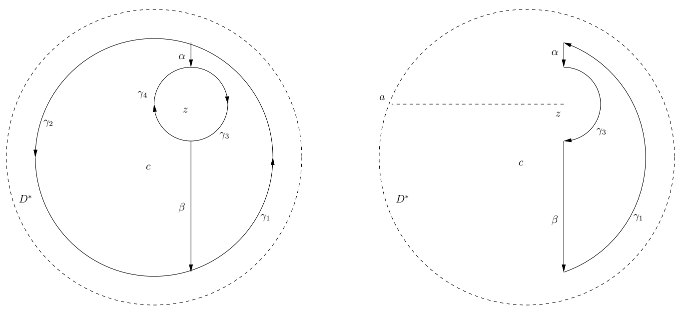
As a first application of Cauchy's theorem we shall prove Cauchy's integral formula for disks. For a general version we refer to (24.6).

15.1. Changing the path of integration. Let $U \subseteq \mathbb{C}$ be a domain and $z \in U$. Let D be a disk containing z and such that $\overline{D} \subseteq U$. Let $f \in \mathcal{H}(U \setminus \{z\})$. In order to compute $\int_{\partial D} f\, d\zeta$ we may replace the path ∂D by a circle centered at z:

Lemma. *If $\gamma^s : [0, 2\pi] \to D$, $\gamma^s(t) = z + se^{it}$, $s > 0$, then*

$$\int_{\partial D} f\, d\zeta = \int_{\gamma^s} f\, d\zeta.$$

Proof. Let $D^* \supseteq \overline{D}$ be a disk with the same center c as D such that $f \in \mathcal{H}(D^* \setminus \{z\})$.



Since $D^* \setminus [a, z]$ is a star-shaped domain we have $0 = \int_{\gamma_1+\alpha+\gamma_3+\beta} f\, d\zeta$, and similarly $0 = \int_{-\beta+\gamma_4-\alpha+\gamma_2} f\, d\zeta$. The assertion follows, since $\partial D = \gamma_1 + \gamma_2$ and $\gamma^s = -\gamma_3 - \gamma_4$. □

Corollary. *If f is bounded near z then $\int_{\partial D} f\, d\zeta = 0$.*

Proof. By (11.2),

$$\Big| \int_{\partial D} f\, d\zeta \Big| = \Big| \int_{\gamma^s} f\, d\zeta \Big| \le 2\pi s \|f\|_{\gamma^s} \to 0 \quad \text{ as } s \to 0,$$

since f is bounded near z. □

15.2. Cauchy's integral formula for disks.

Theorem. *Let $f \in \mathcal{H}(U)$ and let D be a disk such that $\overline{D} \subseteq U$. Then*

$$\boxed{f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z}\, d\zeta, \quad z \in D.} \tag{15.1}$$

Proof. Let $z \in D$ be fixed. The function

$$g(\zeta) := \begin{cases} \dfrac{f(\zeta) - f(z)}{\zeta - z} & \zeta \in D \setminus \{z\} \\ f'(z) & \zeta = z \end{cases}.$$

is continuous in D and holomorphic in $D\backslash\{z\}$. Thus $\int_{\partial D} g\,d\zeta = 0$, by Corollary 15.1, and so

$$0 = \int_{\partial D} g\,d\zeta = \int_{\partial D} \frac{f(\zeta)}{\zeta - z}\,d\zeta - f(z)\int_{\partial D} \frac{d\zeta}{\zeta - z} \overset{(12.6)}{=} \int_{\partial D} \frac{f(\zeta)}{\zeta - z}\,d\zeta - 2\pi i f(z). \qquad \square$$

Note that by (15.1) each value $f(z)$, $z \in D$, is completely determined by the values of f on ∂D. Moreover, on the right-hand side the variable z is no longer tied to f. The function $(\zeta, z) \mapsto 1/(\zeta - z)$ is called the **Cauchy kernel**.

15.3. Mean value property. As a special case of (15.1) we obtain:

Proposition. *If f is holomorphic in a neighborhood of the disk $D_r(c)$ then*

$$\boxed{f(c) = \frac{1}{2\pi}\int_0^{2\pi} f(c + re^{it})\,dt.} \tag{15.2}$$

Proof. Use the parameterization $[0, 2\pi] \ni t \mapsto c + re^{it}$ for the boundary of the disk $D = D_r(c)$ in (15.1). $\square$

Formula (15.2) implies the following estimate

$$\boxed{|f(c)| \le \|f\|_{\partial D_r(c)}.} \tag{15.3}$$

15.4. Cauchy's integral formula for C^1-functions. For C^1-functions, Cauchy's theorem 14.2 is a special case of Stokes' theorem. Let $U \subseteq \mathbb{C}$ be a bounded domain such that the boundary ∂U consists of a finite number of simple closed C^1-paths. If $g \in C^1(\overline{U})$, then by Stokes' theorem,

$$\int_{\partial U} g\,d\zeta = \iint_U dg \wedge d\zeta = \iint_U \big(g_\zeta\,d\zeta + g_{\overline{\zeta}}\,d\overline{\zeta}\big) \wedge d\zeta = \iint_U g_{\overline{\zeta}}\,d\overline{\zeta} \wedge d\zeta, \tag{15.4}$$

where ∂U is oriented such that U lies on the left of ∂U. So if g is also holomorphic in U, then $g_{\overline{\zeta}} = 0$ and hence $\int_{\partial U} g\,d\zeta = 0$.

We shall see now that Cauchy's integral formula 15.1 is a special case of a more general formula for C^1-functions.

Theorem. *Let $U \subseteq \mathbb{C}$ be a bounded domain such that the boundary ∂U consists of a finite number of simple closed C^1-paths. If $f \in C^1(\overline{U})$, then*

$$\boxed{f(z) = \frac{1}{2\pi i}\int_{\partial U} \frac{f(\zeta)}{\zeta - z}\,d\zeta + \frac{1}{2\pi i}\iint_U \frac{f_{\overline{\zeta}}(\zeta)}{\zeta - z}\,d\zeta \wedge d\overline{\zeta}, \quad z \in U.} \tag{15.5}$$

Here ∂U is oriented such that U lies on the left of ∂U.

Proof. For fixed z set $U_\epsilon := \{\zeta \in U : |z - \zeta| > \epsilon\}$, where $\epsilon > 0$ is smaller that the distance of ζ to the complement of U. We apply (15.4) to $g : U_\epsilon \to \mathbb{C}$, $g(\zeta) = \frac{f(\zeta)}{\zeta - z}$, and note that $U_\epsilon \ni \zeta \mapsto (\zeta - z)^{-1}$ is holomorphic,

$$\iint_{U_\epsilon} \frac{f_{\overline{\zeta}}(\zeta)}{\zeta - z}\,d\overline{\zeta} \wedge d\zeta = \int_{\partial U} \frac{f(\zeta)}{\zeta - z}\,d\zeta - \int_0^{2\pi} f(z + \epsilon e^{it}) i\,dt. \tag{15.6}$$

Now $\zeta \mapsto (\zeta - z)^{-1}$ is integrable over U, in fact, if $\zeta = \xi + i\eta = re^{i\varphi}$,

$$\iint_U |\zeta - z|^{-1}\,d(\xi, \eta) = \iint_{U - z} |\zeta|^{-1}\,d(\xi, \eta) \le \int_0^{2\pi}\int_0^R dr d\varphi < \infty,$$

since $U - z$ (being bounded) is contained in a large disk $D_R(0)$. Together with the fact that f and $f_{\overline{\zeta}}$ are continuous, it implies (15.5) by letting $\epsilon \to 0$ in (15.6). $\square$

Power series representation and applications

16. Holomorphic functions are analytic

Cauchy's integral formula (15.1) implies that holomorphic functions admit power series expansions.

16.1. Power series expansion of holomorphic functions.

Theorem. *Let $U \subseteq \mathbb{C}$ be a domain. Let $c \in U$ and let $D_d(c)$ be the largest disk centered at c and contained in U. Then each $f \in \mathcal{H}(U)$ can be expanded into a power series $\sum_{n=0}^{\infty} a_n(z-c)^n$ which converges normally to f in $D_r(c)$ for each $0 < r < d$. The Taylor coefficients a_n are given by*

$$a_n = \frac{f^{(n)}(c)}{n!} = \frac{1}{2\pi i} \int_{\partial D_r(c)} \frac{f(\zeta)}{(\zeta - c)^{n+1}} \, d\zeta, \quad 0 < r < d. \tag{16.1}$$

We have the integral formulas

$$\boxed{f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta, \quad z \in D, \; n \in \mathbb{N},} \tag{16.2}$$

for every open disk D such that $\overline{D} \subseteq U$.

Proof. By Cauchy's integral formula (15.1),

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} \, d\zeta, \quad z \in D,$$

for every open disk D such that $\overline{D} \subseteq U$, in particular, for $D = D_r(c)$ with $0 < r < d$. By Theorem 12.1, $f(z) = \sum_{n=0}^{\infty} a_n(z-c)^n$, for $z \in D_r(c)$, with a_n given by (16.1). Since the coefficients are uniquely given by $a_n = f^{(n)}(c)/n!$, the same power series is obtained for every $r < d$. Thus the power series converges normally to f in each $D_r(c)$, $0 < r < d$; see Theorem 7.3. The integral formulas (16.2) follow from (12.2). $\square$

16.2. Morera's theorem.

A first consequence of Theorem 16.1 is Morera's theorem which is a converse of Cauchy's theorem.

Theorem. *Let $U \subseteq \mathbb{C}$ be a domain. If $f \in C(U)$ satisfies $\int_{\partial\Delta} f(z)\, dz = 0$ for every triangle $\Delta \subseteq U$, then $f \in \mathcal{H}(U)$.*

Proof. It suffices to show that f is holomorphic in every open disk $D \subseteq U$. By Proposition 13.5, f has a primitive in D. By Theorem 16.1, the derivative of a holomorphic function is holomorphic, and so f is holomorphic in D. $\square$

Summarizing we obtain the following characterization of holomorphy.

Corollary. *Let $U \subseteq \mathbb{C}$ be a domain and let $f \in C(U)$. The following are equivalent:*

(1) *f is holomorphic.*
(2) *$\int_{\partial\Delta} f(z)\, dz = 0$ for every triangle $\Delta \subseteq U$.*

(3) *f is locally integrable.*
(4) *For every disk D with $\overline{D} \subseteq U$, $f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)}{\zeta - z}\, d\zeta$ for all $z \in D$.*
(5) *f is analytic.*

16.3. Holomorphy of path integrals. As a corollary of Morera's theorem 16.2 we get a result on holomorphy of path integrals depending on parameters.

Proposition. *Let γ be a path in $\mathbb{C}$ and let $U \subseteq \mathbb{C}$ be a domain. If $g \in C(|\gamma| \times U)$ and $g(w, \cdot) \in \mathcal{H}(U)$ for every $w \in |\gamma|$, then*

$$h(z) := \int_\gamma g(w, z)\, dw, \quad z \in U,$$

is holomorphic in U.

Proof. Let $\Delta \subseteq U$ be a triangle. Then, by Fubini's theorem,

$$\int_{\partial\Delta} h\, dz = \int_{\partial\Delta}\int_\gamma g(w, z)\, dw\, dz = \int_\gamma \int_{\partial\Delta} g(w, z)\, dz\, dw = 0,$$

since $\int_{\partial\Delta} g(w, z)\, dz = 0$ for all $w \in |\gamma|$ (because $g(w, \cdot) \in \mathcal{H}(U)$). The statement follows from Morera's theorem 16.2. $\square$

17. Local properties of holomorphic functions

17.1. The identity theorem.

Theorem. *Let f and g be holomorphic in a region $U \subseteq \mathbb{C}$. The following are equivalent:*

(1) *$f = g$.*
(2) *The set $\{z \in U : f(z) = g(z)\}$ has an accumulation point in U.*
(3) *There is a point $z \in U$ such that $f^{(n)}(z) = g^{(n)}(z)$ for all $n \in \mathbb{N}$.*

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3) Set $h := f - g$ and $A := \{z \in U : h(z) = 0\}$. By assumption, A has an accumulation point a in U. We will show that $h^{(n)}(a) = 0$ for all $n \in \mathbb{N}$. Assume the contrary and let m be the smallest integer such that $h^{(m)}(a) \neq 0$. By Theorem 16.1,

$$h(z) = (z - a)^m h_m(z), \quad h_m(z) := \sum_{k=m}^{\infty} \frac{h^{(k)}(a)}{k!}(z - a)^{k-m}$$

for z in any disk $D \subseteq U$ centered at a. Then h_m is nonzero at a and, by continuity, also in a whole neighborhood of a. That means that a is an isolated zero of h, which contradicts the assumption that a is an accumulation point of A.

(3) $\Rightarrow$ (1) Again let $h = f - g$. The set of points z, where all derivatives of h vanish,

$$Z := \bigcap_{n=0}^{\infty} \{z \in U : h^{(n)}(z) = 0\},$$

is closed in U. On the other hand it is also open: if $z \in Z$ then the Taylor series of h at z is identically zero in each disk $D \subseteq U$ centered at z; thus $D \subseteq Z$. Since U is connected and Z is non-empty by assumption, we may conclude that $Z = U$, that is $f = g$. $\square$

It follows that the zero set of a non-zero holomorphic function h in a region U has no accumulation point in U. For each zero a of h there exists a unique integer m, called the **order** of the zero, such that

$$h(z) = (z-a)^m h_m(z), \quad z \in U,$$

where $h_m \in \mathcal{H}(U)$ and $h_m(a) \neq 0$.

An further consequence of the identity theorem is that a function defined in a real interval I possesses at most one holomorphic extension to some region in $\mathbb{C}$ containing I. In particular, the definition of the functions exp, sin, cos, etc., by their real power series is the only way to extend these functions to the complex domain.

17.2. Singularities. Let U be a domain and $a \in U$. A function $f \in \mathcal{H}(U \setminus \{a\})$ is said to have an **isolated singularity** at a. We shall prove that there are exactly three types of isolated singularities:

- **Removable singularities.** The singularity a of f is called **removable** if f has a holomorphic **extension** to a, i.e., there is a holomorphic function $\tilde{f} : U \to \mathbb{C}$ with $\tilde{f}|_{U\setminus\{a\}} = f$.
- **Poles.** The singularity a of f is called a **pole** of **order** m if there are complex numbers $c_1, \dots, c_m$, where $m > 0$ and $c_m \neq 0$, such that
$$f(z) - \sum_{k=1}^{m} \frac{c_k}{(z-a)^k}$$
has a removable singularity at a.
- **Essential singularities.** Singularities that are neither removable nor poles are called **essential** singularities. For instance, $f(z) = \exp(1/z)$ has an essential singularity at 0. We will see that the image under f of any neighborhood of an essential singularity is dense in $\mathbb{C}$.

We first characterize removable singularities; the following result is known as **Riemann's theorem on removable singularities**.

Proposition. *Let $U \subseteq \mathbb{C}$ be a domain and $a \in U$. For $f \in \mathcal{H}(U \setminus \{a\})$ the following are equivalent:*

(1) *f has a holomorphic extension to a.*
(2) *f has a continuous extension to a.*
(3) *f is bounded near a.*
(4) *$\lim_{z\to a}(z-a)f(z) = 0$.*

Proof. Without loss of generality $a = 0$. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious. Let us prove (4) $\Rightarrow$ (1). Set

$$g(z) := \begin{cases} zf(z) & z \in U \setminus \{0\} \\ 0 & z = 0 \end{cases}; \qquad h(z) := zg(z).$$

By assumption, g is continuous at 0, and so h is $\mathbb{C}$-differentiable at 0 with $h'(0) = g(0) = 0$. It follows that $h \in \mathcal{H}(U)$, since f is holomorphic on $U \setminus \{0\}$. By Theorem 16.1, h has a power series expansion at 0,

$$h(z) = a_2 z^2 + a_3 z^3 + \cdots = z^2(a_2 + a_3 z + \cdots);$$

$a_0 = a_1 = 0$ since $h(0) = h'(0) = 0$. The function $\tilde{f}(z) := a_2 + a_3 z + \cdots$ is a holomorphic extension of f to 0, because $h(z) = z^2 f(z)$ for $z \neq 0$. $\square$

We are now ready to prove the classification of isolated singularities.

Theorem. *Let U be a domain, $a \in U$, and $f \in \mathcal{H}(U \setminus \{a\})$. Then precisely one of the following three cases occurs:*

(1) *f has a removable singularity at a.*
(2) *f has a pole at a.*
(3) *f has an essential singularity at a. If $D \subseteq U$ is any open disk centered at a, then $f(D \setminus \{a\})$ is dense in $\mathbb{C}$.*[15]

Proof. Suppose that there is $\delta > 0$, an open disk $D \subseteq U$ centered at a, and $w \in \mathbb{C}$ such that $|f(z) - w| > \delta$ for all $z \in D \setminus \{a\}$. We need to show that a is either removable or a pole. The function

$$g(z) := \frac{1}{f(z) - w}, \quad z \in D \setminus \{a\}, \tag{17.1}$$

is holomorphic in $D \setminus \{a\}$ and $|g| < 1/\delta$. By the proposition, g extends to a holomorphic function in D.

If $g(a) \neq 0$ then f is bounded near a (by (17.1)), and so a is a removable singularity, by the proposition.

Otherwise g has a zero of order $m \geq 1$ at a and hence

$$g(z) = (z - a)^m g_m(z), \quad z \in D,$$

where $g_m \in \mathcal{H}(D)$ and $g_m(a) \neq 0$. Thus, $h = 1/g_m \in \mathcal{H}(D)$, since g_m does not vanish on D by (17.1), and

$$f(z) - w = \frac{h(z)}{(z - a)^m}, \quad z \in D \setminus \{a\}.$$

Since h has a power series expansion $h(z) = \sum_{n=0}^{\infty} b_n (z - a)^n$ with $b_0 = h(a) \neq 0$, we see that f has a pole of order m at a. □

18. Cauchy's estimates and Liouville's theorem

18.1. Cauchy's estimates.

Theorem. *Let f be holomorphic in a neighborhood of a closed disk $\overline{D} = \overline{D}_r(c)$. Then*

$$\boxed{|f^{(n)}(z)| \leq \frac{n!\, r \|f\|_{\partial D}}{\operatorname{dist}(z, \partial D)^{n+1}}, \quad z \in D,\ n \in \mathbb{N}.} \tag{18.1}$$

Proof. This follows from the integral formulas (16.2) and (11.2),

$$|f^{(n)}(z)| \leq \frac{n!}{2\pi} \int_{\partial D} \frac{|f(\zeta)|}{|\zeta - z|^{n+1}}\, d\zeta \leq n!\, r \max_{\zeta \in \partial D} \frac{|f(\zeta)|}{|\zeta - z|^{n+1}} \leq \frac{n!\, r \|f\|_{\partial D}}{\operatorname{dist}(z, \partial D)^{n+1}}. \qquad \square$$

As an immediate consequence we obtain, for $0 < d < r$,

$$|f^{(n)}(z)| \leq \frac{n!\, r \|f\|_{\partial D}}{d^{n+1}}, \quad z \in \overline{D}_{r-d}(c),\ n \in \mathbb{N}.$$

Letting $d \to r$ we find **Cauchy's inequalities** for the Taylor coefficients.

Corollary. *Let $f(z) = \sum_{n=0}^{\infty} a_n (z - c)^n$ have radius of convergence $> r$. Then*

$$\boxed{|a_n| \leq \frac{\|f\|_{\partial D_r(c)}}{r^n}, \quad n \in \mathbb{N}.} \tag{18.2}$$

[15] By the *big Picard theorem* the image under f of each punctured disk centered at an essential singularity of f is either the whole plane $\mathbb{C}$, or $\mathbb{C}$ with one point missing.

18.2. Gutzmer's formula. Let the power series $f(z) = \sum_{n=0}^{\infty} a_n(z-c)^n$ have radius of convergence $> r$. Then the restriction to the circle $z(t) = c + re^{it}$, $t \in [0, 2\pi]$, is a trigonometric series:

$$f(c+re^{it}) = \sum_{n=0}^{\infty} a_n r^n e^{int} \tag{18.3}$$

which converges normally on $[0, 2\pi]$. Using

$$\frac{1}{2\pi} \int_0^{2\pi} e^{i(m-n)t}\, dt = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

we find by integrating (18.3),

$$a_n r^n = \frac{1}{2\pi} \int_0^{2\pi} f(c+re^{it}) e^{-int}\, dt, \quad n \in \mathbb{N}. \tag{18.4}$$

This implies **Gutzmer's formula**:[16]

Theorem. *Let $f(z) = \sum_{n=0}^{\infty} a_n(z-c)^n$ have radius of convergence $> r$. Then*

$$\boxed{\sum_{n=0}^{\infty} |a_n|^2 r^{2n} = \frac{1}{2\pi} \int_0^{2\pi} |f(c+re^{it})|^2\, dt.} \tag{18.5}$$

Proof. Since $\overline{f(c+re^{it})} = \sum_{n=0}^{\infty} \overline{a_n} r^n e^{-int}$ we have

$$|f(c+re^{it})|^2 = f(c+re^{it}) \sum_{n=0}^{\infty} \overline{a_n} r^n e^{-int}$$

which converges normally on $[0, 2\pi]$, and thus by Proposition 11.4 and by (18.4),

$$\int_0^{2\pi} |f(c+re^{it})|^2\, dt = \sum_{n=0}^{\infty} \overline{a_n} r^n \int_0^{2\pi} f(c+re^{it}) e^{-int}\, dt = 2\pi \sum_{n=0}^{\infty} |a_n|^2 r^{2n}. \qquad \square$$

18.3. Liouville's theorem. Functions that are holomorphic everywhere in $\mathbb{C}$ are called **entire functions**.

Theorem. *Every bounded entire function is constant.*[17]

Proof. If f is entire then $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for all $z \in \mathbb{C}$. If $|f| \le M$ then, by Cauchy's inequalities (18.2) or by Gutzmer's formula (18.5), $|a_n| \le M r^{-n}$ for all r and all n. This is possible only if $a_n = 0$ for all $n \ge 1$. $\square$

19. The open mapping theorem and the maximum modulus principle

19.1. The open mapping theorem. A mapping $f : X \to Y$ between metric spaces is called **open** if the image $f(\Omega)$ of each open set Ω in X is open in Y. We will now prove that non-constant holomorphic functions are open. We need some preparation.

Lemma. *Let D be a disk centered at c and let $f \in \mathcal{H}(U)$, where $\overline{D} \subseteq U$, satisfy $\min_{z \in \partial D} |f(z)| > |f(c)|$. Then there exists $a \in D$ such that $f(a) = 0$.*

[16] This is a special case of Parseval's formula.

[17] The *little Picard theorem* states that the range of every non-constant entire function is either $\mathbb{C}$, or $\mathbb{C}$ with one point missing.

Proof. If $f \neq 0$ on D, then, by the assymption, $f \neq 0$ on an open neighborhood V of $\overline{D}$ in U. Then $1/f \in \mathcal{H}(V)$ and by the mean value property (15.3),

$$\frac{1}{|f(c)|} \leq \max_{z \in \partial D} \frac{1}{|f(z)|} = \frac{1}{\min_{z \in \partial D} |f(z)|}$$

which contradicts the assumption. □

Now we are ready to prove the **open mapping theorem**.

Theorem. *Let $U \subseteq \mathbb{C}$ be a region and let $f \in \mathcal{H}(U)$ be non-constant. Then $f : U \to \mathbb{C}$ is open.*

Proof. Let V be an open neighborhood of c in U. Since f is non-constant, there is an open disk D centered at c with $\overline{D} \subseteq V$ and $f(c) \notin f(\partial D)$, by the identity theorem 17.1. Thus

$$2\delta := \min_{z \in \partial D} |f(z) - f(c)| > 0 \tag{19.1}$$

We claim that $D_\delta(f(c)) \subseteq f(D) \subseteq f(V)$ which implies the theorem. If $b \in D_\delta(f(c))$ and $z \in \partial D$, then by (19.1),

$$|f(z) - b| \geq |f(z) - f(c)| - |b - f(c)| > \delta$$

and so $\min_{z \in \partial D} |f(z) - b| > |f(c) - b|$. By the lemma, there exists $a \in D$ such that $f(a) = b$. Thus $D_\delta(f(c)) \subseteq f(D)$. □

Corollary. *If f is holomorphic in a region U then $f(U)$ is either a region or a point.*

19.2. The maximum modulus principle. We get as a special case of the open mapping theorem:

Corollary. *Let f be holomorphic in a region U and let $c \in U$. Either f is constant in U or each neighborhood of c contains a point b such that $|f(c)| < |f(b)|$.*

Proof. Suppose that $|f(z)| \leq |f(c)|$ for all z in a neighborhood V of c. Then $f(V) \subseteq \{w \in \mathbb{C} : |w| \leq |f(c)|\}$ and so $f(V)$ is not a neighborhood of $f(c)$. That means that f is not open, thus f must be constant. □

If we call the graph of the function $|f|$ on $U \subseteq \mathbb{C} \cong \mathbb{R}^2$ the *analytic landscape* then the maximum modulus principle states roughly that *there are no summits in the analytic landscape of a holomorphic function.*

To put it in another way, if U is a bounded region and $f \in C(\overline{U}) \cap \mathcal{H}(U)$ then $|f|$ attains its maximum at the boundary of U, i.e., $|f(z)| \leq \|f\|_{\partial U}$ for all $z \in \overline{U}$.

20. Convergent sequences of holomorphic functions

20.1. Theorem. *Let $f_n \in \mathcal{H}(U)$ be a sequence of holomorphic functions that converges uniformly on compact sets to a function $f : U \to \mathbb{C}$. Then $f \in \mathcal{H}(U)$ and, for each $k \in \mathbb{N}$, the sequence $f_n^{(k)}$ converges uniformly on compact sets to $f^{(k)}$.*

Proof. The function f is continuous. Let $\Delta \subseteq U$ be a triangle. By Proposition 11.4 and Goursat's lemma 14.1,

$$\int_{\partial \Delta} f \, dz = \lim_{n \to \infty} \int_{\partial \Delta} f_n \, dz = 0.$$

Morera's theorem 16.2 implies that $f \in \mathcal{H}(U)$.

Let us show that $f_n' \to f'$ uniformly on compact sets; the general case follows by iteration. Let $K \subseteq U$ be compact. There exist $r > 0$ such that $L := \bigcup_{z \in K} \overline{D}_r(z)$ is a compact subset of U. Cauchy's estimates (18.1) imply

$$\|f' - f_n'\|_K \le r^{-1} \|f - f_n\|_L \to 0, \qquad \text{as } n \to \infty. \qquad \square$$

Corollary. *Let $\sum f_n$ be a series of holomorphic functions in U that converges uniformly (normally) on compact sets. Then $f = \sum f_n$ is holomorphic in U and, for each $k \in \mathbb{N}$, the series $\sum f_n^{(k)}$ converges uniformly (normally) on compact sets to $f^{(k)}$,*

$$f^{(k)} = \sum f_n^{(k)}.$$

Proof. The statement on uniform convergence on compact sets is a special case of the theorem. Let $K \subseteq U$ be compact and let $L \subseteq U$ be a compact neighborhood of K (as in the proof of the theorem). Cauchy's estimates (18.1) imply that there are constants C_k such that

$$\sum \|f_n^{(k)}\|_K \le C_k \sum \|f_n\|_L.$$

So if $\sum f_n$ converges normally on L then $\sum f_n^{(k)}$ converges normally on K for each k. Since normal convergence implies uniform convergence, we have $f^{(k)} = \sum f_n^{(k)}$. $\square$

Biholomorphic mappings

21. Biholomorphic mappings

A mapping $f : U \to V$ between domains in $\mathbb{C}$ is called **biholomorphic** if f is bijective and f as well as its inverse f^{-1} are holomorphic; in this case we say that U and V are biholomorphic.

21.1. Injective holomorphic mappings are biholomorphic.

Theorem. *If $f : U \to \mathbb{C}$ is holomorphic and injective, then $f : U \to f(U)$ is biholomorphic. Moreover, $f' \neq 0$ on U.*

Proof. Since f is injective, it is nowhere locally constant. By the open mapping theorem 19.1, f is open, thus $V := f(U)$ is open and $f^{-1} : V \to U$ is continuous.

We claim that each point c in the set $Z := \{z \in U : f'(z) = 0\}$ has a neighborhood D in U such that $D \cap Z = \{c\}$. Otherwise, f' would vanish in a neighborhood of c, by the identity theorem 17.1, in contradiction to the fact that f is nowhere locally constant. Since $f : U \to V$ is a homeomorphism, $f(Z)$ has the same property. Moreover, $U \setminus Z$ and $f(U) \setminus f(Z)$ are open sets.

We will show that the restriction $f^{-1} : f(U)\setminus f(Z) \to U\setminus Z$ is holomorphic. This will imply that $f^{-1} : V \to U$ is holomorphic, since the points in $f(Z)$ are removable singularities, by Proposition 17.2. Since f is $\mathbb{C}$-differentiable at $a \in U \setminus Z$, there is a function f_1 continuous at a with $f'(a) = f_1(a) \neq 0$ and

$$f(z) = f(a) + (z - a)f_1(z).$$

Then, as f^{-1} is continuous at $b = f(a)$,

$$z = a + (f(z) - f(a))\frac{1}{f_1(z)}$$

that is

$$f^{-1}(w) = f^{-1}(b) + (w - b)\frac{1}{f_1(f^{-1}(w))}.$$

It follows that f^{-1} is $\mathbb{C}$-differentiable at b with $(f^{-1})'(b) = 1/f'(f^{-1}(b))$. By continuity, $(f^{-1})'(w)f'(f^{-1}(w)) = 1$ for all $w \in V$, and thus $f' \neq 0$ on U. □

21.2. Local biholomorphisms. We are let to a characterization of local biholomorphisms. A mapping f is locally biholomorphic at some point c if there is a neighborhood of c on which f is biholomorphic.

Theorem. *Let $f \in \mathcal{H}(U)$ and $c \in U$. The following are equivalent:*

(1) *f is locally biholomorphic at c.*
(2) *f is locally injective at c.*
(3) $f'(c) \neq 0$.

Proof. Theorem 21.1 yields that (1) and (2) are equivalent, and that they imply (3). Let us show (3) $\Rightarrow$ (2). Since $f' : U \to \mathbb{C}$ is continuous, there is an open disk $D \subseteq U$ centered at c such that $\|f' - f'(c)\|_D < |f'(c)|$. For $z, w \in D$ we have

$$\int_{[z,w]} f'(\zeta) - f'(c)\, d\zeta = f(w) - f(z) - f'(c)(w - z)$$

and thus if $z \neq w$,

$$|f(w) - f(z) - f'(c)(w - z)| < |f'(c)||w - z|.$$

It follows that $f(z) \neq f(w)$. □

In Corollary 5.2 we saw that $f \in C^1(U)$ is conformal if and only if $f \in \mathcal{H}(U)$ and $f' \neq 0$ in U. The theorem implies that conformal mappings are locally biholomorphic, but they need not be biholomorphic, e.g., $z \mapsto z^2$ is a conformal mapping $\mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ that is not injective.

The following corollary is a generalization to the case that $f'(c) = \cdots = f^{(m-1)}(c) = 0$ and $f^{(m)}(c) \neq 0$.

Corollary. *Let $f \in \mathcal{H}(U)$, $c \in U$, and assume that $f'(c) = \cdots = f^{(m-1)}(c) = 0$ and $f^{(m)}(c) \neq 0$ for some $m \geq 1$. Then there exists an open neighborhood V of c in U and a biholomorphic mapping $h : V \to \mathbb{C}$ such that*

$$f(z) = f(c) + h(z)^m, \quad z \in V.$$

Proof. We may assume without loss of generality that $f(c) = 0$. By Theorem 16.1, there is a holomorphic function g with $g(c) \neq 0$ such that

$$f(z) = (z - c)^m g(z).$$

Choose an open neighborhood V of c in U such that $g(V)/g(c) \in \mathbb{C}^-$, and choose ζ such that $\zeta^m = g(c)$. Define $h : V \to \mathbb{C}$ by

$$h(z) := (z - c)\zeta\Big(\frac{g(z)}{g(c)}\Big)^{\frac{1}{m}};$$

cf. Section 10.4. Then h is holomorphic and satisfies $f = h^m$. Since $h'(c) = \zeta \neq 0$ we can achieve that h is biholomorphic by shrinking V; see the theorem. □

22. Fractional linear transformations

22.1. The Riemann sphere. The **Riemann sphere** is the one-point compactification of the complex plane. One extends the complex plane by adding a point called ∞,

$$\boxed{\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\},}$$

and topologizes it in the following way. We define

$$\boxed{D_r(\infty) := \{z \in \mathbb{C} : |z| > r\} \cup \{\infty\}, \quad r > 0,}$$

and declare a subset of $\hat{\mathbb{C}}$ to be open if and only if it is the union of disks $D_r(a)$ where $a \in \hat{\mathbb{C}}$ and $r > 0$. It is clear that this gives the usual topology on $\mathbb{C}$.

The Riemann sphere $\hat{\mathbb{C}}$ is homeomorphic to the Euclidean sphere $S^2 := \{(u, v, w) \in \mathbb{R}^3 : u^2 + v^2 + w^2 = 1\}$. A homeomorphism is given by the **stereographic projection**

$$\Psi : S^2 \to \hat{\mathbb{C}},\ (u, v, w) \mapsto \frac{u}{1 - w} + i\frac{v}{1 - w},\ (0, 0, 1) \mapsto \infty$$

$$\Psi^{-1} : \hat{\mathbb{C}} \to S^2,\ x + iy \mapsto \Big(\frac{2x}{x^2 + y^2 + 1}, \frac{2y}{x^2 + y^2 + 1}, \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}\Big),\ \infty \mapsto (0, 0, 1).$$

If $p = (u, v, w)$ is a point on the sphere S^2 then $z = \Psi(p)$ is the unique point in the equatorial plane that lies on the line through p and the north pole $(0, 0, 1)$.[18]

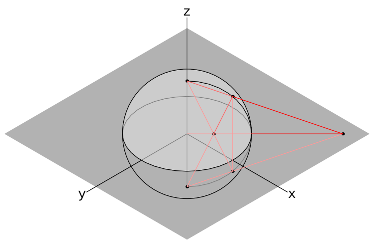


FIGURE 14. The stereographic projection.

Let f be a function that is holomorphic and bounded in $\{z \in \mathbb{C} : |z| > r\}$. Then $\tilde{f}(z) := f(1/z)$ is holomorphic and bounded in $\{z \in \mathbb{C} : 0 < |z| < 1/r\}$, and thus has a holomorphic extension to 0, by Proposition 17.2. That means that the limit $f(\infty) := \lim_{z\to\infty} f(z)$ exists. So we obtain a function f defined in $D_r(\infty)$ and we say that it is holomorphic in $D_r(\infty)$.

22.2. Möbius transformations. A fractional linear transformation

$$\boxed{f(z) = \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{C},\ ad - bc \neq 0,}$$

is called a **Möbius transformation**. Note that $f'(z) = (ad - bc)/(cz + d)^2$. We consider f to be a mapping $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ with the convention that $f(-d/c) = \infty$ and $f(\infty) = a/c$. Then $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is continuous. The composite of Möbius transformations is a Möbius transformation, and the inverse of a Möbius transformation is a Möbius transformation, i.e., the set of all Möbius transformations forms a group.[19]

Multiplying a, b, c, d by the same non-zero constant yields the same Möbius transformation, so we can assume that $ad - bc = 1$. It is easy to see that the mapping

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{az + b}{cz + d}$$

from $\mathrm{SL}(2, \mathbb{C}) := \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : ad - bc = 1\right\}$ to the group of Möbius transformations is a surjective group homomorphism, the kernel of which is $\{\pm\,\mathrm{id}\}$. Thus the group of Möbius transformations is isomorphic to $\mathrm{PSL}(2, \mathbb{C}) := \mathrm{SL}(2, \mathbb{C})/\{\pm\,\mathrm{id}\}$.

Lemma.

(1) *The group of Möbius transformations is generated by translations $z \mapsto z + b$, rotations followed by homotheties $z \mapsto az$, and the inversion $z \mapsto 1/z$.*
(2) *Möbius transformations preserve the family $\mathcal{F}$ consisting of all lines and all circles.*[20]
(3) *For any two triples (a, b, c) and (a', b', c') of distinct points in $\hat{\mathbb{C}}$ there is a unique Möbius transformation f with $f(a) = a'$, $f(b) = b'$, and $f(c) = c'$.*

Proof. (1) This is obvious if $c = 0$ and if $c \neq 0$ it follows from

$$\frac{az + b}{cz + d} = \frac{a}{c} + \frac{\lambda}{cz + d}, \qquad \lambda = \frac{bc - ad}{c}.$$

[18] The Riemann sphere is a complex manifold. The stereographic projections relative to the north and the south pole provide an atlas.

[19] The group of Möbius transformation is the automorphism group of the Riemann sphere.

[20] Via the stereographic projection $\mathcal{F}$ is the family of all circles on S^2.

(2) It is obvious that translations, rotations, and homotheties preserve $\mathcal{F}$. Every member of $\mathcal{F}$ is given by an equation of the form

$$\alpha z\overline{z} + \beta z + \overline{\beta}\overline{z} + \gamma = 0, \quad \alpha, \gamma \in \mathbb{R},\ \beta \in \mathbb{C},\ |\beta|^2 > \alpha\gamma.$$

(If $\alpha \neq 0$ we get a circle, if $\alpha = 0$ a line.) Under the mapping $z \mapsto 1/z$ the equation transforms to

$$\alpha + \beta\overline{z} + \overline{\beta}z + \gamma z\overline{z} = 0$$

which is an equation of the same type.

(3) It suffices to show that there is a unique Möbius transformation that takes the points a, b, c to the points $0, 1, \infty$. The Möbius transformation

$$z \mapsto \frac{(b-c)(z-a)}{(b-a)(z-c)}$$

does the job. □

A consequence of (3) is that every circle can be mapped onto every circle by a Möbius transformation; in this statement *circle* means circle in $\hat{\mathbb{C}}$, that is, circle or line in $\mathbb{C}$. In particular, every open disk can be biholomorphically mapped onto every open half-plane.

22.3. The Cayley mapping. Let $\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ denote the **upper half-plane**.

Proposition. *For each $c \in \mathbb{H}$ the mapping $f : \mathbb{H} \to \mathbb{D}$, $f(z) = \frac{z-c}{z-\overline{c}}$, is biholomorphic with inverse $g : \mathbb{D} \to \mathbb{H}$, $g(z) = \frac{c-\overline{c}z}{1-z}$.*

Proof. Observe that $\mathbb{R} = \{z \in \mathbb{C} : |z-c| = |z-\overline{c}|\}$ and thus $\mathbb{H} = \{z \in \mathbb{C} : |\frac{z-c}{z-\overline{c}}| < 1\}$. This means that f maps $\mathbb{H}$ biholomorphically into $\mathbb{D}$; note that $f \in \mathcal{H}(\mathbb{C} \setminus \{\overline{c}\})$. It is easy to see that g is the inverse of f, $g \in \mathcal{H}(\mathbb{C} \setminus \{1\})$, and

$$\operatorname{Im} g(z) = \operatorname{Im} \frac{c-\overline{c}z}{1-z} = \frac{1}{2i}\Big(\frac{c-\overline{c}z}{1-z} - \frac{\overline{c}-c\overline{z}}{1-\overline{z}}\Big) = \frac{1-|z|^2}{|1-z|^2}\frac{c-\overline{c}}{2i} = \frac{1-|z|^2}{|1-z|^2}\operatorname{Im} c.$$

Thus $g(\mathbb{D}) \subseteq \mathbb{H}$ and the statement is proved. □

In the special case that $c = i$ we get the **Cayley mapping**:

$$\boxed{h : \mathbb{H} \to \mathbb{D},\ z \mapsto \frac{z-i}{z+i}, \quad h^{-1} : \mathbb{D} \to \mathbb{H}, z \mapsto i\frac{1+z}{1-z}.} \tag{22.1}$$

23. Automorphism groups

For a domain $U \subseteq \mathbb{C}$ we denote by $\operatorname{Aut}(U)$ the set of all **automorphims** of U, i.e., biholomorphic mappings $U \to U$. Obviously, $\operatorname{Aut}(U)$ is a group with respect to composition. We shall compute $\operatorname{Aut}(\mathbb{D})$, $\operatorname{Aut}(\mathbb{H})$, and $\operatorname{Aut}(\mathbb{C})$.

23.1. The Schwarz lemma.

Lemma. *Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic and $f(0) = 0$. Then*

$$|f(z)| \le |z| \quad \text{for all } z \in \mathbb{D}, \quad \text{and} \quad |f'(0)| \le 1. \tag{23.1}$$

If for some $c \neq 0$ we have $|f(c)| = |c|$ or if $|f'(0)| = 1$, then f is a rotation, i.e., there is $a \in S^1$ such that $f(z) = az$ for $z \in \mathbb{D}$.

Proof. By Theorem 16.1, we may expand f in a power series $f(z) = \sum_{n=1}^{\infty} a_n z^n$ (note that $a_0 = f(0) = 0$). Then $g(z) := \sum_{n=1}^{\infty} a_n z^{n-1}$ is a holomorphic function in $\mathbb{D}$ such that

$$f(z) = zg(z) \quad \text{and} \quad g(0) = a_1 = f'(0).$$

Since $|f(z)| < 1$ we have $r \max_{|z|=r} |g(z)| \le 1$ for all $0 < r < 1$. By the maximum modulus principle 19.2, we have $|g(z)| < 1/r$ for all $z \in D_r(0)$, $0 < r < 1$. Letting $r \to 1$ implies (23.1).

If $|f(c)| = |c|$ for some $c \neq 0$ or if $|f'(0)| = 1$, then $|g(c)| = 1$ or $|g(0)| = 1$. That is, g attains its maximum in $\mathbb{D}$. By the maximum modulus principle, g must be a constant a with $|a| = 1$. □

23.2. Automorphisms of the unit disk.

Theorem. *The automorphisms of $\mathbb{D}$ are precisely the Möbius transformations of the form $z \mapsto \frac{az+b}{\bar{b}z+\bar{a}}$:*

$$\boxed{\mathrm{Aut}(\mathbb{D}) = \Big\{ z \mapsto \frac{az+b}{\bar{b}z+\bar{a}} : a, b \in \mathbb{C},\ |a|^2 - |b|^2 = 1 \Big\}.}$$

Proof. It is easy to check that the right-hand side is a group. Let $f(z) = \frac{az+b}{\bar{b}z+\bar{a}}$, $|a|^2 - |b|^2 = 1$. Then $f \in \mathcal{H}(\mathbb{D})$ since $|-\frac{\bar{a}}{\bar{b}}| = |\frac{a}{b}| > 1$. To see that $f(\mathbb{D}) \subseteq \mathbb{D}$, observe that

$$1 > \Big| \frac{az+b}{\bar{b}z+\bar{a}} \Big|^2 \quad \Leftrightarrow \quad |a|^2 - |b|^2 > (|a|^2 - |b|^2)|z|^2.$$

The inverse f^{-1} has the same properties, and so $f \in \mathrm{Aut}(\mathbb{D})$.

For the converse inclusion, let $g \in \mathrm{Aut}(\mathbb{D})$. Then $c := g(0) \in \mathbb{D}$ and

$$f(z) := \frac{az+b}{\bar{b}z+\bar{a}}, \quad a := \frac{1}{\sqrt{1-|c|^2}}, \quad b := \frac{c}{\sqrt{1-|c|^2}},$$

is a Möbius transformation of the required form that maps 0 to c. By the first part of the proof, $f \in \mathrm{Aut}(\mathbb{D})$ and thus $h := f^{-1} \circ g \in \mathrm{Aut}(\mathbb{D})$ with $h(0) = 0$. By the Schwarz lemma 23.1 applied to h and h^{-1},

$$|h(z)| \le |z| = |h^{-1}(h(z))| \le |h(z)| \quad \text{for all } z \in \mathbb{D}.$$

By the second part of the Schwarz lemma, there is $\beta \in S^1$ such that $h(z) = \beta z$. If we choose $\alpha \in S^1$ with $\beta = \alpha^2$ then

$$h(z) = \beta z = \frac{\alpha z}{\bar{\alpha}},$$

i.e., h, and hence also $g = f \circ h$, is a Möbius transformation of the required form. □

23.3. Automorphisms of the upper half-plane. Since the unit disk $\mathbb{D}$ and the upper half-plane $\mathbb{H}$ are biholomorphic, we can easily deduce the automorphism group of $\mathbb{H}$.

Theorem. *The automorphisms of $\mathbb{H}$ are precisely the Möbius transformations of the form $z \mapsto \frac{az+b}{cz+d}$ where $a, b, c, d \in \mathbb{R}$:*

$$\boxed{\mathrm{Aut}(\mathbb{H}) = \Big\{ \frac{az+b}{cz+d} : a, b, c, d \in \mathbb{R},\ ad - bc = 1 \Big\}.}$$

Proof. Since the Cayley mapping $h : \mathbb{H} \to \mathbb{D}$ from (22.1) is a biholomorphism, we have, by Theorem 23.2,

$$\begin{aligned}\operatorname{Aut}(\mathbb{H}) = h^{-1} \circ \operatorname{Aut}(\mathbb{D}) \circ h &= \Big\{h^{-1} \circ \frac{az+b}{\bar{b}z+\bar{a}} \circ h : a,b \in \mathbb{C},\ |a|^2 - |b|^2 = 1\Big\} \\ &= \Big\{\frac{\alpha z+\beta}{\gamma z+\delta} : \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})\Big\},\end{aligned}$$

where the last equality is a straight forward computation. □

We may conclude that the groups $\operatorname{Aut}(\mathbb{D})$ and $\operatorname{Aut}(\mathbb{H})$ are both isomorphic to the group $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm \mathrm{id}\}$.

23.4. Automorphisms of the plane.

Theorem. *The automorphisms of $\mathbb{C}$ are precisely the Möbius transformations of the form $z \mapsto az + b$ where $a, b \in \mathbb{C}$, $a \neq 0$:*

$$\boxed{\operatorname{Aut}(\mathbb{C}) = \Big\{az + b : a, b \in \mathbb{C},\ a \neq 0\Big\}.}$$

Proof. It is clear that mappings of the form $z \mapsto az + b$, $a \neq 0$, are automorphisms of $\mathbb{C}$. Conversely, let $f \in \operatorname{Aut}(\mathbb{C})$. Then $g(z) := f(1/z)$ is in $\mathcal{H}(\mathbb{C} \setminus \{0\})$. Since $f : \mathbb{C} \to \mathbb{C}$ is biholomorphic, g can neither have a removable singularity at 0, by Liouville's theorem 18.3, nor an essential singularity, cf. Theorem 17.2. So 0 is a pole of g, i.e.,

$$g(z) - \sum_{k=1}^{m} \frac{c_k}{z^k}$$

is bounded near 0, and thus

$$f(z) - \sum_{k=1}^{m} c_k z^k$$

is bounded near ∞. By Liouville's theorem 18.3, f is a polynomial. Its degree must be one, since otherwise f' is a polynomial of degree ≥ 1 and thus has a zero in $\mathbb{C}$, contradicting Theorem 21.1. □

The global Cauchy theorem

So far we treated Cauchy's theorem and the integral formula only on star-shaped domains. This is adequate for studying local properties of holomorphic functions. But the result is obviously incomplete. Two questions arise:

(1) Given an arbitrary domain U, how can we describe the closed paths in U for which the assertion of Cauchy's theorem is true?
(2) Can we characterize the domains in which Cauchy's theorem has universal validity?

24. Homology and the general form of Cauchy's theorem

We will answer the first question in this section.

24.1. Chains and cycles. Let us generalize the notion of the path integral. To this end we examine the identity

$$\int_{\gamma_1+\cdots+\gamma_n} f\,dz = \int_{\gamma_1} f\,dz + \cdots + \int_{\gamma_n} f\,dz \tag{24.1}$$

which holds if the paths γ_i have matching endpoints, i.e., $z_A(\gamma_{i+1}) = z_\Omega(\gamma_i)$, $1 \le i \le n-1$. The right-hand side of (24.1) has a meaning for *any* finite collection of paths $\gamma_1, \ldots, \gamma_n$. So let us consider arbitrary formal sums $\gamma_1 + \cdots + \gamma_n$ and let us define $\int_{\gamma_1+\cdots+\gamma_n} f\,dz$ by equation (24.1). Such formal sums of paths are called **chains**.[21]

Chains are considered identical if they yield the same path integral for all functions f. Thus two chains are identical if one is obtained from the other by

- permutation of paths,
- subdivison of paths,
- fusion of subpaths,
- reparameterization of paths,
- cancellation of opposite paths.

Chains can be added and (24.1) remains valid for arbitrary chains. If identical chains are added, we denote the sum as a multiple. By allowing $a(-\gamma) = -a\gamma$, every chain can be written as a finite linear combination

$$\gamma = a_1\gamma_1 + \cdots + a_n\gamma_n,$$

where $a_i \in \mathbb{Z}$, all γ_i are different, and no two γ_i are opposite. We allow zero coefficients, in particular, the zero chain 0. Clearly, a chain can be represented as a sum of paths in many ways.

For a formal sum $\gamma = \gamma_1 + \cdots + \gamma_n$ of paths γ_i we set $|\gamma| = \bigcup_{i=1}^n |\gamma_i|$ and $|0| = \emptyset$. Note that $|\gamma|$ depends on the representation of γ (due to cancellation of opposite paths).

[21] More formally, chains can be defined as formal sums $\gamma_1 + \cdots + \gamma_n$ of linear functionals $\gamma_i(f) = \int_{\gamma_i} f\,dz$ for $f \in C(\bigcup_i |\gamma_i|)$, cf. [**6**, 10.34].

A chain is called a **cycle** if it can be represented as a sum of closed paths.

We will consider chains contained in a given domain $U \subseteq \mathbb{C}$. This means that the chains have a representation by paths in U and only such representations are considered.

For a cycle γ and a point $z \notin |\gamma|$ the **index** of z with respect to γ is defined by

$$\operatorname{ind}_\gamma(z) = \frac{1}{2\pi i} \int_\gamma \frac{d\zeta}{\zeta - z}, \tag{24.2}$$

just as in Section 12.2. Clearly,

$$\operatorname{ind}_{\gamma_1+\gamma_2}(z) = \operatorname{ind}_{\gamma_1}(z) + \operatorname{ind}_{\gamma_2}(z), \quad \operatorname{ind}_{-\gamma}(z) = -\operatorname{ind}_\gamma(z). \tag{24.3}$$

24.2. Homology. A cycle γ in a domain $U \subseteq \mathbb{C}$ is said to be **homologous to zero** with respect to U if $\operatorname{ind}_\gamma(z) = 0$ for all $z \in \mathbb{C} \setminus U$; we write $\gamma \sim 0 \pmod U$. Two cycles γ_1 and γ_2 in U are **homologous** in U, in symbols $\gamma_1 \sim \gamma_2$, if $\gamma_1 - \gamma_2 \sim 0$. By (24.3),

$$\boxed{\gamma_1 \sim \gamma_2 \pmod U \;\Leftrightarrow\; \operatorname{ind}_{\gamma_1}(z) = \operatorname{ind}_{\gamma_2}(z) \text{ for all } z \notin U.}$$

This defines an equivalence relation on the set of cycles in U. The set of equivalence classes, called **homology classes**, forms an additive group, the **homology group**. If $\gamma \sim 0 \pmod U$ then $\gamma \sim 0 \pmod{U'}$ for all $U' \supseteq U$.

24.3. The general form of Cauchy's theorem.

Lemma. *If $f \in \mathcal{H}(U)$ then*

$$g : U \times U \to \mathbb{C}, \quad g(z,w) := \begin{cases} \dfrac{f(z) - f(w)}{z - w} & z \neq w \\ f'(z) & z = w \end{cases} \tag{24.4}$$

is continuous.

Proof. We need to check continuity at points on the diagonal $z = w$. Fix $a \in U$ and $\epsilon > 0$. Since f' is continuous, there is a disk $D_r(a) \subseteq U$ such that $|f'(\zeta) - f'(a)| < \epsilon$ if $\zeta \in D_r(a)$. If $z, w \in D_r(a)$, $z \neq w$, then $\zeta(t) := (1-t)z + tw \in D_r(a)$, $t \in [0,1]$, and

$$|g(z,w) - g(a,a)| = \Big|\frac{f(z) - f(w)}{z - w} - f'(a)\Big| = \Big|\int_0^1 (f'(\zeta(t)) - f'(a))\, dt\Big| \le \epsilon.$$

Thus g is continuous at (a,a). $\square$

Theorem. *Let $U \subseteq \mathbb{C}$ be a domain and let $f \in \mathcal{H}(U)$.*

(1) *If γ is a cycle that is homologous to zero in U, then*

$$\boxed{\int_\gamma f\, dz = 0,} \tag{24.5}$$

$$\boxed{\operatorname{ind}_\gamma(z) f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta, \quad z \in U \setminus |\gamma|.} \tag{24.6}$$

(2) *If γ_1 and γ_2 are homologous cycles in U, then*

$$\boxed{\int_{\gamma_1} f\, dz = \int_{\gamma_2} f\, dz.} \tag{24.7}$$

Equation (24.6) is the **general form of Cauchy's integral formula**.

Proof. (1) Consider the continuous function g in (24.4), and define

$$h(z) := \frac{1}{2\pi i}\int_\gamma g(z,w)\,dw, \quad z \in U.$$

For each $w \in U$ we have $g(\cdot, w) \in \mathcal{H}(U)$, since the singularity at $z = w$ is removable by Proposition 17.2. Thus $h \in \mathcal{H}(U)$ by Proposition 16.3.

Our goal is to show that $h(z) = 0$ for $z \in U \setminus |\gamma|$ which is equivalent to (24.6) (by (24.2)). Set $U_1 := \{z \in \mathbb{C} \setminus |\gamma| : \operatorname{ind}_\gamma(z) = 0\}$ and define

$$h_1(z) := \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw, \quad z \in U_1.$$

Since $h_1(z) = h(z)$ for $z \in U \cap U_1$, there exists a function $\varphi \in \mathcal{H}(U \cup U_1)$ such that $\varphi|_U = h$ and $\varphi|_{U_1} = h_1$. Since γ is homologous to zero in U, the set U_1 contains $\mathbb{C} \setminus U$, so $U \cup U_1 = \mathbb{C}$ and φ is entire. By definition U_1 also contains the unbounded connected component of the complement of $|\gamma|$ on which $\operatorname{ind}_\gamma$ vanishes; see Theorem 12.2. Thus

$$\lim_{|z|\to\infty} \varphi(z) = \lim_{|z|\to\infty} h_1(z) = 0.$$

By Liouville's theorem 18.3, $\varphi = 0$ and hence $h = 0$. We proved (24.6).

Let us deduce (24.5) from (24.6). Fix $a \in U \setminus |\gamma|$ and set $F(z) := (z-a)f(z)$. Then, as $F(a) = 0$,

$$\frac{1}{2\pi i}\int_\gamma f\,dz = \frac{1}{2\pi i}\int_\gamma \frac{F(z)}{z-a}\,dz = \operatorname{ind}_\gamma(a)F(a) = 0.$$

(2) Apply (24.5) to $\gamma = \gamma_1 - \gamma_2$. □

24.4. Practical computation of the index.

Theorem. *Let γ be a closed C^1-path in $\mathbb{C}$ and $z \in \mathbb{C} \setminus |\gamma|$. Let $v \in S^1$ be a unit vector such that the ray $R := \{z + rv : r > 0\} = z + (0,\infty)v$ intersects γ only transversally,*[22] *i.e., if $\gamma(t) \in R$ then $\det(v, \gamma'(t)) \neq 0$. Then*

$$\boxed{\operatorname{ind}_\gamma(z) = \sum_{t \in \gamma^{-1}(R)} \operatorname{sgn}\big(\det(v, \gamma'(t))\big).} \tag{24.8}$$

So, in practice, choose a suitable ray, start in the unbounded connected component of the complement of $|\gamma|$ where the index is 0, and move inwards along the ray. At points where γ meets the ray the index increases (decreases) by one if the direction of the ray and the tangent vector of γ are positively (negatively) oriented. It may happen that γ passes through the same intersection point with the ray a finite number of times; the index is counted accordingly.

Proof. Let $\gamma : [0,1] \to \mathbb{C}$. Without loss of generality $z = 0$. The set $\gamma^{-1}(R)$ is finite. Otherwise there exists an accumulation point $t_0 \in \gamma^{-1}(R)$ (since $|\gamma|$ is compact), but $\gamma(t) \notin R$ for $t \neq t_0$ near t_0 by transversality, a contradiction.

If $\gamma^{-1}(R) = \emptyset$ then 0 lies in the unbounded connected component of the complement of $|\gamma|$, and (24.8) is true.

Suppose that $\gamma^{-1}(R) = \{t_1, \ldots, t_n\}$ with $0 < t_1 < \cdots < t_n \le 1$. Let $\gamma : \mathbb{R} \to \mathbb{C} \setminus \{0\}$ also denote the periodic extension of the closed path $\gamma : [0,1] \to \mathbb{C} \setminus \{0\}$.

[22] Sard's theorem implies that almost every ray intersects γ transversally.

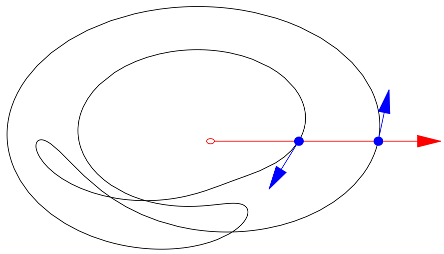


FIGURE 15. Illustration of Formula (24.8).

Then $[t_0, t_n]$, for $t_0 := t_n - 1$, is an interval of periodicity. Let γ be parameterized in polar form $\gamma(t) = r(t)e^{i\varphi(t)}$ by continuous functions $r(t)$ and $\varphi(t)$. By (12.5),

$$\operatorname{ind}_\gamma(0) = \frac{\varphi(t_n) - \varphi(t_0)}{2\pi}.$$

Let θ be such that $v = e^{-i\theta}$. For $t \in (t_{k-1}, t_k)$, we have $\gamma(t) \notin R$ and so $\varphi(t) \in (\theta_k - \pi, \theta_k + \pi)$ for some $\theta_k \in \theta + 2\pi\mathbb{Z}$. Moreover, since $\varphi(t)$ is continuous,

$$\begin{aligned} \varphi(t_k) = \lim_{t \nearrow t_k} \varphi(t) &= \begin{cases} \theta_k + \pi & \text{if } \varphi(t) < \varphi(t_k) \text{ for } t < t_k \\ \theta_k - \pi & \text{if } \varphi(t) > \varphi(t_k) \text{ for } t < t_k \end{cases} \\ &= \theta_k + \operatorname{sgn}\big(\det(v, \gamma'(t_k))\big)\pi, \end{aligned}$$

and analogously

$$\varphi(t_{k-1}) = \theta_k - \operatorname{sgn}\big(\det(v, \gamma'(t_{k-1}))\big)\pi$$

It follows that

$$\begin{aligned} \operatorname{ind}_\gamma(0) &= \frac{\varphi(t_n) - \varphi(t_0)}{2\pi} = \frac{1}{2\pi}\sum_{k=1}^{n}(\varphi(t_k) - \varphi(t_{k-1})) \\ &= \frac{1}{2}\sum_{k=1}^{n}\Big(\operatorname{sgn}\big(\det(v, \gamma'(t_k))\big) + \operatorname{sgn}\big(\det(v, \gamma'(t_{k-1}))\big)\Big) \\ &= \sum_{k=1}^{n}\operatorname{sgn}\big(\det(v, \gamma'(t_k))\big). \end{aligned}$$

Formula (24.8) is proved. □

25. The calculus of residues

25.1. Meromorphic functions and residues. A function f defined in a domain $U \subseteq \mathbb{C}$ is said to be **meromorphic** if there is a set $A \subseteq U$ such that

- A has no accumulation point in U,
- $f \in \mathcal{H}(U \setminus A)$,
- f has a pole at each point of A.

Every function holomorphic in U is meromorphic in U (in this case $A = \emptyset$). Note that A is at most countable.

If $a \in A$ then (cf. 17.2) there exist complex numbers $c_1, \ldots, c_m$, $c_m \neq 0$, such that

$$f(z) - \sum_{k=1}^{m} \frac{c_k}{(z-a)^k} =: f(z) - Q(z) \tag{25.1}$$

has a removable singularity at a; we say that Q is the **principal part** of f at a. The number c_1 is called the **residue** of f at a,

$$\boxed{c_1 = \operatorname{res}(f; a).}$$

If a is a pole of order m of f then, in view of (25.1),

$$\boxed{\operatorname{res}(f; a) = \lim_{z \to a} \frac{1}{(m-1)!} \Big(\frac{d}{dz}\Big)^{m-1} (z-a)^m f(z).}$$

If γ is a cycle and $a \notin |\gamma|$, then

$$\frac{1}{2\pi i} \int_\gamma Q\, dz = c_1 \operatorname{ind}_\gamma(a) = \operatorname{res}(Q; a) \operatorname{ind}_\gamma(a). \tag{25.2}$$

25.2. The residue theorem.

Theorem. *Let f be meromorphic in U and let A be the set of poles of f. Let γ be a cycle in $U \setminus A$ that is homologous to zero in U. Then*

$$\boxed{\frac{1}{2\pi i} \int_\gamma f\, dz = \sum_{a \in A} \operatorname{res}(f; a) \operatorname{ind}_\gamma(a).} \tag{25.3}$$

Proof. Set $B := \{a \in A : \operatorname{ind}_\gamma(a) \neq 0\}$. We claim that B, and therefore the sum in (25.3), is finite. Let V any connected component of $\mathbb{C} \setminus |\gamma|$. If V is unbounded or if $V \cap (\mathbb{C} \setminus U) \neq \emptyset$, then $\operatorname{ind}_\gamma$ vanishes on V, since γ is homologous to zero in U and since $\operatorname{ind}_\gamma$ is locally constant, by Theorem 12.2. Since A has no accumulation point in U, B must be finite.

Let $a_1, \ldots, a_n$ be the points of B and let $Q_1, \ldots, Q_n$ be the principal parts of f at $a_1, \ldots, a_n$. The function $g := f - \sum_{j=1}^n Q_j$ has removable singularities at $a_1, \ldots, a_n$ and thus application of Theorem 24.3 on the domain $U \setminus (A \setminus B)$ gives

$$\int_\gamma g\, dz = 0.$$

(Note that γ is homologous to zero with respect to $U \setminus (A \setminus B)$ since $\operatorname{ind}_\gamma(z) = 0$ for all z in $\mathbb{C} \setminus (U \setminus (A \setminus B)) = (\mathbb{C} \setminus U) \cup (A \setminus B)$ by assumption and by the definition of B.) Consequently,

$$\frac{1}{2\pi i} \int_\gamma f\, dz = \sum_{j=1}^n \frac{1}{2\pi i} \int_\gamma Q_j\, dz \overset{(25.2)}{=} \sum_{j=1}^n \operatorname{res}(Q_j; a_j) \operatorname{ind}_\gamma(a_j),$$

which shows (25.3), because $\operatorname{res}(Q_j; a_j) = \operatorname{res}(f; a_j)$. □

25.3. The argument principle.

Theorem. *Let f be meromorphic in U with zeros a_j and poles b_k, and let γ be a cycle which is homologous to zero in U and does not pass through any of the zeros or poles. Then*

$$\boxed{\frac{1}{2\pi i} \int_\gamma \frac{f'}{f}\, dz = \sum_j \operatorname{ind}_\gamma(a_j) - \sum_k \operatorname{ind}_\gamma(b_k),} \tag{25.4}$$

where multiple zeros or poles are repeated according to their order.

Proof. Suppose that c is a zero of order m of f. By Section 17.1, we can write

$$f(z) = (z-c)^m g(z),$$

where g is holomorphic and nowhere vanishing in a neighborhood of c. Thus,

$$\frac{f'(z)}{f(z)} = \frac{m}{z-c} + \frac{g'(z)}{g(z)},$$

i.e., f'/f has a simple pole with residue m at c. The same arguments show that if f has a pole of order m at c, then f'/f has a simple pole with residue $-m$ at c. So (25.4) follows from (25.3). □

Corollary. *Let f be meromorphic in U with zeros a_j and poles b_k, and let γ be a simple closed positively oriented path in U which does not pass through any of the zeros or poles. Then*

$$\boxed{\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz = \#(\textit{zeros of } f \textit{ inside } \gamma) - \#(\textit{poles of } f \textit{ inside } \gamma),}$$

where the zeros and poles are counted with their multiplicities.

25.4. Evaluation of integrals. The calculus of residues provides a method of computing a wide range of integrals which we will explain by means of two examples.

Example. Consider an integral of the form

$$I = \int_0^{2\pi} R(\cos t, \sin t)\,dt,$$

where $R(x, y)$ is a rational function without a pole on the circle $x^2 + y^2 = 1$. If we set $z = e^{it}$, then

$$\cos t = \frac{e^{it} + e^{-it}}{2} = \frac{1}{2}\Big(z + \frac{1}{z}\Big), \quad \sin t = \frac{e^{it} - e^{-it}}{2i} = \frac{1}{2i}\Big(z - \frac{1}{z}\Big),$$

and thus

$$\begin{aligned} I &= \int_{S^1} \frac{1}{iz} R\Big(\frac{1}{2}\Big(z + \frac{1}{z}\Big), \frac{1}{2i}\Big(z - \frac{1}{z}\Big)\Big)\,dz \\ &= 2\pi \sum \operatorname{res}\Big[\frac{1}{z} R\Big(\frac{1}{2}\Big(z + \frac{1}{z}\Big), \frac{1}{2i}\Big(z - \frac{1}{z}\Big)\Big)\Big], \end{aligned}$$

where the sum is over all poles in $\mathbb{D}$ of the function in the square brackets.

For instance, for $a > 1$,

$$\int_0^{2\pi} \frac{dt}{a + \sin t} = 2\pi \sum \operatorname{res} \frac{2i}{z^2 + 2aiz - 1}.$$

The function on the right-hand side has two simple poles $p_1 := -ia + i\sqrt{a^2-1}$ and $p_2 := -ia - i\sqrt{a^2-1}$, but only the first pole lies in $\mathbb{D}$. Its residue is

$$\lim_{z\to p_1} (z - p_1)\frac{2i}{z^2 + 2aiz - 1} = \lim_{z\to p_1} \frac{2i}{z - p_2} = \frac{1}{\sqrt{a^2-1}}.$$

Therefore,

$$\int_0^{2\pi} \frac{dt}{a + \sin t} = \frac{2\pi}{\sqrt{a^2-1}}.$$

Example. Our goal is to prove

$$\boxed{\int_{-\infty}^{\infty} \frac{e^{ax}}{1 + e^x}\,dx = \frac{\pi}{\sin \pi a}, \quad 0 < a < 1.} \tag{25.5}$$

It is easy to see that the integral converges. Set $f(z) = e^{az}/(1 + e^z)$ and consider the path γ which parameterizes the boundary of the rectangle with vertices $-R_1$,

R_2, $R_2 + 2\pi i$, $-R_1 + 2\pi i$ (positively oriented; $R_1, R_2 > 0$). The only pole of f inside the rectangle is πi. Let us compute its residue,

$$(z - \pi i)f(z) = e^{az}\frac{z - \pi i}{e^z - e^{\pi i}} \to e^{a\pi i}\frac{1}{e^{\pi i}} = -e^{a\pi i}, \qquad \text{as } z \to \pi i.$$

Thus,

$$\int_\gamma f\,dz = 2\pi i \operatorname{res}(f; \pi i) = -2\pi i e^{a\pi i} \tag{25.6}$$

Observe that

$$\begin{aligned}\int_{[R_2+2\pi i, -R_1+2\pi i]} f\,dz &= \int_{R_2}^{-R_1} \frac{e^{a(t+2\pi i)}}{1+e^{t+2\pi i}}\,dt \\ &= -e^{2\pi a i}\int_{-R_1}^{R_2}\frac{e^{at}}{1+e^t}\,dt = -e^{2\pi a i}\int_{[-R_1,R_2]} f\,dz,\end{aligned}$$

and

$$\Big|\int_{[R_2,R_2+2\pi i]} f\,dz\Big| \le \int_0^{2\pi}\Big|\frac{e^{a(R_2+it)}}{1+e^{R_2+it}}\Big|\,dt \le Ce^{(a-1)R_2} \to 0,$$
$$\Big|\int_{[-R_1+2\pi i,-R_1]} f\,dz\Big| \le \int_0^{2\pi}\Big|\frac{e^{a(-R_1+it)}}{1+e^{-R_1+it}}\Big|\,dt \le Ce^{-aR_1} \to 0,$$

as $R_1, R_2 \to \infty$, where C is some constant. Consequently, by (25.6),

$$(1 - e^{2\pi a i})\int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^x}\,dx = -2\pi i e^{a\pi i}$$

which implies (25.5).

26. Homotopy and simply connected domains

Let us turn to the second question.

26.1. Homotopy. Let $\gamma_i : [a,b] \to U \subseteq \mathbb{C}$, $i = 0, 1$, be closed curves in U. We say that γ_0 and γ_1 are **homotopic** in U if there is a continuous mapping $H : [0,1] \times [a,b] \to U$, $H(s,t) = H_s(t) = H^t(s)$, such that

$$H_0 = \gamma_0, \quad H_1 = \gamma_1, \quad H^a = H^b.$$

The mapping H is called a **homotopy**. It defines a one-parameter family of closed curves H_s in U which connects γ_0 and γ_1. This defines an equivalence relation on the set of closed curves in U. Similarly one defines homotopies of non-closed curves, cf. Figure 17, but we will have no use for that.

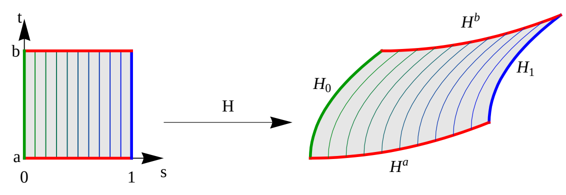


FIGURE 16. A homotopy of non-closed curves; $H^a \neq H^b$.

If γ_0 is homotopic in U to a constant curve γ_1 (i.e., a point), we say that γ_0 is **null-homotopic** in U.

A connected subset $U \subseteq \mathbb{C}$ is said to be **simply connected** if every closed curve in U is null-homotopic in U.

Example. Star-shaped domains U are simply connected. Let $c \in U$ be a center and let $\gamma : [a, b] \to U$ be a closed curve. Then

$$H(s, t) := (1 - s)\gamma(t) + sc$$

defines a homotopy with $H_0 = \gamma$, $H_1 = c$, and $H^a = H^b$.

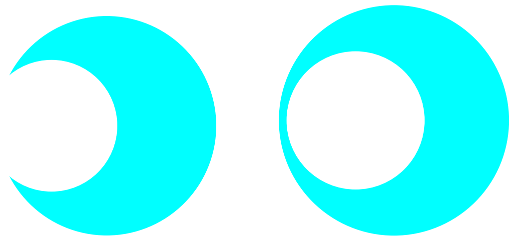


FIGURE 17. The set on the left is simply connected, the set on the right is not simply connected.

26.2. Cauchy's theorem for simply connected domains.

Lemma. *Let γ_0 and γ_1 be closed paths in $\mathbb{C}$ with parameter interval $[a, b]$. If $z \in \mathbb{C}$ and*

$$|\gamma_1(t) - \gamma_0(t)| < |z - \gamma_0(t)|, \quad t \in [a, b], \tag{26.1}$$

then $\operatorname{ind}_{\gamma_0}(z) = \operatorname{ind}_{\gamma_1}(z)$.

Proof. By (26.1), $z \notin |\gamma_0| \cup |\gamma_1|$, and $\gamma := \frac{\gamma_1 - z}{\gamma_0 - z}$ defines a path in $D_1(1)$, thus, $\operatorname{ind}_\gamma(0) = 0$. Moreover,

$$\frac{\gamma'}{\gamma} = \frac{\gamma_1'}{\gamma_1 - z} - \frac{\gamma_0'}{\gamma_0 - z}$$

and therefore $0 = \operatorname{ind}_\gamma(0) = \operatorname{ind}_{\gamma_1}(z) - \operatorname{ind}_{\gamma_0}(z)$. □

Theorem. *Let $U \subseteq \mathbb{C}$ be a domain and let $f \in \mathcal{H}(U)$.*

(1) *If γ_0 and γ_1 are homotopic closed paths in U, then γ_0 and γ_1 are homologous in U; thus,*

$$\int_{\gamma_0} f\,dz = \int_{\gamma_1} f\,dz.$$

(2) *If U is simply connected, then every cycle γ in U is homologous to zero in U; thus,*

$$\int_\gamma f\,dz = 0,$$

$$\operatorname{ind}_\gamma(z) f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta, \quad z \in U \setminus |\gamma|.$$

Proof. (1) We must show that $\operatorname{ind}_{\gamma_0}(z) = \operatorname{ind}_{\gamma_1}(z)$ if $z \notin U$. There exists a homotopy $H : [0, 1] \times [a, b] \to U$ with $H_0 = \gamma_0$, $H_1 = \gamma_1$, and $H^a = H^b$. For simplicity of notation we assume without loss of generality that $[a, b] = [0, 1]$. Since $H([0, 1]^2)$ is compact, there exists $\epsilon > 0$ such that

$$|z - H(s, t)| > 2\epsilon, \quad \text{for all } (s, t) \in [0, 1]^2. \tag{26.2}$$

Since H is uniformly continuous, there is a positive integer $n \in \mathbb{N}$ such that

$$|H(s, t) - H(s', t')| < \epsilon, \quad \text{if } |s - s'| + |t - t'| \le \frac{1}{n}. \tag{26.3}$$

We define polygonal closed paths Γ_k, $k = 0, 1, \ldots, n$, by setting

$$\Gamma_k(t) := H_{\frac{k}{n}}(\tfrac{i}{n})(nt + 1 - i) + H_{\frac{k}{n}}(\tfrac{i-1}{n})(i - nt), \quad t \in [\tfrac{i-1}{n}, \tfrac{i}{n}],\ i = 1, \ldots, n;$$

i.e., we connect the points $H_{\frac{k}{n}}(\frac{i}{n})$, $i = 0, 1, \ldots, n$, on the curve $H_{\frac{k}{n}}$ by line segments. By (26.3),

$$|\Gamma_k(t) - H_{\frac{k}{n}}(t)| < \epsilon, \quad k = 0, \ldots, n,\ t \in [0, 1], \tag{26.4}$$

$$|\Gamma_{k-1}(t) - \Gamma_k(t)| < \epsilon, \quad k = 1, \ldots, n,\ t \in [0, 1]. \tag{26.5}$$

In particular, (26.4) implies

$$|\Gamma_0(t) - \gamma_0(t)| < \epsilon, \quad |\Gamma_n(t) - \gamma_1(t)| < \epsilon. \tag{26.6}$$

By (26.2) and (26.4),

$$|z - \Gamma_k(t)| \ge |z - H_{\frac{k}{n}}(t)| - |\Gamma_k(t) - H_{\frac{k}{n}}(t)| > \epsilon, \quad k = 0, \ldots, n,\ t \in [0, 1]. \tag{26.7}$$

Then (26.5), (26.6), (26.7) and the lemma imply

$$\mathrm{ind}_{\gamma_0}(z) = \mathrm{ind}_{\Gamma_0}(z) = \mathrm{ind}_{\Gamma_1}(z) = \cdots = \mathrm{ind}_{\Gamma_n}(z) = \mathrm{ind}_{\gamma_1}(z).$$

The identity in (1) follows from (24.7).

(2) Every closed path γ in U is null-homotopic in U, thus homologous to zero in U, by (1). The identities follow from (24.5) and (24.6). □

The paths Γ_k were introduced since the curves H_s need not be piecewise C^1, and so the lemma is not applicable directly.

Remark. The converse of (1) is not true: there exist domains U and closed paths in U that are homologous but not homotopic in U. For instance, the **Pochhammer cycle** (see Figure 18) is homologous to zero but not null-homotopic in $\mathbb{C} \setminus \{a, b\}$.

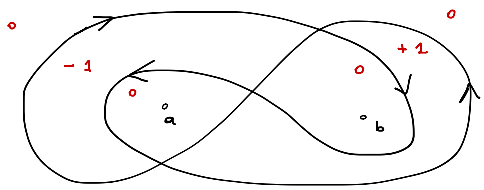


FIGURE 18. The Pochhammer cycle. It is homologous to zero in $\mathbb{C} \setminus \{a, b\}$, $a \neq b$, since the index is zero at a and b. But it is not null-homotopic in $\mathbb{C} \setminus \{a, b\}$.

26.3. Characterization of simply connected domains. We have fully answered the first question and partly the second question posed at the beginning of this chapter. The following theorem gives a full answer to the second question. We state it without proof.

Theorem. *Let $U \subseteq \mathbb{C}$ be a region. The following are equivalent:*

(1) *U is homeomorphic to $\mathbb{D}$.*
(2) *U is simply connected.*
(3) $\mathrm{ind}_\gamma(z) = 0$ *for every closed path γ in U and all $z \in \hat{\mathbb{C}} \setminus U$.*
(4) *$\hat{\mathbb{C}} \setminus U$ is connected.*
(5) *Every $f \in \mathcal{H}(U)$ can be approximated by polynomials, uniformly on compact sets.*
(6) *For every $f \in \mathcal{H}(U)$ and every closed path γ in U, $\int_\gamma f\, dz = 0$.*

(7) *Every* $f \in \mathcal{H}(U)$ *is integrable on* U.
(8) *If* $f \in \mathcal{H}(U)$ *and* $1/f \in \mathcal{H}(U)$, *then there exists* $g \in \mathcal{H}(U)$ *such that* $f = \exp g$.
(9) *If* $f \in \mathcal{H}(U)$ *and* $1/f \in \mathcal{H}(U)$, *then there exists* $g \in \mathcal{H}(U)$ *such that* $f = g^2$.

Items (8) and (9) mean that every unit in the ring $\mathcal{H}(U)$ has a holomorphic logarithm and a holomorphic square root.

The proof of this theorem involves the **Riemann mapping theorem**:

Theorem. *Every simply connected region in the complex plane, other than the plane itself, is biholomorphic to the unit disk.*

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