HÖLDER–ZYGMUND CLASSES ON SMOOTH CURVES

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Abstract. We prove that a function in several variables is in the local Zygmund class $\mathcal{Z}^{m,1}$ if and only if its composite with every smooth curve is of class $\mathcal{Z}^{m,1}$. This complements the well-known analogous result for local Hölder–Lipschitz classes $\mathcal{C}^{m,\alpha}$ which we reprove along the way. We demonstrate that these results generalize to mappings between Banach spaces and use them to study the regularity of the superposition operator $f_* : g \mapsto f \circ g$ acting on the global Zygmund space $\Lambda_{m+1}(\mathbb{R}^d)$. We prove that, for all integers $m, k \geq 1$, the map $f_* : \Lambda_{m+1}(\mathbb{R}^d) \to \Lambda_{m+1}(\mathbb{R}^d)$ is of Lipschitz class $\mathcal{C}^{k-1,1}$ if and only if $f \in \mathcal{Z}^{m+k,1}(\mathbb{R})$.

1. Introduction

The main goal of this note is to prove the following 1-dimensional characterization of the local *Zygmund class* $\mathcal{Z}^{m,1}$ of functions in several variables.

Theorem 1.1. *Let $m \in \mathbb{N}$. Let $f : U \to \mathbb{R}$ be a function defined on an open subset $U \subseteq \mathbb{R}^d$. The following conditions are equivalent:*

(1) *$f \circ c \in \mathcal{Z}^{m,1}(\mathbb{R})$ for each $c \in \mathcal{C}^\infty(\mathbb{R}, U)$.*
(2) *$f \in \mathcal{Z}^{m,1}(U)$.*

By definition, $\mathcal{Z}^{m,1}(U)$ consists of all $\mathcal{C}^m$-functions $f : U \to \mathbb{R}$ such that for all compact subsets $K \subseteq U$ and all multiindices $\gamma \in \mathbb{N}^d$ with $|\gamma| = m$ the set

$$\Big\{ \frac{f^{(\gamma)}(x+h) - 2f^{(\gamma)}(x) + f^{(\gamma)}(x-h)}{h} : x, x \pm h \in K,\, h \neq 0 \Big\}$$

is bounded.

The local Zygmund classes $\mathcal{Z}^{m,1}(U)$ fit in the scale of local *Hölder–Lipschitz classes*: in fact, $\mathcal{C}^{m,1}(U) \subseteq \mathcal{Z}^{m,1}(U) \subseteq \mathcal{C}^{m,\alpha}(U)$ for all $m \in \mathbb{N}$ and $\alpha \in (0,1)$ with strict inclusions. In some respects (e.g. in harmonic analysis, cf. [8]) the Zygmund class $\mathcal{Z}^{m,1}(U)$ is more natural and important than the Lipschitz class $\mathcal{C}^{m,1}(U)$. We speak of the scale of local *Hölder–Zygmund classes*, where for $\alpha = 1$ the Zygmund class replaces the respective Lipschitz class:

$$\{\mathcal{C}^{m,\alpha}(U) : m \in \mathbb{N},\, \alpha \in (0,1)\} \cup \{\mathcal{Z}^{m,1}(U) : m \in \mathbb{N}\}.$$

It corresponds to the scale of global Hölder–Zygmund classes $\{\Lambda_s(\mathbb{R}^d) : s > 0\}$ which are the Besov classes $\mathcal{B}^{s,\infty}_\infty(\mathbb{R}^d)$; see Section 2 for precise definitions.

The Hölder–Lipschitz analogue of Theorem 1.1 is due to Boman [1]:

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Theorem 1.2. *Let $m \in \mathbb{N}$ and $\alpha \in (0,1]$. Let $f : U \to \mathbb{R}$ be a function defined on an open subset $U \subseteq \mathbb{R}^d$. The following conditions are equivalent:*

(1) $f \circ c \in \mathcal{C}^{m,\alpha}(\mathbb{R})$ *for each* $c \in \mathcal{C}^\infty(\mathbb{R}, U)$.
(2) $f \in \mathcal{C}^{m,\alpha}(U)$.

Faure [4] generalized this result to $\mathcal{C}^{m,\omega}$, where ω is any modulus of continuity.

The results of Theorem 1.1 and Theorem 1.2 have a certain similarity with the well-known fact (e.g. [8, Theorem 9.1]) that a function $f : \mathbb{R}^d \to \mathbb{R}$ belongs to $\Lambda_s(\mathbb{R}^d)$ if it does so *uniformly* in each variable separately. There is no uniformity (with respect to c) required in condition (1) of Theorem 1.1 and Theorem 1.2, respectively; on the other hand, f is tested on the much larger set of all $\mathcal{C}^\infty$-curves in the domain (instead of just affine lines parallel to the axes).

We will prove Theorem 1.1 in Section 3. Only a few modifications are necessary to obtain also a proof of Theorem 1.2. The general proof scheme is the one used in [7, Section 4.3] and [10, Section 12] for the Lipschitz case. It is combined with the characterization of Hölder–Zygmund classes in terms of finite differences; see [8].

The characterization in Theorem 1.1 enables us to extend the notion of local Zygmund classes to mappings between convenient vector spaces (i.e. Mackey–complete locally convex spaces) such as has been done for Hölder–Lipschitz classes in [7, 6, 10]. This leads to a version of Theorem 1.1 for maps between Banach spaces; see Corollary 4.4.

We also discuss versions of Theorem 1.1, where the domain of definition of f is not an open set. Then a loss of regularity depending on the geometry of the boundary occurs; see Theorem 4.7, Theorem 4.8, and Theorem 4.9 as well as [9] and [14, 15].

In the final section we utilize a version of Theorem 1.2 for maps between Banach spaces to study the regularity of the nonlinear superposition operator $f_* : g \mapsto f \circ g$ acting on the global Zygmund class $\Lambda_{m+1}(\mathbb{R}^d)$.

Theorem 1.3. *Let $m, k \in \mathbb{N}_{\ge 1}$ and $f : \mathbb{R} \to \mathbb{R}$ a function. Then f_* acts on $\Lambda_{m+1}(\mathbb{R}^d)$ and $f_* : \Lambda_{m+1}(\mathbb{R}^d) \to \Lambda_{m+1}(\mathbb{R}^d)$ is of Lipschitz class $\mathcal{C}^{k-1,1}$ if and only if $f \in \mathcal{Z}^{m+k,1}(\mathbb{R})$.*

This complements results of [2] on the $\mathcal{C}^k$-regularity of $f_* : \Lambda_{m+1}(\mathbb{R}^d) \to \Lambda_{m+1}(\mathbb{R}^d)$ based on totally different methods.

Notation. We denote by $\mathbb{N} = \{0, 1, 2, \ldots\}$ the set of nonnegative integers and set $\mathbb{N}_{\ge k} := \{n \in \mathbb{N} : n \ge k\}$. We will make use of standard multiindex notation. The partial derivative of a function f with respect to the j-th variable is denoted by $\partial_j f$. If $\gamma \in \mathbb{N}^d$ is a multiindex, then we use the notation $f^{(\gamma)} = \partial^\gamma f = \partial_1^{\gamma_1} \cdots \partial_d^{\gamma_d} f$ for the corresponding partial derivative of higher order.

2. Hölder–Zygmund classes

2.1. **Function spaces.** By a *modulus of continuity* we mean an increasing subadditive function $\omega : [0,\infty) \to [0,\infty)$ such that $\lim_{t\to 0} \omega(t) = 0$ and $t \mapsto t/\omega(t)$ is locally bounded.

Let ω be a modulus of continuity. Let $U \subseteq \mathbb{R}^d$ be an open set. Recall that $\mathcal{C}^{0,\omega}(U)$ denotes the set of functions $f : U \to \mathbb{R}$ such that

$$\sup_{x, x+h \in K,\, h \neq 0} \frac{|f(x+h) - f(x)|}{\omega(|h|)} < \infty$$

for all compact subsets $K \subseteq U$. For a positive integer m,

$$\mathcal{C}^{m,\omega}(U) := \{f \in \mathcal{C}^m(U) : f^{(\gamma)} \in \mathcal{C}^{0,\omega}(U) \text{ for all } \gamma \in \mathbb{N}^d \text{ with } |\gamma| = m\}.$$

For $\omega(t) = t^\alpha$, where $\alpha \in (0,1]$, we obtain the *Hölder–Lipschitz classes* $\mathcal{C}^{m,\alpha}(U)$.

The *Zygmund class* $\mathcal{Z}^{0,1}(U)$ is the set of all continuous functions $f : U \to \mathbb{R}$ such that

$$\sup_{x,x\pm h\in K,\, h\neq 0} \frac{|f(x+h) - 2f(x) + f(x-h)|}{|h|} < \infty, \tag{2.1}$$

and if m is a positive integer,

$$\mathcal{Z}^{m,1}(U) := \{f \in \mathcal{C}^m(U) : f^{(\gamma)} \in \mathcal{Z}^{0,1}(U) \text{ for all } \gamma \in \mathbb{N}^d \text{ with } |\gamma| = m\}.$$

Continuity of f does not follow from (2.1) and has to be imposed (cf. [8, Proposition 2.7]). All classes $\mathcal{C}^{m,\omega}(U)$ and $\mathcal{Z}^{m,1}(U)$ are endowed with their natural locally convex topologies.

Note that for all $m \in \mathbb{N}$ and $0 < \alpha < \beta < 1$, we have the strict continuous inclusions

$$\mathcal{C}^{m+1}(U) \subsetneq \mathcal{C}^{m,1}(U) \subsetneq \mathcal{Z}^{m,1}(U) \subsetneq \mathcal{C}^{m,\omega}(U) \subsetneq \mathcal{C}^{m,\beta}(U) \subsetneq \mathcal{C}^{m,\alpha}(U) \subsetneq \mathcal{C}^m(U), \tag{2.2}$$

where $\omega(t) := t \log \frac{1}{t}$. For instance $\mathcal{Z}^{0,1}(\mathbb{R})$ contains the Weierstrass function $t \mapsto \sum_k 2^{-k} \sin(2^k t)$ which is nowhere differentiable and thus not locally Lipschitz.

The spaces $\mathcal{C}^{m,\alpha}(U)$, $0 < \alpha < 1$, and $\mathcal{Z}^{m,1}(U)$ are local versions of the global *Hölder–Zygmund spaces* $\Lambda_s(\mathbb{R}^d)$, $s > 0$, which are Banach spaces defined as follows. For $0 < s \le 1$, $\Lambda_s(\mathbb{R}^d)$ consists of all bounded continuous functions $f : \mathbb{R}^d \to \mathbb{R}$ such that $\|f\|_{\Lambda_s} < \infty$, where

$$\|f\|_{\Lambda_s} := \sup_{x\in\mathbb{R}^d} |f(x)| + \sup_{x,h\in\mathbb{R}^d,\, h\neq 0} \frac{|f(x+h) - f(x)|}{|h|^s}, \quad \text{if } 0 < s < 1,$$

$$\|f\|_{\Lambda_1} := \sup_{x\in\mathbb{R}^d} |f(x)| + \sup_{x,h\in\mathbb{R}^d,\, h\neq 0} \frac{|f(x+h) - 2f(x) + f(x-h)|}{|h|}$$

For $s > 1$ the space $\Lambda_s(\mathbb{R}^d)$ is defined recursively: take the unique $m \in \mathbb{N}$ with $m < s \le m+1$. Then $\Lambda_s(\mathbb{R}^d)$ consists of all functions $f : \mathbb{R}^d \to \mathbb{R}$ of class $\mathcal{C}^m$ such that

$$\|f\|_{\Lambda_s} := \|f\|_{\Lambda_{s-1}} + \sum_{j=1}^d \|\partial_j f\|_{\Lambda_{s-1}} < \infty.$$

For $0 < s < t$ there is a continuous inclusion $\Lambda_t(\mathbb{R}^d) \hookrightarrow \Lambda_s(\mathbb{R}^d)$. For integer $s = m+1 > 0$, $\Lambda_s(\mathbb{R}^d)$ strictly contains the *Lipschitz space*

$$\mathrm{Lip}_s(\mathbb{R}^d) := \{f \in \mathcal{C}^m(\mathbb{R}^d) : \|f\|_{\mathrm{Lip}_s} < \infty\},$$

where (again recursively)

$$\|f\|_{\mathrm{Lip}_1} := \sup_{x\in\mathbb{R}^d} |f(x)| + \sup_{x,h\in\mathbb{R}^d,\, h\neq 0} \frac{|f(x+h) - f(x)|}{|h|},$$

$$\|f\|_{\mathrm{Lip}_s} := \|f\|_{\mathrm{Lip}_{s-1}} + \sum_{j=1}^d \|\partial_j f\|_{\mathrm{Lip}_{s-1}} \quad \text{for } s > 1.$$

Note that $\Lambda_t(\mathbb{R}^d) \hookrightarrow \mathrm{Lip}_s(\mathbb{R}^d) \hookrightarrow \Lambda_s(\mathbb{R}^d)$ if s is an integer and $t > s$.

2.2. **Difference quotients and finite differences.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function. The *difference quotient* $\delta^m f(x_0, \ldots, x_m)$ of order m on the pairwise disjoint points $x_0, \ldots, x_m \in \mathbb{R}$ is recursively defined by $\delta^0 f(x_0) := f(x_0)$ and

$$\delta^m f(x_0, \ldots, x_m) := m \frac{\delta^{m-1} f(x_0, \ldots, x_{m-1}) - \delta^{m-1} f(x_1, \ldots, x_m)}{x_0 - x_m}.$$

It is symmetric in $x_0, \ldots, x_m$. One checks easily that

$$\delta^m f(x_0, \ldots, x_m) = m! \sum_{i=0}^m f(x_i) \prod_{\substack{0 \le j \le m \\ j \ne i}} \frac{1}{x_i - x_j}.$$

We will mainly use the *equidistant difference quotient*

$$\begin{aligned} \delta^m_{\text{eq}} f(x; h) &:= \delta^m f(x, x+h, \ldots, x+mh) \\ &= \frac{1}{h^m} \sum_{i=0}^m (-1)^{m-i} \binom{m}{i} f(x+ih) =: \frac{1}{h^m} \Delta^m_h f(x), \end{aligned}$$

where $\Delta^m_h f(x)$ is the *(forward) finite difference* of order m recursively defined by

$$\Delta^0_h f(x) := f(x), \quad \Delta^1_h f(x) := f(x+h) - f(x), \quad \Delta^m_h f(x) := \Delta^1_h(\Delta^{m-1}_h f(x)).$$

Note that, for $f \in \mathcal{C}^1(\mathbb{R})$,

$$\Delta^2_h f(x) = \int_x^{x+h} \Delta^1_h f'(t)\, dt. \tag{2.3}$$

2.3. **Product and chain rule.** Let $f, g, f_i : \mathbb{R} \to \mathbb{R}$. We have

$$\Delta^m_h (f_1 \cdots f_n)(x) = \sum_{i_1 + \cdots + i_n = m} \binom{m}{i_1, \ldots, i_n} \prod_{j=1}^n \Delta^{i_j}_h f_j\Big(x + \Big(m - \sum_{k=j}^n i_k\Big) h\Big). \tag{2.4}$$

We also need a chain rule for finite differences of order one and two:

$$\begin{aligned} \Delta^1_h(f \circ g)(x) &= \Delta^1_{\Delta^1_h g(x)} f(g(x)), \\ \Delta^2_h(f \circ g)(x) &= \Delta^1_{\Delta^2_h g(x)} f(2g(x+h) - g(x)) + \Delta^2_{\Delta^1_h g(x)} f(g(x)). \end{aligned} \tag{2.5}$$

The validity of these formulas is easily established by expanding the right-hand sides.

2.4. **Hölder–Zygmund classes in terms of difference quotients.** First we recall a description of local Lipschitz classes.

Theorem 2.1 ([10, Lemma 12.4])**.** *Let $m \in \mathbb{N}$. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. The following conditions are equivalent:*

1. *$f \in \mathcal{C}^{m,1}(\mathbb{R})$.*
2. *$(x, h) \mapsto \delta^{m+1}_{\text{eq}} f(x; h)$ is locally bounded on $\mathbb{R} \times (\mathbb{R} \setminus \{0\})$.*

As a consequence we obtain

Corollary 2.2. *Let $m \in \mathbb{N}$. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. That $(x, h) \mapsto \delta^m_{\text{eq}} f(x; h)$ is locally bounded implies that $(x, h) \mapsto \delta^j_{\text{eq}} f(x; h)$ is locally bounded for all $j \le m$.*

To get a similar characterization of local Hölder–Zygmund classes we recall

Theorem 2.3 ([8, Theorem 6.1])**.** *If $f \in L^\infty(\mathbb{R}) \cap C^0(\mathbb{R})$ and $0 < s < n$ with $n \in \mathbb{N}$, then $f \in \Lambda_s(\mathbb{R})$ is equivalent to $|\Delta^n_h f(x)| \le C\, |h|^s$ for all $x, h \in \mathbb{R}$.*

For local Zygmund classes we may infer

Theorem 2.4. *Let $m \in \mathbb{N}$. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. The following conditions are equivalent:*

(1) $f \in \mathcal{Z}^{m,1}(\mathbb{R})$.
(2) $(x,h) \mapsto h\, \delta_{\mathrm{eq}}^{m+2} f(x;h)$ *and* $(x,h) \mapsto \delta_{\mathrm{eq}}^{m} f(x;h)$ *are locally bounded on* $\mathbb{R} \times (\mathbb{R} \setminus \{0\})$.

Proof. (1) ⇒ (2) Let $f \in \mathcal{Z}^{m,1}(\mathbb{R})$ and $I \subseteq \mathbb{R}$ a bounded interval. Then $f|_I$ has an extension $F \in \Lambda_{m+1}(\mathbb{R})$ (such that $\|F\|_{\Lambda_{m+1}(\mathbb{R})} \le C\, \|f\|_{\Lambda_{m+1}(I)}$ for some $C > 0$); see [8, Section 14]. By Theorem 2.3, $|\Delta_h^{m+2} F(x)| \le C\, |h|^{m+1}$ for all $x, h \in \mathbb{R}$. From this it is easy to conclude that $(x,h) \mapsto h\, \delta_{\mathrm{eq}}^{m+2} f(x;h)$ is locally bounded. That also $(x,h) \mapsto \delta_{\mathrm{eq}}^{m} f(x;h)$ is locally bounded is trivial for $m = 0$ and follows from Theorem 2.1 for $m \ge 1$, since $f \in \mathcal{Z}^{m,1}(\mathbb{R}) \subseteq \mathcal{C}^{m-1,1}(\mathbb{R})$ by (2.2).

(2) ⇒ (1) Fix a bounded interval $I \subseteq \mathbb{R}$ and a $\mathcal{C}^\infty$-function $\chi : \mathbb{R} \to [0,1]$ with compact support which is 1 in a neighborhood of I. Then, by the product rule (2.4), $g := \chi f$ satisfies $|\Delta_h^{m+2} g(x)| \le C\, |h|^{m+1}$ for all $x, h \in \mathbb{R}$ and thus $g \in \Lambda_{m+1}(\mathbb{R})$, by Theorem 2.3. Indeed, $|\Delta_h^{m+2} g(x)|$ is bounded by a finite sum of terms which are up to constant factors of the form $|\Delta_h^i \chi(y) \Delta_h^j f(z)|$, where $i + j = m + 2$. If $j \le m$, then $|\Delta_h^j f(z)| \le C|h|^j$ and $|\Delta_h^{m+1} f(z)| \le C|h|^m$, by Corollary 2.2. Thus $|\Delta_h^i \chi(y) \Delta_h^j f(z)| \le C|h|^{m+1}$ for $j \le m + 1$. For $j = m + 2$ this follows from local boundedness of $(x,h) \mapsto h\, \delta_{\mathrm{eq}}^{m+2} f(x;h)$. Since I was arbitrary, we may conclude that $f \in \mathcal{Z}^{m,1}(\mathbb{R})$. □

Remark 2.5. The local boundedness of $(x,h) \mapsto \delta_{\mathrm{eq}}^{m} f(x;h)$ is used for the "local to global" argument in (2) ⇒ (1). It is possible that it can be dropped from the formulation of (2).

For the local Hölder classes we have

Theorem 2.6. *Let $m \in \mathbb{N}$ and $\alpha \in (0,1)$. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. The following conditions are equivalent:*

(1) $f \in \mathcal{C}^{m,\alpha}(\mathbb{R})$.
(2) $(x,h) \mapsto |h|^{1-\alpha} \delta_{\mathrm{eq}}^{m+1} f(x;h)$ *and* $(x,h) \mapsto \delta_{\mathrm{eq}}^{m} f(x;h)$ *are locally bounded on* $\mathbb{R} \times (\mathbb{R} \setminus \{0\})$.

Proof. This follows in analogy to Theorem 2.4 again from Theorem 2.3. □

3. Testing on smooth curves

This section is devoted to the proof of Theorem 1.1. A few adjustments are all that is needed to also prove Theorem 1.2 in one go. These adjustments are indicated in Section 3.7.

3.1. **Composition with smooth curves preserves the class.**

Proposition 3.1. *Let $U \subseteq \mathbb{R}^d$ be an open subset and $c \in \mathcal{C}^\infty(\mathbb{R}, U)$. Then:*

(1) *For each $m \in \mathbb{N}$ and $f \in \mathcal{Z}^{m,1}(U)$ we have $f \circ c \in \mathcal{Z}^{m,1}(\mathbb{R})$.*
(2) *For each $m \in \mathbb{N}$, $\alpha \in (0,1]$, and $f \in \mathcal{C}^{m,\alpha}(U)$ we have $f \circ c \in \mathcal{C}^{m,\alpha}(\mathbb{R})$.*

The proposition is a consequence of sharper versions: e.g. [13], [2], and [3]. For $m \ge 1$ a proof of (1) can be assembled from the arguments in Section 5; see Remark 5.2.

3.2. **Curve lemma.** We will repeatedly use the *general curve lemma* [10, 12.2] which we restate here in a simple form for the convenience of the reader.

Lemma 3.2. *Let $c_n \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R}^d)$ be a sequence of $\mathcal{C}^\infty$-curves that converges fast to 0, i.e., for each $k \in \mathbb{N}$ the sequence $(n^k c_n)_n$ is bounded in $\mathcal{C}^\infty(\mathbb{R},\mathbb{R}^d)$. Let $s_n \ge 0$ be reals with $\sum_n s_n < \infty$. Then there exist a curve $c \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R}^d)$ and a convergent sequence t_n of reals such that $c(t+t_n) = c_n(t)$ for $|t| \le s_n$ and all n.*

3.3. **Degree zero.** The proof of Theorem 1.1 is based on induction on m. The following lemma treats the base case $m = 0$.

Lemma 3.3. *Let $U \subseteq \mathbb{R}^d$ be an open set and $f : U \to \mathbb{R}$ a function. The following conditions are equivalent:*

(1) *$f \circ c \in \mathcal{Z}^{0,1}(\mathbb{R})$ for all $c \in \mathcal{C}^\infty(\mathbb{R}, U)$.*
(2) *$f \in \mathcal{Z}^{0,1}(U)$.*

Proof. That (2) implies (1) follows from Proposition 3.1. Let us assume that (1) holds and suppose for contradiction that $f \notin \mathcal{Z}^{0,1}(U)$. Note that (1) implies that f is continuous (cf. [10, Theorem 4.11]). Thus there is a compact set $K \subseteq U$ and points x_n, $x_n \pm h_n$ in K such that

$$q_n := \frac{|f(x_n + h_n) - 2f(x_n) + f(x_n - h_n)|}{|h_n|}$$

is unbounded. Passing to subsequences, we may assume that $|x_n - x| \le 4^{-n}$, $|h_n| \le 4^{-n}$, and $q_n \ge n2^n$. Consider the curves $c_n(t) := x_n + t\frac{h_n}{2^n|h_n|}$ and note that $c_n - x$ converges fast to 0. By Lemma 3.2, there is a $\mathcal{C}^\infty$-curve $c : \mathbb{R} \to U$ and a convergent sequence of reals t_n such that $c(t+t_n) = c_n(t)$ for all $|t| \le s_n := 2^n|h_n|$. Then

$$\frac{|(f\circ c)(t_n + s_n) - 2(f \circ c)(t_n) + (f \circ c)(t_n - s_n)|}{s_n} = \frac{q_n}{2^n} \ge n$$

contradicting (1). □

Lemma 3.4. *Let $\alpha \in (0,1]$, $U \subseteq \mathbb{R}^d$ an open set, and $f : U \to \mathbb{R}$ a function. The following conditions are equivalent:*

(1) *$f \circ c \in \mathcal{C}^{0,\alpha}(\mathbb{R})$ for all $c \in \mathcal{C}^\infty(\mathbb{R}, U)$.*
(2) *$f \in \mathcal{C}^{0,\alpha}(U)$.*

Proof. Repeat the proof of Lemma 3.3 with $q_n := |f(x_n + h_n) - f(x_n)|/|h_n|^\alpha$; cf. [5], [10, Lemma 12.7], or [11]. □

3.4. **Proof of Theorem 1.1.** The key step is the following proposition.

Proposition 3.5. *Let $m \in \mathbb{N}_{\ge 1}$. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be such that $f \circ c \in \mathcal{Z}^{m,1}(\mathbb{R})$ for all $c \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R}^2)$. Then $\partial_2 f(\cdot, 0) \in \mathcal{Z}^{m-1,1}(\mathbb{R})$.*

Let us now use Proposition 3.5 to complete the proof of Theorem 1.1. The proof of Proposition 3.5 will be given in Section 3.6.

Proposition 3.6. *Let $m \in \mathbb{N}_{\ge 1}$. Let $U \subseteq \mathbb{R}^d$ be an open set. Let $f : U \to \mathbb{R}$ be a function such that $f \circ c \in \mathcal{Z}^{m,1}(\mathbb{R})$ for all $c \in \mathcal{C}^\infty(\mathbb{R}, U)$. Then $d_v f(x) := \partial_t|_{t=0} f(x+tv)$ exists for all $(x, v) \in U \times \mathbb{R}^d$ and defines a mapping $df : U \times \mathbb{R}^d \to \mathbb{R}$ such that $df \circ (x, v) \in \mathcal{Z}^{m-1,1}(\mathbb{R})$ for each $(x, v) \in \mathcal{C}^\infty(\mathbb{R}, U \times \mathbb{R}^d)$.*

Proof. The directional derivative $d_v f(x)$ exists, since $s \mapsto f(x+sv)$ belongs to $\mathcal{Z}^{m,1} \subseteq \mathcal{C}^1$ for s near 0. Let $(x,v) \in \mathcal{C}^\infty(\mathbb{R}, U \times \mathbb{R}^d)$ and consider the $\mathcal{C}^\infty$-map $g(t,s) := x(t) + sv(t)$. Then the open set $\Omega := g^{-1}(U)$ contains $\mathbb{R} \times \{0\}$. Fix $R, r > 0$ such that $[-R,R] \times [-r,r] \subseteq \Omega$. For any $u > 0$ choose a $\mathcal{C}^\infty$-function $\varphi_u : \mathbb{R} \to [-u,u]$ such that $\varphi_u(x) = x$ for all $x \in [-\frac{u}{2}, \frac{u}{2}]$. Then $\tilde{g}(t,s) := g(\varphi_R(t), \varphi_r(s))$ maps $\mathbb{R}^2$ to U and coincides with g on $[-\frac{R}{2}, \frac{R}{2}] \times [-\frac{r}{2}, \frac{r}{2}]$. Thus $f \circ \tilde{g} : \mathbb{R}^2 \to \mathbb{R}$ has the property that $f \circ \tilde{g} \circ c \in \mathcal{Z}^{m,1}(\mathbb{R})$ for all $c \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^2)$, by assumption. By Proposition 3.5, $t \mapsto \partial_2(f \circ \tilde{g})(t,0)$ belongs to $\mathcal{Z}^{m-1,1}(\mathbb{R})$. For $t \in [-\frac{R}{2}, \frac{R}{2}]$ we have

$$\partial_2(f \circ \tilde{g})(t,0) = \partial_s|_{s=0}\big(f(x(t)+sv(t))\big) = d_{v(t)} f(x(t)) = (df \circ (x,v))(t).$$

Since $R > 0$ was arbitrary, we may conclude that $df \circ (x,v) \in \mathcal{Z}^{m-1,1}(\mathbb{R})$. □

Now we may complete the proof of Theorem 1.1. One implication simply follows from Proposition 3.1. For the other implication suppose that $f : U \to \mathbb{R}$ has the property that $f \circ c \in \mathcal{Z}^{m,1}(\mathbb{R})$ for all $c \in \mathcal{C}^\infty(\mathbb{R}, U)$. We proceed by induction on m. The case $m = 0$ follows from Lemma 3.3. Suppose that $m \geq 1$. We may infer from Proposition 3.6 that the partial derivatives $\partial_j f(x)$, $j = 1, \ldots, d$, of first order exist at all $x \in U$ and $\partial_j f \circ c \in \mathcal{Z}^{m-1,1}(\mathbb{R})$ for each $c \in C^\infty(\mathbb{R}, U)$ and all j. The induction hypothesis implies that $\partial_j f \in \mathcal{Z}^{m-1,1}(U)$ for all j, that is, $f \in \mathcal{Z}^{m,1}(U)$. This ends the proof of Theorem 1.1.

3.5. **Auxiliary results.** We need some preparatory results for the proof of Proposition 3.5. First we derive some properties of functions in $\mathcal{Z}^{m,1}(\mathbb{R})$.

We use the Landau O-notation at 0 in the following way. Let $I \subseteq \mathbb{R}$ be a bounded interval. If $\varphi_x(h)$ is a function in h which may also depend on $x \in I$ and $\psi(h) > 0$ is a function of h, then $\varphi_x(h) = O_I(\psi(h))$ shall mean that there is a constant $C = C(I) > 0$ such that $|\varphi_x(h)| \leq C\,\psi(h)$ for all $x \in I$ and all sufficiently small h.

Of crucial importance will be a Taylor formula for functions in $\mathcal{Z}^{m,1}(\mathbb{R})$:

Lemma 3.7. *Let $m \in \mathbb{N}$ and $f \in \mathcal{Z}^{m,1}(\mathbb{R})$. Then, for each bounded interval $I \subseteq \mathbb{R}$,*

$$\frac{1}{2^m} f(x+2h) - 2f(x+h) + \sum_{j=0}^{m} \Big(2 - \frac{1}{2^{m-j}}\Big) \frac{f^{(j)}(x)}{j!} h^j = O_I(|h|^{m+1}). \tag{3.1}$$

Proof. We proceed by induction on m. For $m = 0$ the statement follows from the definition. Let us assume that the identity holds for m and show it for $m+1$. We suppose that $h > 0$; if $h < 0$ the arguments are similar. Let $F \in \mathcal{Z}^{m+1,1}(\mathbb{R})$ with $F' = f \in \mathcal{Z}^{m,1}(\mathbb{R})$. Integrating (3.1) in h yields

$$\begin{aligned}
&\frac{1}{2^{m+1}}\big(F(x+2h) - F(x)\big) - 2\big(F(x+h) - F(x)\big) + \sum_{j=0}^{m} \Big(2 - \frac{1}{2^{m-j}}\Big) \frac{F^{(j+1)}(x)}{(j+1)!} h^{j+1} \\
&\qquad = \frac{1}{2^{m+1}} F(x+2h) - 2F(x+h) + \sum_{i=0}^{m+1} \Big(2 - \frac{1}{2^{m+1-i}}\Big) \frac{F^{(i)}(x)}{i!} h^i \\
&\qquad = O_I(h^{m+2})
\end{aligned}$$

completing the induction. □

Let $f : \mathbb{R} \to \mathbb{R}$ and $m \in \mathbb{N}$. We set

$$D_m(f)(x;h) := \frac{1}{2^m} f(x+2h) - 2f(x+h), \quad x, h \in \mathbb{R}.$$

If $m \ge 1$ we can approximate $hf'(x)$ to order $m+1$ in h by a suitable linear combination of $D_m(f)(x;jh)$, for $j = 0,1,\dots,m$, in a uniform way for $f \in \mathcal{Z}^{m,1}(\mathbb{R})$ and $x \in I$:

Lemma 3.8. *Let $m \in \mathbb{N}_{\ge 1}$. There exist constants $a_0,\dots,a_m \in \mathbb{R}$ such that for all $f \in \mathcal{Z}^{m,1}(\mathbb{R})$ we have, for each bounded interval $I \subseteq \mathbb{R}$,*

$$hf'(x) - \sum_{j=0}^m a_j D_m(f)(x;jh) = O_I(|h|^{m+1}).$$

Proof. By Lemma 3.7, for any choice of $a_0,\dots,a_m \in \mathbb{R}$ we have

$$\begin{aligned}\sum_{j=0}^m a_j D_m(f)(x;jh) &= \sum_{j=0}^m a_j \Big(\sum_{i=0}^m \Big(\frac{1}{2^{m-i}} - 2\Big) \frac{f^{(i)}(x)}{i!} (jh)^i + O_I(|h|^{m+1})\Big) \\ &= \sum_{i=0}^m \Big(\sum_{j=0}^m a_j j^i\Big) \Big(\frac{1}{2^{m-i}} - 2\Big) \frac{f^{(i)}(x)}{i!} h^i + O_I(|h|^{m+1}).\end{aligned}$$

To obtain the assertion it suffices to choose $a_0,\dots,a_m$ such that

$$\begin{aligned}\sum_{j=0}^m a_j j &= \Big(\frac{1}{2^{m-1}} - 2\Big)^{-1}, \\ \sum_{j=0}^m a_j j^i &= 0, \quad 0 \le i \le m,\ i \ne 1,\end{aligned}$$

which is possible, since the coefficients of this linear system of equations form a Vandermonde matrix. □

Let us set

$$A_m(f)(x;h) := \sum_{j=0}^m a_j D_m(f)(x;jh),$$

where $a_0,\dots,a_m \in \mathbb{R}$ are the constants provided by Lemma 3.8.

3.6. **Proof of Proposition 3.5.** For $f : \mathbb{R}^2 \to \mathbb{R}$ we define

$$\mathbf{D}_m(f)(x,h) := \frac{1}{2^m} f(x,2h) - 2f(x,h)$$

and

$$\mathbf{A}_m(f)(x,h) := \sum_{j=0}^m a_j \mathbf{D}_m(f)(x,jh),$$

where $a_0,\dots,a_m \in \mathbb{R}$ are the constants provided by Lemma 3.8.

From Lemma 3.8 we get an approximation result for functions in two variables:

Lemma 3.9. *Let $m \in \mathbb{N}_{\ge 1}$. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be such that $f \circ c \in \mathcal{Z}^{m,1}(\mathbb{R})$ for all $c \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R}^2)$. For each compact interval $I \subseteq \mathbb{R}$ there is a constant $C > 0$ such that for all $x,h \in I$ we have*

$$\big|h\partial_2 f(x,0) - \mathbf{A}_m(f)(x,h)\big| \le C|h|^{m+1}.$$

Proof. In the case $0 \notin I$ so that h is bounded away from 0, it is enough to check that $x \mapsto \partial_2 f(x,0)$ is bounded on I. (That f and consequently $(x,h) \mapsto \mathbf{A}_m(f)(x,h)$ is bounded on $I \times I$ follows easily from Lemma 3.2 or [10, 2.8].) Suppose, for contradiction, that there exist $x_n, x \in I$ with $|x_n - x| \le 4^{-n}$ and $|\partial_2 f(x_n,0)| \ge n2^n$. Applying Lemma 3.2 to $c_n(t) := (x_n, 2^{-n}t)$ and $s_n := 2^{-n}$ gives a $\mathcal{C}^\infty$-curve $c : \mathbb{R} \to \mathbb{R}^2$ and a convergent sequence of reals t_n such that $c(t+t_n) = c_n(t)$ for $|t| \le s_n$. Thus, $|(f \circ c)'(t_n)| = |(f \circ c_n)'(0)| = 2^{-n}|\partial_2 f(x_n,0)| \ge n$, contradicting that $f \circ c \in \mathcal{Z}^{m,1}(\mathbb{R}) \subseteq \mathcal{C}^1(\mathbb{R})$.

Now assume that $0 \in I$. Suppose, for contradiction, that there are $x_n, h_n \in I$ such that

$$\big|h_n \partial_2 f(x_n,0) - \mathbf{A}_m(f)(x_n,h_n)\big| \ge n2^{n(m+1)}|h_n|^{m+1}.$$

By passing to subsequences, we may assume that $x_n \to x$ and $h_n \to 0$ (by the first paragraph) and in turn that $|x_n - x| \le 4^{-n}$ and $|h_n| \le 4^{-n}$. Applying Lemma 3.2 to $c_n(t) := (x_n, 2^{-n}t)$ and $s_n = 2m \cdot 2^{-n}$ we find a $\mathcal{C}^\infty$-curve $c : \mathbb{R} \to \mathbb{R}^2$ and a convergent sequence of reals t_n with $c(t+t_n) = c_n(t)$ for $|t| \le s_n$. Then

$$\begin{aligned}
&\Big|2^n h_n (f \circ c)'(t_n) - A_m(f \circ c)(t_n; 2^n h_n)\Big| \\
&\quad = \Big|2^n h_n (f \circ c_n)'(0) - A_m(f \circ c_n)(0; 2^n h_n)\Big| \\
&\quad = \big|h_n \partial_2 f(x_n,0) - \mathbf{A}_m(f)(x_n,h_n)\big| \\
&\quad \ge n(2^n|h_n|)^{m+1}.
\end{aligned}$$

But this contradicts Lemma 3.8. □

Now we are ready to show Proposition 3.5, assuming the validity of Theorem 1.2 which will be (re)proved in Section 3.7. Let $m \in \mathbb{N}_{\ge 1}$ and suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies $f \circ c \in \mathcal{Z}^{m,1}(\mathbb{R})$ for all $c \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R}^2)$. Then $g := \partial_2 f(\cdot,0)$ is well-defined. We have to prove that $g \in \mathcal{Z}^{m-1,1}(\mathbb{R})$. By Theorem 2.4, it suffices to check the following three claims.

Claim (i). *g is continuous.*

Since $f \circ c \in \mathcal{C}^{1,\alpha}(\mathbb{R})$ for all $c \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R}^2)$ and all $\alpha \in (0,1)$, by (2.2), we may invoke the result for $\mathcal{C}^{1,\alpha}$; cf. Theorem 1.2 and its proof in Section 3.7. Claim (i) follows.

Claim (ii). *$\delta_{\mathrm{eq}}^{m-1} g(x;h)$ is locally bounded in $(x,h) \in \mathbb{R} \times (\mathbb{R} \setminus \{0\})$.*

By (2.2), we have $f \circ c \in \mathcal{C}^{m-1,1}(\mathbb{R})$ for all $c \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R}^2)$. By Theorem 1.2, $g \in \mathcal{C}^{m-2,1}(\mathbb{R})$ if $m \ge 2$ and thus $\delta_{\mathrm{eq}}^{m-1} g(x;h)$ is locally bounded in $(x,h) \in \mathbb{R} \times (\mathbb{R} \setminus \{0\})$, in view of Theorem 2.1. If $m = 1$ this is trivially true by Claim (i).

Claim (iii). *$h\delta_{\mathrm{eq}}^{m+1} g(x;h)$ is locally bounded in $(x,h) \in \mathbb{R} \times (\mathbb{R} \setminus \{0\})$.*

Suppose, for contradiction, that Claim (iii) is not true, say for x in a neighborhood of 0. Then there exist x_n and h_n with $|x_n| \le 4^{-n}$ and $0 < h_n < 4^{-n}$ (if necessary replace $f(x,y)$ by $f(-x,y)$) such that

$$|h_n \delta_{\mathrm{eq}}^{m+1} g(x_n; h_n)| \ge n2^{n(m+1)}.$$

Let $c_n(t) := (x_n - h_n + 2^{-n}t, 2^{-n}t)$ and $s_n := (m+2)2^{-n}$. Lemma 3.2 gives a $\mathcal{C}^\infty$-curve $c : \mathbb{R} \to \mathbb{R}^2$ and a convergent sequence of reals t_n with $c(t+t_n) = c_n(t)$

for $|t| \le s_n$. Set

$$f_1(x,h) := hg(x),$$
$$f_2(x,h) := hg(x) - \mathbf{A}_m(f)(x,h).$$

Then the sequence

$$\begin{aligned}
T_n :&= 2^n h_n \delta_{\text{eq}}^{m+2}(f_1 \circ c)(t_n; 2^n h_n) \\
&= \frac{2^n h_n}{(2^n h_n)^{m+2}} \sum_{i=0}^{m+2} (-1)^{m+2-i} \binom{m+2}{i} f_1(x_n - h_n + ih_n, ih_n) \\
&= \frac{1}{(2^n h_n)^{m+1}} \sum_{i=0}^{m+2} (-1)^{m+2-i} \binom{m+2}{i} ih_n g(x_n + (i-1)h_n) \\
&= \frac{1}{2^{n(m+1)}} \frac{h_n}{h_n^{m+1}} \sum_{i=1}^{m+2} (-1)^{m+2-i} \binom{m+2}{i} ig(x_n + (i-1)h_n) \\
&= \frac{m+2}{2^{n(m+1)}} \frac{h_n}{h_n^{m+1}} \sum_{j=0}^{m+1} (-1)^{m+1-j} \binom{m+1}{j} g(x_n + jh_n) \\
&= \frac{m+2}{2^{n(m+1)}} h_n \delta_{\text{eq}}^{m+1} g(x_n; h_n)
\end{aligned} \tag{3.2}$$

satisfies $|T_n| \ge (m+2)n$. On the other hand, $\mathbf{A}_m(f) \circ c \in \mathcal{Z}^{m,1}(\mathbb{R})$, by the assumption on f, so that $2^n h_n \delta_{\text{eq}}^{m+2}(\mathbf{A}_m(f) \circ c)(t_n; 2^n h_n)$ is bounded, by Theorem 2.4. By Lemma 3.9, there is $C > 0$ such that $|f_2(x,h)| \le C|h|^{m+1}$ for all $x, h \in [-(m+2), m+2]$ which implies that

$$\begin{aligned}
2^n h_n |\delta_{\text{eq}}^{m+2}(f_2 \circ c)(t_n; 2^n h_n)| &\le \frac{2^n h_n}{(2^n h_n)^{m+2}} \sum_{i=0}^{m+2} \binom{m+2}{i} |f_2(x_n - h_n + ih_n, ih_n)| \\
&\le C\Big(\frac{m+2}{2^n}\Big)^{m+1} \sum_{i=0}^{m+2} \binom{m+2}{i}.
\end{aligned}$$

Consequently, T_n is bounded, a contradiction. Thus, Claim (iii) is shown and the proof of Proposition 3.5 is complete.

3.7. **Proof of Theorem 1.2.** We will indicate how to show

Proposition 3.10. *Let $m \in \mathbb{N}$ and $\alpha \in (0,1]$. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be such that $f \circ c \in \mathcal{C}^{m+1,\alpha}(\mathbb{R})$ for all $c \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^2)$. Then $\partial_2 f(\cdot, 0) \in \mathcal{C}^{m,\alpha}(\mathbb{R})$.*

Then it is easy to finish the proof of Theorem 1.2 using a variant of Proposition 3.6 as well as Lemma 3.4 and Proposition 3.1.

We replace Lemma 3.7 by the following easy consequence of Taylor's formula.

Lemma 3.11. *Let $m \in \mathbb{N}$, $\alpha \in (0,1]$, and $f \in C^{m,\alpha}(\mathbb{R})$. Then, for each bounded interval $I \subseteq \mathbb{R}$,*

$$f(x+h) = \sum_{j=0}^{m} \frac{f^{(j)}(x)}{j!} h^j + O_I(|h|^{m+\alpha}).$$

As in Lemma 3.8 we conclude: If $b_0, \dots, b_m \in \mathbb{R}$ is the solution of the system

$$\sum_{j=0}^{m} b_j j = 1,$$
$$\sum_{j=0}^{m} b_j j^i = 0, \quad 0 \le i \le m,\ i \ne 1,$$

then for all $f \in \mathcal{C}^{m,\alpha}(\mathbb{R})$, $m \ge 1$, and for each bounded interval $I \subseteq \mathbb{R}$,

$$hf'(x) = \sum_{j=0}^{m} b_j f(x+jh) + O_I(|h|^{m+\alpha}). \tag{3.3}$$

Now suppose that $m \in \mathbb{N}$, $\alpha \in (0,1]$, and $f : \mathbb{R}^2 \to \mathbb{R}$ is such that $f \circ c \in \mathcal{C}^{m+1,\alpha}(\mathbb{R})$ for all $c \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R}^2)$. Then we infer from (3.3), in analogy to Lemma 3.9, that on each compact interval I there is $C > 0$ such that

$$|hg(x) - \mathbf{B}_m(f)(x,h)| \le C|h|^{m+1+\alpha}, \quad x, h \in I, \tag{3.4}$$

where $g := \partial_2 f(\cdot, 0)$ and $\mathbf{B}_m(f)(x,h) := \sum_{j=0}^m b_j f(x, jh)$.

To complete the proof of Proposition 3.10 we have to show that $g \in \mathcal{C}^{m,\alpha}(\mathbb{R})$. By Theorem 2.6, it is enough to check the following three claims.

Claim (I)**.** *g is continuous.*

To see that g is continuous it suffices to finish the proof of Proposition 3.10 in the case $m = 0$. Theorem 2.6 is trivial for $m = 0$ and holds without the a priori assumption that g is continuous: local boundedness of $|h|^{1-\alpha}\delta^1_{\text{eq}} g(x;h) = |h|^{-\alpha}(g(x+h) - g(x))$ in x and h is equivalent to $g \in \mathcal{C}^{0,\alpha}$. That means for $m = 0$ only Claim (III) must be shown.

Claim (II)**.** *$\delta^m_{\text{eq}} g(x;h)$ is locally bounded in $(x,h) \in \mathbb{R} \times (\mathbb{R} \setminus \{0\})$.*

By (2.2), we have $f \circ c \in \mathcal{C}^{m,1}(\mathbb{R})$ for all $c \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R}^2)$. Thus it suffices to finish the proof of Proposition 3.10 in the case $\alpha = 1$. Indeed, by Theorem 2.1, Claim (II) holds if and only if $g \in \mathcal{C}^{m-1,1}(\mathbb{R})$ as g is continuous by Claim (I).

Claim (III)**.** *$|h|^{1-\alpha}|\delta^{m+1}_{eq} g(x;h)|$ is locally bounded in $(x,h) \in \mathbb{R} \times (\mathbb{R} \setminus \{0\})$.*

We may assume that there exist x_n and h_n with $|x_n| \le 4^{-n}$ and $0 < h_n < 4^{-n}$ such that

$$h_n^{1-\alpha} |\delta^{m+1}_{\text{eq}} g(x_n; h_n)| \ge n 2^{n(m+1+\alpha)}.$$

With the same choices of c_n and s_n as in Section 3.6 we find a $\mathcal{C}^\infty$-curve c and a convergent sequence of reals t_n such that $c(t + t_n) = c_n(t)$ for $|t| \le s_n$. Using $f_1(x,h) := hg(x)$ and T_n from (3.2), we find

$$\begin{aligned}(2^n h_n)^{-\alpha}|T_n| &= (2^n h_n)^{1-\alpha} |\delta^{m+2}_{\text{eq}} (f_1 \circ c)(t_n; 2^n h_n)| \\ &= \frac{m+2}{2^{n(m+1+\alpha)}} h_n^{1-\alpha} |\delta^{m+1}_{\text{eq}} g(x_n; h_n)| \ge (m+2)n.\end{aligned}$$

On the other hand, (3.4) implies that for $f_2 := f_1 - \mathbf{B}_m(f)$ the sequence $(2^n h_n)^{1-\alpha}|\delta^{m+2}_{\text{eq}}(f_2 \circ c)(t_n; 2^n h_n)|$ is bounded. Since $\mathbf{B}_m(f) \circ c \in \mathcal{C}^{m+1,\alpha}(\mathbb{R})$ also $(2^n h_n)^{1-\alpha}|\delta^{m+2}_{\text{eq}}(\mathbf{B}_m(f) \circ c)(t_n; 2^n h_n)|$ is bounded, by Theorem 2.6, so that $(2^n h_n)^{-\alpha}|T_n|$ is bounded, a contradiction.

4. On Banach spaces and beyond

In this section we use the characterization in Theorem 1.1 to extend the local Zygmund classes to convenient vector spaces. We shall see in Corollary 4.4 that a version of Theorem 1.1 holds for maps between Banach spaces. For background on convenient analysis we refer to [10] and also [7]. We will also discuss versions of Theorem 1.1 and Theorem 1.2 where the domain of f is not an open set.

4.1. **Convenient analysis of local Hölder–Zygmund classes.** Recall that a *convenient vector space* E is a Mackey-complete locally convex space. The c^∞-*topology* on E is the final topology with respect to all $\mathcal{C}^\infty$-curves (equivalently, all Mackey-convergent sequences) in E; it is not a linear topology.

Let $m \in \mathbb{N}$ and $\alpha \in (0,1]$.

Definition 4.1. Let E, F be convenient vector spaces, U a c^∞-open subset of E. A map $f : U \to F$ is said to be of class Zyg^m (resp. Höl^m_α) if for each $c \in \mathcal{C}^\infty(\mathbb{R}, U)$ and each $\ell \in F'$ the composite $\ell \circ f \circ c$ belongs to $\mathcal{Z}^{m,1}(\mathbb{R})$ (resp. $\mathcal{C}^{m,\alpha}(\mathbb{R})$).

For Höl^m_α the results we are aiming for have already been established in [6] (for $\alpha = 1$ also in [7]). So we will exploit the fact that, by (2.2), if f is of class Zyg^m then it is of class Höl^m_α for any $\alpha \in (0,1)$.

Let $f : U \to F$ be of class Zyg^1. Then, by [6, Lemma 7], f is *weakly differentiable*, i.e., for all $x \in U$, $v \in E$, the limit

$$df(x,v) := \lim_{t \to 0} \frac{f(x+tv) - f(x)}{t}$$

exists with respect to the weak topology. By [6, Proposition 9], f is also *strictly differentiable*, i.e., for each Mackey-compact $K \subseteq U$ and bounded $B \subseteq E$,

$$\frac{f(x+sv) - f(x+tv)}{s-t}$$

is Mackey-convergent to $df(x,v)$ as $s,t \to 0$ uniformly for $x \in K$ and $v \in B$ and $f'(x) := df(x,\cdot) \in L(E,F)$ for every $x \in U$.

Lemma 4.2. *Let $m \in \mathbb{N}$. Let $f : U \to F$ be of class Zyg^{m+1}. Then:*

1. *$df : U \times E \to F$ is of class Zyg^m.*
2. *$f' : U \to L(E,F)$ is of class Zyg^m.*

Proof. (1) follows from the proof of Proposition 3.6: consider $(t,s) \mapsto \ell(f(x(t) + sv(t)))$ where $\ell \in F'$. By the uniform boundedness principle, to see (2) it suffices to check that $\operatorname{ev}_v \circ f' : U \to F$ is of class Zyg^m for all $v \in E$ which holds by (1) since $\operatorname{ev}_v \circ f' = df(\cdot, v)$. Indeed, a curve $c : \mathbb{R} \to G$ in a convenient vector space G is of class Zyg^m if and only if $h\delta^{m+2}_{\text{eq}} c(t;h)$ and $\delta^m_{\text{eq}} c(t;h)$ are locally bounded in $(t,h) \in \mathbb{R} \times (\mathbb{R} \setminus \{0\})$, by Theorem 2.4 and testing with $\ell \in G'$. □

Assume that $f : U \to F$ is weakly differentiable and df is of class Zyg^0. Let $c \in \mathcal{C}^\infty(\mathbb{R}, U)$. Then $f \circ c$ is differentiable and $(f \circ c)'(t) = df(c(t), c'(t))$, by [6, Proposition 11].

Theorem 4.3. *Let $m \in \mathbb{N}$. Let E, F be convenient vector spaces, U a c^∞-open subset of E, and $f : U \to F$ a map. The following conditions are equivalent:*

1. *f is of class Zyg^{m+1}.*
2. *f is strictly differentiable and $f' : U \mapsto L(E,F)$ is of class Zyg^m.*

(3) *f is weakly differentiable and $df : U \times E \to F$ is of class* Zyg^m.

Proof. (1) $\Rightarrow$ (2) follows from Lemma 4.2 and the preceding remarks.

(2) $\Rightarrow$ (3) The map $f' \times \mathrm{id} : U \times E \mapsto L(E, F) \times E$ is of class Zyg^m. The evaluation map $\mathrm{ev} : L(E, F) \times E \to F$ is bilinear and smooth. It follows that $df = \mathrm{ev} \circ (f' \times \mathrm{id})$ is of class Zyg^m.

(3) $\Rightarrow$ (1) Let $c \in \mathcal{C}^\infty(\mathbb{R}, U)$. Then $(f \circ c)'(t) = df(c(t), c'(t))$ (see remark before the theorem) is of class Zyg^m, whence f is of class Zyg^{m+1}. □

Corollary 4.4. *Let $m \in \mathbb{N}$. Let E, F be Banach spaces, U open in E, and $f : U \to F$ a map. Then f is of class Zyg^m if and only if f is m-times Fréchet differentiable such that $f^{(m)} \in \mathcal{Z}^{0,1}(U, L^m(E, F))$.*

It is straightforward to adapt the definition of $\mathcal{Z}^{0,1}$ to maps between Banach spaces. The proof of Lemma 3.3 shows that such a map is of class $\mathcal{Z}^{0,1}$ if and only if it is of class Zyg^0; note that Lemma 3.2 is valid in convenient vector spaces. So, for maps between Banach spaces, Zyg^m coincides with the naive notion of local Zygmund regularity.

The definition of difference quotient and finite difference of arbitrary order obviously makes sense for functions $f : \mathbb{R} \to E$ with values in a vector space E. It turns out that Theorem 2.4 (as well as Theorem 2.1 and Theorem 2.6) are still valid if E is a convenient vector space:

Theorem 4.5. *Let $m \in \mathbb{N}$. Let E be a convenient vector space. Let $f : \mathbb{R} \to E$ be a function such that $\ell \circ f$ is continuous for all $\ell \in E'$. The following conditions are equivalent:*

(1) *f is of class* Zyg^m.
(2) *$(x, h) \mapsto h\delta_{\mathrm{eq}}^{m+2} f(x; h)$ and $(x, h) \mapsto \delta_{\mathrm{eq}}^m f(x; h)$ are bounded on bounded subsets of $\mathbb{R} \times (\mathbb{R} \setminus \{0\})$.*

Proof. This is immediate from Theorem 2.4, since (1) and (2) can be tested by composing with $\ell \in E'$. □

Remark 4.6. Let E, F be convenient vector spaces and $U \subseteq E$ a c^∞-open subset. Let us endow $\operatorname{Zyg}^m(\mathbb{R}, F)$ with the initial structure with respect to all maps $c \mapsto ((x, h) \mapsto \delta_{\mathrm{eq}}^j c(x; h))$, for $j = 0, 1, \ldots, m$, and $c \mapsto ((x, h) \mapsto h\delta_{\mathrm{eq}}^{m+2} c(x; h))$ into the space of all maps $\mathbb{R} \times (\mathbb{R} \setminus \{0\}) \to F$ that are bounded on bounded sets, where the latter space carries the locally convex topology of uniform convergence on bounded sets. Then the space $\operatorname{Zyg}^m(U, F)$ of all maps $f : U \to F$ of class Zyg^m endowed with the initial structure with respect to all maps $c^* : \operatorname{Zyg}^m(U, F) \to \operatorname{Zyg}^m(\mathbb{R}, F)$ for $c \in \mathcal{C}^\infty(\mathbb{R}, U)$, is a convenient vector space and it satisfies the uniform boundedness principle with respect to the point evaluations $\mathrm{ev}_x : \operatorname{Zyg}^m(U, F) \to F$ for $x \in U$. This can be seen in analogy to [10, 12.11].

4.2. **Functions on non-open domains.** Let E, F be convenient vector spaces and let $X \subseteq E$ be a convex subset with non-empty c^∞-interior X°. We say that a map $f : X \to F$ is of class Zyg^m (resp. Höl_α^m) if for each $c \in \mathcal{C}^\infty(\mathbb{R}, X)$ (i.e. $c \in \mathcal{C}^\infty(\mathbb{R}, E)$ with $c(\mathbb{R}) \subseteq X$) and each $\ell \in F'$ the composite $\ell \circ f \circ c$ belongs to $\mathcal{Z}^{m,1}(\mathbb{R})$ (resp. $\mathcal{C}^{m,\alpha}$).

Theorem 4.7. *Let E, F be convenient vector spaces and let $X \subseteq E$ be a convex subset with non-empty c^∞-interior X°. Let $f : X \to F$ be of class Zyg^{2m} (resp. Höl_α^{2m})*

for $m \in \mathbb{N}_{\ge 1}$ (and $\alpha \in (0,1]$). Then $f|_{X^\circ}$ is of class Zyg^{2m} *(resp.* Höl^{2m}_α*) and for $j \le m$ the derivatives $(f|_{X^\circ})^{(j)}$ extend uniquely to maps $f^{(j)} : X \to L^j(E,F)$ of class* $\mathrm{Zyg}^{2(m-j)}$ *(resp.* $\text{Höl}^{2(m-j)}_\alpha$*).*

Proof. This can be shown in analogy to [9] and [10, Theorem 24.5]. The crucial ingredient is an application of Theorem 1.1 (resp. Theorem 1.2) in dimension two. □

In finite dimensions this has been generalized in the $\mathcal{C}^\infty$- and the Höl^m_α-setting by [14, 15] to a large class of closed sets $X \subseteq \mathbb{R}^d$ with $X = \overline{X^\circ}$ admitting cusps. Note that in $\mathbb{R}^d$ the c^∞-topology coincides with the classical topology. In the Höl^m_α-setting, a loss of regularity becomes apparent which is directly related to the sharpness of the cusps. More precisely, for $\beta \in (0,1]$ let $\mathscr{H}^\beta(\mathbb{R}^d)$ denote the family of closed subsets $X \subseteq \mathbb{R}^d$ with $X = \overline{X^\circ}$ such that X° has the *uniform β-cusp property*: for each $x \in \partial X$ there exist $\epsilon > 0$, a cusp

$$\Gamma = \Big\{(x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R} : |x'| < r,\ h\Big(\frac{|x'|}{r}\Big)^\beta < x_d < h\Big\}$$

for some $r, h > 0$, and an orthogonal linear map $A : \mathbb{R}^d \to \mathbb{R}^d$ such that $y + A\Gamma \subseteq X^\circ$ for all $y \in X \cap B(x, \epsilon)$. Consider the functions $p, q : (0,1] \to \mathbb{N}$ defined by

$$p(\beta) := \Big\lceil \frac{2}{\beta} \Big\rceil \quad \text{and} \quad q(\beta) := \Big\lceil \frac{1}{\beta} \Big\rceil.$$

Theorem 4.8 ([15, Theorem A])**.** *Let $m \in \mathbb{N}$ and $\alpha, \beta \in (0,1]$. Let $X \in \mathscr{H}^\beta(\mathbb{R}^d)$. If $f : X \to \mathbb{R}$ is of class* $\text{Höl}^{mp(\beta)}_\alpha$*, then all partial derivatives of f of order $j \le m$ extend continuously from X° to X and are of class* $\text{Höl}^{(m-j)p(\beta)}_\alpha$*, and the partial derivatives of order m are locally $\frac{\alpha\beta}{2q(\beta)}$-Hölder continuous on X.*

Combining Theorem 1.1 with the proof of [15, Proposition 3.3 and 3.4] gives

Theorem 4.9. *Let $m \in \mathbb{N}$ and $\beta \in (0,1]$. Let $X \in \mathscr{H}^\beta(\mathbb{R}^d)$. If $f : X \to \mathbb{R}$ is of class* $\mathrm{Zyg}^{mp(\beta)}$*, then all partial derivatives of f of order $j \le m$ extend continuously from X° to X and are of class* $\mathrm{Zyg}^{(m-j)p(\beta)}$*.*

By the inclusion $\mathcal{Z}^{0,1} \subseteq \mathcal{C}^{0,\alpha}$ for each $\alpha \in (0,1)$ (cf. (2.2)), also in this case the partial derivatives of order m satisfy a local $\frac{\alpha\beta}{2q(\beta)}$-Hölder condition on X for each $\alpha \in (0,1)$.

5. Regularity of superposition on Zygmund spaces

The goal of this section is to prove Theorem 1.3 which characterizes the Lipschitz regularity of the superposition operator

$$f_* : g \mapsto f \circ g$$

acting on the global Zygmund spaces $\Lambda_{m+1}(\mathbb{R}^d)$ for $m \in \mathbb{N}_{\ge 1}$.

To put our result in perspective we recall the characterization of the $\mathcal{C}^k$-regularity of f_*: Let $m \in \mathbb{N}_{\ge 1}$ and $f : \mathbb{R} \to \mathbb{R}$ a function. Then:

(1) $f_* \Lambda_{m+1}(\mathbb{R}^d) \subseteq \Lambda_{m+1}(\mathbb{R}^d)$ if and only if $f \in \mathcal{Z}^{m,1}(\mathbb{R})$; see [2, Theorem 1] or Remark 5.2.

(2) For $k \in \mathbb{N}$ the map $f_* : \Lambda_{m+1}(\mathbb{R}^d) \to \Lambda_{m+1}(\mathbb{R}^d)$ is of class $\mathcal{C}^k$ if and only if $f \in \mathcal{C}^{m+k}(\mathbb{R})$ and

$$f^{(m+k)}(x+h) - 2f^{(m+k)}(x) + f^{(m+k)}(x-h) = o(h) \quad \text{as } h \to 0^+$$

uniformly on compact subsets of $\mathbb{R}$; cf. [2, Theorem 7].

Theorem 1.3 will follow from Proposition 5.1 and Proposition 5.3.

5.1. **Sufficiency.** We first show that $f \in \mathcal{Z}^{m+k,1}(\mathbb{R})$ implies that f_* acts on $\Lambda_{m+1}(\mathbb{R}^d)$ and is of class $\mathcal{C}^{k-1,1}$. Our approach is based on a version of Theorem 1.2 for maps between Banach spaces (analogous to Corollary 4.4); see [7, 10]. This allows for a simple lucid proof. Similar results hold for f_* acting on Hölder–Lipschitz spaces; see e.g. [12, Theorem 2.14].

Proposition 5.1. *Let $m, k \in \mathbb{N}_{\ge 1}$ and $f \in \mathcal{Z}^{m+k,1}(\mathbb{R})$. Then f_* acts on $\Lambda_{m+1}(\mathbb{R}^d)$ and $f_* : \Lambda_{m+1}(\mathbb{R}^d) \to \Lambda_{m+1}(\mathbb{R}^d)$ is of class $\mathcal{C}^{k-1,1}$.*

Proof. Let $r := m+1$. For simplicity let $d = 1$. For the fact that $f_*\Lambda_r(\mathbb{R}) \subseteq \Lambda_r(\mathbb{R})$ we refer to [2, Theorem 1], but see also Remark 5.2. By [7, Theorem 4.3.27] or [6, Corollary 15] (i.e. the Lipschitz analogue of Corollary 4.4), it suffices to check that $f_* : \Lambda_r(\mathbb{R}) \to \Lambda_r(\mathbb{R})$ maps $\mathcal{C}^\infty$-curves to $\mathcal{C}^{k-1,1}$-curves. That $t \mapsto g(t, \cdot)$ is $\mathcal{C}^\infty$ in $\Lambda_r(\mathbb{R})$ means that, for all $\ell \in \mathbb{N}$, $\|\partial_1^\ell g(t, \cdot)\|_{\Lambda_r}$ is locally bounded in t (cf. [7, 4.1.19]).

We first prove the case $k = 1$. Set $h(t, x) := f(g(t, x))$. In the following we denote, for clarity, by h_t, g_t, etc. the partial derivatives with respect to t and write ∂_x for partial derivatives with respect to x.

Then,

$$h(t,x) - h(s,x) = \int_s^t h_t(\tau, x)\, d\tau = \int_s^t (f' \circ g)(\tau, x) g_t(\tau, x)\, d\tau$$

and, for $\ell \le m$,

$$\partial_x^\ell h(t,x) - \partial_x^\ell h(s,x) = \sum_{j=0}^{\ell} \binom{\ell}{j} \int_s^t \partial_x^j (f' \circ g)(\tau, x) \partial_x^{\ell-j} g_t(\tau, x)\, d\tau.$$

By Faà di Bruno's formula,

$$\partial_x^j (f' \circ g)(\tau, x) = \sum_{i=1}^{j} \sum_{\gamma \in \Gamma(i,j)} c_\gamma\, (f^{(i+1)} \circ g)(\tau, x) \partial_x^{\gamma_1} g(\tau, x) \cdots \partial_x^{\gamma_i} g(\tau, x), \quad (j \ge 1),$$

where $\Gamma(i,j) := \{\gamma \in (\mathbb{N}_{\ge 1})^i : |\gamma| = j\}$ and $c_\gamma := \frac{j!}{i!\gamma!}$, it is readily checked that $t \mapsto h(t, \cdot)$ is locally Lipschitz into $\mathcal{C}_b^m(\mathbb{R}) := \{u \in \mathcal{C}^m(\mathbb{R}) : \sup_{\ell \le m} \|u^{(\ell)}\|_{L^\infty(\mathbb{R})} < \infty\}$.

To see that $t \mapsto h(t, \cdot)$ is locally Lipschitz into $\Lambda_r(\mathbb{R})$ it remains to show

Claim. *For each bounded interval $I \subseteq \mathbb{R}$ the set*

$$\Big\{ \frac{\Delta_v^2 \partial_x^m h(t,x) - \Delta_v^2 \partial_x^m h(s,x)}{|v||t-s|} : x, v \in \mathbb{R},\, v \ne 0,\, s \ne t \in I \Big\},$$

is bounded, where the second finite difference Δ_v^2 acts in the x-variable.

In view of the above, it is enough to show that for all $0 \le i \le j \le m$ and $\gamma \in \Gamma(i,j)$,

$$\Delta_v^2 \big[(f^{(i+1)} \circ g)(\tau, x) \partial_x^{\gamma_1} g(\tau, x) \cdots \partial_x^{\gamma_i} g(\tau, x) \partial_x^{m-j} g_t(\tau, x) \big] = O(|v|), \tag{5.1}$$

uniformly in $x \in \mathbb{R}$ and $\tau \in I$. Let us from now on suppress the dependence on τ in the notation. We can assume that $|v|$ is small; otherwise the result follows from the fact that $t \mapsto h(t, \cdot)$ is locally Lipschitz into $\mathcal{C}^m_b(\mathbb{R})$.

Each of the factors in the product

$$(f^{(i+1)} \circ g)(x)\partial_x^{\gamma_1} g(x) \cdots \partial_x^{\gamma_l} g(x)\partial_x^{m-j} g_t(x) \tag{5.2}$$

is globally bounded in x, locally in τ. Thus, by the product rule (2.4), in order to prove (5.1) it suffices to show the following two facts.

Fact 1: For each factor Π in the product (5.2) we have $\Delta_v^2 \Pi = O(|v|)$, uniformly in $x \in \mathbb{R}$ and $\tau \in I$.

Fact 2: For any two factors Π_1 and Π_2 in the product (5.2) we have $\Delta_v^1 \Pi_1 \cdot \Delta_v^1 \Pi_2 = O(|v|)$, uniformly in $x \in \mathbb{R}$ and $\tau \in I$.

Fact 1. For $\Pi = \partial_x^\ell g(x)$ and $\Pi = \partial_x^\ell g_t(x)$, where $\ell \le m$, the assertion $\Delta_v^2 \Pi = O(|v|)$ holds; either by assumption if $\ell = m$ or using (2.3) if $\ell < m$. It remains to consider $\Pi = (f^{(\ell+1)} \circ g)(x)$ for $\ell \le m$. In order to estimate $\Delta_v^2 (f^{(m+1)} \circ g)(x)$ we have, by (2.5), to deal with terms of the form

$$\Delta^1_{\Delta_v^2 g(x)} f^{(m+1)}(y) \quad \text{and} \quad \Delta^2_{\Delta_v^1 g(x)} f^{(m+1)}(y), \tag{5.3}$$

where y ranges over a bounded set. By assumption and (2.2), $f^{(m+1)} \in \mathcal{Z}^{0,1}(\mathbb{R}) \subseteq \mathcal{C}^{0,\alpha}(\mathbb{R})$ for all $\alpha \in (0,1)$ so that

$$\Delta^1_{\Delta_v^2 g(x)} f^{(m+1)}(y) = O(|\Delta_v^2 g(x)|^\alpha).$$

Because $g \in \Lambda_r(\mathbb{R}) \hookrightarrow \Lambda_{1+\beta}(\mathbb{R})$ for any $\beta \in (0,1)$ (as $r \ge 2$), Theorem 2.3 implies

$$\Delta_v^2 g(x) = O(|v|^{1+\beta}).$$

Taking $\alpha := (1+\beta)^{-1}$ we conclude that

$$\Delta^1_{\Delta_v^2 g(x)} f^{(m+1)}(y) = O(|v|).$$

For the second term in (5.3) we have

$$\Delta^2_{\Delta_v^1 g(x)} f^{(m+1)}(y) = O(|\Delta_v^1 g(x)|) = O(|v|),$$

since g is globally Lipschitz. For $\ell < m$ use (2.3) and similar arguments.

Fact 2. By (2.5) and since $f^{(m+1)} \in \mathcal{Z}^{0,1}(\mathbb{R}) \subseteq \mathcal{C}^{0,\omega}(\mathbb{R})$ where $\omega(t) := t \log \frac{1}{t}$, we find

$$\Delta_v^1 (f^{(m+1)} \circ g)(x) = \Delta^1_{\Delta_v^1 g(x)} f^{(m+1)}(g(x)) = O(\omega(|\Delta_v^1 g(x)|)) = O(\omega(|v|))$$

as well as $\Delta_v^1 \partial_x^m g(x) = O(\omega(|v|))$ and $\Delta_v^1 \partial_x^m g_t(x) = O(\omega(|v|))$. If the order of differentiation in x is lower than m, then all these terms are actually $O(|v|)$. In any case it follows that $\Delta_v^1 \Pi_1 \cdot \Delta_v^1 \Pi_2 = O(|v|)$.

This ends the proof for $k = 1$. Now we argue by induction on k. Let $k > 1$ and $f \in \mathcal{Z}^{m+k,1}(\mathbb{R})$. Then $t \mapsto h_t(t, \cdot) = (f' \circ g(t, \cdot)) g_t(t, \cdot)$ is of class $\mathcal{C}^{k-2,1}$ into $\Lambda_r(\mathbb{R})$, since $(f')_*$ is of class $\mathcal{C}^{k-2,1}$ by induction hypothesis. Consequently, $t \mapsto h(t, \cdot)$ is of class $\mathcal{C}^{k-1,1}$ into $\Lambda_r(\mathbb{R})$ (cf. [7, Theorem 4.3.24]). □

Remark 5.2. It is not difficult to build a proof of the fact that $f_* \Lambda_{m+1}(\mathbb{R}) \subseteq \Lambda_{m+1}(\mathbb{R})$ if $f \in \mathcal{Z}^{m,1}(\mathbb{R})$ from the arguments used above. To see that, conversely, $f_* \Lambda_{m+1}(\mathbb{R}) \subseteq \Lambda_{m+1}(\mathbb{R})$ implies $f \in \mathcal{Z}^{m,1}(\mathbb{R})$ consider $f \circ g$, where g is a $\mathcal{C}^\infty$-function with compact support and $g(x) = x$ on a compact interval.

5.2. **Necessity.**

Proposition 5.3. *Let $m, k \in \mathbb{N}_{\geq 1}$ and $f : \mathbb{R} \to \mathbb{R}$ a function. Suppose that f_* acts on $\Lambda_{m+1}(\mathbb{R})$ and $f_* : \Lambda_{m+1}(\mathbb{R}) \to \Lambda_{m+1}(\mathbb{R})$ is of class $\mathcal{C}^{k-1,1}$. Then $f \in \mathcal{Z}^{m+k,1}(\mathbb{R})$.*

Proof. By Remark 5.2, $f \in \mathcal{Z}^{m,1}(\mathbb{R})$, in particular, $f \in \mathcal{C}^m(\mathbb{R})$.

For any compact interval $I \subseteq \mathbb{R}$ let $\rho_I : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^\infty$-function with compact support such that $\rho_I(x) = x$ for all $x \in I$. Then $g(x) := \rho_I(x)$ belongs to $\Lambda_{m+1}(\mathbb{R})$ and $c(t) := g + \rho_{[-1,1]}(t)$ defines a $\mathcal{C}^\infty$-curve in $\Lambda_{m+1}(\mathbb{R})$ with $c(0) = g$ and $c(t) = g+t$ if $t \in [-1, 1]$. By assumption, f_* maps $\mathcal{C}^\infty$-curves in $\Lambda_{m+1}(\mathbb{R})$ to $\mathcal{C}^{k-1,1}$-curves. Thus $f_*(c)$ is a $\mathcal{C}^{k-1,1}$-curve in $\Lambda_{m+1}(\mathbb{R})$ and $(f_*(c))(t)(x) = f(x + t)$ if $x \in I$ and $t \in [-1, 1]$. Thus, by the Höl_1^{k-1}-version of Theorem 4.5, see also [10, Lemma 12.4],

$$\frac{\delta_{\text{eq}}^k \Delta_v^2 (f_*(c))^{(m)}(x;t)}{v} = \frac{\Delta_t^k \Delta_v^2 f^{(m)}(x)}{t^k v}$$

is bounded for all $x \in I$ and all small $v, t \in \mathbb{R} \setminus \{0\}$. For $t = v$ we see that

$$\frac{\Delta_t^{k+2} f^{(m)}(x)}{t^{k+1}} = t\delta_{\text{eq}}^{k+2} f^{(m)}(x;t)$$

is bounded for all $x \in I$ and all small $t \neq 0$. Since $\Lambda_{m+1}(\mathbb{R}) \hookrightarrow \text{Lip}_m(\mathbb{R})$, $f_*(c)$ also is a $\mathcal{C}^{k-1,1}$-curve in $\text{Lip}_m(\mathbb{R})$ whence similar considerations give that

$$\frac{\Delta_t^k \Delta_v^1 f^{(m-1)}(x)}{t^k v}$$

is bounded for all $x \in I$ and all small v, t. We see that $\delta_{\text{eq}}^{k+1} f^{(m-1)}(x;t)$ is bounded for $x \in I$ and small t. Invoking Theorem 2.1 twice, we find that $f^{(m-1)} \in \mathcal{C}^{k,1}(\mathbb{R})$ which entails $f \in \mathcal{C}^{m+k-1,1}(\mathbb{R})$, and so $\delta_{\text{eq}}^k f^{(m)}(x;t)$ is locally bounded in $(x,t) \in \mathbb{R} \times (\mathbb{R} \setminus \{0\})$. By Theorem 2.4, we have $f^{(m)} \in \mathcal{Z}^{k,1}(\mathbb{R})$ and consequently $f \in \mathcal{Z}^{m+k,1}(\mathbb{R})$. □

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