PERTURBATION OF COMPLEX POLYNOMIALS AND NORMAL OPERATORS

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Abstract. We study the regularity of the roots of complex monic polynomials $P(t)$ of fixed degree depending smoothly on a real parameter t. We prove that each continuous parameterization of the roots of a generic C^∞ curve $P(t)$ (which always exists) is locally absolutely continuous. Generic means that no two of the continuously chosen roots meet of infinite order of flatness. Simple examples show that one cannot expect a better regularity than absolute continuity. This result will follow from the proposition that for any t_0 there exists a positive integer N such that $t \mapsto P(t_0 \pm (t-t_0)^N)$ admits smooth parameterizations of its roots near t_0. We show that C^n curves $P(t)$ (where $n = \deg P$) admit differentiable roots if and only if the order of contact of the roots is ≥ 1. We give applications to the perturbation theory of normal matrices and unbounded normal operators with compact resolvents and common domain of definition: The eigenvalues and eigenvectors of a generic C^∞ curve of such operators can be arranged locally in an absolutely continuous way.

1. Introduction

Let us consider a curve of polynomials

$$P(t)(z) = z^n + \sum_{j=1}^{n} (-1)^j a_j(t) z^{n-j},$$

where the coefficients $a_j : I \to \mathbb{C}$, $1 \leq j \leq n$, are complex valued functions defined on an interval $I \subseteq \mathbb{R}$. Given that the coefficients a_j possess some regularity, it is natural to ask whether the roots of P can be arranged in a regular way as well, i.e., whether it is possible to find n regular functions $\lambda_j : I \to \mathbb{C}$, $1 \leq j \leq n$, such that $\lambda_1(t), \dots, \lambda_n(t)$ represent the roots of $P(t)$ for each $t \in I$.

This problem has been extensively studied under the additional assumption that the polynomials $P(t)$ are hyperbolic, i.e., all roots of $P(t)$ are real. By a classical theorem due to Rellich [26], there exist real analytic parameterizations of the roots of P if its coefficients are real analytic. Bronshtein [5] proved that if all a_j are of class C^n, then there exists a differentiable parameterization of the roots of P with locally bounded derivative (see also Wakabayashi [34] for a different proof). It has been shown in [1] that if all a_j are smooth (C^∞) and no two of the increasingly ordered (thus continuous) roots meet of infinite order of flatness, then there exist smooth parameterizations of the roots. Moreover, by [12], the roots may always be chosen twice differentiable provided that the a_j are C^{3n}. The conclusion in this statement is best possible as shown by an example in [4]. Recently, also the best possible assumptions have been found by [7]: If the coefficients a_j are C^n (resp. C^{2n}), the roots allow C^1 (resp. twice differentiable) parameterizations. For further results on this problem see also [10], [8], [22], [6], [21], [20].

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If the hyperbolicity assumption is dropped, then there is the following general result (e.g. [11, II §5 5.2]): There exist continuous functions $\lambda_j : I \to \mathbb{C}$, $1 \le j \le n$, which parameterize the roots of a curve of polynomials P with continuous coefficients $a_j : I \to \mathbb{C}$, $1 \le j \le n$. Note that, in contrast to the hyperbolic case, there is no hope that the roots of a polynomial P which depends regularly on more than one parameter might be parameterized even continuously; just take $P(t,s)(z) = z^2 - (t + is)$, where $t, s \in \mathbb{R}$ and $i = \sqrt{-1}$. Of course, in that example the roots are given as 2-valued analytic function with branching point 0 in terms of a Puiseux series, e.g. [3, Appendix], but we do not go into that in this note. Here we restrict our attention to the one parameter case. In the absence of hyperbolicity the roots of a Lipschitz curve $t \mapsto P(t)$ of polynomials of degree n may still be parameterized in a $C^{0,1/n}$ way, locally, which follows from a result of Ostrowski [23], but we cannot expect that the roots of P are locally Lipschitz continuous even when the coefficients a_j are real analytic; for instance, consider $P(t)(z) = z^2 - t$, $t \in \mathbb{R}$. However the roots of P may possess a weaker regularity: They may be parameterized by locally absolutely continuous functions. In fact, Spagnolo [33] proved that there exist absolutely continuous parameterizations of the roots of P on compact intervals I if one of the following conditions holds:

(1) $n = 2$ and the coefficients a_j belong to C^5,
(2) $n = 3$ and the coefficients a_j belong to C^{25} (the case $n = 4$ is announced),
(3) $P(t)(z) = z^n - f(t)$ and f belongs to C^{2n+1}.

The proof makes essential use of the explicit formulas for the roots of P available in those cases.

In this paper we extend this result to generic smooth curves of polynomials P of arbitrary degree n. We say that P is generic if no two of the continuously arranged roots of P meet of infinite order of flatness. We show in section 3 that, if the a_j are smooth, then there exists an absolutely continuous parameterization of the roots of P on each compact interval I; actually, any continuous parameterization of the roots is locally absolutely continuous. In particular, these conditions are satisfied if the coefficients a_j are real analytic or, more generally, belong to a quasianalytic class of C^∞ functions. The main ingredient in the proof is the proposition 3.2 that for any $t_0 \in I$ there exists a positive integer N such that $t \mapsto P(t_0 \pm (t - t_0)^N)$ admits smooth parameterizations of its roots near t_0. It is not known whether the roots of P may be arranged in a locally absolutely continuous way if P is not generic. That problem requires different methods.

In section 4 we find conditions for the existence of differentiable parameterizations of the roots of P. Evidently, a necessary condition is that there exists a continuous choice of the roots such that whenever two of them meet they meet of order ≥ 1. We show that this condition is also sufficient, provided that the coefficients a_j of P belong to C^n.

In section 5 we discuss a reformulation of the problem in terms of a lifting problem which has been discussed in [2] and [14, 15, 13]. This yields implicit sufficient conditions for a curve of polynomials P to allow smooth, C^1, or twice differentiable parameterizations of its roots. As application we discuss the quadratic case.

Applications to the perturbation theory of normal matrices are given in section 6. The eigenvalues and eigenvectors of a generic smooth curve $t \mapsto A(t)$ of normal complex matrices may be parameterized locally in an absolutely continuous way. The curve $t \mapsto A(t)$ is called generic if the associated characteristic polynomial $t \mapsto \chi_{A(t)}$ is generic. Examples show that without genericity or normality of $A(t)$ the eigenvectors need not admit continuous arrangements. We also prove that, for each t_0 there exists a positive integer N such that $t \mapsto A(t_0 \pm (t - t_0)^N)$ allows a smooth parameterization of its eigenvalues and eigenvectors near t_0. If A is real analytic, then the eigenvalues and eigenvectors of $t \mapsto A(t_0 \pm (t-t_0)^N)$ may be arranged real analytic as well.

In section 7 we obtain analogous results for curves $t \mapsto A(t)$ of unbounded normal operators in a Hilbert space with common domain of definition and with compact resolvents.

For more on the perturbation theory of linear operators consider Rellich [26, 27, 28, 29, 30, 31], Kato [11], Baumgärtel [3], and also [1], [18], and [19].

2. Preliminaries

2.1. Let

$$P(z) = z^n + \sum_{j=1}^{n} (-1)^j a_j z^{n-j} = \prod_{j=1}^{n} (z - \lambda_j)$$

be a monic polynomial with coefficients $a_1, \dots, a_n \in \mathbb{C}$ and roots $\lambda_1, \dots, \lambda_n \in \mathbb{C}$. By Vieta's formulas, $a_i = \sigma_i(\lambda_1, \dots, \lambda_n)$, where $\sigma_1, \dots, \sigma_n$ are the elementary symmetric functions in n variables:

$$\sigma_i(\lambda_1, \dots, \lambda_n) = \sum_{1 \le j_1 < \cdots < j_i \le n} \lambda_{j_1} \cdots \lambda_{j_i}. \tag{2.1.1}$$

Denote by s_i, $i \in \mathbb{N}$, the Newton polynomials $\sum_{j=1}^n \lambda_j^i$ which are related to the elementary symmetric functions by

$$s_k - s_{k-1}\sigma_1 + s_{k-2}\sigma_2 - \cdots + (-1)^{k-1} s_1 \sigma_{k-1} + (-1)^k k \sigma_k = 0, \quad (k \ge 1). \tag{2.1.2}$$

Let us consider the so-called Bezoutiant

$$B := \begin{pmatrix} s_0 & s_1 & \dots & s_{n-1} \\ s_1 & s_2 & \dots & s_n \\ \vdots & \vdots & \ddots & \vdots \\ s_{n-1} & s_n & \dots & s_{2n-2} \end{pmatrix} = (s_{i+j-2})_{1 \le i,j \le n}.$$

Since the entries of B are symmetric polynomials in $\lambda_1, \dots, \lambda_n$, we find a unique symmetric $n \times n$ matrix $\tilde{B}$ with $B = \tilde{B} \circ \sigma$, where $\sigma = (\sigma_1, \dots, \sigma_n)$.

Let B_k denote the minor formed by the first k rows and columns of B. Then we find

(2.1.3)

$$\Delta_k(\lambda) := \det B_k(\lambda) = \sum_{i_1 < i_2 < \cdots < i_k} (\lambda_{i_1} - \lambda_{i_2})^2 \cdots (\lambda_{i_1} - \lambda_{i_k})^2 \cdots (\lambda_{i_{k-1}} - \lambda_{i_k})^2.$$

Since the polynomials Δ_k are symmetric, we have $\Delta_k = \tilde{\Delta}_k \circ \sigma$ for unique polynomials $\tilde{\Delta}_k$.

From (2.1.3) follows that the number of distinct roots of P equals the maximal k such that $\tilde{\Delta}_k(P) \neq 0$.

2.2. **Multiplicity.** For a continuous real or complex valued function f defined near 0 in $\mathbb{R}$ let the *multiplicity* (or *order of flatness*) $m(f)$ at 0 be the supremum of all integers p such that $f(t) = t^p g(t)$ near 0 for a continuous function g. We define in the obvious way the *multiplicity* $m_{t_0}(f)$ of f at any $t_0 \in \mathbb{R}$ (if f is defined near t_0). Note that, if f is of class C^n and $m(f) < n$, then $f(t) = t^{m(f)} g(t)$ near 0, where now g is $C^{n-m(f)}$ and $g(0) \neq 0$.

If f is a continuous function on the space of polynomials, then for a fixed continuous curve P of polynomials we will denote by $m(f)$ the multiplicity at 0 of $t \mapsto f(P(t))$.

We shall say that two functions f and g *meet of order* $\ge p$ *at* 0 or *have order of contact* $\ge p$ *at* 0 iff $m(f-g) \ge p$.

Lemma. *Let $I \subseteq \mathbb{R}$ be an interval containing* 0. *Consider a curve of polynomials*

$$P(t)(z) = z^n + \sum_{j=2}^{n} (-1)^j a_j(t) z^{n-j},$$

with $a_j : I \to \mathbb{C}$, $2 \le j \le n$, smooth. Then, for integers r, the following conditions are equivalent:

(1) *$m(a_k) \ge kr$, for all $2 \le k \le n$;*
(2) *$m(\tilde{\Delta}_k) \ge k(k-1)r$, for all $2 \le k \le n$.*

Proof. Without loss of generality we can assume $r > 0$.

(1) $\Rightarrow$ (2): From (2.1.2) we deduce $m(\tilde{s}_k) \ge kr$ for all k, where $s_k = \tilde{s}_k \circ \sigma$. By the definition of $\tilde{\Delta}_k = \det(\tilde{B}_k)$ we obtain (2).

(2) $\Rightarrow$ (1): It is easy to see that $\tilde{\Delta}_k(0) = 0$ for all $2 \le k \le n$ implies $\tilde{s}_k(0) = 0$ for all $2 \le k \le n$. Then by (2.1.2) we have $a_k(0) = 0$ for all $2 \le k \le n$. So near 0 we have $a_2(t) = t^{2r} a_{2,2r}(t)$ and $a_k(t) = t^{m_k} a_{k,m_k}(t)$ for $3 \le k \le n$, where the m_k are positive integers and $a_{2,2r}, a_{3,m_3}, \ldots, a_{n,m_n}$ are smooth functions. Let us suppose for contradiction that for some $k > 2$ we have $m_k = m(a_k) < kr$. We put

$$m := \min\left\{r, \frac{m_3}{3}, \ldots, \frac{m_n}{n}\right\} < r, \tag{2.2.1}$$

and consider the following continuous curve of polynomials for (small) $t \ge 0$:

$$P_{(m)}(t)(z) := z^n + a_{2,2r}(t) t^{2r-2m} z^{n-2} - \cdots + (-1)^n a_{n,m_n}(t) t^{m_n - nm}.$$

We have $\tilde{\Delta}_k(P_{(m)}(t)) = t^{-k(k-1)m} \tilde{\Delta}_k(P(t))$. By (2.2.1), $r - m > 0$, whence $t \mapsto \tilde{\Delta}_k(P_{(m)}(t))$, $2 \le k \le n$, vanish at $t = 0$. We may conclude as before that all coefficients of $t \mapsto P_{(m)}(t)$ vanish for $t = 0$. But this is a contradiction for those k with $m(a_k) = m_k = km$. $\square$

Remark. If the coefficients a_j of P in lemma 2.2 are just of class C^n, the conclusion remains true for $r = 1$. The proof is the same with the slight modification that we define $m_k := \min\{k, m(a_k)\}$ for all k.

2.3. **Genericity condition.** Let $I \subseteq \mathbb{R}$ be an interval. We call a curve of monic polynomials

$$P(t)(z) = z^n + \sum_{j=1}^{n} (-1)^j a_j(t) z^{n-j}$$

with continuous coefficients $a_j : I \to \mathbb{C}$, $1 \le j \le n$, *generic* if the following equivalent conditions are satisfied at any $t_0 \in I$:

(1) If two of the continuously parameterized roots of P meet of infinite order of flatness at t_0, then their germs at t_0 are equal.
(2) Let k be maximal with the property that the germ at t_0 of $t \mapsto \tilde{\Delta}_k(P(t))$ is not 0. Then $t \mapsto \tilde{\Delta}_k(P(t))$ is not infinitely flat at t_0.

The equivalence of (1) and (2) follows easily from (2.1.3). For instance, P is generic, if its coefficients are real analytic, or more generally, belong to a quasianalytic class of C^∞ functions.

2.4. **Splitting lemma.** [1, 3.4] *Let $P_0 = z^n + \sum_{j=1}^n (-1)^j a_j z^{n-j}$ be a polynomial satisfying $P_0 = P_1 \cdot P_2$, where P_1 and P_2 are polynomials without common root. Then for P near P_0 we have $P = P_1(P) \cdot P_2(P)$ for real analytic mappings of monic polynomials $P \mapsto P_1(P)$ and $P \mapsto P_2(P)$, defined for P near P_0, with the given initial values.*

2.5. **Absolutely continuous functions.** Let $I \subseteq \mathbb{R}$ be an interval. A function $f : I \to \mathbb{C}$ is called *absolutely continuous*, or $f \in AC(I)$, if for all $\epsilon > 0$ there exists a $\delta > 0$ such that $\sum_{i=1}^N (b_i - a_i) < \delta$ implies $\sum_{i=1}^N |f(b_i) - f(a_i)| < \epsilon$, for all sequences of pairwise disjoint subintervals $(a_i, b_i) \subseteq I$, $1 \le i \le N$. By the fundamental theorem of calculus for the Lebesgue integral, $f \in AC([a,b])$ if and only if there is a function $g \in L^1([a,b])$ such that

$$f(t) = f(a) + \int_a^t g(s)ds \quad \text{for all } t \in [a,b].$$

Then $f' = g$ almost everywhere. Every Lipschitz function is absolutely continuous.

Gluing finitely many absolutely continuous functions provides an absolutely continuous function: Let $f_1 \in AC([a,b])$, $f_2 \in AC([b,c])$, and $f_1(b) = f_2(b)$. Then $f : [a,c] \to \mathbb{C}$, defined by $f(t) = f_1(t)$ if $t \in [a,b]$ and $f(t) = f_2(t)$ if $t \in [b,c]$, belongs to $AC([a,c])$. Similarly for more than two functions.

Let $\varphi : I \to I$ be bijective, strictly increasing, and Lipschitz continuous. If $f \in AC(I)$ then also $f \circ \varphi \in AC(I)$. Furthermore:

Lemma. *Let $r > 0$ and $n \in \mathbb{N}_{>0}$. Let $f \in AC([0,r])$ (resp. $f \in AC([-r,0])$) and set $h(t) = f(\sqrt[n]{t})$ (resp. $h(t) = f(-\sqrt[n]{|t|})$). Then $h \in AC([0,r^n])$ (resp. $h \in AC([-r^n,0])$).*

Proof. There exists a function $g \in L^1([0,r])$ such that

$$f(t) = f(0) + \int_0^t g(s)ds$$

for all $t \in [0,r]$. The function $(0,r^n] \to (0,r], t \mapsto \sqrt[n]{t}$, is smooth and bijective, so

$$\int_0^{r^n} |g(\sqrt[n]{s})|(\sqrt[n]{s})'ds = \int_0^r |g(s)|ds$$

and $t \mapsto g(\sqrt[n]{t})(\sqrt[n]{t})'$ belongs to $L^1([0,r^n])$. Thus $h(t) = f(\sqrt[n]{t})$ is in $AC([0,r^n])$.

For the second statement consider the absolutely continuous function $f \circ S|_{[0,r]}$, where $S : \mathbb{R} \to \mathbb{R}, t \mapsto -t$. By the above, $h_S(t) = (f \circ S|_{[0,r]})(\sqrt[n]{t})$ is in $AC([0,r^n])$, and so $h(t) = h_S(S^{-1}|_{[-r^n,0]}(t)) = f(-\sqrt[n]{-t}) = f(-\sqrt[n]{|t|})$ is in $AC([-r^n,0])$. □

3. Absolutely continuous parameterization of the roots

3.1. **Lemma.** *Let $I \subseteq \mathbb{R}$ be an interval. Consider a curve of monic polynomials*

$$P(t)(z) = z^n + \sum_{j=1}^n (-1)^j a_j(t) z^{n-j}$$

such that the coefficients $a_j : I \to \mathbb{C}$, $1 \le j \le n$, are continuous. If there is a Lipschitz parameterization of the roots of $P(t)$, then any continuous parameterization is Lipschitz.

Proof. Let $\mu_1, \dots, \mu_n$ be a Lipschitz parameterization of the roots of P on I with common Lipschitz constant C. Assume that $t \mapsto \lambda(t)$ is any continuous root of $t \mapsto P(t)$ for $t \in I$. Let $t < s$ be in I. Then there is an i_0 such that $\lambda(t) = \mu_{i_0}(t)$. Now let t_1 be the maximum of all $r \in [t,s]$ such that $\lambda(r) = \mu_{i_0}(r)$. If $t_1 < s$ then $\mu_{i_0}(t_1) = \mu_{i_1}(t_1)$ for some $i_1 \ne i_0$. Let t_2 be the maximum of all $r \in [t_1, s]$ such that $\lambda(r) = \mu_{i_1}(r)$. If $t_2 < s$ then $\mu_{i_1}(t_2) = \mu_{i_2}(t_2)$ for some $i_2 \notin \{i_0, i_1\}$. And so on until $s = t_k$ for some $k \le n$. Then we have (where $t_0 = t$)

$$\frac{|\lambda(s) - \lambda(t)|}{s - t} \le \sum_{j=0}^{k-1} \frac{|\mu_{i_j}(t_{j+1}) - \mu_{i_j}(t_j)|}{t_{j+1} - t_j} \cdot \frac{t_{j+1} - t_j}{s - t} \le C. \quad \square$$

3.2. **Proposition.** *Let $I \subseteq \mathbb{R}$ be an interval. Consider a generic curve of monic polynomials*

$$P(t)(z) = z^n + \sum_{j=1}^{n} (-1)^j a_j(t) z^{n-j},$$

with smooth coefficients $a_j : I \to \mathbb{C}$, $1 \le j \le n$. For any $t_0 \in I$, there exists a positive integer N such that the roots of $t \mapsto P(t_0 \pm (t - t_0)^N)$ can be parameterized smoothly near t_0. If the coefficients a_i are real analytic, then the roots of $t \mapsto P(t_0 \pm (t - t_0)^N)$ can be parameterized real analytically near t_0.

Proof. It is no restriction to assume that $0 \in I$ and $t_0 = 0$.

We use the following:

Algorithm. (1) If all roots of $P(0)$ are pairwise different, the roots of $t \to P(\pm t)$ may be parameterized smoothly near 0, by the implicit function theorem. Then $N = 1$.

(2) If there are distinct roots of $P(0)$, we put them into two subsets which factors $P(t) = P_1(t)P_2(t)$ by the splitting lemma 2.4. Suppose that $t \mapsto P_1(\pm t^{N_1})$ and $t \mapsto P_2(\pm t^{N_2})$ are smoothly solvable near 0, then $t \mapsto P(\pm t^{N_1 N_2})$ is smoothly solvable near 0 as well.

(3) If all roots of $P(0)$ are equal, we reduce to the case $a_1 = 0$, by replacing z with $z - a_1(t)/n$. Then all roots of $P(0)$ are equal to 0, hence, $a_k(0) = 0$ for all k. Let $m := \min\{m(a_k)/k : 2 \le k \le n\}$ which exists since P is generic (by lemma 2.2). Let d be a minimal integer such that $dm \ge 1$. Then for the multiplicity of $t \mapsto a_k(\pm t^d)$, $2 \le k \le n$, we find

$$m(a_k(\pm t^d)) = dm(a_k) \ge dmk \ge k.$$

Hence we may write $a_k(\pm t^d) = t^k \tilde{a}_k^{\pm}(t)$ near 0 with $\tilde{a}_k^{\pm}$ smooth, for all k. Consider

$$\tilde{P}^{\pm}(t)(z) = z^n + \sum_{j=2}^{n} (-1)^j \tilde{a}_j^{\pm}(t) z^{n-j}.$$

If $t \to \tilde{P}^{\pm}(t)$ is smoothly solvable and $t \mapsto \lambda_j^{\pm}(t)$ are its smooth roots, then $t \mapsto t\lambda_j^{\pm}(t)$ are smooth parameterizations of the roots of $t \mapsto P(\pm t^d)$.

Note that $m(\tilde{a}_k^{\pm}) = dm(a_k) - k$, for $2 \le k \le n$, and thus

$$\tilde{m} := \min_{2 \le k \le n} \frac{m(\tilde{a}_k^{\pm})}{k} = dm - 1 < m, \tag{3.2.1}$$

by the minimality of d.

If $\tilde{m} = 0$ there exists some k such that $\tilde{a}_k^{\pm}(0) \ne 0$, and not all roots of $\tilde{P}^{\pm}(0)$ are equal. We feed $\tilde{P}^{\pm}$ into step (2). Otherwise we feed $\tilde{P}^{\pm}$ into step (3).

Step (1) and (2) either provide a required parameterization or reduce the problem to a lower degree n. Since $\tilde{m}$ is of the form p/k where $2 \le k \le n$ and $p \in \mathbb{N}$ and by (3.2.1), also step (3) is visited only finitely many times. So the algorithm stops after finitely many steps and it provides an integer N and a smooth parameterization of the roots of $t \mapsto P(\pm t^N)$ near 0. The real analytic case is analogous. $\square$

3.3. **Theorem.** *Let $I \subseteq \mathbb{R}$ be an interval. Consider a generic curve of monic polynomials*

$$P(t)(z) = z^n + \sum_{j=1}^{n} (-1)^j a_j(t) z^{n-j},$$

with smooth coefficients $a_j : I \to \mathbb{C}$, $1 \le j \le n$. Any continuous parameterization $\lambda = (\lambda_1, \ldots, \lambda_n) : I \to \mathbb{C}^n$ of the roots of P is locally absolutely continuous.

Proof. It suffices to show that each $t_0 \in I$ has a neighborhood on which λ is absolutely continuous. Without restriction we assume that $0 \in I$ and $t_0 = 0$. By proposition 3.2, there is an integer N and a neighborhood J_N of 0 such that $t \mapsto P(\pm t^N)$ allows a smooth parameterization $\mu^\pm = (\mu_1^\pm, \ldots, \mu_n^\pm)$ of its roots on J_N. Another continuous parameterization is provided by $t \mapsto \lambda(\pm t^N) = (\lambda_1(\pm t^N), \ldots, \lambda_n(\pm t^N))$. By lemma 3.1, the parameterization $t \mapsto \lambda(\pm t^N)$ is actually Lipschitz (by shrinking J_N if necessary), in particular, absolutely continuous. Let $J = \{t \in I : \pm \sqrt[N]{|t|} \in J_N\}$, $J_{\geq 0} = \{t \in J : t \geq 0\}$, and $J_{\leq 0} = \{t \in J : t \leq 0\}$. By lemma 2.5, we find that λ is absolutely continuous on $J_{\geq 0}$. In order to see that λ is absolutely continuous on $J_{\leq 0}$ we apply lemma 2.5 to $t \mapsto \lambda(-t^N)$, if N is even, and to $t \mapsto \lambda(t^N)$, if N is odd. Hence λ is absolutely continuous on J. This completes the proof. $\square$

3.4. **Corollary.** *Any continuous parameterization of the roots of a real analytic, or more generally quasianalytic, curve $I \ni t \mapsto P(t)$ of monic polynomials is locally absolutely continuous.* $\square$

3.5. **Remark.** The conclusion in theorem 3.3 is best possible. In general the roots cannot be chosen with first derivative in L^p_{loc} for any $1 < p \leq \infty$. A counter example is given by

$$P(t)(z) = z^n - t, \quad t \in \mathbb{R},$$

if $n \geq \frac{p}{p-1}$, for $1 < p < \infty$, and if $n \geq 2$, for $p = \infty$.

On the other hand, finding the optimal assumptions on P for admitting locally absolutely continuous roots is an open problem.

4. Differentiable parameterization of the roots

4.1. **Lemma.** [14, 4.3] *Consider a continuous curve $c : (a, b) \to X$ in a compact metric space X. Then the set of all accumulation points of $c(t)$ as $t \searrow a$ is connected.*

4.2. **Proposition.** *Let $I \subseteq \mathbb{R}$ be an interval containing 0. Consider a curve of monic polynomials*

$$P(t)(z) = z^n + \sum_{j=1}^{n} (-1)^j a_j(t) z^{n-j},$$

such that the coefficients $a_j : I \to \mathbb{C}$, $1 \leq j \leq n$, are of class C^n. Then the following conditions are equivalent:

1. *There exists a local continuous parameterization of the roots of P near 0 which is differentiable at 0.*
2. *There exists a local continuous parameterization λ_i of the roots of P near 0 such that $\lambda_i(0) = \lambda_j(0)$ implies $m(\lambda_i - \lambda_j) \geq 1$, for all $i \neq j$.*
3. *Split $P(t) = P_1(t) \cdots P_l(t)$ according to lemma 2.4, where l is the number of distinct roots of $P(0)$. Then $m(\tilde{\Delta}_k(P_i)) \geq k(k-1)$, for all $1 \leq i \leq l$ and $2 \leq k \leq \deg P_i$.*

Proof. (1) $\Rightarrow$ (2) is obvious and (2) $\Rightarrow$ (3) follows immediately from (2.1.3).

(3) $\Rightarrow$ (1): Using the splitting $P(t) = P_1(t) \cdots P_l(t)$, we may suppose that all roots of $P(0)$ coincide. We can reduce to the case $a_1 = 0$ by replacing the variable z with $z - a_1(t)/n$. Then all roots of $P(0)$ are equal to 0. By assumption and remark 2.2, we find $m(a_k) \geq k$, for all $2 \leq k \leq n$. So, for t near 0, we can write $a_k(t) = t^k a_{k,k}(t)$ for continuous $a_{k,k}$ and $2 \leq k \leq n$. The continuous curve of polynomials $P_{(1)}(t)(z) := z^n + \sum_{j=2}^n (-1)^j a_{j,j}(t) z^{n-j}$ admits a continuous parameterization $\tilde{\lambda} = (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n)$ of its roots near 0. Then $\lambda(t) := t\tilde{\lambda}(t)$ parameterizes the roots of P, locally near 0, and is differentiable at 0. $\square$

4.3. **Theorem.** *Let $I \subseteq \mathbb{R}$ be an open interval. Consider a curve of monic polynomials*

$$P(t)(z) = z^n + \sum_{j=1}^{n} (-1)^j a_j(t) z^{n-j},$$

such that the coefficients $a_j : I \to \mathbb{C}$, $1 \le j \le n$, are of class C^n. Then the following conditions are equivalent:

(1) *There exists a global differentiable parameterization of the roots of P.*
(2) *There exists a global continuous parameterization of the roots of P with order of contact ≥ 1 (i.e. if any two roots meet they meet of order ≥ 1).*
(3) *Let $t_0 \in I$. Split $P(t) = P_1(t) \cdots P_l(t)$ near t_0 according to lemma 2.4, where l is the number of distinct roots of $P(t_0)$. Then $m_{t_0}(\tilde{\Delta}_k(P_i)) \ge k(k-1)$, for all $1 \le i \le l$ and $2 \le k \le \deg P_i$.*

Proof. By proposition 4.2, it just remains to check (3) $\Rightarrow$ (1).

We use induction on n. There is nothing to prove if $n = 1$. So let us assume that (3) $\Rightarrow$ (1) holds for degrees strictly less than n.

We may suppose that $a_1 = 0$ by replacing z with $z - a_1(t)/n$. Consider the set F of all $t \in I$ such that all roots of $P(t)$ coincide. Then F is closed and its complement $I \backslash F$ is a countable union of open subintervals whose boundary points lie in F.

Let J denote one such interval. For each $t_0 \in J$, the polynomial $P(t_0)$ has distinct roots which may be put into distinct subsets, and, by lemma 2.4, we obtain a local splitting $P(t) = P_1(t)P_2(t)$ near t_0, where both P_1 and P_2 have degree less than n. Clearly, P_1 and P_2 satisfy (3) as well. By induction hypothesis, we find differentiable parameterizations of the roots of P, locally near any $t_0 \in J$.

Let λ be a differentiable parameterization of the roots of P defined on a maximal subinterval $J' \subseteq J$. Suppose that the right (say) endpoint t_1 of J' belongs to J. Then there exists a differentiable parameterization $\bar{\lambda}$ of the roots of P, locally near t_1, and there is a $t_0 < t_1$ such that both λ and $\bar{\lambda}$ are defined near t_0. Let (t_m) be a sequence with $t_m \to t_0$. For each m there exists a permutation τ_m such that $\lambda(t_m) = \tau_m.\bar{\lambda}(t_m)$. By passing to a subsequence, again denoted by (t_m), we have $\lambda(t_m) = \tau.\bar{\lambda}(t_m)$ for a fixed permutation τ and for all m. Then $\lambda(t_0) = \lim_{t_m \to t_0} \lambda(t_m) = \tau.(\lim_{t_m \to t_0} \bar{\lambda}(t_m)) = \tau.\bar{\lambda}(t_0)$ and

$$\lambda'(t_0) = \lim_{t_m \to t_0} \frac{\lambda(t_m) - \lambda(t_0)}{t_m - t_0} = \lim_{t_m \to t_0} \frac{\tau.\bar{\lambda}(t_m) - \tau.\bar{\lambda}(t_0)}{t_m - t_0} = \tau.\bar{\lambda}'(t_0).$$

Hence, the differentiable parameterization λ of the roots of P was not maximal: we can extend it differentiably by defining $\tilde{\lambda}(t) := \lambda(t)$ for $t \le t_0$ and $\tilde{\lambda}(t) := \tau.\bar{\lambda}(t)$ for $t \ge t_0$. This shows that there exists a differentiable parameterization λ of the roots of P defined on J.

Let us extend λ to the closure of J, by setting it 0 at the endpoints of J. Since $a_1 = 0$, then λ still parameterizes the roots of P on the closure of J. Let t_0 denote the right (say) endpoint of J. By proposition 4.2, there exists a local continuous parameterization $\bar{\lambda}$ of the roots of P near t_0 which is differentiable at t_0. Let (t_m) be a sequence with $t_m \nearrow t_0$. By passing to a subsequence, we may assume that $\lambda(t_m) = \tau.\bar{\lambda}(t_m)$ for a fixed permutation τ and for all m. Then $\lim_{t_m \nearrow t_0} \lambda(t_m) = \tau.(\lim_{t_m \nearrow t_0} \bar{\lambda}(t_m)) = \tau.0 = 0$ and

$$\lim_{t_m \nearrow t_0} \frac{\lambda(t_m)}{t_m - t_0} = \lim_{t_m \nearrow t_0} \frac{\tau.\bar{\lambda}(t_m)}{t_m - t_0} = \tau.\bar{\lambda}'(t_0).$$

It follows that the set of accumulation points of $\lambda(t)/(t - t_0)$, as $t \nearrow t_0$, lies in the S_n-orbit through $\bar{\lambda}'(t_0)$ of the symmetric group S_n. Since this orbit is finite, we may conclude from lemma 4.1 that the limit $\lim_{t \nearrow t_0} \lambda(t)/(t - t_0)$ exits. Thus the one-sided derivative of λ at t_0 exists.

For isolated points in F, it follows from the discussion in the previous paragraph that we can apply a fixed permutation to one of the neighboring differentiable parameterizations of the roots in order to glue them differentiably. Therefore, we have found a differentiable parameterization λ of the roots of P defined on $I \backslash F'$, where F' denotes the set of accumulation points of F.

Let us extend λ by 0 on F'. Then it provides a global differentiable parameterization of the roots of P, since any parameterization is differentiable at points $t' \in F'$. For: It is clear that the derivative at t' of any differentiable parameterization has to be 0. Let $\bar{\lambda}$ be the local parameterization near t', provided by proposition 4.2. As above we may conclude that the set of accumulation points of $\lambda(t)/(t - t')$, as $t \to t'$, lies in the S_n-orbit through $\bar{\lambda}'(t') = 0$. □

5. Reformulation of the problem

5.1. **Lifting curves over invariants.** Let G be a compact Lie group and let $\rho : G \to \mathrm{O}(V)$ be an orthogonal representation in a real finite dimensional Euclidean vector space V. By a classical theorem of Hilbert and Nagata, the algebra $\mathbb{R}[V]^G$ of invariant polynomials on V is finitely generated. So let $\sigma_1, \ldots, \sigma_n$ be a system of homogeneous generators of $\mathbb{R}[V]^G$ of positive degrees $d_1, \ldots, d_n$. Consider the orbit map $\sigma = (\sigma_1, \ldots, \sigma_n) : V \to \mathbb{R}^n$. The image $\sigma(V)$ is a semialgebraic subset of $\{y \in \mathbb{R}^n : P(y) = 0 \text{ for all } P \in I\}$, where I is the ideal of relations between $\sigma_1, \ldots, \sigma_n$. Since G is compact, σ is proper and separates orbits of G, it thus induces a homeomorphism between V/G and $\sigma(V)$.

Let $H = G_v$ be the isotropy group of $v \in V$ and (H) the conjugacy class of H in G which is called the type of the orbit $G.v$. The union $V_{(H)}$ of orbits of type (H) is called an orbit type submanifold of the representation ρ, and $V_{(H)}/G$ is called an orbit type submanifold of the orbit space V/G. The collection of connected components of the manifolds $\{V_{(H)}/G\}$ forms a stratification of V/G called orbit type stratification, see e.g. [24, 4.3].

Let $c : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ be a smooth curve in the orbit space; smooth as curve in $\mathbb{R}^n$. A curve $\bar{c} : \mathbb{R} \to V$ is called *lift* of c to V, if $c = \sigma \circ \bar{c}$ holds. The problem of lifting smooth curves over invariants is independent of the choice of a system of homogeneous generators of $\mathbb{R}[V]^G$, see [14, 2.2].

Let $s \in \mathbb{N}$. Denote by A_s the union of all strata X of the orbit space V/G with $\dim X \le s$, and by I_s the ideal of $\mathbb{R}[V]^G$ consisting of all polynomials vanishing on A_{s-1}. Let $c : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ be a smooth curve, $t \in \mathbb{R}$, and $s = s(c, t)$ a minimal integer such that, for a neighborhood J of t in $\mathbb{R}$, we have $c(J) \subseteq A_s$. The curve c is called *normally nonflat at* t if there is $f \in I_s$ such that $f \circ c$ is not infinitely flat at t. A smooth curve $c : \mathbb{R} \to \sigma(V) \subseteq \mathbb{R}^n$ is called *generic*, if c is normally nonflat at t for each $t \in \mathbb{R}$.

Let $G = \mathrm{S}_n$, the symmetric group, and let ρ be the standard representation of S_n in $\mathbb{R}^n$ by permuting the coordinates. The elementary symmetric functions σ_i in (2.1.1) generate the algebra of symmetric polynomials $\mathbb{R}[\mathbb{R}^n]^{\mathrm{S}_n}$. Hence the image $\sigma(\mathbb{R}^n)$ may be identified with the space of monic hyperbolic polynomials of degree n. Recall that a polynomial is called hyperbolic if all its roots are real. A lift to $\mathbb{R}^n$ of a curve P in $\sigma(\mathbb{R}^n)$ represents a parameterization of the roots of P. A curve P of hyperbolic polynomials is generic in the sense of the last paragraph if and only if it is generic in the sense of 2.3, see e.g. [21, 2.6].

The following theorem generalizes the main results on the one dimensional perturbation theory of hyperbolic polynomials. It collects the main results of [2] and [14, 15, 13].

Theorem. *Let $c : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ be a curve in the orbit space and let $d = \max\{d_1, \dots, d_n\}$. Then:*

(1) *If c is real analytic, then it allows a real analytic lift, locally.*
(2) *If c is smooth and generic, then there exists a global smooth lift.*
(3) *If c is C^d, then there exists a global differentiable lift.*

If G is finite, write $V = V_1 \oplus \cdots \oplus V_l$ as orthogonal direct sum of irreducible subspaces V_i and define $k = \max\{d, k_1, \dots, k_l\}$, where $k_i = \min\{|G.v| : v \in V_i \backslash \{0\}\}$. Then:

(4) *If c is C^k, then each differentiable lift is C^1.*
(5) *If c is C^{d+k}, then there exists a global twice differentiable lift.*

5.2. Let us consider the standard action of the symmetric group S_n on $\mathbb{C}^n$ by permuting the coordinates and the diagonal action of S_n on $\mathbb{R}^n \times \mathbb{R}^n$ by permuting the coordinates in each factor. Write $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ where $z_k = x_k + iy_k$, $1 \le k \le n$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, and $y = (y_1, \dots, y_n) \in \mathbb{R}^n$. The mapping

$$T : \mathbb{C}^n \longrightarrow \mathbb{R}^n \times \mathbb{R}^n : z \longmapsto (x, y)$$

is an equivariant $\mathbb{R}$-linear homeomorphism. Consequently, it descends to a homeomorphism $\hat{T}$ such that the following diagram commutes

$$\begin{tikzcd} \mathbb{C}^n \arrow[r, "T"] \arrow[d] & \mathbb{R}^n \times \mathbb{R}^n \arrow[d] \\ \mathbb{C}^n/\,\mathrm{S}_n \arrow[r, "\hat{T}"'] & (\mathbb{R}^n \times \mathbb{R}^n)/\,\mathrm{S}_n \end{tikzcd} \tag{5.2.1}$$

Consider the respective orbit type stratifications of the S_n-modules $\mathbb{C}^n$ and $\mathbb{R}^n \times \mathbb{R}^n$ and of its orbit spaces. It is evident that T, and thus also $\hat{T}$, maps strata onto strata. Note that, while the orbit type stratification of $\mathbb{C}^n/\,\mathrm{S}_n \cong \mathbb{C}^n$ is finer than its stratification as affine variety, the orbit type stratification of $(\mathbb{R}^n \times \mathbb{R}^n)/\,\mathrm{S}_n$ is its coarsest stratification, e.g. [24, 4.4.6].

Let $P : \mathbb{R} \to \mathbb{C}^n/\,\mathrm{S}_n = \mathbb{C}^n$ be a curve of monic polynomials of degree n. Then $\hat{T} \circ P$ is a curve in $(\mathbb{R}^n \times \mathbb{R}^n)/\,\mathrm{S}_n \subseteq \mathbb{R}^N$. It follows that P allows a regular lift to $\mathbb{C}^n$, i.e., a regular parameterization of its roots, if and only if $\hat{T} \circ P$ allows a lift of the same regularity to $\mathbb{R}^n \times \mathbb{R}^n$. Theorem 5.1 provides sufficient conditions for $\hat{T} \circ P$ to be liftable regularly, and hence for P to admit regular parameterizations of its roots.

As generators for the algebra $\mathbb{C}[\mathbb{C}^n]^{\mathrm{S}_n}$ we may choose the Newton polynomials $s_i(z) = \sum_{j=1}^n z_j^i$, for $1 \le i \le n$. By the first fundamental theorem of invariant theory for S_n (e.g. [32, 3.4.1]), the algebra $\mathbb{R}[\mathbb{R}^n \times \mathbb{R}^n]^{\mathrm{S}_n}$ is generated by the polarizations of the s_i:

$$\tau_{i,j}(x, y) = \sum_{k=1}^n x_k^i y_k^j, \quad (i, j \in \mathbb{N}, 1 \le i + j \le n).$$

We may then identify the orbit projections

$$\mathbb{C}^n \longrightarrow \mathbb{C}^n/\,\mathrm{S}_n \quad \text{and} \quad \mathbb{R}^n \times \mathbb{R}^n \longrightarrow (\mathbb{R}^n \times \mathbb{R}^n)/\,\mathrm{S}_n$$

with the mappings

$$s = (s_i) : \mathbb{C}^n \longrightarrow s(\mathbb{C}^n) = \mathbb{C}^n \quad \text{and} \quad \tau = (\tau_{i,j}) : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \tau(\mathbb{R}^n \times \mathbb{R}^n) \subseteq \mathbb{R}^N,$$

respectively. Here $N = \binom{n+2}{n} - 1 = \frac{1}{2}n(n+3)$. The image $\tau(\mathbb{R}^n \times \mathbb{R}^n)$ is a semialgebraic subset of $\mathbb{R}^N$. Since it is homeomorphic with $s(\mathbb{C}^n) = \mathbb{C}^n$, its dimension is $2n$. It follows that there are at least $\frac{1}{2}n(n-1)$ independent non-trivial polynomial relations between the $\tau_{i,j}$.

The homeomorphism $\hat{T}$ from the diagram (5.2.1) is then determined by:

$$\hat{T}^{-1} : \mathbb{R}^N \supseteq \tau(\mathbb{R}^n \times \mathbb{R}^n) \longrightarrow \mathbb{C}^n : (\tau_{i,j}) \longmapsto \left(\sum_{k=0}^{m} \binom{m}{k} i^k \tau_{m-k,k} \right)_{1 \le m \le n}.$$

5.3. **The quadratic case.** Without loss it suffices to consider $P(t)(z) = z^2 - f(t)$ with $f : I \to \mathbb{C}$. Let us consider the curve $\hat{T} \circ P$ in $(\mathbb{R}^2 \times \mathbb{R}^2)/\,\mathrm{S}_2$ whose coordinates $\tau_{i,j}(P)$ have to satisfy:

$$\tau_{1,0}(P) = \tau_{0,1}(P) = 0,\ \tau_{2,0}(P) - \tau_{0,2}(P) = 2\,\mathrm{Re}(f),\ \tau_{1,1}(P) = \mathrm{Im}(f),$$
$$\tau_{2,0}(P)\tau_{0,2}(P) = \tau_{1,1}^2.$$

It is easy to compute

$$\hat{T} \circ P = (0, 0, |f| + \mathrm{Re}(f), |f| - \mathrm{Re}(f), \mathrm{Im}(f)). \tag{5.3.1}$$

In the following a square root of f is any function g satisfying $g^2 = f$. Applying 5.1 and 5.2, we obtain:

(1) *If f is smooth and nowhere infinitely flat and $|f|$ is smooth, then there exist smooth square roots of f.*
(2) *If f and $|f|$ are of class C^4, then there exist twice differentiable square roots of f.*

Theorem 3.3 and theorem 4.3 give:

(3) *If f is smooth and nowhere infinitely flat, then any continuous choice of the square roots of f is locally absolutely continuous.*
(4) *Assume that f is C^2. Then there exist differentiable square roots of f if and only if f vanishes of order ≥ 2 at all its zeros.*

Let us assume that f is real valued. Then (5.3.1) reduces to:

$$(\hat{T} \circ P)(t) = \begin{cases} (0, 0, 2f(t), 0, 0) & \text{if } f(t) \ge 0, \\ (0, 0, 0, -2f(t), 0) & \text{if } f(t) \le 0. \end{cases}$$

Suppose further that f is C^2 and that $f(t_0) = 0$ implies $f'(t_0) = f''(t_0) = 0$. It follows that $\hat{T} \circ P$ is of class C^2. By 5.1 and 5.2, there exist C^1 parameterizations of the square root of f. So:

(5) *If f is real valued, C^2, and $f(t_0) = 0$ implies $f'(t_0) = f''(t_0) = 0$, then there exist C^1 square roots of f.*

Combining (3) and (5) we obtain:

(6) *If f is real valued and smooth, then each continuous choice of square roots of f is locally absolutely continuous.*

6. Regular diagonalization of normal matrices

6.1. Let $I \subseteq \mathbb{R}$ be an interval. A smooth curve of normal complex $n \times n$ matrices $I \ni t \mapsto A(t) = (A_{ij}(t))_{1 \le i,j \le n}$ is called *generic*, if $I \ni t \mapsto \chi_{A(t)}$ is generic, where $\chi_{A(t)}(\lambda) = \det(A(t) - \lambda \mathbb{I})$ is the characteristic polynomial of $A(t)$.

6.2. **Theorem.** *Let $I \subseteq \mathbb{R}$ be an interval. Let $I \ni t \mapsto A(t) = (A_{ij}(t))_{1 \le i,j \le n}$ be a generic smooth curve of normal complex matrices acting on a complex vector space $V = \mathbb{C}^n$. Then:*

(1) *For each $t_0 \in I$ there exists an integer N such that $t \mapsto A(t_0 \pm (t - t_0)^N)$ allows a smooth parameterization of its eigenvalues and eigenvectors near t_0. If A is real analytic, then the eigenvalues and eigenvectors of $t \mapsto A(t_0 \pm (t - t_0)^N)$ may be parameterized real analytically near t_0.*
(2) *There exist locally absolutely continuous parameterizations of the eigenvalues and the eigenvectors of A.*

Proof. We adapt the proof of [1, 7.6].

By theorem 3.3, the characteristic polynomial

$$\chi_{A(t)}(\lambda) = \det(A(t) - \lambda\mathbb{I}) = \sum_{j=0}^{n} (-1)^{n-j} \operatorname{Trace}(\Lambda^j A(t))\lambda^{n-j} \tag{6.2.1}$$
$$= (-1)^n \Big(\lambda^n + \sum_{j=1}^{n} (-1)^j a_j(t)\lambda^{n-j}\Big)$$

admits a continuous, locally absolutely continuous parameterization $\lambda_1, \ldots, \lambda_n$ of its roots. This shows the first part of (2).

Let us show (1). Without loss we may assume that $t_0 = 0$. By proposition 3.2, there is an integer N_0 such that the eigenvalues of $t \mapsto A(\pm t^{N_0})$ can be parameterized by smooth functions $t \mapsto \mu_j^{\pm}(t)$ near 0. Consider the following algorithm:

(a) Not all eigenvalues of $A(0)$ agree. Let $\nu_1, \ldots, \nu_l$ denote the pairwise distinct eigenvalues of $A(0)$ with respective multiplicities $m_1, \ldots, m_l$. Assume without loss that

$$\begin{aligned}
\nu_1 &= \mu_1^{\pm}(0) = \cdots = \mu_{m_1}^{\pm}(0),\\
\nu_2 &= \mu_{m_1+1}^{\pm}(0) = \cdots = \mu_{m_1+m_2}^{\pm}(0),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\nu_l &= \mu_{n-m_l}^{\pm}(0) = \cdots = \mu_n^{\pm}(0).
\end{aligned}$$

This defines a partition into subsets of smooth eigenvalues such that, for t near 0, they do not meet each other if they belong to different subsets. For $1 \le j \le l$ consider

$$\begin{aligned}
V_t^{(j),\pm} &:= \bigoplus_{\{i\,:\,\nu_j = \mu_i^{\pm}(0)\}} \ker(A(\pm t^{N_0}) - \mu_i^{\pm}(t))\\
&= \ker\big(\circ_{\{i\,:\,\nu_j=\mu_i^{\pm}(0)\}}\,(A(\pm t^{N_0}) - \mu_i^{\pm}(t))\big).
\end{aligned}$$

Note that the order of the compositions in the above expression is not relevant. So $V_t^{(j),\pm}$ is the kernel of a smooth vector bundle homomorphism $B^{\pm}(t)$ of constant rank, and thus is a smooth vector subbundle of the trivial bundle $(-\epsilon, \epsilon) \times V \to (-\epsilon, \epsilon)$. This can be seen as follows: Choose a basis of V such that $A(0)$ is diagonal. By the elimination procedure one can construct a basis for the kernel of $B^{\pm}(0)$. For t near 0, the elimination procedure (with the same choices) gives then a basis of the kernel of $B^{\pm}(t)$. The elements of this basis are then smooth in t near 0.

It follows that it suffices to find smooth eigenvectors in each subbundle $V^{(j),\pm}$ separately, expanded in the constructed smooth frame field. But in this frame field the vector subbundle looks again like a constant vector space. So feed each of these parts ($t \to A(\pm t^{N_0})$ restricted to $V^{(j),\pm}$, as matrix with respect to the frame field) into step (b) below.

(b) All eigenvalues of $A(0)$ coincide and are equal to $a_1(0)/n$, according to (6.2.1). Eigenvectors of $A(t)$ are also eigenvectors of $A(t) - (a_1(t)/n)\mathbb{I}$, thus we may replace $A(t)$ by $A(t) - (a_1(t)/n)\mathbb{I}$ and assume that the first coefficient of the characteristic polynomial (6.2.1) vanishes identically. Then $A(0) = 0$.

If $A(t) = 0$ for t near 0 we choose the eigenvectors constant.

Otherwise write $A_{ij}(t) = t^m A_{ij}^{(m)}(t)$, where $m := \min\{m(A_{ij}) : 1 \le i, j \le n\}$ which exists by assumption. It follows from (6.2.1) that the characteristic polynomial of $A^{(m)}(t)$ is

$$\chi_{A^{(m)}(t)}(\lambda) = (-1)^n \Big(\lambda^n + \sum_{j=2}^{n} (-1)^j t^{-mj} a_j(t)\lambda^{n-j}\Big),$$

Hence $m(a_k) \geq mk$ for all k. By proposition 3.2, there exists an integer N_1 such that $t \mapsto \chi_{A^{(m)}(\pm t^{N_1})}$ admits smooth parameterizations of its roots (eigenvalues of $t \mapsto A^{(m)}(\pm t^{N_1})$) for t near 0. Eigenvectors of $A^{(m)}(\pm t^{N_1})$ are also eigenvectors of $A(\pm t^{N_1})$. There exist $1 \leq i, j \leq n$ such that $A_{ij}^{(m)}(0) \neq 0$ and thus not all eigenvalues of $A^{(m)}(0)$ are equal. Feed $t \mapsto A^{(m)}(\pm t^{N_1})$ into (a).

By assumption, this algorithm stops after finitely many steps and shows (1). The real analytic case is analogous.

Now we finish the proof of (2). By (1), we find an integer N such that $t \mapsto A(\pm t^N)$ allows smooth parameterizations $t \mapsto \mu_j^{\pm}(t)$ and $t \mapsto v_j^{\pm}(t)$ of its eigenvalues and eigenvectors near 0. In a similar way as in the proof of theorem 3.3, we can compose $t \mapsto \mu_j^{\pm}(t)$ and $t \mapsto v_j^{\pm}(t)$ with $t \mapsto \sqrt[N]{t}$ and $t \mapsto -\sqrt[N]{|t|}$ in order to obtain absolutely continuous parameterizations of the eigenvalues and eigenvectors of A near 0. □

Remark. The condition that $A(t)$ is normal cannot be omitted. Any choice of eigenvectors of the following real analytic curve A of 2×2 matrices has a pole at 0. Hence there does not exist an integer N such that $t \mapsto A(\pm t^N)$ allows regular eigenvectors near 0.

$$A(t) = \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix}.$$

The following smooth curve A of symmetric real matrices allows smooth eigenvalues, but the eigenvectors cannot be chosen continuously. This example (due to [26, §2]) shows that the assumption that A is generic is essential in theorem 6.2.

$$A(t) = e^{-\frac{1}{t^2}} \begin{pmatrix} \cos\frac{2}{t} & \sin\frac{2}{t} \\ \sin\frac{2}{t} & -\cos\frac{2}{t} \end{pmatrix}, \quad A(0) = 0.$$

7. Perturbation of unbounded normal operators

7.1. **Theorem.** *Let $I \subseteq \mathbb{R}$ be an interval. Let $I \ni t \mapsto A(t)$ be a generic smooth curve of unbounded normal operators in a Hilbert space with common domain of definition and with compact resolvents. Let $t_0 \in I$ and let z_0 be an eigenvalue of $A(t_0)$. Let n be the multiplicity of z_0. Then:*

(1) *There exists an integer N such that the n eigenvalues of $t \mapsto A(t_0 \pm (t-t_0)^N)$ passing through z_0 and the corresponding eigenvectors allow smooth parameterizations, locally near t_0. If A is real analytic, then the n eigenvalues of $t \mapsto A(t_0 \pm (t-t_0)^N)$ passing through z_0 and its eigenvectors may be arranged real analytically, locally near t_0.*
(2) *There exist locally absolutely continuous parameterizations of the n eigenvalues of A passing through z_0 and its eigenvectors, locally near t_0.*

That $A(t)$ is a smooth (resp. real analytic) curve of unbounded operators means the following: There is a dense subspace V of the Hilbert space H such that V is the domain of definition of each $A(t)$, and such that each $A(t)$ is closed and $A(t)^*A(t) = A(t)A(t)^*$, where the adjoint operator $A(t)^*$ is defined as usual by $\langle A(t)u, v\rangle = \langle u, A(t)^*v\rangle$ for all v for which the left-hand side is bounded as function in $u \in H$. Note that the domain of definition of $A(t)^*$ is V. Moreover, we require that $t \mapsto \langle A(t)u, v\rangle$ is smooth (resp. real analytic) for each $u \in V$ and $v \in H$. This implies that $t \mapsto A(t)u$ is of the same class $\mathbb{R} \to H$ for each $u \in V$, by [17, 2.3] or [9, 2.6.2].

We call the curve $I \ni t \mapsto A(t)$ *generic*, if no two unequal continuously parameterized eigenvalues meet of infinite order at any $t \in I$.

Proof. We use the resolvent lemma in [18] (see also [1]): *If $A(t)$ is smooth (resp. real analytic), then also the resolvent $(A(t)-z)^{-1}$ is smooth (resp. real analytic) into $L(H,H)$ in t and z jointly.*

Let z be an eigenvalue of $A(s)$ of multiplicity n for s fixed. Choose a simple closed curve γ in the resolvent set of $A(s)$ enclosing only z among all eigenvalues of $A(s)$. Since the global resolvent set $\{(t,z) \in \mathbb{R} \times \mathbb{C} : (A(t)-z) : V \to H \text{ is invertible}\}$ is open, no eigenvalue of $A(t)$ lies on γ, for t near s. Consider

$$t \mapsto -\frac{1}{2\pi i}\int_\gamma (A(t)-z)^{-1} dz =: P(t),$$

a smooth (resp. real analytic) curve of projections (on the direct sum of all eigenspaces corresponding to eigenvalues in the interior of γ) with finite dimensional ranges and constant ranks (see [1] or [18]). So for t near s, there are equally many eigenvalues in the interior of γ. Let us call them $\lambda_i(t)$, $1 \le i \le n$, (repeated with multiplicity) and let us denote by $e_i(t)$, $1 \le i \le n$, a corresponding system of eigenvectors of $A(t)$. Then by the residue theorem we have

$$\sum_{i=1}^n \lambda_i(t)^p e_i(t)\langle e_i(t),\ \rangle = -\frac{1}{2\pi i}\int_\gamma z^p (A(t)-z)^{-1} dz$$

which is smooth (resp. real analytic) in t near s, as a curve of operators in $L(H,H)$ of rank n.

Recall claim 2 in [1, 7.8]: *Let $t \mapsto T(t) \in L(H,H)$ be a smooth (resp. real analytic) curve of operators of rank n in Hilbert space such that $T(0)T(0)(H) = T(0)(H)$. Then $t \mapsto \mathrm{Trace}(T(t))$ is smooth (resp. real analytic) near 0.*

We conclude that the Newton polynomials

$$\sum_{i=1}^n \lambda_i(t)^p = -\frac{1}{2\pi i}\,\mathrm{Trace}\int_\gamma z^p (A(t)-z)^{-1} dz,$$

are smooth (resp. real analytic) for t near s, and thus also the elementary symmetric functions

$$\sum_{i_1<\cdots<i_p} \lambda_{i_1}(t)\cdots\lambda_{i_p}(t).$$

It follows that $\{\lambda_i(t) : 1 \le i \le n\}$ represents the set of roots of a polynomial of degree n with smooth (resp. real analytic) coefficients. The statement of the theorem follows then from proposition 3.2, theorem 3.3, and theorem 6.2, since the image of $t \mapsto P(t)$, for t near s describes a finite dimensional smooth (resp. real analytic) vector subbundle of $\mathbb{R} \times H \to \mathbb{R}$ and the $\lambda_i(t)$, $1 \le i \le n$, form the set of eigenvalues of $P(t)A(t)|_{P(t)(H)}$. $\square$

References

[1] D. Alekseevsky, A. Kriegl, M. Losik, and P. W. Michor, *Choosing roots of polynomials smoothly*, Israel J. Math. **105** (1998), 203–233.

[2] ———, *Lifting smooth curves over invariants for representations of compact Lie groups*, Transform. Groups **5** (2000), no. 2, 103–110.

[3] H. Baumgärtel, *Endlichdimensionale analytische Störungstheorie*, Akademie-Verlag, Berlin, 1972, Mathematische Lehrbücher und Monographien, II. Abteilung. Mathematische Monographien, Band 28.

[4] J.-M. Bony, F. Broglia, F. Colombini, and L. Pernazza, *Nonnegative functions as squares or sums of squares*, J. Funct. Anal. **232** (2006), no. 1, 137–147.

[5] M. D. Bronshtein, *Smoothness of roots of polynomials depending on parameters*, Sibirsk. Mat. Zh. **20** (1979), no. 3, 493–501, 690, English transl. in Siberian Math. J. **20** (1980), 347–352. See also MR0537355 (82c:26018).

[6] J. Chaumat and A.-M. Chollet, *Division par un polynôme hyperbolique*, Canad. J. Math. **56** (2004), no. 6, 1121–1144.

[7] F. Colombini, N. Orrú, and L. Pernazza, *On the regularity of the roots of hyperbolic polynomials*, Preprint.
[8] J. Dieudonné, *Sur un théorème de Glaeser*, J. Analyse Math. **23** (1970), 85–88.
[9] A. Frölicher and A. Kriegl, *Linear spaces and differentiation theory*, Pure and Applied Mathematics (New York), John Wiley & Sons Ltd., Chichester, 1988, A Wiley-Interscience Publication.
[10] G. Glaeser, *Racine carrée d'une fonction différentiable*, Ann. Inst. Fourier (Grenoble) **13** (1963), no. fasc. 2, 203–210.
[11] T. Kato, *Perturbation theory for linear operators*, second ed., Grundlehren der Mathematischen Wissenschaften, vol. 132, Springer-Verlag, Berlin, 1976.
[12] A. Kriegl, M. Losik, and P. W. Michor, *Choosing roots of polynomials smoothly. II*, Israel J. Math. **139** (2004), 183–188.
[13] A. Kriegl, M. Losik, P. W. Michor, and A. Rainer, *Addendum to: "Lifting smooth curves over invariants for representations of compact Lie groups. III"[J. Lie Theory 16 (2006), no. 3, 579–600]*, Preprint, `http://www.mat.univie.ac.at/~armin/publ/add-rep-lift4.pdf`.
[14] ———, *Lifting smooth curves over invariants for representations of compact Lie groups. II*, J. Lie Theory **15** (2005), no. 1, 227–234.
[15] ———, *Lifting smooth curves over invariants for representations of compact Lie groups. III*, J. Lie Theory **16** (2006), no. 3, 579–600.
[16] ———, *Lifting mappings over invariants of finite groups*, Acta Math. Univ. Comenian. (N.S.) **77** (2008), no. 1, 93–122.
[17] A. Kriegl and P. W. Michor, *The convenient setting of global analysis*, Mathematical Surveys and Monographs, vol. 53, American Mathematical Society, Providence, RI, 1997, `http://www.ams.org/online_bks/surv53/`.
[18] ———, *Differentiable perturbation of unbounded operators*, Math. Ann. **327** (2003), no. 1, 191–201.
[19] A. Kriegl, P. W. Michor, and A. Rainer, *Many parameter Lipschitz perturbation of unbounded operators*, Preprint, `arXiv:math.FA/0611506`.
[20] K. Kurdyka and L. Paunescu, *Hyperbolic polynomials and multiparameter real-analytic perturbation theory*, Duke Math. J. **141** (2008), no. 1, 123–149.
[21] M. Losik and A. Rainer, *Choosing roots of polynomials with symmetries smoothly*, Rev. Mat. Complut. **20** (2007), no. 2, 267–291.
[22] T. Mandai, *Smoothness of roots of hyperbolic polynomials with respect to one-dimensional parameter*, Bull. Fac. Gen. Ed. Gifu Univ. (1985), no. 21, 115–118.
[23] A. Ostrowski, *Recherches sur la méthode de Graeffe et les zéros des polynomes et des séries de Laurent*, Acta Math. **72** (1940), 99–155.
[24] M. J. Pflaum, *Analytic and geometric study of stratified spaces*, Lecture Notes in Mathematics, vol. 1768, Springer-Verlag, Berlin, 2001.
[25] A. Rainer, *Perturbation of hyperbolic polynomials and related lifting problems*, Preprint, `http://www.mat.univie.ac.at/~armin/publ/roots-lifts.pdf`.
[26] F. Rellich, *Störungstheorie der Spektralzerlegung*, Math. Ann. **113** (1937), no. 1, 600–619.
[27] ———, *Störungstheorie der Spektralzerlegung*, Math. Ann. **113** (1937), no. 1, 677–685.
[28] ———, *Störungstheorie der Spektralzerlegung*, Math. Ann. **116** (1939), no. 1, 555–570.
[29] ———, *Störungstheorie der Spektralzerlegung. IV*, Math. Ann. **117** (1940), 356–382.
[30] ———, *Störungstheorie der Spektralzerlegung. V*, Math. Ann. **118** (1942), 462–484.
[31] ———, *Perturbation theory of eigenvalue problems*, Assisted by J. Berkowitz. With a preface by Jacob T. Schwartz, Gordon and Breach Science Publishers, New York, 1969.
[32] L. Smith, *Polynomial invariants of finite groups*, Research Notes in Mathematics, vol. 6, A K Peters Ltd., Wellesley, MA, 1995.
[33] S. Spagnolo, *On the absolute continuity of the roots of some algebraic equations*, Ann. Univ. Ferrara Sez. VII (N.S.) **45** (1999), no. suppl., 327–337 (2000), Workshop on Partial Differential Equations (Ferrara, 1999).
[34] S. Wakabayashi, *Remarks on hyperbolic polynomials*, Tsukuba J. Math. **10** (1986), no. 1, 17–28.

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