# COMPOSITION IN ULTRADIFFERENTIABLE CLASSES

ARMIN RAINER AND GERHARD SCHINDL

Abstract. We characterize stability under composition of ultradifferentiable classes defined by weight sequences $M$, by weight functions $\omega$, and, more generally, by weight matrices $\mathfrak{M}$, and investigate continuity of composition $(g, f) \mapsto f \circ g$. In addition, we represent the Beurling space $\mathcal{E}^{(\omega)}$ and the Roumieu space $\mathcal{E}^{\{\omega\}}$ as intersection and union of spaces $\mathcal{E}^{(M)}$ and $\mathcal{E}^{\{M\}}$ for associated weight sequences, respectively.

## 1. Introduction

This paper arose from our wish to characterize stability under composition of Denjoy–Carleman classes $\mathcal{E}^{\{M\}}$ and $\mathcal{E}^{(M)}$. For these classes we have developed a calculus in infinite dimensions beyond Banach spaces in [24, 26, 25] which is heavily based on composition: A smooth mapping $f$ is of class $\mathcal{E}^{\{M\}}$ if and only if $f \circ p \in \mathcal{E}^{\{M\}}$ for all $\mathcal{E}^{\{M\}}$ Banach plots (i.e., mappings defined in open subsets of Banach spaces); accordingly for $\mathcal{E}^{(M)}$. Sometimes curves suffice.

Denjoy–Carleman differentiable functions form classes of smooth functions that are described by growth conditions on the Taylor expansion. The growth is prescribed in terms of a sequence $M = (M_k)$ of positive real numbers which serves as a weight for the iterated derivatives: for compact $K$ the sets

$$\Big\{ \frac{f^{(k)}(x)}{\rho^k \, k! \, M_k} : x \in K, k \in \mathbb{N} \Big\}$$

are required to be bounded. The positive real number $\rho$ is subject to either a universal or an existential quantifier, thereby dividing the Denjoy–Carleman classes into those of Beurling type $\mathcal{E}^{(M)}$ and those of Roumieu type $\mathcal{E}^{\{M\}}$, respectively. We write $\mathcal{E}^{[M]}$ for either $\mathcal{E}^{(M)}$ or $\mathcal{E}^{\{M\}}$.

It is well-known that $\mathcal{E}^{[M]}$ is stable under composition, if $M$ is log-convex, see [34], [20], [13], and usually in the literature log-convexity is assumed in order to have stability under composition; but is log-convexity also necessary? Actually, when proving stability under composition with Faá di Bruno's formula one needs a weaker condition that we call (FdB)-property. We prove that the (FdB)-property (for the weakly log-convex minorant $M^{\flat(c)}$) is also a necessary condition for stability under composition, if $\mathcal{E}^{[M]}$ is stable under derivation, see Theorem 3.2. More precisely, if $\mathcal{E}^{[M]}$ is stable under derivation, then stability under composition is in turn equivalent to being holomorphically closed, being inverse closed, $(M_k^{\flat(c)})^{\frac{1}{k}}$ being almost increasing, and $M^{\flat(c)}$ having the (FdB)-property. For further equivalent stability properties we refer to [33]. Inverse closedness has been studied intensively,

*Date*: November 24, 2013.
2010 *Mathematics Subject Classification.* 26E10, 30D60, 46E10, 47B33.
*Key words and phrases.* Ultradifferentiable functions, composition.
AR was supported by FWF-Project P 22218-N13 and GS by FWF-Project P 23028-N13.

e.g. [35], [10], [39]. In this context we prove that, as in the Roumieu case [11], one has $\mathcal{E}^{(M)} = \mathcal{E}^{(M^{b(c)})}$ if $C^\omega \subseteq \mathcal{E}^{(M)}$, see Theorem 2.15. Finally, we demonstrate that log-convexity is not necessary for stability under composition: We construct classes $\mathcal{E}^{[M]}$ which are stable under composition and such that there is no log-convex $N = (N_k)$ with $\mathcal{E}^{[M]} = \mathcal{E}^{[N]}$, see Example 3.6.

Another common way to define ultradifferentiable classes is by means of a weight function $\omega$ which controls the decay of the Fourier transform, see [5] and [6]; we shall use the following equivalent description due to [9]: for compact $K$ the sets

$$\Big\{f^{(k)}(x)\exp(-\tfrac{1}{\rho}\varphi^*(\rho k)) : x \in K, k \in \mathbb{N}\Big\},$$

where $\varphi^*$ is the Young conjugate of $\varphi(t) = \omega(e^t)$, are required to be bounded either for all $\rho > 0$ in the Beurling case $\mathcal{E}^{(\omega)}$ or for some $\rho > 0$ in the Roumieu case $\mathcal{E}^{\{\omega\}}$. Again $\mathcal{E}^{[\omega]}$ stands for either $\mathcal{E}^{(\omega)}$ or $\mathcal{E}^{\{\omega\}}$. For these classes stability under composition was characterized in [16] under the additional assumption of non-quasianalyticity. Note that the sets $\{\mathcal{E}^{[M]} : M \text{ weight sequence}\}$ and $\{\mathcal{E}^{[\omega]} : \omega \text{ weight function}\}$ have a *large* intersection but neither of them contains the other, see [8]. We want to stress the fact that the usual requirements on the weight function $\omega$ ensure that the spaces $\mathcal{E}^{[\omega]}$ come with incorporated stability properties, for instance stability under derivation, see Corollary 5.15.

We prove that $\mathcal{E}^{(\omega)}$ and $\mathcal{E}^{\{\omega\}}$ can be represented (as locally convex spaces with their natural topologies) as intersections and unions of ultradifferentiable classes defined by means of associated weight sequences, see Theorem 5.14: For each open subset $U \subseteq \mathbb{R}^n$, compact $K \subseteq U$, and for $\Omega^\rho = (\Omega^\rho_k)$ defined by $\Omega^\rho_k := \frac{1}{k!}\exp(\frac{1}{\rho}\varphi^*(\rho k))$ we have

$$\mathcal{E}^{(\omega)}(U) = \bigcap_{\rho>0} \mathcal{E}^{(\Omega^\rho)}(U) \quad \text{and} \quad \mathcal{E}^{\{\omega\}}(U) = \bigcap_{K\subseteq U}\bigcup_{\rho>0} \mathcal{E}^{\{\Omega^\rho\}}(K). \tag{1.1}$$

We use this representation for characterizing stability under composition, and believe that it is also of independent interest.

In fact, inspired by (1.1), we characterize stability under composition for more general ultradifferentiable classes defined by weight matrices $\mathfrak{M} = \{M^\lambda \in \mathbb{R}^{\mathbb{N}}_{>0} : \lambda \in \Lambda\}$, where $\Lambda$ is an ordered subset of $\mathbb{R}$:

$$\mathcal{E}^{(\mathfrak{M})}(U) := \bigcap_{\lambda\in\Lambda} \mathcal{E}^{(M^\lambda)}(U) \quad \text{and} \quad \mathcal{E}^{\{\mathfrak{M}\}}(U) := \bigcap_{K\subseteq U}\bigcup_{\lambda\in\Lambda} \mathcal{E}^{\{M^\lambda\}}(K),$$

endowed with their natural topologies. Among the spaces $\mathcal{E}^{(\mathfrak{M})}$ and $\mathcal{E}^{\{\mathfrak{M}\}}$, commonly denoted by $\mathcal{E}^{[\mathfrak{M}]}$, are all the spaces defined by means of weight sequences and weight functions, but not exclusively, see Theorem 5.22. For instance, the intersection, resp. the union, of all non-quasianalytic Gevrey classes is an autonomous $\mathcal{E}^{(\mathfrak{M})}$-space, resp. $\mathcal{E}^{\{\mathfrak{M}\}}$-space, with suitable $\mathfrak{M}$. Intersections of non-quasianalytic ultradifferentiable classes have been studied by Rudin [35], Boman [7], Chaumat and Chollet [12], Beaugendre [3, 4], and Schmets and Valdivia [37, 38] (among others). It seems, however, that unions of ultradifferentiable classes have not been investigated before.

Given that $\mathcal{E}^{[\mathfrak{M}]}$ is stable under composition, the nonlinear composition operators

$$\operatorname{comp}^{(\mathfrak{M})} : \mathcal{E}^{(\mathfrak{M})}(\mathbb{R}^p,\mathbb{R}^q) \times \mathcal{E}^{(\mathfrak{M})}(\mathbb{R}^q,\mathbb{R}^r) \to \mathcal{E}^{(\mathfrak{M})}(\mathbb{R}^p,\mathbb{R}^r) : (g,f) \mapsto f\circ g$$

$$\mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R}^p,f) : \mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R}^p,\mathbb{R}^q) \to \mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R}^p,\mathbb{R}^r) : g \mapsto f\circ g, \quad f \in \mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R}^q,\mathbb{R}^r),$$

turn out to be continuous. This is proved in Theorem 4.13. The special case $\mathcal{E}^{[\omega]}$ was treated in [16], see also [1]. Under suitable assumptions we expect $\text{comp}^{[\mathfrak{M}]}$ to be of class $\mathcal{E}^{[\mathfrak{M}]}$, see Remark 4.23.

The paper is structured as follows: We first treat the weight sequence case in Section 2 and Section 3. In Section 4 we introduce ultradifferentiable classes defined by weight matrices $\mathfrak{M}$, characterize their stability under composition, and show that composition is continuous. We discuss classes defined by weight functions $\omega$ and identify them as classes defined by weight matrices $\mathfrak{M}$ in Section 5, and characterize their stability under composition in Section 6.

**Notation and conventions.** The notation $\mathcal{E}^{[*]}$ for $* \in \{M, \omega, \mathfrak{M}\}$ stands for either $\mathcal{E}^{(*)}$ or $\mathcal{E}^{\{*\}}$ with the following restriction: Statements that involve more than one $\mathcal{E}^{[*]}$ symbol must not be interpreted by mixing $\mathcal{E}^{(*)}$ and $\mathcal{E}^{\{*\}}$. This convention will be used broadly, but self-evidently: For example, $\mathfrak{M}[\preceq]\mathfrak{N} \Leftrightarrow \mathcal{E}^{[\mathfrak{M}]} \subseteq \mathcal{E}^{[\mathfrak{N}]}$ in Proposition 4.6 means $\mathfrak{M}(\preceq)\mathfrak{N} \Leftrightarrow \mathcal{E}^{(\mathfrak{M})} \subseteq \mathcal{E}^{(\mathfrak{N})}$ and $\mathfrak{M}\{\preceq\}\mathfrak{N} \Leftrightarrow \mathcal{E}^{\{\mathfrak{M}\}} \subseteq \mathcal{E}^{\{\mathfrak{N}\}}$.

Let $\mathbb{N} = \mathbb{N}_{>0} \cup \{0\}$. For $\alpha = (\alpha_1, \ldots, \alpha_q) \in \mathbb{N}^q$ and $x = (x_1, \ldots, x_q) \in \mathbb{R}^q$ we write $\alpha! = \alpha_1! \cdots \alpha_q!$, $|\alpha| = \alpha_1 + \cdots + \alpha_q$, and $x^\alpha = x_1^{\alpha_1} \cdots x_q^{\alpha_q}$. We use $\partial_i = \partial/\partial x_i$, $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_q^{\alpha_q}$ and write $d^k f$ or $f^{(k)}$ for the $k$th order Fréchet derivative of $f$, and $d_v f$ for the directional derivative in direction $v$. For sequences of reals $M = (M_k)$ and $N = (N_k)$ we write $M \le N$ if $M_k \le N_k$ for all $k$.

$L(E_1, \ldots, E_k; F)$ is the space of $k$-linear bounded mappings $E_1 \times \cdots \times E_k \to F$ (between topological vector spaces); if $E_i = E$ for all $i$, we also write $L^k(E, F)$.

Let $\mathcal{F}$ and $\mathcal{G}$ denote classes of mappings. We write $\mathcal{F} \subseteq \mathcal{G}$ if $\mathcal{F}(U, \mathbb{R}^m) \subseteq \mathcal{G}(U, \mathbb{R}^m)$ for all open subsets $U \subseteq \mathbb{R}^n$ and all $n, m \in \mathbb{N}_{>0}$. We say that $\mathcal{F}$ *is stable under composition* if $g \in \mathcal{F}(U, V)$ and $f \in \mathcal{F}(V, W)$ implies $f \circ g \in \mathcal{F}(U, W)$, for all open subsets $U \subseteq \mathbb{R}^p$, $V \subseteq \mathbb{R}^q$, $W \subseteq \mathbb{R}^r$, and all $p, q, r \in \mathbb{N}_{>0}$. A class $\mathcal{F}$ is called *holomorphically closed* if $f \circ g \in \mathcal{F}(U, \mathbb{C})$ for each $g \in \mathcal{F}(U) = \mathcal{F}(U, \mathbb{R})$ and each $f$ which is holomorphic in a complex neighborhood of the range of $g$, and $\mathcal{F}$ is *inverse closed* if $1/f \in \mathcal{F}(U)$ for each non-vanishing $f \in \mathcal{F}(U)$. That $\mathcal{F}$ *is derivation closed* means that $f \in \mathcal{F}(U)$ implies $\partial_i f \in \mathcal{F}(U)$ for all open $U \subseteq \mathbb{R}^n$, $n \in \mathbb{N}_{>0}$, and $1 \le i \le n$. A class $\mathcal{F}$ of smooth mappings is *quasianalytic* if for each open connected $U \subseteq \mathbb{R}^n$ and each $x \in U$ the Borel mapping $\mathcal{F}(U) \ni f \mapsto (\partial^\alpha f(x))_\alpha$ is injective.

## 2. Weight sequences and $[M]$-ultradifferentiable functions

2.1. **Weight sequences.** A sequence $M = (M_k) \in \mathbb{R}_{>0}^{\mathbb{N}}$ of positive real numbers is said to

($M_{\text{lc}}$) be *log-convex* if $k \mapsto \log M_k$ is convex, i.e., $\forall k : M_k^2 \le M_{k-1}\, M_{k+1}$;
($M_{\text{wlc}}$) be *weakly log-convex* if $(k!\, M_k)_k$ is log-convex;
($M_{\text{mg}}$) be of *moderate growth* if $\exists C > 0\ \forall j, k \ge 1 : M_{j+k} \le C^{j+k} M_j\, M_k$;
($M_{\text{dc}}$) be *derivation closed* if $\exists C > 0\ \forall k \ge 1 : M_{k+1} \le C^k M_k$;
($M_{\text{ai}}$) be *almost increasing* if $\exists C > 0\ \forall j \le k : M_j \le C M_k$;
($M_{\text{FdB}}$) have the *(FdB)-property* if $\exists C > 0\ \forall \alpha_i \in \mathbb{N}_{>0},\ \alpha_1 + \cdots + \alpha_j = k : M_j M_{\alpha_1} \cdots M_{\alpha_j} \le C^k M_k$;
($M_{\text{qa}}$) be *quasianalytic* if $\sum_{k=1}^\infty (k! M_k)^{-\frac{1}{k}} = \infty$.

Obviously ($M_{\text{lc}}$) implies ($M_{\text{wlc}}$) and ($M_{\text{mg}}$) implies ($M_{\text{dc}}$). If $M$ is log-convex, we further have $M_j M_k \le M_0 M_{j+k}$ for all $j, k$ and $(M_k/M_0)^{\frac{1}{k}}$ is increasing. Moreover:

2.2. **Lemma.** *For $M \in \mathbb{R}_{>0}^{\mathbb{N}}$ having the (FdB)-property, each of the following conditions is sufficient:*

(1) *$M$ is log-convex.*
(2) *$M$ is derivation closed and $(M_k^{\frac{1}{k}})_k$ is almost increasing.*
(3) *$M_j M_k \le M_1 M_{j+k-1}$ for all $j, k \ge 1$.*

**Proof.** (1) We show $(\text{M}_{\text{FdB}})$ with $C := \max\{M_1, 1\}$ by induction on $k$. The assertion is trivial for $k = j$. Assume that $j < k$. Then $\alpha_i' := \alpha_i - 1 \ge 1$ for some $i$, and we have

$$M_j M_{\alpha_1} \cdots M_{\alpha_j} = M_j M_{\alpha_1} \cdots M_{\alpha_i'} \cdots M_{\alpha_j} \frac{M_{\alpha_i}}{M_{\alpha_i'}} \le C^{k-1} M_{k-1} \frac{M_k}{M_{k-1}} \le C^k M_k,$$

by induction hypothesis and by $(\text{M}_{\text{lc}})$.

(2) This is proved in more general terms in 4.9[(3) ⇒ (4)] and 4.11[(3) ⇒ (4)].

(3) This is readily seen by iteration. □

For $M, N \in \mathbb{R}_{>0}^{\mathbb{N}}$ we define:

$$\begin{aligned} M \preceq N \quad &:\Leftrightarrow \quad \exists C, \rho > 0 \ \forall k : M_k \le C\rho^k N_k \quad \Leftrightarrow \quad \sup_k \Big(\frac{M_k}{N_k}\Big)^{\frac{1}{k}} < \infty \\ M \approx N \quad &:\Leftrightarrow \quad M \preceq N \text{ and } N \preceq M \\ M \lhd N \quad &:\Leftrightarrow \quad \forall \rho > 0 \ \exists C > 0 \ \forall k : M_k \le C\rho^k N_k \quad \Leftrightarrow \quad \lim_{k\to\infty} \Big(\frac{M_k}{N_k}\Big)^{\frac{1}{k}} = 0 \end{aligned}$$

The following lemma is a variant of [21, Lemma 6].

2.3. **Lemma.** *Let $L, M \in \mathbb{R}_{>0}^{\mathbb{N}}$ satisfy $L \lhd M$ and $M_k^{\frac{1}{k}} \to \infty$. Then there exist sequences $N^i \in \mathbb{R}_{>0}^{\mathbb{N}}$, $i = 1, 2$, satisfying $(N_k^i)^{\frac{1}{k}} \to \infty$ such that $L \le N^1 \lhd N^2 \lhd M$.*

**Proof.** It suffices to show that there exists $N^1 \in \mathbb{R}_{>0}^{\mathbb{N}}$ with $L \le N^1 \lhd M$ and $(N_k^1)^{\frac{1}{k}} \to \infty$; for $N^2 = (N_k^2)$ we may then choose $N_k^2 := \sqrt{N_k^1 M_k}$.

The sequence $N^1 = (N_k^1)$ defined by $N_k^1 := \max\{\sqrt{M_k}, L_k\}$ is as required: We have $L \le N^1 \lhd M$, since

$$\Big(\frac{N_k^1}{M_k}\Big)^{\frac{1}{k}} = \max\Big\{M_k^{-\frac{1}{2k}}, \Big(\frac{L_k}{M_k}\Big)^{\frac{1}{k}}\Big\} \to 0$$

as $M_k^{\frac{1}{k}} \to \infty$ and $L \lhd M$. Moreover, $N_k^1 \ge \sqrt{M_k}$ implies $(N_k^1)^{\frac{1}{k}} \to \infty$. □

2.4. **Remark.** The lemma remains true if we replace $M_k^{\frac{1}{k}} \to \infty$ by $(k!M_k)^{\frac{1}{k}} \to \infty$ and $(N_k^i)^{\frac{1}{k}} \to \infty$ by $(k!N_k^i)^{\frac{1}{k}} \to \infty$; set $N_k^1 := \max\{\sqrt{M_k/k!}, L_k\}$ in the above proof. But in this case it is unclear if $\underline{\lim}\, M_k^{\frac{1}{k}} > 0$ implies $\underline{\lim} (N_k^i)^{\frac{1}{k}} > 0$ which we need in Theorem 2.15.

2.5. **Regularizations.** Cf. [2], [27], or [22]. For $M \in \mathbb{R}_{>0}^{\mathbb{N}}$ with $(k!M_k)^{\frac{1}{k}} \to \infty$ set

$$T_M(t) := \sup_{k\in\mathbb{N}} \frac{t^k}{k!M_k}, \ t > 0, \quad \text{and} \quad M_k^{\flat(c)} := \frac{1}{k!} \sup_{t>0} \frac{t^k}{T_M(t)}.$$

Then $T_M = T_{M^{\flat(c)}}$. The sequence $(k!M_k^{\flat(c)})_k$ is the largest log-convex minorant of $(k!M_k)_k$; in particular, $M$ is weakly log-convex if and only if $M = M^{\flat(c)}$. The condition $(k!M_k)^{\frac{1}{k}} \to \infty$ guarantees that $M_k = M_k^{\flat(c)}$ for infinitely many $k$.

We shall also use

$$S_M(t) := \max_{k\le t} \frac{t^k}{k! M_k} \quad \text{and} \quad M_k^{\flat(o)} := \frac{1}{k!} \sup_{t\ge k} \frac{t^k}{S_M(t)},$$

and again have $S_M = S_{M^{\flat(o)}}$.

2.6. **Lemma.** *Let $M, N \in \mathbb{R}_{>0}^{\mathbb{N}}$ satisfy $(k!M_k)^{\frac{1}{k}} \to \infty$ and $(k!N_k)^{\frac{1}{k}} \to \infty$. Then $M \preceq N$ implies $M^{\flat(c)} \preceq N^{\flat(c)}$ and $M \lhd N$ implies $M^{\flat(c)} \lhd N^{\flat(c)}$.*

**Proof.** For $\rho > 0$ set $N^\rho = (N_k^\rho) := (\rho^k N_k)$. Easy computations show $T_{N^\rho}(t) = T_N(\frac{t}{\rho})$ and thus $(N^\rho)^{\flat(c)} = (N^{\flat(c)})^\rho$. Both assertions follow immediately. □

2.7. $[M]$**-ultradifferentiable functions.** Let $M \in \mathbb{R}_{>0}^{\mathbb{N}}$ and let $U \subseteq \mathbb{R}^n$ be open. Define

$$\mathcal{E}^{(M)}(U) := \Big\{ f \in C^\infty(U,\mathbb{R}) : \forall K \subseteq U \text{ compact } \forall \rho > 0 : \|f\|_{K,\rho}^M < \infty \Big\}$$
$$\mathcal{E}^{\{M\}}(U) := \Big\{ f \in C^\infty(U,\mathbb{R}) : \forall K \subseteq U \text{ compact } \exists \rho > 0 : \|f\|_{K,\rho}^M < \infty \Big\}$$
$$\|f\|_{K,\rho}^M := \sup \Big\{ \frac{\|f^{(k)}(x)\|_{L^k(\mathbb{R}^n,\mathbb{R})}}{k!\rho^k M_k} : x \in K, k \in \mathbb{N} \Big\}$$

and endow $\mathcal{E}^{(M)}(U)$ with its natural Fréchet space topology and $\mathcal{E}^{\{M\}}(U)$ with the projective limit topology over $K$ of the inductive limit topology over $\rho$; note that it suffices to take countable limits. We write $\mathcal{E}^{[M]}$ for either $\mathcal{E}^{(M)}$ or $\mathcal{E}^{\{M\}}$. The elements of $\mathcal{E}^{[M]}(U)$ are called $[M]$-*ultradifferentiable functions*; an $(M)/\{M\}$-ultradifferentiable function is said to be of Beurling/Roumieu type, respectively. For compact $K \subseteq U$ with smooth boundary,

$$\mathcal{E}_\rho^M(K) := \{ f \in C^\infty(K) : \|f\|_{K,\rho}^M < \infty \}$$

is a Banach space, and we have

$$\mathcal{E}^{(M)}(U) = \varprojlim_{K\subseteq U} \varprojlim_{m\in\mathbb{N}} \mathcal{E}_{\frac{1}{m}}^M(K) \quad \text{and} \quad \mathcal{E}^{\{M\}}(U) = \varprojlim_{K\subseteq U} \varinjlim_{m\in\mathbb{N}} \mathcal{E}_m^M(K);$$

we also set

$$\mathcal{E}^{(M)}(K) := \Big\{ f \in C^\infty(K) : \forall \rho > 0 : \|f\|_{K,\rho}^M < \infty \Big\} = \varprojlim_{m\in\mathbb{N}} \mathcal{E}_{\frac{1}{m}}^M(K)$$
$$\mathcal{E}^{\{M\}}(K) := \Big\{ f \in C^\infty(K) : \exists \rho > 0 : \|f\|_{K,\rho}^M < \infty \Big\} = \varinjlim_{m\in\mathbb{N}} \mathcal{E}_m^M(K).$$

The definitions work as well for mappings $f : U \to \mathbb{R}^m$, and so we shall use also $\mathcal{E}^{[M]}(U,V)$, $\mathcal{E}^{[M]}(K,V)$, and $\mathcal{E}_\rho^M(K,V)$, for open subsets $V \subseteq \mathbb{R}^m$.

By the Denjoy–Carleman theorem, $\mathcal{E}^{[M]}$ is quasianalytic if and only if $M^{\flat(c)}$ satisfies $(\mathrm{M_{qa}})$; this is in turn equivalent to

$$\sum_{k=0}^\infty \frac{M_k^{\flat(c)}}{(k+1)M_{k+1}^{\flat(c)}} = \infty \quad \text{and} \quad \int_1^\infty \frac{\log T_M(t)}{t^2} dt = \infty$$

For contemporary proofs see for instance [18, 1.3.8], [36, 19.11], and [20, 4.2].

2.8. **Examples.** For $s \in \mathbb{R}_{\ge 0}$ the sequence $G^s = (G^s_k) = ((k!)^s)$ is log-convex and has moderate growth; it is quasianalytic if and only if $s = 0$. The elements of $\mathcal{E}^{\{G^0\}}(U)$ are exactly the real analytic functions $C^\omega(U)$ and the elements of $\mathcal{E}^{(G^0)}(U)$ are exactly the restrictions of entire functions $\mathcal{H}(\mathbb{C}^n)$. The class $\mathcal{E}^{\{G^s\}}$ coincides with the Gevrey class $\mathcal{G}^{1+s}$.

2.9. **Lemma.** *Let $M \in \mathbb{R}^{\mathbb{N}}_{>0}$ be weakly log-convex. Then there exists a function $f \in \mathcal{E}^{\{M\}}_{\text{global}}(\mathbb{R}) := \{f \in C^\infty(\mathbb{R}) : \exists \rho > 0 : \|f\|^M_{\mathbb{R},\rho} < \infty\}$ such that $|f^{(k)}(0)| \ge k! M_k$ for all $k$.*

Such a function is called a *characteristic $\mathcal{E}^{\{M\}}$-function.*

**Proof.** The complex valued function

$$g(t) := \sum_{k=0}^\infty \frac{k! M_k}{(2\mu_k)^k} e^{2i\mu_k t}, \quad \text{where} \quad \mu_k := \frac{(k+1)M_{k+1}}{M_k}, \tag{2.10}$$

belongs to $\mathcal{E}^{\{M\}}_{\text{global}}(\mathbb{R},\mathbb{C})$ and satisfies

$$g^{(j)}(0) = i^j h_j, \quad \text{where} \quad h_j \ge j! M_j, \tag{2.11}$$

thus

$$|g^{(j)}(0)| \ge j! M_j,$$

for all $j$; see [40, Thm. 1]. Setting $f := \operatorname{Re} g + \operatorname{Im} g$ we obtain a real valued function with the required properties. □

2.12. **Proposition.** *Let $L, M, N \in \mathbb{R}^{\mathbb{N}}_{>0}$, let $U \subseteq \mathbb{R}^n$ be open, and let $K \subseteq U$ be compact. We have:*

(1) *$M \preceq N \Rightarrow \mathcal{E}^{[M]} \subseteq \mathcal{E}^{[N]}$ and $M \lhd N \Rightarrow \mathcal{E}^{\{M\}} \subseteq \mathcal{E}^{(N)}$ with continuous inclusions. If $M$ is weakly log-convex, then also the converse implications hold; more precisely, $\mathcal{E}^{[M]}(\mathbb{R}) \subseteq \mathcal{E}^{[N]}(\mathbb{R}) \Rightarrow M \preceq N$ and $\mathcal{E}^{\{M\}}(\mathbb{R}) \subseteq \mathcal{E}^{(N)}(\mathbb{R}) \Rightarrow M \lhd N$.*

(2) *We have*

$$\mathcal{E}^{\{M\}}(U,\mathbb{R}^m) = \bigcap_{M \lhd N} \mathcal{E}^{(N)}(U,\mathbb{R}^m) = \bigcap_{M \lhd N} \mathcal{E}^{\{N\}}(U,\mathbb{R}^m).$$

*If $M$ is (weakly) log-convex, then the intersections may be taken over all (weakly) log-convex $M \lhd N$.*

(3) *If $M_k^{\frac{1}{k}} \to \infty$ then*

$$\mathcal{E}^{(M)}(K,\mathbb{R}^m) = \bigcup_{\substack{L \lhd M \\ L_k^{\frac{1}{k}} \to \infty}} \mathcal{E}^{\{L\}}(K,\mathbb{R}^m) = \bigcup_{\substack{L \lhd M \\ L_k^{\frac{1}{k}} \to \infty}} \mathcal{E}^{(L)}(K,\mathbb{R}^m).$$

*If $(k! M_k)^{\frac{1}{k}} \to \infty$ then the unions may be taken over all $L \lhd M$ with $(k! L_k)^{\frac{1}{k}} \to \infty$. If $M$ is log-convex and $\frac{M_{k+1}}{M_k} \to \infty$ then the unions may be taken over all log-convex $L \lhd M$ with $\frac{L_{k+1}}{L_k} \to \infty$.*

**Proof.** (1) The directions "⇒" are clear by definition, see also [24, 2.3]. If $M$ is weakly log-convex, then the implications $\mathcal{E}^{\{M\}} \subseteq \mathcal{E}^{\{N\}} \Rightarrow M \preceq N$ and $\mathcal{E}^{\{M\}} \subseteq \mathcal{E}^{(N)} \Rightarrow M \lhd N$ follow from the existence of a characteristic $\mathcal{E}^{\{M\}}$-function, see Lemma 2.9. That $\mathcal{E}^{(M)} \subseteq \mathcal{E}^{(N)}$ implies $M \preceq N$ is shown in [10, Thm. 2.2] and in more general terms in Proposition 4.6.

(2) See [24, 2.4 and 8.2].

(3) follows from (1), Lemma 2.3, Remark 2.4, and [21, Lemma 6]. □

As the elements of $\mathcal{E}^{\{1\}}(U)$ are exactly the real analytic functions $C^\omega(U)$ and the elements of $\mathcal{E}^{(1)}(U)$ are exactly the restrictions of entire functions $\mathcal{H}(\mathbb{C}^n)$, we may conclude:

(4) $C^\omega \subseteq \mathcal{E}^{\{M\}} \Leftrightarrow \mathcal{H}(\mathbb{C}^n) \subseteq \mathcal{E}^{(M)}(U)\ \forall U \subseteq \mathbb{R}^n \Leftrightarrow \underline{\lim}\, M_k^{\frac{1}{k}} > 0$

(5) $C^\omega \subseteq \mathcal{E}^{(M)}$ if and only if $\lim M_k^{\frac{1}{k}} = \infty$.

(6) $\mathcal{E}^{[M]}$ is derivation closed if $M$ satisfies $(\mathrm{M_{dc}})$. If $M$ is weakly log-convex, then $(\mathrm{M_{dc}})$ is also necessary for $\mathcal{E}^{[M]}$ being derivation closed; indeed for $M^{+1} = (M_k^{+1}) := (M_{k+1})$ we have $\mathcal{E}^{[M^{+1}]}(\mathbb{R}) = \{f' : f \in \mathcal{E}^{[M]}(\mathbb{R})\}$.

In particular, if $L \lhd M$ with $\underline{\lim}\, L_k^{\frac{1}{k}} > 0$ then necessarily $\lim M_k^{\frac{1}{k}} = \infty$, by (1), (4), and (5).

Note that $\lim \frac{M_{k+1}}{M_k} = \infty$ implies $\lim M_k^{\frac{1}{k}} = \infty$ and thus $C^\omega \subseteq \mathcal{E}^{(M)}$. Indeed, there exists $k_0$ with $M_{k_0} \ge 1$, and for every $C > 0$ there exists $k_1 \ge k_0$ so that $M_k \ge C M_{k-1}$ for all $k > k_1$, whence $M_k^{\frac{1}{k}} \ge M_{k_0}^{\frac{1}{k}} C^{1-\frac{k_0}{k}} \ge C^{\frac{1}{2}}$ as $k > 2k_1$. If $M_k^{\frac{1}{k}}$ is increasing, we have also the converse: $\lim M_k^{\frac{1}{k}} = \infty$ implies $\lim \frac{M_{k+1}}{M_k} = \infty$.

2.13. **Lemma** ([11, Lemme 3]). *Let $M \in \mathbb{R}_{>0}^{\mathbb{N}}$ and $\lambda > 0$. If $M_0 \le \lambda^k M_k$ for all $k$ and*

$$|f^{(k)}(t)| \le k! M_k \quad \textit{for all} \quad t \in [-\lambda, \lambda], k \in \mathbb{N},$$

*then*

$$|f^{(k)}(0)| \le 2 e^k k! M_k^{\flat(c)} \quad \textit{for all} \quad k \in \mathbb{N}.$$

2.14. **Proposition.** *Let $M \in \mathbb{R}_{>0}^{\mathbb{N}}$ satisfy $\underline{\lim}\, M_k^{\frac{1}{k}} > 0$ and $M_0 = 1$, let $K \subseteq \mathbb{R}^n$ be compact, and let $K_\lambda := \bigcup_{x \in K} \overline{B_\lambda(x)}$, $\lambda > 0$, be a $\lambda$-neighborhood of $K$. Then we have $\mathcal{E}^{\{M\}}(K_\lambda) \subseteq \mathcal{E}^{\{M^{\flat(c)}\}}(K)$ via restriction.*

**Proof.** By the assumption $\underline{\lim}\, M_k^{\frac{1}{k}} > 0$ there exists $\tau > 0$ so that $M_k \ge \tau^k$ for all $k$. If $f \in \mathcal{E}^{\{M\}}(K_\lambda)$, then $C := \|f\|^M_{K_\lambda,\rho} < \infty$, where we may assume that $\rho$ is such that $\rho\lambda\tau \ge 1$. The function $f_{x,v}(t) := f(x + tv)$ satisfies $\|f_{x,v}\|^M_{[-\lambda,\lambda],\rho} \le \|f\|^M_{K_\lambda,\rho} = C$ for all $x \in K$ and $v \in S^{n-1}$. By Lemma 2.13, we have

$$|d_v^k f(x)| = |f_{x,v}^{(k)}(0)| \le 2C(e\rho)^k k! M_k^{\flat(c)} \quad \text{for all} \quad x \in K, v \in S^{n-1}, k \in \mathbb{N},$$

since $(C\rho^k M_k)^{\flat(c)} = C\rho^k M_k^{\flat(c)}$ (see 2.6). Thus $f|_K \in \mathcal{E}^{\{M^{\flat(c)}\}}(K)$, see e.g. [23, 7.13.1]. □

2.15. **Theorem.** *Let $M \in \mathbb{R}_{>0}^{\mathbb{N}}$ and let $U \subseteq \mathbb{R}^n$ be open. We have:*

(1) *If $\underline{\lim}\, M_k^{\frac{1}{k}} > 0$ then $\mathcal{E}^{\{M\}}(U) = \mathcal{E}^{\{M^{\flat(c)}\}}(U)$.*

(2) *If $\lim M_k^{\frac{1}{k}} = \infty$ then $\mathcal{E}^{(M)}(U) = \mathcal{E}^{(M^{\flat(c)})}(U)$.*

*Under these assumptions $\mathcal{E}^{[M]}(U)$ is an algebra.*

(1) is due to [11, Thm. I & Appendix].

**Proof.** (1) Apply Proposition 2.12(1) and Proposition 2.14.

(2) Proposition 2.12(1) implies $\mathcal{E}^{(M^{\flat(c)})}(U) \subseteq \mathcal{E}^{(M)}(U)$. Conversely, let $K \subseteq U$ be compact and let $K_\lambda := \bigcup_{x \in K} \overline{B_\lambda(x)} \subseteq U$ be a $\lambda$-neighborhood of $K$ in $U$. By Proposition 2.12(3), Proposition 2.14, and Lemma 2.6,

$$\mathcal{E}^{(M)}(K_\lambda) = \bigcup \mathcal{E}^{\{L\}}(K_\lambda) \subseteq \bigcup \mathcal{E}^{\{L^{\flat(c)}\}}(K) \subseteq \mathcal{E}^{(M^{\flat(c)})}(K),$$

where the unions are taken over all $L \lhd M$ with $L_k^{\frac{1}{k}} \to \infty$. As $K$ was arbitrary, we have $\mathcal{E}^{(M)}(U) \subseteq \mathcal{E}^{(M^{\flat(c)})}(U)$.

The supplement is a well-known consequence of weak log-convexity. □

As a consequence $C^\omega \subseteq \mathcal{E}^{\{M\}} = \mathcal{E}^{(N)}$ is impossible. Assume the contrary. Then, by 2.12(4)&(5) and Theorem 2.15, we may assume that $M$ and $N$ are weakly log-convex, and by Proposition 2.12(1), we have $M \lhd N$. Setting $L = (L_k)$ with $L_k := \sqrt{M_k N_k}$ we obtain $M \lhd L \lhd N$, and, by Lemma 2.6, we may assume that $L$ is weakly log-convex. But then $\mathcal{E}^{\{M\}} \subseteq \mathcal{E}^{(L)} \subseteq \mathcal{E}^{\{L\}} \subseteq \mathcal{E}^{(N)} = \mathcal{E}^{\{M\}}$ and thus $M \approx L \approx N$, a contradiction.

## 3. Stability under composition of $\mathcal{E}^{[M]}$

For $M \in \mathbb{R}_{>0}^{\mathbb{N}}$ we define $M^\circ = (M_k^\circ)$ by setting

$$M_k^\circ := \max\{M_j M_{\alpha_1} \dots M_{\alpha_j} : \alpha_i \in \mathbb{N}_{>0}, \alpha_1 + \dots + \alpha_j = k\}, \quad M_0^\circ := 1.$$

Clearly, $M \preceq M^\circ$. We have $M^\circ \preceq M$ if and only if $M$ has the (FdB)-property.

3.1. **Proposition.** *Let $M \in \mathbb{R}_{>0}^{\mathbb{N}}$ and let $U \subseteq \mathbb{R}^p$, $V \subseteq \mathbb{R}^q$, and $W \subseteq \mathbb{R}^r$ be open.*

(1) *If $g \in \mathcal{E}^{[M]}(U,V)$ and $f \in \mathcal{E}^{[M]}(V,W)$, then $f \circ g \in \mathcal{E}^{[M^\circ]}(U,W)$*

(2) *If $M$ has the (FdB)-property, then $\mathcal{E}^{[M]}$ is stable under composition.*

**Proof.** (1) Let $K \subseteq U$ be compact. There exist $C_g, \rho_g > 0$ (resp. for each $\rho_g > 0$ there exists $C_g > 0$) such that

$$\frac{\|g^{(k)}(x)\|_{L^k(\mathbb{R}^p,\mathbb{R}^q)}}{k!} \le C_g \rho_g^k M_k \quad \text{for all } x \in K, k \in \mathbb{N},$$

and there exist $C_f, \rho_f > 0$ (resp. for each $\rho_f > 0$ there exists $C_f > 0$) such that

$$\frac{\|f^{(k)}(y)\|_{L^k(\mathbb{R}^q,\mathbb{R}^r)}}{k!} \le C_f \rho_f^k M_k \quad \text{for all } y \in g(K), k \in \mathbb{N}.$$

By Faà di Bruno's formula ([15] for the 1-dimensional version; the second sum is over all $\alpha \in \mathbb{N}_{>0}^j$ with $\alpha_1 + \dots + \alpha_j = k$)

$$\frac{\|(f \circ g)^{(k)}(x)\|_{L^k(\mathbb{R}^p,\mathbb{R}^q)}}{k!} \le \sum_{j \ge 1} \sum_{\alpha} \frac{\|f^{(j)}(g(x))\|_{L^j(\mathbb{R}^q,\mathbb{R}^r)}}{j!} \prod_{i=1}^{j} \frac{\|g^{(\alpha_i)}(x)\|_{L^{\alpha_i}(\mathbb{R}^p,\mathbb{R}^q)}}{\alpha_i!}$$

$$\le \sum_{j \ge 1} \sum_{\alpha} C_f \rho_f^j C_g^j \rho_g^k M_j \prod_{i=1}^{j} M_{\alpha_i} \le C_f \rho_g^k \Big( \sum_{j \ge 1} \binom{k-1}{j-1} (\rho_f C_g)^j \Big) M_k^\circ$$

$$\le C_f C_g \rho_f (\rho_g (1 + \rho_f C_g))^k M_k^\circ.$$

This implies the assertion in the Roumieu case. For the Beurling case, let $\tau > 0$ be arbitrary, and choose $\sigma > 0$ such that $\tau = \sqrt{\sigma} + \sigma$. If we set $\rho_g = \sqrt{\sigma}$ and $\rho_f = \sqrt{\sigma}/C_g$, then $\|f \circ g\|_{K,\tau}^{M^\circ} < \infty$.

(2) follows immediately from (1) and Proposition 2.12(1). □

We get a nice characterization of stability under composition if we assume that $\mathcal{E}^{[M]}$ is stable under derivation.

3.2. **Theorem.** *Let $M \in \mathbb{R}_{>0}^{\mathbb{N}}$ and assume that $\mathcal{E}^{[M]}$ is stable under derivation. Consider the following conditions:*

(1) *$\mathcal{E}^{[M]}$ is stable under composition.*
(2) *$\mathcal{E}^{[M]}$ is holomorphically closed.*
(3) *$\mathcal{E}^{[M]}$ is inverse closed.*
(4) *$(M_k^{\flat(c)})^{\frac{1}{k}}$ is almost increasing.*
(5) *$(M_k^{\flat(o)})^{\frac{1}{k}}$ is almost increasing.*
(6) *$M^{\flat(c)}$ has the (FdB)-property.*
(7) *$M^{\flat(o)}$ has the (FdB)-property.*

*If $\underline{\lim} M_k^{\frac{1}{k}} > 0$ then all conditions are equivalent in the Roumieu case $\mathcal{E}^{[M]} = \mathcal{E}^{\{M\}}$. If $\lim M_k^{\frac{1}{k}} = \infty$ then all conditions are equivalent in any case.*

**Proof.** Under the assumption $\underline{\lim} M_k^{\frac{1}{k}} > 0$ we have $\mathcal{E}^{\{M\}} = \mathcal{E}^{\{M^{\flat(c)}\}}$, by Theorem 2.15. The equivalences (4) $\Leftrightarrow$ (5) and and (6) $\Leftrightarrow$ (7) follow from the fact that $\mathcal{E}^{\{M\}}(I) = \mathcal{E}^{\{M^{\flat(o)}\}}(I)$ for open intervals $I$, see [27, 6.5.1], which implies $M^{\flat(c)} \approx M^{\flat(o)}$, by [39, Lemma II]. Lemma 2.2 and 2.12(6) imply (4) $\Rightarrow$ (6).

Let us prove the remaining implications in the Roumieu case $\mathcal{E}^{[M]} = \mathcal{E}^{\{M\}}$: Since $C^\omega \subseteq \mathcal{E}^{\{M\}}$ by 2.12(4), we clearly have (1) $\Rightarrow$ (2) $\Rightarrow$ (3). The implication (3) $\Rightarrow$ (5) follows from [39], and (6) $\Rightarrow$ (1) follows from Proposition 3.1. Note that (3) $\Rightarrow$ (4) is shown in greater generality in the proof of Theorem 4.9 below.

Now let us assume the stronger condition $\lim M_k^{\frac{1}{k}} = \infty$ and show the remaining implications in the Beurling case $\mathcal{E}^{[M]} = \mathcal{E}^{(M)}$: Since $C^\omega \subseteq \mathcal{E}^{(M)}$ by 2.12(5), we have (1) $\Rightarrow$ (2) $\Rightarrow$ (3). The implication (3) $\Rightarrow$ (4) follows from [10] since $\mathcal{E}^{(M)}(\mathbb{R}) = \mathcal{E}^{(M^{\flat(c)})}(\mathbb{R})$ is a Fréchet algebra, by Theorem 2.15, and (6) $\Rightarrow$ (1) follows from Proposition 3.1. □

3.3. **Log-convexity is not necessary for stability under composition.** There exist classes $\mathcal{E}^{[M]}$ (containing $C^\omega$) which are closed under composition and there is no log-convex $N \in \mathbb{R}_{>0}^{\mathbb{N}}$ such that $\mathcal{E}^{[M]} = \mathcal{E}^{[N]}$. We need the following lemma.

3.4. **Lemma.** *Let $M \in \mathbb{R}_{>0}^{\mathbb{N}}$ be such that $C^\omega \subseteq \mathcal{E}^{[M]}$ (i.e., $\underline{\lim} M_k^{\frac{1}{k}} > 0$ in the Roumieu case and $\lim M_k^{\frac{1}{k}} = \infty$ in the Beurling case). If there exists a log-convex $N \in \mathbb{R}_{>0}^{\mathbb{N}}$ such that $\mathcal{E}^{[M]} = \mathcal{E}^{[N]}$, then the sequence $k_{i+1}/k_i$ is bounded, where the $k_i$ are precisely those $k$ with $M_k = M_k^{\flat(c)}$.*

**Proof.** This is a special case of [11, Appendix Prop. 3]. For the reader's convenience we give a short proof. By Theorem 2.15, we have $\mathcal{E}^{[M^{\flat(c)}]} = \mathcal{E}^{[N]}$ and thus $M^{\flat(c)} \approx N$, by Proposition 2.12(1). Since $N$ is weakly log-convex, we have $N \le M^{\flat(c)} \le M$. Set

$$L := \begin{pmatrix} N_k & k = k_i \\ +\infty & \text{otherwise} \end{pmatrix}^{\flat(c)}.$$

For $M \in \mathbb{R}_{>0}^{\mathbb{N}}$ consider the graph $\Gamma_M := \{(k, \log(k! M_k)) : k \in \mathbb{N}\}$. Then $\Gamma_{M^{\flat(c)}}$ and $\Gamma_L$ lie on piecewise linear curves with vertices $\{(k_i, \log(k_i! M_{k_i})) : i \in \mathbb{N}\}$ and $\{(k_i, \log(k_i! N_{k_i})) : i \in \mathbb{N}\}$, respectively. Since $N$ is weakly log-convex and since

$\Gamma_L$ lies below $\Gamma_{M^{\flat(c)}}$, we have $N \le L \le M^{\flat(c)} \approx N$ and hence $L \approx N$. As $N$ is log-convex, we have, for $k_i \le k \le k_{i+1}$,

$$\begin{aligned}\log(k!L_k) &= \tfrac{k_{i+1}-k}{k_{i+1}-k_i} \log(k_i!N_{k_i}) + \tfrac{k-k_i}{k_{i+1}-k_i} \log(k_{i+1}!N_{k_{i+1}}) \\ &\ge \tfrac{k_{i+1}-k}{k_{i+1}-k_i} \log k_i! + \tfrac{k-k_i}{k_{i+1}-k_i} \log k_{i+1}! + \log N_k,\end{aligned}$$

and therefore

$$\log\Big(\frac{k!L_k}{k!N_k}\Big)^{\frac{1}{k}} \ge \tfrac{1}{k}\tfrac{k_{i+1}-k}{k_{i+1}-k_i} \log k_i! + \tfrac{1}{k}\tfrac{k-k_i}{k_{i+1}-k_i} \log k_{i+1}! - \tfrac{1}{k}\log k!. \tag{3.5}$$

By Stirling's formula, for $k_{i+1}/k_i =: a_i$ and $k := 2k_i$ the right-hand side of (3.5) is greater than

$$\tfrac{1}{2}\tfrac{a_i-2}{a_i-1}(\log k_i - 1) + \tfrac{1}{2}\tfrac{a_i}{a_i-1}(\log a_i + \log k_i - 1) - \log(2k_i) = \tfrac{1}{2}\tfrac{a_i}{a_i-1}\log a_i - \log 2 - 1,$$

and so $L \approx N$ implies that $a_i$ is bounded. □

3.6. **Example.** Choose $r \in \mathbb{R}_{\ge 4}$. Set $k_i := k_{i-1}\lceil \log(i+1)\rceil$, $i \ge 2$, $k_1 := 3$, where $\lceil x \rceil$ denotes the smallest integer $n \ge x$, and define

$$\mu_k = \mu(r)_k := \begin{cases} 1 & k = 1,2 \\ r^k & k = k_i \\ r^{k_i - 1} & k_i < k < k_{i+1} \end{cases}, \qquad M_k = M(r)_k := \frac{1}{k!}\prod_{j=1}^k \mu_j.$$

Then $M = (M_k)$ is derivation closed, since $\frac{\mu_k}{k} \le r^k$ for all $k$, and $M$ is not weakly log-convex, since $\mu = (\mu_k)$ is not increasing. By construction we have $M_j M_k \le M_1 M_{j+k-1}$ for all $j,k \ge 1$, i.e.,

$$\frac{\mu_1}{1}\cdots\frac{\mu_k}{k} \le \frac{\mu_1}{1}\frac{\mu_{j+1}}{j+1}\cdots\frac{\mu_{j+k-1}}{j+k-1}, \quad j,k \ge 1.$$

Indeed, since $\frac{\mu_k}{k}$ is decreasing for $k_i \le k < k_{i+1}$ and since $\frac{\mu_{k_i+1}}{k_i+1} \le \frac{\mu_{k_{i+2}-1}}{k_{i+2}-1}$ for all $i$, it suffices to check that, for all $i$,

$$\frac{\mu_{k_{i+1}-1}}{k_{i+1}-1}\frac{\mu_{k_{i+1}}}{k_{i+1}} \le \frac{\mu_{k_{i+2}-2}}{k_{i+2}-2}\frac{\mu_{k_{i+2}-1}}{k_{i+2}-1}$$

which is a straightforward computation. By Lemma 2.2(3) and Proposition 3.1, $\mathcal{E}^{[M]}$ is stable under composition.

Consider the graph $\Gamma_M := \{P_k := (k, \log(k!M_k)) : k \in \mathbb{N}\}$. The subset $\{P_k : k_i \le k < k_{i+1}\}$ lies on an affine line with slope $(k_i - 1)\log r$. The line that connects the two points $P_{k_i-1}$ and $P_{k_i}$ has slope $k_i \log r$, and the line that connects the two points $P_{k_i-1}$ and $P_{k_{i+1}-1}$ has slope $(k_i - 1 + (k_{i+1}-k_i)^{-1})\log r$. All these slopes are strictly increasing to infinity in $i$. We may conclude that the graph $\Gamma_{M^{\flat(c)}} := \{(k, \log(k!M_k^{\flat(c)})) : k \in \mathbb{N}\}$ lies on the piecewise linear curve with vertices $\{P_{k_i-1} : i \in \mathbb{N}\}$ and that $\{k_i - 1\}$ is precisely the set of $k$ with $M_k = M_k^{\flat(c)}$.

As $\frac{M_k}{M_{k-1}} = \frac{\mu_k}{k} \to \infty$ we have $M_k^{\frac{1}{k}} \to \infty$ (see the remarks after 2.12), and, by Lemma 3.4, there is no log-convex $N \in \mathbb{R}_{>0}^{\mathbb{N}}$ such that $\mathcal{E}^{[M]} = \mathcal{E}^{[N]}$. It is easy to see that the mapping $r \mapsto \mathcal{E}^{[M(r)]}$ is injective.

## 4. More general spaces of ultradifferentiable functions

4.1. **Weight matrices.** A *weight matrix* $\mathfrak{M} = \{M^\lambda \in \mathbb{R}^{\mathbb{N}}_{>0} : \lambda \in \Lambda\}$ is a family of weakly log-convex sequences $M^\lambda = (M^\lambda_k)$ satisfying $M^\lambda_0 = 1$, $\lim_k (k! M^\lambda_k)^{\frac{1}{k}} = \infty$, and $M^\lambda \le M^\mu$ if $\lambda \le \mu$, where $\Lambda$ is a directed partially ordered set. Let $\mathscr{M} = \mathscr{M}(\Lambda)$ be the set of all weight matrices $\mathfrak{M}$ parameterized by the same set $\Lambda$. Consider the following conditions:

$(\mathfrak{M}_{\mathcal{H}})$ $\quad \forall \lambda \in \Lambda : \underline{\lim} (M^\lambda_k)^{\frac{1}{k}} > 0.$
$(\mathfrak{M}_{(C^\omega)})$ $\quad \forall \lambda \in \Lambda : \lim (M^\lambda_k)^{\frac{1}{k}} = \infty.$
$(\mathfrak{M}_{\{C^\omega\}})$ $\quad \exists \lambda \in \Lambda : \underline{\lim} (M^\lambda_k)^{\frac{1}{k}} > 0.$
$(\mathfrak{M}_{(\mathrm{dc})})$ $\quad \forall \lambda \in \Lambda\ \exists \mu \in \Lambda\ \exists C > 0\ \forall k \in \mathbb{N} : M^\mu_{k+1} \le C^k M^\lambda_k.$
$(\mathfrak{M}_{\{\mathrm{dc}\}})$ $\quad \forall \lambda \in \Lambda\ \exists \mu \in \Lambda\ \exists C > 0\ \forall k \in \mathbb{N} : M^\lambda_{k+1} \le C^k M^\mu_k.$
$(\mathfrak{M}_{(\mathrm{mg})})$ $\quad \forall \lambda \in \Lambda\ \exists \mu \in \Lambda\ \exists C > 0\ \forall j, k \in \mathbb{N} : M^\mu_{j+k} \le C^{j+k} M^\lambda_j M^\lambda_k.$
$(\mathfrak{M}_{\{\mathrm{mg}\}})$ $\quad \forall \lambda \in \Lambda\ \exists \mu \in \Lambda\ \exists C > 0\ \forall j, k \in \mathbb{N} : M^\lambda_{j+k} \le C^{j+k} M^\mu_j M^\mu_k.$
$(\mathfrak{M}_{(\mathrm{alg})})$ $\quad \forall \lambda \in \Lambda\ \exists \mu \in \Lambda\ \exists C > 0\ \forall j, k \in \mathbb{N} : M^\mu_j M^\mu_k \le C^{j+k} M^\lambda_{j+k}.$
$(\mathfrak{M}_{\{\mathrm{alg}\}})$ $\quad \forall \lambda \in \Lambda\ \exists \mu \in \Lambda\ \exists C > 0\ \forall j, k \in \mathbb{N} : M^\lambda_j M^\lambda_k \le C^{j+k} M^\mu_{j+k}.$
$(\mathfrak{M}_{(\mathrm{FdB})})$ $\quad \forall \lambda \in \Lambda\ \exists \mu \in \Lambda : (M^\mu)^\circ \preceq M^\lambda.$
$(\mathfrak{M}_{\{\mathrm{FdB}\}})$ $\quad \forall \lambda \in \Lambda\ \exists \mu \in \Lambda : (M^\lambda)^\circ \preceq M^\mu.$
$(\mathfrak{M}_{(\mathrm{L})})$ $\quad \forall \lambda \in \Lambda\ \forall \rho > 0\ \exists \mu \in \Lambda\ \exists C > 0\ \forall k \in \mathbb{N} : \rho^k M^\mu_k \le C M^\lambda_k.$
$(\mathfrak{M}_{\{\mathrm{L}\}})$ $\quad \forall \lambda \in \Lambda\ \forall \rho > 0\ \exists \mu \in \Lambda\ \exists C > 0\ \forall k \in \mathbb{N} : \rho^k M^\lambda_k \le C M^\mu_k.$
$(\mathfrak{M}_{(\mathrm{BR})})$ $\quad \forall \lambda \in \Lambda\ \exists \mu \in \Lambda : M^\mu \lhd M^\lambda.$
$(\mathfrak{M}_{\{\mathrm{BR}\}})$ $\quad \forall \lambda \in \Lambda\ \exists \mu \in \Lambda : M^\lambda \lhd M^\mu.$

Obviously, $(\mathfrak{M}_{(C^\omega)}) \Rightarrow (\mathfrak{M}_{\mathcal{H}}) \Rightarrow (\mathfrak{M}_{\{C^\omega\}})$ and $(\mathfrak{M}_{[\mathrm{mg}]}) \Rightarrow (\mathfrak{M}_{[\mathrm{dc}]})$. Both conditions $(\mathfrak{M}_{(\mathrm{alg})})$ and $(\mathfrak{M}_{\{\mathrm{alg}\}})$ are trivially satisfied since all $M^\lambda$ are weakly log-convex, but see Remarks 4.5.

Henceforth we assume that $\Lambda$ is $\mathbb{R}$ or any ordered subset of $\mathbb{R}$. This will enable us to assume that the limits over $\lambda \in \Lambda$ in the definition of $[\mathfrak{M}]$-ultradifferentiable functions in 4.2 are countable. Then $\mathfrak{M}$ is in fact an infinite matrix, and the name *weight matrix* is justified. On the other hand it is convenient to admit uncountable index sets $\Lambda$.

4.2. **$[\mathfrak{M}]$-ultradifferentiable functions.** Let $\mathfrak{M}$ be a weight matrix, let $U \subseteq \mathbb{R}^n$ be open, and let $K \subseteq U$ be compact. We define

$$\mathcal{E}^{(\mathfrak{M})}(K) := \bigcap_{\lambda \in \Lambda} \mathcal{E}^{(M^\lambda)}(K) \quad \text{and} \quad \mathcal{E}^{\{\mathfrak{M}\}}(K) := \bigcup_{\lambda \in \Lambda} \mathcal{E}^{\{M^\lambda\}}(K),$$
$$\mathcal{E}^{(\mathfrak{M})}(U) := \bigcap_{\lambda \in \Lambda} \mathcal{E}^{(M^\lambda)}(U) \quad \text{and} \quad \mathcal{E}^{\{\mathfrak{M}\}}(U) := \bigcap_{K \subseteq U} \bigcup_{\lambda \in \Lambda} \mathcal{E}^{\{M^\lambda\}}(K),$$

and endow these spaces with their natural topologies:

$$\mathcal{E}^{(\mathfrak{M})}(U) := \varprojlim_{\lambda \in \Lambda} \mathcal{E}^{(M^\lambda)}(U) \quad \text{and} \quad \mathcal{E}^{\{\mathfrak{M}\}}(U) := \varprojlim_{K \subseteq U} \varinjlim_{\lambda \in \Lambda} \mathcal{E}^{\{M^\lambda\}}(K).$$

It is no loss of generality to assume that the limits are countable. We write $\mathcal{E}^{[\mathfrak{M}]}$ for either $\mathcal{E}^{(\mathfrak{M})}$ or $\mathcal{E}^{\{\mathfrak{M}\}}$. The elements of $\mathcal{E}^{[\mathfrak{M}]}(U)$ are called $[\mathfrak{M}]$-*ultradifferentiable functions*. Note that $\mathcal{E}^{[\mathfrak{M}]}(U)$ forms an algebra, since all $M^\lambda$ are weakly log-convex.

We shall use also $\mathcal{E}^{[\mathfrak{M}]}(U, V)$ and $\mathcal{E}^{[\mathfrak{M}]}(K, V)$, for open subsets $V \subseteq \mathbb{R}^m$.

The inductive limit

$$\mathcal{E}^{\{\mathfrak{M}\}}(K,\mathbb{R}^m) = \varinjlim_{\lambda\in\Lambda} \varinjlim_{\rho>0} \mathcal{E}^{M^\lambda}_\rho(K,\mathbb{R}^m) = \varinjlim_{(\lambda,\rho)} \mathcal{E}^{M^\lambda}_\rho(K,\mathbb{R}^m),$$

where $(\lambda,\rho)\le(\mu,\sigma)$ if and only if $\lambda\le\mu$ and $\rho\le\sigma$, is a Silva space. Indeed, if $\lambda\le\mu$ and $\rho<\sigma$ then the inclusion

$$\mathcal{E}^{M^\lambda}_\rho(K,\mathbb{R}^m) \longrightarrow \mathcal{E}^{M^\mu}_\rho(K,\mathbb{R}^m) \longrightarrow \mathcal{E}^{M^\mu}_\sigma(K,\mathbb{R}^m)$$

is compact, since the first inclusion is bounded and the second inclusion is compact, by [20, Prop. 2.2].

If $\mathfrak{M}$ satisfies $(\mathfrak{M}_{(\mathrm{L})})$, respectively $(\mathfrak{M}_{\{\mathrm{L}\}})$, we have

$$\begin{aligned} \mathcal{E}^{(\mathfrak{M})}(K,\mathbb{R}^m) &= \varprojlim_{(\lambda,\rho)} \mathcal{E}^{M^\lambda}_\rho(K,\mathbb{R}^m) = \varprojlim_{\lambda} \mathcal{E}^{M^\lambda}_1(K,\mathbb{R}^m), \quad \text{respectively} \\ \mathcal{E}^{\{\mathfrak{M}\}}(K,\mathbb{R}^m) &= \varinjlim_{(\lambda,\rho)} \mathcal{E}^{M^\lambda}_\rho(K,\mathbb{R}^m) = \varinjlim_{\lambda} \mathcal{E}^{M^\lambda}_1(K,\mathbb{R}^m) \end{aligned} \tag{4.3}$$

as locally convex spaces, where the latter is a Silva space. Indeed, for $1<\rho$ and by $(\mathfrak{M}_{\{\mathrm{L}\}})$ the inclusion

$$\mathcal{E}^{M^\lambda}_1(K,\mathbb{R}^m) \longrightarrow \mathcal{E}^{M^\lambda}_\rho(K,\mathbb{R}^m) \longrightarrow \mathcal{E}^{M^\mu}_1(K,\mathbb{R}^m)$$

is compact. If $(\mathfrak{M}_{(\mathrm{L})})$ then for each $\lambda\in\Lambda$ and each $\rho>0$ we find $\mu\in\Lambda$ such that $\mathcal{E}^{M^\mu}_1(K,\mathbb{R}^m)\subseteq\mathcal{E}^{M^\lambda}_\rho(K,\mathbb{R}^m)$ with continuous inclusion.

If $\mathfrak{M}$ satisfies $(\mathfrak{M}_{(\mathrm{BR})})$, respectively $(\mathfrak{M}_{\{\mathrm{BR}\}})$, we have

$$\begin{aligned} \mathcal{E}^{(\mathfrak{M})}(U,\mathbb{R}^m) &= \varprojlim_{\lambda\in\Lambda} \mathcal{E}^{(M^\lambda)}(U,\mathbb{R}^m) = \varprojlim_{\lambda\in\Lambda} \mathcal{E}^{\{M^\lambda\}}(U,\mathbb{R}^m), \quad \text{respectively} \\ \mathcal{E}^{\{\mathfrak{M}\}}(K,\mathbb{R}^m) &= \varinjlim_{\lambda\in\Lambda} \mathcal{E}^{\{M^\lambda\}}(K,\mathbb{R}^m) = \varinjlim_{\lambda\in\Lambda} \mathcal{E}^{(M^\lambda)}(K,\mathbb{R}^m) \end{aligned} \tag{4.4}$$

as locally convex spaces.

Among the spaces $\mathcal{E}^{[\mathfrak{M}]}$ we recover the spaces $\mathcal{E}^{[M]}$ defined by weight sequences, if $\mathfrak{M}=\{M\}$ consists just of a single $M\in\mathbb{R}^{\mathbb{N}}_{>0}$, and the spaces $\mathcal{E}^{[\omega]}$ defined by weight functions, see Corollary 5.15 below. We shall see in Theorem 5.22 that in general $\mathcal{E}^{[\mathfrak{M}]}$ is different from $\mathcal{E}^{[M]}$ and from $\mathcal{E}^{[\omega]}$.

4.5. **Remarks.** (1) One can replace the condition that the $M^\lambda\in\mathfrak{M}$ are weakly log-convex by the condition $(\mathfrak{M}_{\mathcal{H}})$ (resp. $(\mathfrak{M}_{(C^\omega)})$), and work with the log-convex minorants $(M^\lambda)^{\flat(c)}$ without changing the space $\mathcal{E}^{\{\mathfrak{M}\}}(U)$ (resp. $\mathcal{E}^{(\mathfrak{M})}(U)$), see Proposition 2.14 and Theorem 2.15. Alternatively, assuming $(\mathfrak{M}_{[\mathrm{alg}]})$ makes $\mathcal{E}^{[\mathfrak{M}]}(U)$ into an algebra as well. The condition $M^\lambda\le M^\mu$ if $\lambda\le\mu$ may be relaxed to $M^\lambda\preceq M^\mu$.

(2) Assuming that $(M^\lambda_k/M^\mu_k)^{\frac{1}{k}}$ is (ultimately) monotonic in $k$ for all $\lambda,\mu$, we have either $M^\lambda\approx M^\mu$ for all $\lambda,\mu$ or $M^\lambda\lhd M^\mu$ for all $\lambda<\mu$. That is either $\mathcal{E}^{[\mathfrak{M}]}=\mathcal{E}^{[M^\lambda]}$ for all $\lambda$ or we have the representations in (4.4).

For $\mathfrak{M}, \mathfrak{N} \in \mathscr{M}$ we define

$$\begin{aligned}
\mathfrak{M}(\preceq)\mathfrak{N} \quad &:\Leftrightarrow \quad \forall \lambda \in \Lambda \ \exists \mu \in \Lambda : M^\mu \preceq N^\lambda \\
\mathfrak{M}\{\preceq\}\mathfrak{N} \quad &:\Leftrightarrow \quad \forall \lambda \in \Lambda \ \exists \mu \in \Lambda : M^\lambda \preceq N^\mu \\
\mathfrak{M}[\approx]\mathfrak{N} \quad &:\Leftrightarrow \quad \mathfrak{M}[\preceq]\mathfrak{N} \text{ and } \mathfrak{N}[\preceq]\mathfrak{M} \\
\mathfrak{M}(\preceq\}\mathfrak{N} \quad &:\Leftrightarrow \quad \exists \lambda \in \Lambda \ \exists \mu \in \Lambda : M^\lambda \preceq N^\mu \\
\mathfrak{M}\{\lhd)\mathfrak{N} \quad &:\Leftrightarrow \quad \forall \lambda \in \Lambda \ \forall \mu \in \Lambda : M^\lambda \lhd N^\mu
\end{aligned}$$

4.6. **Proposition.** *For $\mathfrak{M}, \mathfrak{N} \in \mathscr{M}$ we have:*
(1) $\mathfrak{M}[\preceq]\mathfrak{N} \Rightarrow \mathcal{E}^{[\mathfrak{M}]} \subseteq \mathcal{E}^{[\mathfrak{N}]}$ *and* $\mathcal{E}^{[\mathfrak{M}]}(\mathbb{R}) \subseteq \mathcal{E}^{[\mathfrak{N}]}(\mathbb{R}) \Rightarrow \mathfrak{M}[\preceq]\mathfrak{N}$.
(2) $\mathfrak{M}\{\lhd)\mathfrak{N} \Rightarrow \mathcal{E}^{\{\mathfrak{M}\}} \subseteq \mathcal{E}^{(\mathfrak{N})}$ *and* $\mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R}) \subseteq \mathcal{E}^{(\mathfrak{N})}(\mathbb{R}) \Rightarrow \mathfrak{M}\{\lhd)\mathfrak{N}$.
(3) $\mathfrak{M}(\preceq\}\mathfrak{N} \Rightarrow \mathcal{E}^{(\mathfrak{M})} \subseteq \mathcal{E}^{\{\mathfrak{N}\}}$ *and* $\mathcal{E}^{(\mathfrak{M})}(\mathbb{R}) \subseteq \mathcal{E}^{\{\mathfrak{N}\}}(\mathbb{R}) \Rightarrow \mathfrak{M}(\preceq\}\mathfrak{N}$.
*All inclusions are continuous.*

**Proof.** (1) That $\mathfrak{M}[\preceq]\mathfrak{N}$ implies $\mathcal{E}^{[\mathfrak{M}]} \subseteq \mathcal{E}^{[\mathfrak{N}]}$ is clear by definition. If $\mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R}) \subseteq \mathcal{E}^{\{\mathfrak{N}\}}(\mathbb{R})$ then $\mathfrak{M}\{\preceq\}\mathfrak{N}$ follows from the existence of characteristic $\mathcal{E}^{\{M^\lambda\}}$-functions, by Lemma 2.9. If $\mathcal{E}^{(\mathfrak{M})}(\mathbb{R}) \subseteq \mathcal{E}^{(\mathfrak{N})}(\mathbb{R})$ then this inclusion is continuous, by the closed graph theorem since convergence in $\mathcal{E}^{(\mathfrak{M})}(\mathbb{R})$ implies pointwise convergence; here we follow [10, Thm. 2.2]. Thus for each $\lambda \in \Lambda$, each compact $I \subseteq \mathbb{R}$, and each $\tau > 0$ there exist $\mu \in \Lambda$, $J \subseteq \mathbb{R}$ compact, and constants $C, \rho > 0$ such that

$$\|f\|^{N^\lambda}_{I,\tau} \le C \|f\|^{M^\mu}_{J,\rho} \quad \text{ for } \quad f \in \mathcal{E}^{(\mathfrak{M})}(\mathbb{R}).$$

In particular, for $f_t(x) = e^{itx}$ and $\tau = 1$, we obtain

$$T_{N^\lambda}(t) = \sup_{k \in \mathbb{N}} \frac{t^k}{k! N^\lambda_k} \le C \sup_{k \in \mathbb{N}} \frac{t^k}{k! \rho^k M^\mu_k} = C T_{M^\mu}(\tfrac{t}{\rho}),$$

and thus

$$k! N^\lambda_k = \sup_{t > 0} \frac{t^k}{T_{N^\lambda}(t)} \ge \sup_{t>0} \frac{t^k}{C T_{M^\mu}(\frac{t}{\rho})} = k! \frac{\rho^k}{C} M^\mu_k,$$

that is $\mathfrak{M}(\preceq)\mathfrak{N}$.

(2) That $\mathfrak{M}\{\lhd)\mathfrak{N}$ implies $\mathcal{E}^{\{\mathfrak{M}\}} \subseteq \mathcal{E}^{(\mathfrak{N})}$ is clear by definition. The converse follows from the existence of characteristic $\mathcal{E}^{\{M^\lambda\}}$-functions.

(3) That $\mathfrak{M}(\preceq\}\mathfrak{N}$ implies $\mathcal{E}^{(\mathfrak{M})} \subseteq \mathcal{E}^{\{\mathfrak{N}\}}$ is clear by definition. Conversely, if $\mathcal{E}^{(\mathfrak{M})}(\mathbb{R}) \subseteq \mathcal{E}^{\{\mathfrak{N}\}}(\mathbb{R})$ then the closed graph theorem (cf. [19, 5.4.1]) implies that this inclusion is continuous. Indeed $\mathcal{E}^{(\mathfrak{M})}(\mathbb{R})$ is a Fréchet space, $\mathcal{E}^{\{\mathfrak{N}\}}(\mathbb{R})$ is projective limit of Silva spaces, hence webbed, and convergence implies pointwise convergence. This and Grothendieck's factorization theorem (e.g. [28, 24.33]) imply that for each compact $I \subseteq \mathbb{R}$ there exist $\lambda \in \Lambda$, $\tau > 0$, $\mu \in \Lambda$, $J \subseteq \mathbb{R}$ compact, and constants $C, \rho > 0$ such that

$$\|f\|^{N^\lambda}_{I,\tau} \le C \|f\|^{M^\mu}_{J,\rho} \quad \text{ for } \quad f \in \mathcal{E}^{(\mathfrak{M})}(\mathbb{R}).$$

Applying this to $f_t(x) = e^{itx}$ we obtain, similarly as in (1),

$$M^\mu_k \le C (\tfrac{\tau}{\rho})^k N^\lambda_k,$$

that is $\mathfrak{M}(\preceq\}\mathfrak{N}$. □

We may conclude:
(4) $\mathcal{H}(\mathbb{C}^n) \subseteq \mathcal{E}^{(\mathfrak{M})}(U)$, for all open $U \subseteq \mathbb{R}^n$, if and only if $(\mathfrak{M}_\mathcal{H})$.

(5) $C^\omega \subseteq \mathcal{E}^{[\mathfrak{M}]}$ if and only if $(\mathfrak{M}_{[C^\omega]})$.
(6) $\mathcal{E}^{[\mathfrak{M}]}$ is derivation closed if and only if $(\mathfrak{M}_{[\mathrm{dc}]})$.

Note that for $L \in \mathbb{R}_{>0}^{\mathbb{N}}$ we have $L(\preceq)\mathfrak{M}$ if and only if $L\{\preceq\}\mathfrak{M}$; in particular, $\mathcal{H}(\mathbb{C}^n) \subseteq \mathcal{E}^{\{\mathfrak{M}\}}(U)$ if and only if $C^\omega(U) \subseteq \mathcal{E}^{\{\mathfrak{M}\}}(U)$, for all open $U \subseteq \mathbb{R}^n$. Moreover:

4.7. **Corollary.** *Let $M \in \mathbb{R}_{>0}^{\mathbb{N}}$ with $\lim M_k^{\frac{1}{k}} = \infty$. Then there is no $N \in \mathbb{R}_{>0}^{\mathbb{N}}$ such that $\mathcal{E}^{(M)}(\mathbb{R}) \subsetneq \mathcal{E}^{[N]}(\mathbb{R}) \subsetneq \mathcal{E}^{\{M\}}(\mathbb{R})$.*

**Proof.** This follows from Proposition 4.6 and Theorem 2.15. □

4.8. **Remark.** It is easy to see that $\mathcal{E}^{\{\mathfrak{M}\}}$ is non-quasianalytic if and only if there is some $\lambda \in \Lambda$ such that $M^\lambda$ is non-quasianalytic. Likewise if $\mathcal{E}^{(\mathfrak{M})}$ is non-quasianalytic then $M^\lambda$ is non-quasianalytic for all $\lambda \in \Lambda$. Intersections $\bigcap_M \mathcal{E}^{[M]}$, where $M$ runs through a *large* family of non-quasianalytic weakly log-convex weight sequences, can be quasianalytic, see [26] and references therein. But we do not know whether $\mathcal{E}^{(\mathfrak{M})}$ can be quasianalytic if all $M^\lambda$ are non-quasianalytic and $\Lambda$ is restricted to a 1-parameter family (as assumed in this paper).

4.9. **Theorem.** *For a weight matrix $\mathfrak{M}$ satisfying $(\mathfrak{M}_{\{\mathrm{dc}\}})$ and $(\mathfrak{M}_{\{C^\omega\}})$ the following are equivalent:*

(1) *$\mathcal{E}^{\{\mathfrak{M}\}}$ is stable under composition.*
(2) *$\mathcal{E}^{\{\mathfrak{M}\}}$ is holomorphically closed.*
(3) *For all $\lambda \in \Lambda$ there are $\mu \in \Lambda$ and $C > 0$ so that $(M_j^\lambda)^{\frac{1}{j}} \le C(M_k^\mu)^{\frac{1}{k}}$ if $j \le k$.*
(4) *$\mathfrak{M}$ satisfies $(\mathfrak{M}_{\{\mathrm{FdB}\}})$.*

$(\mathfrak{M}_{\{C^\omega\}})$ is only needed for (1) ⇒ (2); $(\mathfrak{M}_{\{\mathrm{dc}\}})$ is only needed for (3) ⇒ (4).

**Proof.** (1) ⇒ (2) This is obvious, by 4.6(5).

(2) ⇒ (3) We prove that (3) holds if $\mathcal{E}^{\{\mathfrak{M}\}}$ is inverse closed and follow the idea of [39]. Let $\lambda \in \Lambda$ be fixed and let $g$ be the function in $\mathcal{E}^{\{M^\lambda\}}(\mathbb{R},\mathbb{C})$ defined by (2.10) (with $M = (M_k)$ replaced by $M^\lambda = (M_k^\lambda)$). Choose $H > 0$ such that $H > 1 + \sup_{t\in\mathbb{R}} |g(t)|$. We have $H - g \in \mathcal{E}^{\{M^\lambda\}}(\mathbb{R},\mathbb{C})$, and thus $f := (H-g)^{-1} \in \mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R},\mathbb{C})$, as $\mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R},\mathbb{C})$ is inverse closed, by assumption. Thus, there exist $\mu \in \Lambda$ and constants $C, \rho > 0$ so that

$$\|f\|^{M^\mu}_{[-1,1],\rho} < C. \tag{4.10}$$

By Faá di Bruno's formula and using (2.11), for $k \ge 1$,

$$\begin{aligned}\frac{f^{(k)}(0)}{k!} &= \sum_{j\ge1} \sum_{\substack{\alpha_1+\dots+\alpha_j=k\\ a_\ell>0}} \frac{1}{(H-g(0))^{j+1}} \prod_{\ell=1}^{j} \frac{g^{(\alpha_\ell)}(0)}{\alpha_\ell!}\\ &= i^k \sum_{j\ge1} \sum_{\substack{\alpha_1+\dots+\alpha_j=k\\ a_\ell>0}} \frac{1}{(H-g(0))^{j+1}} \prod_{\ell=1}^{j} \frac{h_{\alpha_\ell}}{\alpha_\ell!}.\end{aligned}$$

By (4.10),

$$C\rho^k M_k^\mu \ge \frac{|f^{(k)}(0)|}{k!} = \sum_{j\ge 1} \sum_{\substack{\alpha_1+\cdots+\alpha_j=k\\ a_\ell>0}} \frac{1}{(H-g(0))^{j+1}} \prod_{\ell=1}^j \frac{h_{\alpha_\ell}}{\alpha_\ell!}$$
$$\ge \sum_{j\ge 1} \sum_{\substack{\alpha_1+\cdots+\alpha_j=k\\ a_\ell>0}} \frac{1}{(H-g(0))^{j+1}} \prod_{\ell=1}^j M^\lambda_{\alpha_\ell}$$
$$\ge \frac{1}{(H-g(0))^{k+1}} \prod_{\ell=1}^j M^\lambda_{\alpha_\ell}.$$

In particular, for $\alpha_1 = \cdots = \alpha_j = p$, $p \in \mathbb{N}_{>0}$, we have

$$C_1 \rho_1^{pj} M^\mu_{pj} \ge (M^\lambda_p)^j$$

and hence, for all $j$ and $p$,

$$C_2 (M^\mu_{pj})^{\frac{1}{pj}} \ge (M^\lambda_p)^{\frac{1}{p}}.$$

For arbitrary $p \le k$ choose $j$ so that $jp \le k < (j+1)p$. Then

$$(M^\mu_k)^{\frac{1}{k}} \ge (M^\mu_{jp})^{\frac{1}{jp}} \frac{(jp)!^{\frac{1}{jp}}}{k!^{\frac{1}{k}}} \ge C_2^{-1} (M^\lambda_p)^{\frac{1}{p}} \frac{(jp)!^{\frac{1}{jp}}}{k!^{\frac{1}{k}}} \ge C_2^{-1} (M^\lambda_p)^{\frac{1}{p}},$$

since $(k! M^\mu_k)^{1/k}$ is non-decreasing.

(3) $\Rightarrow$ (4) By $(\mathfrak{M}_{\{\mathrm{dc}\}})$, for $\lambda \in \Lambda$ there exist $\mu \in \Lambda$ and $D > 0$ so that $M^\lambda_{k+1} \le D^k M^\mu_k$ for all $k \ge 1$. The assumption implies that there is $\nu \in \Lambda$ so that $M^\mu_{\beta_1} \cdots M^\mu_{\beta_j} \le C^k M^\nu_k$ for all $\beta_i \in \mathbb{N}_{>0}$ with $\beta_1 + \cdots + \beta_j = k$. Let $I := \{i : \alpha_i \ge 2\}$ and set $\alpha'_i := \alpha_i - 1$. Then, as $\mu \ge \lambda$,

$$M^\lambda_j M^\lambda_{\alpha_1} \cdots M^\lambda_{\alpha_j} = M^\lambda_j (M^\lambda_1)^{j-|I|} \prod_{i\in I} M^\lambda_{\alpha_i} \le D^{k-j} M^\lambda_j (M^\lambda_1)^{j-|I|} \prod_{i\in I} M^\mu_{\alpha'_i}$$
$$\le D^{k-j} (M^\lambda_1)^{j-|I|} C^k M^\nu_k \le \tilde{C}^k M^\nu_k,$$

which shows (4).

(4) $\Rightarrow$ (1) Let $g \in \mathcal{E}^{\{\mathfrak{M}\}}(U,V)$ and $f \in \mathcal{E}^{\{\mathfrak{M}\}}(V,W)$, for open subsets $U \subseteq \mathbb{R}^p$, $V \subseteq \mathbb{R}^q$, $W \subseteq \mathbb{R}^r$, and let $K \subseteq U$ be compact. By definition, there exist $\lambda_i \in \Lambda$, $i = 1, 2$, such that $g \in \mathcal{E}^{\{M^{\lambda_1}\}}(K,V)$ and $f \in \mathcal{E}^{\{M^{\lambda_2}\}}(g(K),W)$, and there exists $\lambda \ge \lambda_i$, $i = 1, 2$. By $(\mathfrak{M}_{\{\mathrm{FdB}\}})$, there exists $\mu \in \Lambda$ such that $(M^\lambda)^\circ \preceq M^\mu$, and thus, by Proposition 3.1, we have $f \circ g \in \mathcal{E}^{\{M^\mu\}}(K,W)$ which implies the assertion. □

4.11. **Theorem.** *For a weight matrix $\mathfrak{M}$ satisfying $(\mathfrak{M}_{(\mathrm{dc})})$ and $(\mathfrak{M}_{\mathcal{H}})$ the following are equivalent:*

(1) *$\mathcal{E}^{(\mathfrak{M})}$ is stable under composition.*
(2) *$\mathcal{E}^{(\mathfrak{M})}$ is holomorphically closed.*
(3) *For all $\lambda \in \Lambda$ there are $\mu \in \Lambda$ and $C > 0$ so that $(M^\mu_j)^{\frac{1}{j}} \le C (M^\lambda_k)^{\frac{1}{k}}$ if $j \le k$.*
(4) *$\mathfrak{M}$ satisfies $(\mathfrak{M}_{(\mathrm{FdB})})$.*

$(\mathfrak{M}_{\mathcal{H}})$ is only needed for (1) $\Rightarrow$ (2); $(\mathfrak{M}_{(\mathrm{dc})})$ is only needed for (3) $\Rightarrow$ (4).

**Proof.** (1) $\Rightarrow$ (2) This is obvious, by 4.6(4).

(2) $\Rightarrow$ (3) We follow [10]. Since all $M^\lambda$ are weakly log-convex, $\mathcal{E}^{(\mathfrak{M})}(\mathbb{R})$ is a Fréchet algebra which is locally m-convex, by [29], i.e., $\mathcal{E}^{(\mathfrak{M})}(\mathbb{R})$ has an equivalent

seminorm system $\{p\}$ such that $p(fg) \le p(f)p(g)$ for all $f, g \in \mathcal{E}^{(\mathfrak{M})}(\mathbb{R})$. So for each $\lambda \in \Lambda$, compact $K \subseteq \mathbb{R}$, and $\rho > 0$ there exist $p$, $\mu \in \Lambda$, compact $L \subseteq \mathbb{R}$, $\sigma > 0$ and constants $C, D > 0$ such that

$$\|f^m\|^{M^\lambda}_{K,\rho} \le Cp(f^m) \le C(p(f))^m \le CD^m(\|f\|^{M^\mu}_{L,\sigma})^m, \quad f \in \mathcal{E}^{(\mathfrak{M})}(\mathbb{R}), m \in \mathbb{N},$$

in particular, for $f_t(x) = e^{itx}$ and $\rho = 1$, we find

$$T_{M^\lambda}(mt) \le CD^m(T_{M^\mu}(\tfrac{t}{\sigma}))^m.$$

Let $j \le k$ and suppose that $k = jl$ with $l \in \mathbb{N}$. We have, for some constant $\tilde{C}$,

$$(T_{M^\lambda}(t))^{\frac{1}{k}} = (T_{M^\lambda}(l\tfrac{t}{l}))^{\frac{1}{k}} \le C^{\frac{1}{k}} D^{\frac{1}{j}} (T_{M^\mu}(\tfrac{t}{\sigma l}))^{\frac{1}{j}} \le \tilde{C}(T_{M^\mu}(\tfrac{t}{\sigma l}))^{\frac{1}{j}},$$

thus

$$(k!M^\lambda_k)^{\frac{1}{k}} = \sup_{t>0} \frac{t}{(T_{M^\lambda}(t))^{\frac{1}{k}}} \ge \sup_{t>0} \frac{t}{\tilde{C}(T_{M^\mu}(\frac{t}{\sigma l}))^{\frac{1}{j}}} = \frac{\sigma l}{\tilde{C}}(j!M^\mu_j)^{\frac{1}{j}}.$$

In general choose $l \in \mathbb{N}$ such that $lj \le k < (l+1)j$. Then, as $(k!M^\lambda_k)^{\frac{1}{k}}$ is increasing,

$$(k!M^\lambda_k)^{\frac{1}{k}} \ge ((lj)!M^\lambda_{lj})^{\frac{1}{lj}} \ge \frac{\sigma l}{\tilde{C}}(j!M^\mu_j)^{\frac{1}{j}} \ge \frac{\sigma(l+1)}{2\tilde{C}}(j!M^\mu_j)^{\frac{1}{j}},$$

and, by Stirling's formula, there is a constant $\bar{C} > 0$ such that $(M^\mu_j)^{\frac{1}{j}} \le \bar{C}(M^\lambda_k)^{\frac{1}{k}}$ for all $j \le k$.

(3) $\Rightarrow$ (4) The assumption implies that $M^\mu_{\beta_1} \cdots M^\mu_{\beta_j} \le C^k M^\lambda_k$ for all $\beta_i \in \mathbb{N}_{>0}$ with $\beta_1 + \cdots + \beta_j = k$. By $(\mathfrak{M}_{(\mathrm{dc})})$, there exist $\nu \in \Lambda$ and $D > 0$ so that $M^\nu_{k+1} \le D^k M^\mu_k$ for all $k \ge 1$. Let $I := \{i : \alpha_i \ge 2\}$ and set $\alpha'_i := \alpha_i - 1$. Then, as $\nu \le \mu$,

$$\begin{aligned} M^\nu_j M^\nu_{\alpha_1} \cdots M^\nu_{\alpha_j} &= M^\nu_j (M^\nu_1)^{j-|I|} \prod_{i\in I} M^\nu_{\alpha_i} \le D^{k-j} M^\nu_j (M^\nu_1)^{j-|I|} \prod_{i \in I} M^\mu_{\alpha'_i} \\ &\le D^{k-j}(M^\nu_1)^{j-|I|} C^k M^\lambda_k \le \tilde{C}^k M^\lambda_k, \end{aligned}$$

which shows (4).

(4) $\Rightarrow$ (1) Let $g \in \mathcal{E}^{(\mathfrak{M})}(U,V)$ and $f \in \mathcal{E}^{(\mathfrak{M})}(V,W)$, for open subsets $U \subseteq \mathbb{R}^p$, $V \subseteq \mathbb{R}^q$, $W \subseteq \mathbb{R}^r$, and let $K \subseteq U$ be compact. By definition, for each $\mu \in \Lambda$ we have $g \in \mathcal{E}^{(M^\mu)}(K,V)$ and $f \in \mathcal{E}^{(M^\mu)}(g(K),W)$. By $(\mathfrak{M}_{(\mathrm{FdB})})$ and by Proposition 3.1, we obtain $f \circ g \in \mathcal{E}^{(M^\lambda)}(K,W)$ for each $\lambda \in \Lambda$ which implies the assertion. $\square$

4.12. **Composition operators.** Let $\mathfrak{M}$ be a weight matrix. If $\mathfrak{M}$ satisfies $(\mathfrak{M}_{[\mathrm{FdB}]})$, we may consider the nonlinear composition operators

$$\mathrm{comp}^{[\mathfrak{M}]} : \mathcal{E}^{[\mathfrak{M}]}(\mathbb{R}^p, \mathbb{R}^q) \times \mathcal{E}^{[\mathfrak{M}]}(\mathbb{R}^q, \mathbb{R}^r) \to \mathcal{E}^{[\mathfrak{M}]}(\mathbb{R}^p, \mathbb{R}^r) : (g, f) \mapsto f \circ g$$

$$\mathcal{E}^{[\mathfrak{M}]}(\mathbb{R}^p, f) : \mathcal{E}^{[\mathfrak{M}]}(\mathbb{R}^p, \mathbb{R}^q) \to \mathcal{E}^{[\mathfrak{M}]}(\mathbb{R}^p, \mathbb{R}^r) : g \mapsto f \circ g, \quad f \in \mathcal{E}^{[\mathfrak{M}]}(\mathbb{R}^q, \mathbb{R}^r),$$

by Theorem 4.9 and Theorem 4.11.

4.13. **Theorem.** *We have:*

(1) *If* $\mathfrak{M}$ *satisfies* $(\mathfrak{M}_{(\mathrm{FdB})})$, *then* $\mathrm{comp}^{(\mathfrak{M})}$ *is continuous.*
(2) *If* $\mathfrak{M}$ *satisfies* $(\mathfrak{M}_{\{\mathrm{FdB}\}})$, *then* $\mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R}^p, f)$, *for* $f \in \mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R}^q, \mathbb{R}^r)$, *is continuous and* $\mathrm{comp}^{\{\mathfrak{M}\}}$ *is sequentially continuous.*

**Proof.** We follow [1] and subdivide the proof into several claims.

4.14. **Claim.** *If* $\mathfrak{M}$ *satisfies* $(\mathfrak{M}_{[\mathrm{FdB}]})$, *then* $\mathrm{comp}^{[\mathfrak{M}]}$ *is bounded.*

We treat the cases $\mathcal{E}^{(\mathfrak{M})}$ and $\mathcal{E}^{\{\mathfrak{M}\}}$ separately.

($\mathcal{E}^{[\mathfrak{M}]}=\mathcal{E}^{\{\mathfrak{M}\}}$) Let $\mathcal{B}_1 \subseteq \mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R}^p,\mathbb{R}^q)$ and $\mathcal{B}_2 \subseteq \mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R}^q,\mathbb{R}^r)$ be bounded subsets. Let $K \subseteq \mathbb{R}^p$ be an arbitrary, but fixed, compact subset. Then $\mathcal{B}_1$ is bounded in $\mathcal{E}^{\{\mathfrak{M}\}}(K,\mathbb{R}^q)$. Since the inductive limit $\mathcal{E}^{\{\mathfrak{M}\}}(K,\mathbb{R}^q) = \varinjlim_{(\lambda,\rho)} \mathcal{E}^{M^\lambda}_\rho(K,\mathbb{R}^q)$ is regular, see 4.2, $\mathcal{B}_1$ is contained and bounded in some step $\mathcal{E}^{M^{\lambda_1}}_{\rho_1}(K,\mathbb{R}^q)$, i.e., there exist $\lambda_1 \in \Lambda$ and $\rho_1 > 0$ such that $\sup_{g\in\mathcal{B}_1} \|g\|^{M^{\lambda_1}}_{K,\rho_1} < \infty$. In particular, the closure

$$L := \overline{\bigcup_{g\in\mathcal{B}_1} g(K)} \tag{4.15}$$

is a compact subset of $\mathbb{R}^q$, and $\mathcal{B}_2$ is bounded in $\mathcal{E}^{\{\mathfrak{M}\}}(L,\mathbb{R}^r) = \varinjlim_{(\lambda,\rho)} \mathcal{E}^{M^\lambda}_\rho(L,\mathbb{R}^r)$. So there exist $\lambda_2 \in \Lambda$ and $\rho_2 > 0$ such that $\sup_{f\in\mathcal{B}_2} \|f\|^{M^{\lambda_2}}_{L,\rho_2} < \infty$. For $\lambda := \max \lambda_i$, we have

$$C_1 := \sup_{g\in\mathcal{B}_1} \|g\|^{M^\lambda}_{K,\rho_1} < \infty \quad \text{and} \quad C_2 := \sup_{f\in\mathcal{B}_2} \|f\|^{M^\lambda}_{L,\rho_2} < \infty. \tag{4.16}$$

By the proof of Proposition 3.1 we find that

$$\sup_{(g,f)\in\mathcal{B}_1\times\mathcal{B}_2} \|f\circ g\|^{(M^\lambda)^\circ}_{K,\sigma} \le C_1 C_2 \rho_2 < \infty, \qquad \text{with } \sigma := \rho_1(1+\rho_2 C_1), \tag{4.17}$$

and by ($\mathfrak{M}_{\{\mathrm{FdB}\}}$) there exist $\mu \in \Lambda$ and $C > 0$ such that

$$\sup_{(g,f)\in\mathcal{B}_1\times\mathcal{B}_2} \|f\circ g\|^{M^\mu}_{K,C\sigma} \le \sup_{(g,f)\in\mathcal{B}_1\times\mathcal{B}_2} \|f\circ g\|^{(M^\lambda)^\circ}_{K,\sigma} < \infty. \tag{4.18}$$

Since $K$ was arbitrary, $\mathrm{comp}^{\{\mathfrak{M}\}}(\mathcal{B}_1\times\mathcal{B}_2)$ is bounded in $\mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R}^p,\mathbb{R}^r)$.

($\mathcal{E}^{[\mathfrak{M}]}=\mathcal{E}^{(\mathfrak{M})}$) Let $\mathcal{B}_1 \subseteq \mathcal{E}^{(\mathfrak{M})}(\mathbb{R}^p,\mathbb{R}^q)$ and $\mathcal{B}_2 \subseteq \mathcal{E}^{(\mathfrak{M})}(\mathbb{R}^q,\mathbb{R}^r)$ be bounded. Let $\mu \in \Lambda$, let $K \subseteq \mathbb{R}^p$ be compact, and let $\tau > 0$. By ($\mathfrak{M}_{(\mathrm{FdB})}$), we find $\lambda \in \Lambda$ and $C > 0$ such that $(M^\lambda)^\circ_k \le C^k M^\mu_k$ for all $k$. Choose $\rho > 0$ so that $\tau/C = \sqrt{\rho}+\rho$ and set $\rho_1 = \sqrt{\rho}$. Let $C_1$ be defined by (4.16); $\mathcal{B}_1$ is bounded in $\mathcal{E}^{M^\lambda}_{\rho_1}(K,\mathbb{R}^q)$. Set $\rho_2 = \sqrt{\rho}/C_1$ and let $C_2$ be defined by (4.16); $\mathcal{B}_2$ is bounded in $\mathcal{E}^{M^\lambda}_{\rho_2}(L,\mathbb{R}^r)$, where $L$ is defined by (4.15). As before we may conclude (4.17) and (4.18), where $\sigma = \tau/C$, which completes the proof of the claim.

4.19. **Claim.** *If $\mathfrak{M}$ satisfies ($\mathfrak{M}_{[\mathrm{FdB}]}$), then* $\mathrm{comp}^{[\mathfrak{M}]}$ *is sequentially continuous.*

($\mathcal{E}^{[\mathfrak{M}]}=\mathcal{E}^{\{\mathfrak{M}\}}$) Let $(g_n,f_n)\to(g,f)$ in $\mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R}^p,\mathbb{R}^q)\times\mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R}^q,\mathbb{R}^r)$. Then the sets $\mathcal{B}_1 := \{g_n : n\in\mathbb{N}\}$, $\mathcal{B}_2 := \{f_n : n\in\mathbb{N}\}$, and $\{f_n\circ g_n : n \in\mathbb{N}\} \subseteq \mathrm{comp}^{\{\mathfrak{M}\}}(\mathcal{B}_1\times\mathcal{B}_2)$ are bounded, by Claim 4.14. Let $K \subseteq \mathbb{R}^p$ be an arbitrary, but fixed, compact subset, and let $L$ be given by (4.15). By regularity of the inductive limit $\mathcal{E}^{\{\mathfrak{M}\}}(K,\mathbb{R}^r) = \varinjlim_{(\lambda,\rho)} \mathcal{E}^{M^\lambda}_\rho(K,\mathbb{R}^r)$, the set $\{f_n\circ g_n : n\in\mathbb{N}\}$ is contained and bounded in some step $\mathcal{E}^{M^\lambda}_\rho(K,\mathbb{R}^r)$ and hence is precompact in $\mathcal{E}^{M^\mu}_\sigma(K,\mathbb{R}^r)$, where $\lambda\le\mu$ and $\rho<\sigma$, see 4.2, and so it has an accumulation point $h \in \mathcal{E}^{M^\mu}_\sigma(K,\mathbb{R}^r)$. It is well-known that composition of continuous mappings, i.e., $\mathrm{comp}^0 : C^0(K,L)\times C^0(L,\mathbb{R}^r)\to C^0(K,\mathbb{R}^r)$, is continuous, see e.g. [14, Thm. 3.4.2], and thus $f_n\circ g_n \to f\circ g$ in $C^0(K,\mathbb{R}^r)$. It follows that $f\circ g = h$. As $K$ was arbitrary the assertion follows.

($\mathcal{E}^{[\mathfrak{M}]}=\mathcal{E}^{(\mathfrak{M})}$) The proof is analogous; note that here $\{f_n\circ g_n : n\in\mathbb{N}\}$ is precompact in every step $\mathcal{E}^{M^\mu}_\sigma(K,\mathbb{R}^r)$.

4.20. **Claim.** *If* $\mathfrak{M}$ *satisfies* $(\mathfrak{M}_{(\mathrm{FdB})})$, *then* $\mathrm{comp}^{(\mathfrak{M})}$ *is continuous.*

This follows from Claim 4.19, since $\mathcal{E}^{(\mathfrak{M})}(\mathbb{R}^p,\mathbb{R}^q)\times\mathcal{E}^{(\mathfrak{M})}(\mathbb{R}^q,\mathbb{R}^r)$ is metrizable.

4.21. **Claim.** *If* $\mathfrak{M}$ *satisfies* $(\mathfrak{M}_{\{\mathrm{FdB}\}})$, *then* $\mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R}^p,f)$ *is continuous.*

Arguments similar as in the proof of Claim 4.19 show that the restricted mapping $\mathcal{E}^{\{\mathfrak{M}\}}(K,f):\mathcal{E}^{\{\mathfrak{M}\}}(K,\mathbb{R}^q)\to\mathcal{E}^{\{\mathfrak{M}\}}(K,\mathbb{R}^r)$ is sequentially continuous, thus continuous, for each compact subset $K\subseteq\mathbb{R}^p$, since $\mathcal{E}^{\{\mathfrak{M}\}}(K,\mathbb{R}^q)$ is sequential, by 4.2 and e.g. [31, 8.5.28]. The projective structure of $\mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R}^p,\mathbb{R}^q)=\varprojlim_K\mathcal{E}^{\{\mathfrak{M}\}}(K,\mathbb{R}^q)$ implies that the mapping $\mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R}^p,f):\mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R}^p,\mathbb{R}^q)\to\mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R}^p,\mathbb{R}^r)$ is continuous. □

4.22. **Corollary.** *Let* $M\in\mathbb{R}_{>0}^{\mathbb{N}}$ *satisfy* $(\mathrm{M}_{\mathrm{FdB}})$. *Then* $\mathrm{comp}^{(M)}$ *is continuous,* $\mathcal{E}^{\{M\}}(\mathbb{R}^p,f)$, *for* $f\in\mathcal{E}^{\{M\}}(\mathbb{R}^q,\mathbb{R}^r)$, *is continuous, and* $\mathrm{comp}^{\{M\}}$ *is sequentially continuous.*

**Proof.** This is a special case of Theorem 4.13; weak log-convexity of $M$ is not needed here. □

4.23. **Remark.** If $M$ additionally has moderate growth, then the mapping $\mathrm{comp}^{[M]}$ is even $\mathcal{E}^{[M]}$ which is a consequence of the $\mathcal{E}^{[M]}$-exponential law, see [25, 5.5]. We expect that more generally $\mathrm{comp}^{[\mathfrak{M}]}$ is $\mathcal{E}^{[\mathfrak{M}]}$, if $\mathfrak{M}$ satisfies $(\mathfrak{M}_{[\mathrm{FdB}]})$ and $(\mathfrak{M}_{[\mathrm{mg}]})$. This is work in progress and will appear in a forthcoming paper.

## 5. Weight functions and $[\omega]$-ultradifferentiable functions

5.1. **Weight functions.** Let $\mathscr{W}$ be the set of all continuous increasing functions $\omega:[0,\infty)\to[0,\infty)$ with $\omega|_{[0,1]}=0$, $\lim_{t\to\infty}\omega(t)=\infty$, and so that the following assumptions $(\omega_1)$, $(\omega_2)$, and $(\omega_3)$ are satisfied:

$(\omega_1)$ $\omega(2t)=O(\omega(t))$ as $t\to\infty$.
$(\omega_2)$ $\log(t)=o(\omega(t))$ as $t\to\infty$.
$(\omega_3)$ $\varphi:t\mapsto\omega(e^t)$ is convex on $[0,\infty)$.

Occasionally, we shall also consider the following conditions:

$(\omega_4)$ $\omega(t)=O(t)$ as $t\to\infty$.
$(\omega_5)$ $\omega(t)=o(t)$ as $t\to\infty$.
$(\omega_6)$ $\exists H\ge 1\ \forall t\ge 0: 2\omega(t)\le\omega(Ht)+H$.
$(\omega_7)$ $\exists C>0\ \exists t_0>0\ \forall\lambda\ge 1\ \forall t\ge t_0:\omega(\lambda t)\le C\lambda\omega(t)$.
$(\omega_8)$ $\exists C>0\ \exists H>0\ \forall t\ge 0:\omega(t^2)\le C\omega(Ht)+C$.

Then $\mathscr{W}$ forms an abelian semigroup with respect to pointwise addition, which also preserves all conditions $(\omega_4)$–$(\omega_8)$.

For $\omega\in\mathscr{W}$ the *Young conjugate* of $\varphi$, given by

$$\varphi^*(t):=\sup\{st-\varphi(s):s\ge 0\},\quad t\ge 0,$$

is convex, increasing, and satisfies $\varphi^*(0)=0$, $\varphi^{**}=\varphi$, and $\lim_{t\to\infty}\frac{t}{\varphi^*(t)}=0$. Moreover, the functions $t\mapsto\frac{\varphi(t)}{t}$ and $t\mapsto\frac{\varphi^*(t)}{t}$ are increasing. Cf. [9]. Convexity of $\varphi^*$ and $\varphi^*(0)=0$ implies

$$\varphi^*(t)+\varphi^*(s)\le\varphi^*(t+s)\le\tfrac12\varphi^*(2t)+\tfrac12\varphi^*(2s),\quad t,s\ge 0. \tag{5.2}$$

Note that $\omega(t):=\max\{0,(\log t)^s\}$, $s>1$, belongs to $\mathscr{W}$ and satisfies all listed conditions except $(\omega_6)$.

For $\omega, \sigma \in \mathscr{W}$ we define:

$$\begin{aligned} \omega \preceq \sigma \quad &:\Leftrightarrow \quad \sigma(t) = O(\omega(t)) \text{ as } t \to \infty \\ \omega \approx \sigma \quad &:\Leftrightarrow \quad \omega \preceq \sigma \text{ and } \sigma \preceq \omega \\ \omega \lhd \sigma \quad &:\Leftrightarrow \quad \sigma(t) = o(\omega(t)) \text{ as } t \to \infty \end{aligned}$$

5.3. $[\omega]$**-ultradifferentiable functions.** Let $\omega \in \mathscr{W}$ and let $U \subseteq \mathbb{R}^n$ be open. Define

$$\mathcal{E}^{(\omega)}(U) := \Big\{ f \in C^\infty(U, \mathbb{R}) : \forall K \subseteq U \text{ compact } \forall \rho > 0 : \|f\|^\omega_{K,\rho} < \infty \Big\}$$
$$\mathcal{E}^{\{\omega\}}(U) := \Big\{ f \in C^\infty(U, \mathbb{R}) : \forall K \subseteq U \text{ compact } \exists \rho > 0 : \|f\|^\omega_{K,\rho} < \infty \Big\}$$
$$\|f\|^\omega_{K,\rho} := \sup \Big\{ \|f^{(k)}(x)\|_{L^k(\mathbb{R}^n,\mathbb{R})} \exp(-\tfrac{1}{\rho}\varphi^*(\rho k)) : x \in K, k \in \mathbb{N} \Big\}$$

and endow $\mathcal{E}^{(\omega)}(U)$ with its natural Fréchet space topology and $\mathcal{E}^{\{\omega\}}(U)$ with the projective limit topology over $K$ of the inductive limit topology over $\rho$; note that it suffices to take countable limits. We write $\mathcal{E}^{[\omega]}$ for either $\mathcal{E}^{(\omega)}$ or $\mathcal{E}^{\{\omega\}}$. The elements of $\mathcal{E}^{[\omega]}(U)$ are called $[\omega]$*-ultradifferentiable functions*; an $(\omega)/\{\omega\}$-ultradifferentiable function is said to be of Beurling/Roumieu type, respectively. For compact $K \subseteq U$ with smooth boundary, set

$$\mathcal{E}^\omega_\rho(K) := \Big\{ f \in C^\infty(K) : \|f\|^\omega_{K,\rho} < \infty \Big\}$$
$$\mathcal{E}^{(\omega)}(K) := \Big\{ f \in C^\infty(K) : \forall \rho > 0 : \|f\|^\omega_{K,\rho} < \infty \Big\} = \varprojlim_{m \in \mathbb{N}} \mathcal{E}^\omega_{\frac{1}{m}}(K)$$
$$\mathcal{E}^{\{\omega\}}(K) := \Big\{ f \in C^\infty(K) : \exists \rho > 0 : \|f\|^\omega_{K,\rho} < \infty \Big\} = \varinjlim_{m \in \mathbb{N}} \mathcal{E}^\omega_m(K).$$

We shall also use $\mathcal{E}^{[\omega]}(U, V)$, $\mathcal{E}^{[\omega]}(K, V)$, and $\mathcal{E}^\omega_\rho(K, V)$ for open subsets $V \subseteq \mathbb{R}^m$.

Note that $\mathcal{E}^{[\omega]}$ is quasianalytic if and only if

$$\int_1^\infty \frac{\omega(t)}{t^2} dt = \infty$$

(e.g., by Corollary 5.8 and Theorem 5.14 below), and in this case we say that $\omega$ is quasianalytic.

5.4. **Examples.** For $s \in \mathbb{R}_{\ge 0}$ the weight function $\gamma^s(t) = t^{\frac{1}{1+s}}$ has all properties listed in 5.1 except $(\omega_8)$ and $(\omega_5)$ if $s = 0$; it is quasianalytic if and only if $s = 0$. The elements of $\mathcal{E}^{\{\gamma^0\}}(U)$ are exactly the real analytic functions $C^\omega(U)$ and the elements of $\mathcal{E}^{(\gamma^0)}(U)$ are exactly the restrictions of entire functions $\mathcal{H}(\mathbb{C}^n)$. The class $\mathcal{E}^{\{\gamma^s\}}$ coincides with the Gevrey class $\mathcal{G}^{1+s}$.

5.5. **Associated sequences.** For $\omega \in \mathscr{W}$ and each $\rho > 0$ consider the sequence $\Omega^\rho \in \mathbb{R}^{\mathbb{N}}_{>0}$ defined by

$$\Omega^\rho_k := \tfrac{1}{k!} \exp(\tfrac{1}{\rho}\varphi^*(\rho k)).$$

By the properties of $\varphi^*$, each $\Omega^\rho$ is weakly log-convex, $(k!\Omega^\rho_k)^{\frac{1}{k}} \nearrow \infty$, and $\Omega^\rho \le \Omega^\sigma$ if $\rho \le \sigma$. By (5.2),

$$j!\Omega^\rho_j k!\Omega^\rho_k \le (j+k)!\Omega^\rho_{j+k} \le j!\Omega^{2\rho}_j k!\Omega^{2\rho}_k, \quad j, k \in \mathbb{N}. \tag{5.6}$$

In particular, $\Omega^\rho_{k+1} \le C\Omega^{2\rho}_k$ for all $k$, where $C > 0$ is a constant depending on $\rho$.

With $\Omega^\rho$ we may associate the function $\omega_\rho := \log \circ T_{\Omega^\rho}$, cf. [20, (3.1)]; then

$$\omega_\rho(t) = \sup_{k\in\mathbb{N}}(k\log t - \tfrac{1}{\rho}\varphi^*(\rho k)) \le \sup_{s\ge 0}(s\log t - \tfrac{1}{\rho}\varphi^*(\rho s)) = \tfrac{1}{\rho}\omega(t).$$

5.7. **Lemma.** *For $\omega \in \mathscr{W}$ we have $\omega \approx \omega_\rho$ for all $\rho > 0$.*

**Proof.** It suffices to show that $\omega \approx \omega_1$; for arbitrary $\rho > 0$ replace $\omega$ by $\frac{1}{\rho}\omega$. By [27, 1.8.III],

$$\omega_1(t) = \sup_{k\in\mathbb{N}}(k\log t - \varphi^*(k)) = k_t \log t - \varphi^*(k_t),$$

where $k_t \in \mathbb{N}$ is such that $\varpi_{k_t} \le t < \varpi_{k_t+1}$ and $\varpi_k := k\Omega^1_k/\Omega^1_{k-1} \nearrow \infty$. Consider the function $f_t : [0,\infty) \to \mathbb{R}$ given by $f_t(s) = s\log t - \varphi^*(s)$, which is concave (for $t \ge 1$) since $\varphi^*$ is convex. Concavity of $f_t$ shows that $\omega(t) = \sup_{s\ge 0} f_t(s) = f_t(s_t)$ for a point $s_t \in (k_t - 1, k_t + 1)$.

Assume that $s_t \in (k_t, k_t + 1)$. By concavity of $f_t$ and since $f_t(0) = 0$, we find

$$f_t(s_t) \le \tfrac{f_t(k_t)}{k_t} s_t \le \tfrac{f_t(k_t)}{k_t}(k_t + 1) \le 2 f_t(k_t)$$

and hence $\omega(t) \le 2\omega_1(t)$ for sufficiently large $t$. The case $s_t \in (k_t - 1, k_t)$ is similar. □

5.8. **Corollary.** *For $\omega \in \mathscr{W}$ we have:*

(1) *$\omega$ is quasianalytic if and only if each (equivalently, some) $\Omega^\rho$ is quasianalytic.*
(2) *$\omega$ satisfies $(\omega_6)$ if and only if each (equivalently, some) $\Omega^\rho$ has moderate growth.*

**Proof.** This follows from Lemma 5.7, [20, Lemma 4.1], and [20, Prop. 3.6]. □

5.9. **Lemma.** *For $\omega \in \mathscr{W}$ we have*

$$\forall \sigma > 0\ \exists H \ge 1\ \forall \rho > 0\ \exists C \ge 1\ \forall k \in \mathbb{N} : \sigma^k \Omega^\rho_k \le C\Omega^{H\rho}_k. \tag{5.10}$$

*Moreover, $\omega \in \mathscr{W}$ satisfies $(\omega_6)$ if and only if*

$$\forall \rho > 0\ \forall \tau > 0 : \Omega^\rho \approx \Omega^\tau. \tag{5.11}$$

*If $\omega \in \mathscr{W}$ satisfies $(\omega_8)$ then*

$$\exists C > 1\ \forall \rho > 0 : \Omega^{\frac{\rho}{C}} \lhd \Omega^\rho. \tag{5.12}$$

It follows that $(\omega_8)$ is an obstruction for $(\omega_6)$.

**Proof.** The following inequality is well-known (see e.g. [16, p. 404]):

$$\exists L \ge 1\ \forall t \ge 0\ \forall s \in \mathbb{N} : L^s\varphi^*(t) + sL^s t \le \varphi^*(L^s t) + \sum_{i=1}^{s} L^i. \tag{5.13}$$

For the reader's convenience we give a short proof. By $(\omega_1)$, there exists $L_1 \ge 1$ such that $\omega(2t) \le L_1\omega(t) + L_1$ for all $t \ge 0$ and hence there exists $L \ge 1$ such that $\varphi(t+1) \le L\varphi(t) + L$ for all $t \ge 0$. Thus, for $t \ge 0$,

$$\varphi^*(Lt) + L = \sup_{s\ge 0}(Lts - (\varphi(s) - L)) \ge \sup_{s\ge 1}(Lts - L\varphi(s-1)) = L\varphi^*(t) + Lt,$$

and (5.13) follows by iteration.

By choosing $s$ such that $e^s \ge \sigma$ and by setting $t := \rho k$, $H := L^s$ and $C := \exp(\frac{1}{H\rho}\sum_{i=1}^s L^i)$, we see that (5.13) implies (5.10).

Let us prove that $(\omega_6)$ implies (5.11). By $(\omega_6)$ there exists a constant $H \geq 1$ such that $2\omega(t) \leq \omega(Ht) + H$ for all $t \geq 0$, and, consequently, as $\omega|_{[0,1]} = 0$,

$$\begin{aligned}\varphi^*(t) &= \sup_{s\geq 0}(ts - \omega(e^s)) = \sup_{s\in\mathbb{R}}(ts - \omega(e^s)) = \sup_{u\geq 0}(t\log u - \omega(u)) \\ &\geq \sup_{u\geq 0}(t\log u - \tfrac{1}{2}\omega(Hu)) - \tfrac{1}{2}H = \tfrac{1}{2}\varphi^*(2t) - t\log H - \tfrac{1}{2}H.\end{aligned}$$

By setting $t := \rho k$, we may conclude that

$$\exists H \geq 1\ \forall \rho > 0\ \forall k \in \mathbb{N} : \Omega^{2\rho}_k \leq e^{\frac{H}{2\rho}} H^k \Omega^\rho_k$$

which implies $\Omega^{2\rho} \preceq \Omega^\rho$ for all $\rho > 0$. Iteration and the fact that $\Omega^\rho \leq \Omega^{2\rho}$ yield $\Omega^{2^n\rho} \approx \Omega^\rho$ for all $\rho > 0$ and all $n \in \mathbb{N}$, and (5.11) follows.

Conversely assume (5.11) which means that

$$\forall \rho > 0\ \forall \tau > 0\ \exists C > 0\ \forall k \in \mathbb{N} : \tfrac{1}{\rho}\varphi^*(\rho k) \leq Ck + \tfrac{1}{\tau}\varphi^*(\tau k).$$

By (5.2), we may conclude that

$$\forall \rho > 0\ \forall \tau > 0\ \exists D > 0\ \forall t \geq 0 : \tfrac{1}{\rho}\varphi^*(\rho t) \leq Dt + D + \tfrac{1}{2\tau}\varphi^*(2\tau t).$$

Thus

$$\tfrac{1}{2\tau}\varphi(t) = \sup_{s\geq 0}(ts - \tfrac{1}{2\tau}\varphi^*(2\tau s)) \leq \sup_{s\geq 0}(ts + Ds - \tfrac{1}{\rho}\varphi^*(\rho s)) + D = \tfrac{1}{\rho}\varphi(t + D) + D,$$

and, hence,

$$\tfrac{1}{2\tau}\omega(t) \leq \tfrac{1}{\rho}\omega(e^D t) + D.$$

Setting $\rho = 4$ and $\tau = 1$ implies $(\omega_6)$.

Let us prove (5.12). By $(\omega_8)$ there exist constants $C, H > 0$ such that

$$\begin{aligned}C\varphi^*(\tfrac{2t}{C}) &= \sup_{u\geq 0}(2t\log u - C\omega(u)) = \sup_{u\geq 0}(2t\log u - C\omega(Hu)) + 2t\log H \\ &\leq \sup_{u\geq 0}(2t\log u - \omega(u^2)) + 2t\log H + C = \varphi^*(t) + 2t\log H + C.\end{aligned}$$

By setting $t := \rho k$ we find that for all $\rho > 0$ and all $k \in \mathbb{N}$

$$(2k)!\Omega^{\frac{\rho}{C}}_{2k} \leq e^{\frac{C}{\rho}} H^{2k} k!\Omega^\rho_k.$$

Thus the sequence $L = (L_k)$ defined by $k!L_k := (2k)!\Omega^{\frac{\rho}{C}}_{2k} \geq (k!\Omega^{\frac{\rho}{C}}_k)^2$ satisfies $\Omega^{\frac{\rho}{C}} \lhd L \preceq \Omega^\rho$, which implies (5.12). □

5.14. **Theorem.** *Let $\omega \in \mathscr{W}$, let $U \subseteq \mathbb{R}^n$ be open, and let $K \subseteq U$ be compact. Then:*

1. *For each $\rho > 0$ we have $\mathcal{E}^{\{\Omega^\rho\}}(U) \subseteq \mathcal{E}^{\{\omega\}}(U)$ and $\mathcal{E}^{(\omega)}(U) \subseteq \mathcal{E}^{(\Omega^\rho)}(U)$ with continuous inclusion.*
2. *We have as locally convex spaces*

$$\mathcal{E}^{(\omega)}(U) = \varprojlim_{\rho>0} \mathcal{E}^{(\Omega^\rho)}(U) \quad \text{and} \quad \mathcal{E}^{\{\omega\}}(K) = \varinjlim_{\rho>0} \mathcal{E}^{\{\Omega^\rho\}}(K).$$

3. *$\omega$ satisfies $(\omega_6)$ if and only if $\mathcal{E}^{[\Omega^\rho]}(U) = \mathcal{E}^{[\omega]}(U)$, for each $\rho > 0$, as locally convex spaces.*

(4) *If $\omega$ satisfies $(\omega_8)$, then also*

$$\mathcal{E}^{(\omega)}(U) = \varprojlim_{\rho>0} \mathcal{E}^{(\Omega^\rho)}(U) = \varprojlim_{\rho>0} \mathcal{E}^{\{\Omega^\rho\}}(U) \quad \textit{and}$$
$$\mathcal{E}^{\{\omega\}}(K) = \varinjlim_{\rho>0} \mathcal{E}^{\{\Omega^\rho\}}(K) = \varinjlim_{\rho>0} \mathcal{E}^{(\Omega^\rho)}(K)$$

*as locally convex spaces.*

**Proof.** (1) Let $\rho > 0$ be fixed. If $f \in \mathcal{E}^{\{\Omega^\rho\}}(U)$ then for each compact $K \subseteq U$ there exists $\sigma > 0$ such that $\|f\|^{\Omega^\rho}_{K,\sigma} < \infty$. By (5.10), there exist constants $H, C \ge 1$ such that

$$\infty > C\|f\|^{\Omega^\rho}_{K,\sigma} \ge \|f\|^{\Omega^{H\rho}}_{K,1} = \|f\|^{\omega}_{K,H\rho},$$

whence $f \in \mathcal{E}^{\{\omega\}}(U)$.

Assume that $f \in \mathcal{E}^{(\omega)}(U)$. Let $\rho > 0$ and $\sigma > 0$ be fixed. By (5.10), there exist constants $H, C \ge 1$ such that $\Omega^\rho_k \le C\sigma^k \Omega^{H\rho}_k$ for all $k$. Since $f \in \mathcal{E}^{(\omega)}(U)$, for each compact $K \subseteq U$ we have $\|f\|^{\omega}_{K,\frac{\rho}{H}} < \infty$, and, thus,

$$\infty > C\|f\|^{\omega}_{K,\frac{\rho}{H}} = C\|f\|^{\Omega^{\frac{\rho}{H}}}_{K,1} \ge \|f\|^{\Omega^\rho}_{K,\sigma}.$$

Since $\sigma > 0$ was arbitrary, we may conclude that $f \in \mathcal{E}^{(\Omega^\rho)}(U)$.

(2) follows from (1), since the inclusions $\mathcal{E}^{(\omega)}(U) \supseteq \varprojlim_{\rho>0} \mathcal{E}^{(\Omega^\rho)}(U)$ and $\mathcal{E}^{\{\omega\}}(K) \subseteq \varinjlim_{\rho>0} \mathcal{E}^{\{\Omega^\rho\}}(K)$ are clear and continuous by definition.

(3) follows from (2), (5.11), and Proposition 2.12(1).

(4) is a direct consequence of (2), (5.12), and Proposition 2.12(1). □

5.15. **Corollary.** *Let $\omega \in \mathscr{W}$ and let $U \subseteq \mathbb{R}^n$ be open. Then $\mathcal{E}^{[\omega]}(U) = \mathcal{E}^{[\mathfrak{W}]}(U)$ as locally convex spaces, where the weight matrix $\mathfrak{W} := \{\Omega^\rho : \rho > 0\}$ satisfies*

- $(\mathfrak{M}_{(\mathrm{mg})})$ *and* $(\mathfrak{M}_{\{\mathrm{mg}\}})$,
- $(\mathfrak{M}_{(\mathrm{alg})})$ *and* $(\mathfrak{M}_{\{\mathrm{alg}\}})$,
- $(\mathfrak{M}_{(\mathrm{L})})$ *and* $(\mathfrak{M}_{\{\mathrm{L}\}})$.

*If $\omega$ satisfies $(\omega_4)$, respectively $(\omega_5)$, then $\mathfrak{W}$ satisfies $(\mathfrak{M}_{\mathcal{H}})$, respectively $(\mathfrak{M}_{(C^\omega)})$. If $\omega$ satisfies $(\omega_8)$, then $\mathfrak{W}$ satisfies $(\mathfrak{M}_{(\mathrm{BR})})$ and $(\mathfrak{M}_{\{\mathrm{BR}\}})$.*

**Proof.** This is an immediate consequence of Theorem 5.14, (5.6), and (5.10).

For $\omega(t) = \max\{0, t-1\} \approx t$ we find $\varphi^*(t) = t\log t - t + 1$, for $t \ge 1$, $\varphi^*|_{[0,1]} = 0$, and it is easy to see that $(\omega_4)$ implies $(\mathfrak{M}_{\mathcal{H}})$ and $(\omega_5)$ implies $(\mathfrak{M}_{(C^\omega)})$, by Lemma 5.16. That $(\omega_8)$ implies $(\mathfrak{M}_{(\mathrm{BR})})$ and $(\mathfrak{M}_{\{\mathrm{BR}\}})$ follows from (5.12). □

5.16. **Lemma.** *For $\omega, \sigma \in \mathscr{W}$ we have:*

(1) *If $\omega \preceq \sigma$ then $\exists H \ge 1\ \forall \rho > 0\ \exists C > 0 : \Omega^\rho \le C\Sigma^{H\rho}$.*
(2) *If $\omega \lhd \sigma$ then $\forall H > 0\ \forall \rho > 0\ \exists C > 0 : \Omega^\rho \le C\Sigma^{H\rho}$.*

*Here $\Sigma^\rho$ are the sequences associated with $\sigma$.*

**Proof.** (1) If $\omega \preceq \sigma$ then there exists $H \ge 1$ such that $\sigma(t) \le H\omega(t) + H$ for all $t \ge 0$, and thus also $\varphi_\sigma(t) \le H\varphi_\omega(t) + H$ and finally $H\varphi^*_\omega(t) \le \varphi^*_\sigma(Ht) + H$. Setting $t = \rho k$ gives the assertion.

(2) If $\omega \lhd \sigma$ then for all $H > 0$ there exists $D > 0$ such that $\sigma(t) \le H\omega(t) + D$ for all $t \ge 0$, and thus $H\varphi^*_\omega(t) \le \varphi^*_\sigma(Ht) + D$ as in (1). Setting $t = \rho k$ gives the assertion. □

5.17. **Corollary.** *For $\omega,\sigma \in \mathscr{W}$ we have:*
(1) $\omega \preceq \sigma \Rightarrow \mathcal{E}^{[\omega]} \subseteq \mathcal{E}^{[\sigma]}$ *and* $\mathcal{E}^{[\omega]}(\mathbb{R}) \subseteq \mathcal{E}^{[\sigma]}(\mathbb{R}) \Rightarrow \omega \preceq \sigma$.
(2) $\omega \lhd \sigma \Rightarrow \mathcal{E}^{\{\omega\}} \subseteq \mathcal{E}^{(\sigma)}$ *and* $\mathcal{E}^{\{\omega\}}(\mathbb{R}) \subseteq \mathcal{E}^{(\sigma)}(\mathbb{R}) \Rightarrow \omega \lhd \sigma$.
(3) *There is no $\sigma \in \mathscr{W}$ such that* $\mathcal{E}^{(\omega)}(\mathbb{R}) \subsetneq \mathcal{E}^{[\sigma]}(\mathbb{R}) \subsetneq \mathcal{E}^{\{\omega\}}(\mathbb{R})$.

**Proof.** (1) If $\mathfrak{W} := \{\Omega^\rho : \rho > 0\}$ and $\mathfrak{S} := \{\Sigma^\rho : \rho > 0\}$, where $\Sigma^\rho$ are the sequences associated with $\sigma$, then in view of Proposition 4.6 and Corollary 5.15 it suffices to show

(1′) $\omega \preceq \sigma$ if and only if $\mathfrak{W}[\preceq]\mathfrak{S}$.

If $\omega \preceq \sigma$ then Lemma 5.16 implies $\mathfrak{W}(\preceq)\mathfrak{S}$ as well as $\mathfrak{W}\{\preceq\}\mathfrak{S}$.

Conversely, assume $\mathfrak{W}\{\preceq\}\mathfrak{S}$, i.e., using (5.10),

$$\forall \rho > 0\ \exists \tau > 0\ \exists C > 0\ \forall k \in \mathbb{N} : \tfrac{1}{\rho}\varphi^*_\omega(\rho k) \le \tfrac{1}{\tau}\varphi^*_\sigma(\tau k) + C,$$

and, by (5.2),

$$\forall \rho > 0\ \exists \tau > 0\ \exists D > 0\ \forall t \ge 0 : \tfrac{1}{\rho}\varphi^*_\omega(\rho t) \le \tfrac{1}{2\tau}\varphi^*_\sigma(2\tau t) + D.$$

Thus

$$\tfrac{1}{2\tau}\varphi_\sigma(t) = \sup_{s\ge 0}(ts - \tfrac{1}{2\tau}\varphi^*_\sigma(2\tau s)) \le \sup_{s\ge 0}(ts - \tfrac{1}{\rho}\varphi^*_\omega(\rho s)) + D = \tfrac{1}{\rho}\varphi_\omega(t) + D,$$

and, hence,

$$\tfrac{1}{2\tau}\sigma(t) \le \tfrac{1}{\rho}\omega(t) + D, \tag{5.18}$$

which implies $\sigma(t) = O(\omega(t))$ as $t \to \infty$, i.e., $\omega \preceq \sigma$.

If $\mathfrak{W}(\preceq)\mathfrak{S}$, then the same arguments yield (5.18), but with swapped quantifiers:

$$\forall \tau > 0\ \exists \rho > 0\ \exists D > 0\ \forall t \ge 0 : \tfrac{1}{2\tau}\sigma(t) \le \tfrac{1}{\rho}\omega(t) + D.$$

Again this implies $\omega \preceq \sigma$.

(2) If $\omega \lhd \sigma$ then Lemma 5.16 implies $\mathcal{E}^{\{\omega\}} \subseteq \mathcal{E}^{(\sigma)}$. Conversely, if $\mathcal{E}^{\{\omega\}}(\mathbb{R}) \subseteq \mathcal{E}^{(\sigma)}(\mathbb{R})$, then $\mathcal{E}^{\{\Omega^\rho\}}$ admits a characteristic function and is contained in $\mathcal{E}^{(\sigma)}$, thus

$$\forall \rho > 0\ \forall \tau > 0\ \exists C > 0\ \forall k \in \mathbb{N} : \tfrac{1}{\rho}\varphi^*_\omega(\rho k) \le \tfrac{1}{\tau}\varphi^*_\sigma(\tau k) + C.$$

As in (1) we may derive that for all $\rho, \tau > 0$ there is $D > 0$ such that (5.18) for all $t \ge 0$, hence $\sigma(t) = o(\omega(t))$ as $t \to \infty$, i.e., $\omega \lhd \sigma$.

(3) If $\mathcal{E}^{(\omega)}(\mathbb{R}) \subseteq \mathcal{E}^{\{\sigma\}}(\mathbb{R})$, then $\mathfrak{W}(\preceq\}\mathfrak{S}$, by Corollary 5.15 and Proposition 4.6. Similarly as in (1) we may derive that there exist $\rho, \tau > 0$ such that (5.18), and so $\omega \preceq \sigma$. This and (1) imply the assertion. □

As $\mathcal{E}^{\{t\}}(U) = C^\omega(U)$ and $\mathcal{E}^{(t)}(U) = \mathcal{H}(\mathbb{C}^n)$ (via restriction), condition $(\omega_4)$ is equivalent to $C^\omega \subseteq \mathcal{E}^{\{\omega\}}$ and condition $(\omega_5)$ is equivalent to $C^\omega \subseteq \mathcal{E}^{(\omega)}$.

5.19. **Intersection and union of all non-quasianalytic Gevrey classes.** For the weight matrix $\mathfrak{G} = \{G^s : s > 0\}$ with $G^s = (G^s_k) = ((k!)^s)$

$$\mathcal{E}^{(\mathfrak{G})}(U) = \bigcap_{s>0} \mathcal{G}^{1+s}(U), \quad U \subseteq \mathbb{R}^n \text{ open}, \tag{5.20}$$

is the intersection and

$$\mathcal{E}^{\{\mathfrak{G}\}}(K) = \bigcup_{s>0} \mathcal{G}^{1+s}(K), \quad K \subseteq \mathbb{R}^n \text{ compact}, \tag{5.21}$$

is the union of all non-quasianalytic Gevrey classes $\mathcal{G}^{1+s} = \mathcal{E}^{\{G^s\}}$ (as locally convex spaces). Indeed $G^s \lhd G^{s'}$ for all $s < s'$ (so $\mathfrak{G}$ satisfies $(\mathfrak{M}_{(\mathrm{BR})})$ and $(\mathfrak{M}_{\{\mathrm{BR}\}})$), and hence we get (5.20)

$$\mathcal{E}^{(\mathfrak{G})}(U) = \bigcap_{s>0} \mathcal{E}^{(G^s)}(U) = \bigcap_{s>0} \mathcal{E}^{\{G^s\}}(U) = \bigcap_{s>0} \mathcal{G}^{1+s}(U),$$

while (5.21) is evident by definition. Note that $\mathcal{E}^{(\mathfrak{G})}$, and hence also $\mathcal{E}^{\{\mathfrak{G}\}}$, is non-quasianalytic; in fact, the sequence $L = (L_k)$ defined by $k!L_k := k^k(\log(k+e))^{2k}$ is non-quasianalytic and satisfies $L \lhd G^s$ for all $s > 0$, and, as $(k!L_k)^{\frac{1}{k}}$ is increasing, $\mathcal{E}^{[L]}$ is non-quasianalytic, by the Denjoy–Carleman theorem.

The following theorem shows that there exist spaces $\mathcal{E}^{[\mathfrak{M}]}$ that are different from $\mathcal{E}^{[M]}$ as well as from $\mathcal{E}^{[\omega]}$.

5.22. **Theorem.** *Neither $\mathcal{E}^{(\mathfrak{G})}(\mathbb{R})$ nor $\mathcal{E}^{\{\mathfrak{G}\}}(\mathbb{R})$ coincides (as vector space) with $\mathcal{E}^{(M)}(\mathbb{R})$, $\mathcal{E}^{\{M\}}(\mathbb{R})$, $\mathcal{E}^{(\omega)}(\mathbb{R})$, or $\mathcal{E}^{\{\omega\}}(\mathbb{R})$ for any weight sequence $M$ or weight function $\omega$.*

**Proof.** We show first that, given a weight matrix $\mathfrak{M} = \{M^\lambda : \lambda \in \Lambda\}$ with $M^\lambda \not\approx M^\mu$ for all $\lambda \ne \mu$, there cannot exist a weakly log-convex $M \in \mathbb{R}_{>0}^{\mathbb{N}}$ such that $\mathcal{E}^{[\mathfrak{M}]}(\mathbb{R}) = \mathcal{E}^{[M]}(\mathbb{R})$. Indeed, if there is such $M$, Proposition 4.6 implies $M \approx M^\lambda$ for some $\lambda$. Then, by Proposition 2.12(1),

$$\mathcal{E}^{(M)}(\mathbb{R}) = \mathcal{E}^{(\mathfrak{M})}(\mathbb{R}) = \bigcap_{\lambda} \mathcal{E}^{(M^\lambda)}(\mathbb{R}) \subsetneq \mathcal{E}^{(M)}(\mathbb{R}),$$

and, for compact $K \subseteq \mathbb{R}$,

$$\begin{aligned} \mathcal{E}^{\{M\}}(\mathbb{R}) = \mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R}) &= \bigcap_{K} \mathcal{E}^{\{\mathfrak{M}\}}(K) = \bigcap_{K}\bigcup_{\lambda} \mathcal{E}^{\{M^\lambda\}}(K) \\ &\supseteq \bigcup_{\lambda}\bigcap_{K} \mathcal{E}^{\{M^\lambda\}}(K) = \bigcup_{\lambda} \mathcal{E}^{\{M^\lambda\}}(\mathbb{R}) \supsetneq \mathcal{E}^{\{M\}}(\mathbb{R}), \end{aligned}$$

which contradicts the assumption in both cases.

As $\mathcal{E}^{(\mathfrak{G})}(\mathbb{R})$ contains $C^\omega(\mathbb{R})$ it cannot coincide with $\mathcal{E}^{(M)}(\mathbb{R})$ for any weight sequence $M$, by Theorem 2.15 and the first paragraph; neither can $\mathcal{E}^{\{\mathfrak{G}\}}(\mathbb{R})$ coincide with $\mathcal{E}^{\{M\}}(\mathbb{R})$.

If there exists $\omega \in \mathscr{W}$ such that $\mathcal{E}^{(\mathfrak{G})}(\mathbb{R}) = \mathcal{E}^{(\omega)}(\mathbb{R})$, then Proposition 4.6 implies that for each $\rho > 0$ there exist $s, \rho' > 0$ such that

$$\Omega^{\rho'} \preceq G^s \preceq \Omega^\rho, \tag{5.23}$$

and thus, by Proposition 2.12(1),

$$\mathcal{E}^{\{\Omega^{\rho'}\}} \subseteq \mathcal{G}^{1+s} \subseteq \mathcal{E}^{\{\Omega^\rho\}}.$$

Since $\mathcal{G}^{1+s} = \mathcal{E}^{\{\gamma\}}$ with $\gamma(t) = t^{\frac{1}{1+s}}$, using the fact that there exist characteristic $\mathcal{E}^{\{\Omega^\rho\}}$- and $\mathcal{E}^{\{\Gamma^\tau\}}$-functions (where $\Gamma^\tau$ are the sequences associated with $\gamma$), and by (5.10), we may conclude that, for all $k$,

$$\tfrac{1}{\rho'}\varphi^*_\omega(\rho' k) \le \tfrac{1}{\tau}\varphi^*_\gamma(\tau k) + C \quad \text{and} \quad \tfrac{1}{\tau}\varphi^*_\gamma(\tau k) \le \tfrac{1}{H\rho}\varphi^*_\omega(H\rho k) + D,$$

for suitable constants $\tau, C, D, H$. As in the derivation of (5.18) this implies $\omega \approx \gamma$ and hence $\mathcal{E}^{(\mathfrak{G})}(\mathbb{R}) = \mathcal{E}^{(\omega)}(\mathbb{R}) = \mathcal{E}^{(\gamma)}(\mathbb{R}) = \mathcal{E}^{(G^s)}(\mathbb{R})$, a contradiction. Thus there is no $\omega \in \mathscr{W}$ with $\mathcal{E}^{(\mathfrak{G})}(\mathbb{R}) = \mathcal{E}^{(\omega)}(\mathbb{R})$.

If there exists $\omega \in \mathscr{W}$ such that $\mathcal{E}^{\{\mathfrak{G}\}}(\mathbb{R}) = \mathcal{E}^{\{\omega\}}(\mathbb{R})$, then Proposition 4.6 implies that for each $\rho' > 0$ there exist $s, \rho > 0$ such that (5.23). Then the same arguments show $\omega \approx \gamma$ and hence $\mathcal{E}^{\{\mathfrak{G}\}}(\mathbb{R}) = \mathcal{E}^{\{\omega\}}(\mathbb{R}) = \mathcal{E}^{\{\gamma\}}(\mathbb{R}) = \mathcal{G}^{1+s}(\mathbb{R})$, a contradiction. Thus there is no $\omega \in \mathscr{W}$ with $\mathcal{E}^{\{\mathfrak{G}\}}(\mathbb{R}) = \mathcal{E}^{\{\omega\}}(\mathbb{R})$.

For the remaining cases note that $\mathfrak{M}(\preceq\}\mathfrak{N}\{\lhd)\mathfrak{M}$ as well as $\mathfrak{M}\{\lhd)\mathfrak{N}(\preceq\}\mathfrak{M}$ is impossible for any two weight matrices $\mathfrak{M}, \mathfrak{N} \in \mathscr{M}$. This fact together with Proposition 4.6 (and Theorem 2.15) implies that there is no weight sequence $M$ and no weight function $\omega$ so that $\mathcal{E}^{(\mathfrak{G})}(\mathbb{R}) = \mathcal{E}^{\{M\}}(\mathbb{R})$, $\mathcal{E}^{(\mathfrak{G})}(\mathbb{R}) = \mathcal{E}^{\{\omega\}}(\mathbb{R})$, $\mathcal{E}^{\{\mathfrak{G}\}}(\mathbb{R}) = \mathcal{E}^{(M)}(\mathbb{R})$, or $\mathcal{E}^{\{\mathfrak{G}\}}(\mathbb{R}) = \mathcal{E}^{(\omega)}(\mathbb{R})$. The proof is complete. □

5.24. **Corollary.** *Composition is continuous on the intersection of all non-quasianalytic Gevrey classes. More precisely,* $\operatorname{comp}^{(\mathfrak{G})}$ *is continuous,* $\mathcal{E}^{\{\mathfrak{G}\}}(\mathbb{R}^p, f)$*, for* $f \in \mathcal{E}^{\{\mathfrak{G}\}}(\mathbb{R}^q, \mathbb{R}^r)$*, is continuous, and* $\operatorname{comp}^{\{\mathfrak{G}\}}$ *is sequentially continuous.*

**Proof.** This follows from Theorem 4.13 and Theorem 5.22. □

We expect that $\operatorname{comp}^{[\mathfrak{G}]}$ is even $\mathcal{E}^{[\mathfrak{G}]}$, see Remark 4.23.

5.25. **Remark.** More autonomous spaces $\mathcal{E}^{[\mathfrak{M}]}$ can be produced by choosing the weight matrix $\mathfrak{M} := \{M^\lambda : \lambda > 0\}$ such that each $M^\lambda$ has moderate growth, satisfies $\underline{\lim}(M_k^\lambda)^{\frac{1}{k}} > 0$ and $\underline{\lim}\, \mu_{nk}^\lambda/\mu_k^\lambda > 1$ for some $n \in \mathbb{N}$ with $\mu_k^\lambda = kM_k^\lambda/M_{k-1}^\lambda$, and $M^\lambda \not\approx M^\mu$ for $\lambda \ne \mu$. Here we may use the comparison theorems in [8] and argue as above.

## 6. Stability under composition of $\mathcal{E}^{[\omega]}$

Stability under composition of $\mathcal{E}^{[\omega]}$ was characterized in [16] for non-quasianalytic weights $\omega$. In this section we apply the characterization obtained by means of the associated weight matrix $\mathfrak{W} = \{\Omega^\rho : \rho > 0\}$ and relate it to the results of [16].

6.1. **Lemma.** *If* $\omega \in \mathscr{W}$ *is sub-additive, then for each* $\rho > 0$ *we have* $(\Omega^\rho)^\circ \preceq \Omega^{2\rho}$.

Then the weight matrix $\mathfrak{W}$ satisfies $(\mathfrak{M}_{(\mathrm{FdB})})$ and $(\mathfrak{M}_{\{\mathrm{FdB}\}})$.

**Proof.** Sub-additivity of $\omega$ implies

$$\Omega_j^\rho \Omega_k^\rho \le \Omega_{j+k}^\rho, \quad j,k \in \mathbb{N}, \tag{6.2}$$

see [17, Lemma 3.3]. Indeed, we have $\exp(\frac{1}{\rho}\varphi^*(\rho k)) = \sup_{s \ge 1} s^k \exp(-\frac{1}{\rho}\omega(s))$ and hence, using sub-additivity of $\omega$,

$$\Omega_j^\rho \Omega_k^\rho \le \sup_{s,t \ge 1} \frac{s^j t^k}{j!k!} \exp(-\tfrac{1}{\rho}\omega(s+t)) \le \sup_{s,t\ge 1} \frac{(s+t)^{j+k}}{(j+k)!} \exp(-\tfrac{1}{\rho}\omega(s+t)) \le \Omega_{j+k}^\rho.$$

By (5.6), (6.2) and since $\Omega^\rho \le \Omega^{2\rho}$, we get, for $\alpha_i \in \mathbb{N}_{>0}$ with $\alpha_1 + \cdots + \alpha_j = k$,

$$\Omega_j^\rho \Omega_{\alpha_1}^\rho \cdots \Omega_{\alpha_j}^\rho \le C^j \Omega_j^{2\rho} \Omega_{\alpha_1 - 1}^{2\rho} \cdots \Omega_{\alpha_j - 1}^{2\rho} \le C^j \Omega_k^{2\rho}$$

which implies the assertion. □

6.3. **Theorem.** *For* $\omega \in \mathscr{W}$ *satisfying* $(\omega_4)$ *the following are equivalent:*

(1) $\mathcal{E}^{\{\omega\}}$ *is stable under composition.*
(2) *For each* $\rho > 0$ *there is* $\tau > 0$ *so that* $(\Omega^\rho)^\circ \preceq \Omega^\tau$*, i.e.,* $\mathfrak{W}$ *satisfies* $(\mathfrak{M}_{\{\mathrm{FdB}\}})$.
(3) *There exists a sub-additive* $\tilde\omega \in \mathscr{W}$ *so that* $\omega \approx \tilde\omega$.
(4) $\omega$ *satisfies* $(\omega_7)$.

**Proof.** (1) ⇔ (2) follows from Theorem 4.9 and Corollary 5.15.

(3) ⇔ (4) See [32, Prop. 1.1] and [30, Lemma 1].

(3) ⇒ (2) follows from Lemma 6.1.

(2) ⇒ (3) The proof is inspired by [16, Prop. 2.3] which treats the non-quasianalytic case. We do not assume non-quasianalyticity (or quasianalyticity) and use Claim 4.14 to remedy the lack of $\mathcal{E}^{\{\omega\}}$-functions of compact support. If $\omega$ does not satisfy $(\omega_7)$, then there exist increasing sequences $(k_n) \in \mathbb{N}^{\mathbb{N}}$ and $(t_n) \in \mathbb{R}^{\mathbb{N}}_{>0}$ so that

$$\omega(k_n t_n) \ge n^2 k_n \omega(t_n). \tag{6.4}$$

Set $a_n := e^{-n\omega(t_n)}$ and $f_n(x) := a_n e^{it_n x}$, $x \in \mathbb{R}$. Then

$$\begin{aligned}\|f_n\|^{\omega}_{\mathbb{R},\rho} &= a_n \sup_{j\in\mathbb{N}} (t_n^j \exp(-\tfrac{1}{\rho}\varphi^*(\rho j))) = a_n \exp \sup_{j\in\mathbb{N}} (j \log t_n - \tfrac{1}{\rho}\varphi^*(\rho j)) \\ &= e^{-n\omega(t_n)} e^{\omega_\rho(t_n)} \le e^{-(n-\frac{1}{\rho})\omega(t_n)}\end{aligned}$$

and so $\{f_n : n \in \mathbb{N}\}$ is bounded in $\mathcal{E}^{\{\omega\}}(\mathbb{R},\mathbb{C})$ (even in $\mathcal{E}^{(\omega)}(\mathbb{R},\mathbb{C})$). The set $\{\mathbb{C} \ni z \mapsto z^k : k \in \mathbb{N}\}$ forms a bounded subset of $\mathcal{E}^{\{\omega\}}(\mathbb{D},\mathbb{C})$, where $\mathbb{D} \subseteq \mathbb{C}$ is the unit disk and where we identify $\mathbb{C} \cong \mathbb{R}^2$). Indeed, for $|z| \le r < 1$ choose $\rho > 0$ such that $r + \frac{1}{\rho} < 1$, and thus

$$\sup_{j\in\mathbb{N}} \frac{|\partial_z^j z^k|}{\rho^j j!} \le \sup_{j\le k} \binom{k}{j} r^{k-j} \frac{1}{\rho^j} \le \Big(r + \frac{1}{\rho}\Big)^k.$$

So $\{z \mapsto z^k : k \in \mathbb{N}\}$ is bounded in $C^\omega(\mathbb{D},\mathbb{C})$ and, by $(\omega_4)$, in $\mathcal{E}^{\{\omega\}}(\mathbb{D},\mathbb{C})$. Since $\mathfrak{W}$ satisfies $(\mathfrak{M}_{\{\mathrm{FdB}\}})$ by assumption (2), we may conclude, from Claim 4.14, that the set $\{f_n^k : n,k \in \mathbb{N}\}$ is bounded in $\mathcal{E}^{\{\omega\}}(\mathbb{R},\mathbb{C})$. Thus there exists $\rho > 0$ such that

$$\begin{aligned}\infty &> \sup_{n,k,j\in\mathbb{N}} |(f_n^k)^{(j)}(0)| \exp(-\tfrac{1}{\rho}\varphi^*(\rho j)) = \sup_{n,k,j\in\mathbb{N}} a_n^k (t_n k)^j \exp(-\tfrac{1}{\rho}\varphi^*(\rho j)) \\ &= \sup_{n,k\in\mathbb{N}} a_n^k e^{\omega_\rho(t_n k)} \ge D \sup_{n,k\in\mathbb{N}} a_n^k e^{\frac{1}{C}\omega(t_n k)} = D \sup_{n,k\in\mathbb{N}} e^{-nk\omega(t_n) + \frac{1}{C}\omega(t_n k)},\end{aligned}$$

for constants $C, D > 0$, by Lemma 5.7, which contradicts (6.4). □

6.5. **Theorem.** *For $\omega \in \mathscr{W}$ satisfying $(\omega_4)$ the following are equivalent:*

1. *$\mathcal{E}^{(\omega)}$ is stable under composition.*
2. *$\mathcal{E}^{(\omega)}$ is holomorphically closed.*
3. *For each $\rho > 0$ there exists $\tau > 0$ so that $(\Omega^\tau)^\circ \preceq \Omega^\rho$, i.e., $\mathfrak{W}$ satisfies $(\mathfrak{M}_{(\mathrm{FdB})})$.*
4. *There exists $H \ge 1$ so that for each $\rho > 0$ we have $(\Omega^\rho)^\circ \preceq \Omega^{H\rho}$.*
5. *There exists a sub-additive $\tilde\omega \in \mathscr{W}$ so that $\omega \approx \tilde\omega$.*
6. *$\omega$ satisfies $(\omega_7)$.*

Note that $(\omega_4)$ is needed only for (1) ⇒ (2).

**Proof.** (1) ⇔ (2) ⇔ (3) follows from Theorem 4.11 and Corollary 5.15.

(2) ⇒ (6) follows from an argument due to [10], see [16, p. 405].

(5) ⇔ (6) See [32, Prop. 1.1] and [30, Lemma 1].

(5) ⇒ (4) follows from Lemma 6.1 and Lemma 5.16.

(4) ⇒ (3) is evident. □

6.6. **Corollary.** *For $\omega \in \mathscr{W}$ satisfying $(\omega_4)$ the following are equivalent:*

1. *For each $\rho > 0$ there exists $\tau > 0$ such that $(\Omega^\rho)^\circ \preceq \Omega^\tau$.*

(2) *For each* $\rho > 0$ *there exists* $\tau > 0$ *such that* $(\Omega^\tau)^\circ \preceq \Omega^\rho$.
(3) *There exists* $H \ge 1$ *so that for each* $\rho > 0$ *we have* $(\Omega^\rho)^\circ \preceq \Omega^{H\rho}$.

**Proof.** Combine Theorem 6.3 and Theorem 6.5. □

Special cases of Theorem 4.13 were proven in [16, 4.2 and 4.4]:

6.7. **Corollary.** *Let* $\omega \in \mathscr{W}$ *satisfy* $(\omega_7)$. *Then* $\operatorname{comp}^{(\omega)}$ *is continuous,* $\mathcal{E}^{\{\omega\}}(\mathbb{R}^p, f)$, *for* $f \in \mathcal{E}^{\{\omega\}}(\mathbb{R}^q, \mathbb{R}^r)$, *is continuous, and* $\operatorname{comp}^{\{\omega\}}$ *is sequentially continuous.*

**Proof.** This is a special case of Theorem 4.13, by Corollary 5.15, Theorem 6.3, and Theorem 6.5. □

We expect that the mapping $\operatorname{comp}^{[\omega]}$ is even $\mathcal{E}^{[\omega]}$, see Remark 4.23.

## References


[1] J. Appell, V. I. Nazarov, and P. P. Zabrejko, *Composing infinitely differentiable functions*, Math. Z. **206** (1991), no. 4, 659–670.

[2] T. Bang, *Om quasi-analytiske Funktioner*, Ph.D. thesis, University of Copenhagen, 1946.

[3] P. Beaugendre, *Extensions de jets dans des intersections de classes non quasi-analytiques*, Ann. Polon. Math. **76** (2001), no. 3, 213–243. MR 1841528 (2002d:46030)

[4] ———, *Intersections de classes non quasi-analytiques*, Ph.D. thesis, Université de Paris-Sud, 2002.

[5] A. Beurling, *Quasi-analyticity and general distributions*, Lecture notes, AMS Summer Institute, Stanford, 1961.

[6] G. Björck, *Linear partial differential operators and generalized distributions*, Ark. Mat. **6** (1966), 351–407.

[7] J. Boman, *On the intersection of classes of infinitely differentiable functions*, Ark. Mat. **5** (1963/1965), 301–309.

[8] J. Bonet, R. Meise, and S. N. Melikhov, *A comparison of two different ways to define classes of ultradifferentiable functions*, Bull. Belg. Math. Soc. Simon Stevin **14** (2007), 424–444.

[9] R. W. Braun, R. Meise, and B. A. Taylor, *Ultradifferentiable functions and Fourier analysis*, Results Math. **17** (1990), no. 3-4, 206–237.

[10] J. Bruna, *On inverse-closed algebras of infinitely differentiable functions*, Studia Math. **69** (1980/81), no. 1, 59–68.

[11] H. Cartan, *Sur les classes de fonctions définies par des inégalités portant sur leurs dérivées successives*, Actual. Sci. Ind., no. 867, Hermann et Cie., Paris, 1940.

[12] J. Chaumat and A.-M. Chollet, *Propriétés de l'intersection des classes de Gevrey et de certaines autres classes*, Bull. Sci. Math. **122** (1998), no. 6, 455–485.

[13] E. M. Dyn'kin, *Pseudoanalytic extension of smooth functions. The uniform scale.*, Transl., Ser. 2, Am. Math. Soc. **115** (1980), 33–58 (English).

[14] R. Engelking, *General topology*, second ed., Sigma Series in Pure Mathematics, vol. 6, Heldermann Verlag, Berlin, 1989, Translated from the Polish by the author. MR 1039321 (91c:54001)

[15] C. F. Faà di Bruno, *Note sur une nouvelle formule du calcul différentielle*, Quart. J. Math. **1** (1855), 359–360.

[16] C. Fernández and A. Galbis, *Superposition in classes of ultradifferentiable functions*, Publ. Res. Inst. Math. Sci. **42** (2006), no. 2, 399–419.

[17] C. Fernández, A. Galbis, and D. Jornet, *ω-hypoelliptic differential operators of constant strength*, J. Math. Anal. Appl. **297** (2004), no. 2, 561–576, Special issue dedicated to John Horváth. MR 2088680 (2006f:35038)

[18] L. Hörmander, *The analysis of linear partial differential operators. I*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 256, Springer-Verlag, Berlin, 1983, Distribution theory and Fourier analysis.

[19] H. Jarchow, *Locally convex spaces*, B. G. Teubner, Stuttgart, 1981, Mathematische Leitfäden. [Mathematical Textbooks].

[20] H. Komatsu, *Ultradistributions. I. Structure theorems and a characterization*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **20** (1973), 25–105.

[21] ———, *An analogue of the Cauchy-Kowalevsky theorem for ultradifferentiable functions and a division theorem for ultradistributions as its dual*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **26** (1979), no. 2, 239–254.
[22] P. Koosis, *The logarithmic integral. I*, Cambridge Studies in Advanced Mathematics, vol. 12, Cambridge University Press, Cambridge, 1998, Corrected reprint of the 1988 original.
[23] A. Kriegl and P. W. Michor, *The convenient setting of global analysis*, Mathematical Surveys and Monographs, vol. 53, American Mathematical Society, Providence, RI, 1997, `http://www.ams.org/online_bks/surv53/`.
[24] A. Kriegl, P. W. Michor, and A. Rainer, *The convenient setting for non-quasianalytic Denjoy–Carleman differentiable mappings*, J. Funct. Anal. **256** (2009), 3510–3544.
[25] ———, *The convenient setting for Denjoy–Carleman differentiable mappings of Beurling and Roumieu type*, (2011), arXiv:1111.1819.
[26] ———, *The convenient setting for quasianalytic Denjoy–Carleman differentiable mappings*, J. Funct. Anal. **261** (2011), 1799–1834.
[27] S. Mandelbrojt, *Séries adhérentes, régularisation des suites, applications*, Gauthier-Villars, Paris, 1952.
[28] R. Meise and D. Vogt, *Introduction to functional analysis*, Oxford Graduate Texts in Mathematics, vol. 2, The Clarendon Press Oxford University Press, New York, 1997. MR 1483073 (98g:46001)
[29] B. Mitiagin, S. Rolewicz, and W. Żelazko, *Entire functions in $B_0$-algebras*, Studia Math. **21** (1961/1962), 291–306.
[30] J. Peetre, *Concave majorants of positive functions*, Acta Math. Acad. Sci. Hungar. **21** (1970), 327–333.
[31] P. Pérez Carreras and J. Bonet, *Barrelled locally convex spaces*, North-Holland Mathematics Studies, vol. 131, North-Holland Publishing Co., Amsterdam, 1987, Notas de Matemática [Mathematical Notes], 113. MR 880207 (88j:46003)
[32] H.-J. Petzsche and D. Vogt, *Almost analytic extension of ultradifferentiable functions and the boundary values of holomorphic functions*, Math. Ann. **267** (1984), no. 1, 17–35.
[33] A. Rainer and G. Schindl, *Equivalence of stability properties for ultradifferentiable function classes*, (2014), arXiv:1407.6673.
[34] C. Roumieu, *Ultra-distributions définies sur $R^n$ et sur certaines classes de variétés différentiables*, J. Analyse Math. **10** (1962/1963), 153–192.
[35] W. Rudin, *Division in algebras of infinitely differentiable functions*, J. Math. Mech. **11** (1962), 797–809.
[36] ———, *Real and complex analysis*, third ed., McGraw-Hill Book Co., New York, 1987.
[37] J. Schmets and M. Valdivia, *Extension properties in intersections of non quasi-analytic classes*, Note Mat. **25** (2005/06), no. 2, 159–185.
[38] ———, *Intersections of non quasi-analytic classes of ultradifferentiable functions*, Bull. Soc. Roy. Sci. Liège **77** (2008), 29–43.
[39] J. A. Siddiqi, *Inverse-closed Carleman algebras of infinitely differentiable functions*, Proc. Amer. Math. Soc. **109** (1990), no. 2, 357–367.
[40] V. Thilliez, *On quasianalytic local rings*, Expo. Math. **26** (2008), no. 1, 1–23.

A. Rainer: Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, A-1090 Wien, Austria
*E-mail address*: `armin.rainer@univie.ac.at`

G. Schindl: Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, A-1090 Wien, Austria
*E-mail address*: `a0304518@unet.univie.ac.at`