NONLINEAR CONDITIONS FOR ULTRADIFFERENTIABILITY: A UNIFORM APPROACH

DAVID NICOLAS NENNING, ARMIN RAINER, AND GERHARD SCHINDL

ABSTRACT. Recent work showed that a theorem of Joris (that a function f is smooth if two coprime powers of f are smooth) is valid in a wide variety of ultradifferentiable classes $\mathcal{C}$. The core of the proof was essentially 1-dimensional. In certain cases a multidimensional version resulted from subtle reduction arguments, but general validity, notably in the quasianalytic setting, remained open. In this paper we give a uniform proof which works in all cases and dimensions. It yields the result even on infinite dimensional Banach spaces and convenient vector spaces. We also consider more general nonlinear conditions, namely general analytic germs Φ instead of the powers, and characterize when $\Phi \circ f \in \mathcal{C}$ implies $f \in \mathcal{C}$.

1. INTRODUCTION

A function f is smooth provided that two relatively prime powers or, equivalently, two consecutive powers of f are smooth, by a theorem of Joris [4]. In the recent paper [17] Thilliez showed that this result carries over to Denjoy–Carleman classes of Roumieu type and, by refining Thilliez's method, we proved in [11] that it is valid in a wide variety of ultradifferentiable classes. See the introduction in [11] for more on the historical development.

In this analysis the dimension of the domain of f had some substantial significance. The proof in dimension one was based on holomorphic approximation and utilized tools of complex analysis in one variable. Multidimensional versions could be obtained only by some subtle reduction arguments (i.e., variants of Boman's theorem [1] and Beurling-to-Roumieu reduction). Consequently, general validity of the result in all dimensions remained open in cases, most notably the quasianalytic case, where the suitable reduction tools are not available. See the table in the introduction of [11] for a summary of the known results (in that paper the result in question was called property $(\mathscr{D})$).

In the present paper we prove general validity in all dimensions and all cases. In fact, the uniformity of our proof allows us to extend the result even to infinite dimensional Banach spaces and convenient vector spaces. The crucial observation is that, in arbitrary dimension, holomorphic approximation in dimension one can be replaced by uniform unidirectional holomorphic approximation combined with the polarization inequality.

Having proved that $\Phi \circ f \in \mathcal{C}$ implies $f \in \mathcal{C}$ for $\Phi(t) = (t^j, t^{j+1})$, $j \in \mathbb{N}_{\geq 1}$, and $\mathcal{C}$ a suitable ultradifferentiable class, it is natural to ask what other Φ have this

property. This question was studied by [2] and [5] in the C^∞-setting. By combining slightly adjusted arguments of [5] with our result, we obtain a full characterization of the analytic germs $\Phi : (\mathbb{K}, 0) \to (\mathbb{K}^n, 0)$ (where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$) having this property in terms of the support of the Taylor series of Φ. For example, it shows that $f \in \mathcal{C}$ provided that $f^2 + f^3 \in \mathcal{C}$ and $f^p \in \mathcal{C}$, where p is any positive integer.

The outline of the paper is the following. Our main result is Theorem 1.1. For its formulation we first need to define ultradifferentiable classes (Section 1.1) and specify the required assumptions (Section 1.2). The proof of Theorem 1.1 is given in Section 2, and in Section 3 we show that it extends to Banach spaces and even to convenient vector spaces. Finally, in Section 4 we prove the characterization alluded to above.

1.1. Ultradifferentiable classes.

Let $\mathfrak{M}$ be a family of positive sequences $M = (M_k)_{k \geq 0}$. Let $U \subseteq \mathbb{R}^d$ be an open subset. The class $\mathcal{E}^{\{\mathfrak{M}\}}(U)$ consists, by definition, of all complex valued functions $f \in C^\infty(U)$ with the following property: for all compact subsets $K \subseteq U$ there exist $M \in \mathfrak{M}$ and $\rho > 0$ such that

$$\|f\|_{K,\rho}^M := \sup_{x \in K} \sup_{k \in \mathbb{N}} \frac{\|f^{(k)}(x)\|_{L^k(\mathbb{R}^d, \mathbb{C})}}{\rho^k M_k} < \infty. \tag{1}$$

Moreover, $\mathcal{E}^{(\mathfrak{M})}(U)$ consists of all complex valued functions $f \in C^\infty(U)$ such that (1) holds for all compact $K \subseteq U$, all $M \in \mathfrak{M}$, and all $\rho > 0$. Here $f^{(k)}$ denotes the Fréchet derivative of order k of f and $\|f^{(k)}(x)\|_{L^k(\mathbb{R}^d, \mathbb{C})} := \sup\{|f^{(k)}(x)(v_1, \ldots, v_k)| : \|v_i\| \leq 1\}$ is the operator norm. In view of the polarization inequality (cf. [7, 7.13.1])

$$\sup_{\|v\| \leq 1} |d_v^k f(x)| \leq \|f^{(k)}(x)\|_{L^k(\mathbb{R}^d, \mathbb{C})} \leq (2e)^k \sup_{\|v\| \leq 1} |d_v^k f(x)|, \tag{2}$$

where $d_v^k f(x) := \partial_t^k f(x + tv)|_{t=0}$, we can equivalently use

$$\sup_{x \in K} \sup_{k \in \mathbb{N}} \sup_{\|v\| \leq 1} \frac{|d_v^k f(x)|}{\rho^k M_k} < \infty \tag{3}$$

instead of (1) in the definition of $\mathcal{E}^{\{\mathfrak{M}\}}(U)$ and $\mathcal{E}^{(\mathfrak{M})}(U)$.

The global classes $\mathcal{B}^{\{\mathfrak{M}\}}(U)$ and $\mathcal{B}^{(\mathfrak{M})}(U)$ are defined by taking the supremum over all $x \in U$ in (1) (that is $\|f\|_{U,\rho}^M < \infty$) or equivalently (3). We shall also need the Banach space $\mathcal{B}_\rho^M(U) := \{f \in C^\infty(U) : \|f\|_{U,\rho}^M < \infty\}$.

The classes $\mathcal{E}^{\{\mathfrak{M}\}}$ and $\mathcal{B}^{\{\mathfrak{M}\}}$ are said to be of *Roumieu type*, $\mathcal{E}^{(\mathfrak{M})}$ and $\mathcal{B}^{(\mathfrak{M})}$ of *Beurling type*. By convention, we use $[\cdot]$ as a placeholder for $\{\cdot\}$ and $(\cdot)$.

1.2. Admissible weight matrices.

We need to impose some conditions on $\mathfrak{M}$.

Definition. A family $\mathfrak{M}$ of positive sequences $M = (M_k)_{k \geq 0}$ is said to be a *weight matrix* if $\mathfrak{M}$ is totally ordered with respect to the pointwise order relation on sequences, and each $M \in \mathfrak{M}$ is log-convex with $M_0 = 1 \leq M_1$ and $M_k^{1/k} \to \infty$. A positive sequence M with these properties is called a *weight sequence*.

With M we associate the sequence m given by $k! \, m_k = M_k$. Assuming that $m_k^{1/k} \to \infty$ we consider the function

$$h_m(t) := \inf_{k \in \mathbb{N}} m_k t^k, \quad \text{for } t > 0, \quad \text{and} \quad h_m(0) := 0,$$

which is increasing, continuous on $[0, \infty)$, positive for $t > 0$ and equals 1 for large t. For $t > 0$ we put

$$\overline{\Gamma}_m(t) := \min\{k : h_m(t) = m_k t^k\}$$

and

$$\underline{\Gamma}_m(t) := \min\left\{k : \frac{m_{k+1}}{m_k} \geq \frac{1}{t}\right\}.$$

Trivially, $\underline{\Gamma}_m \leq \overline{\Gamma}_m$. Equality $\overline{\Gamma}_m = \underline{\Gamma}_m$ holds, if m is log-convex.

For positive sequences M, N we set

$$\mathrm{mg}(M, N) := \sup_{j,k \geq 0,\, j+k \geq 1} \left(\frac{M_{j+k}}{N_j N_k}\right)^{1/(j+k)} \in (0, \infty].$$

Note that $\mathrm{mg}(M, M) < \infty$ is the condition (M.2) of Komatsu [6] often called *moderate growth*.

Definition. A weight matrix $\mathfrak{M}$ satisfying $m_k^{1/k} \to \infty$ for all $M \in \mathfrak{M}$ is called

- $\{admissible\}$ or *R-admissible* if
  - $\forall M \in \mathfrak{M} \; \exists N \in \mathfrak{M} \; \exists C \geq 1 \; \forall t > 0 : \overline{\Gamma}_n(Ct) \leq \underline{\Gamma}_m(t)$,
  - $\forall M \in \mathfrak{M} \; \exists N \in \mathfrak{M} : \mathrm{mg}(M, N) < \infty$.
- $(admissible)$ or *B-admissible* if
  - $\forall M \in \mathfrak{M} \; \exists N \in \mathfrak{M} \; \exists C \geq 1 \; \forall t > 0 : \overline{\Gamma}_m(Ct) \leq \underline{\Gamma}_n(t)$,
  - $\forall M \in \mathfrak{M} \; \exists N \in \mathfrak{M} : \mathrm{mg}(N, M) < \infty$.

By our convention, [admissible] stands for {admissible} (i.e. R-admissible) in the Roumieu case and (admissible) (i.e. B-admissible) in the Beurling case.

Remark. In the terminology of [11] a weight matrix is [admissible] if and only if it is [regular] and has [moderate growth].

These are the minimal requirements needed for the tools used in our proof; see also Remark 4.3 in which the necessity of these assumptions is discussed. Note that [admissibility] of the weight matrix $\mathfrak{M}$ neither implies nor obstructs quasianalyticity of the corresponding class $\mathcal{E}^{[\mathfrak{M}]}$. In fact, (non-)quasianalyticity plays no role in our analysis.

1.3. **Main results.**

Theorem 1.1. *Let $\mathfrak{M}$ be an [admissible] weight matrix. Let $U \subseteq \mathbb{R}^d$ be an open subset. Then a function $f : U \to \mathbb{C}$ belongs to $\mathcal{E}^{[\mathfrak{M}]}(U)$ if $f^j, f^{j+1} \in \mathcal{E}^{[\mathfrak{M}]}(U)$ for some positive integer j.*

If we apply the theorem to the weight matrix which consists but of a single weight sequence M (with $m_k^{1/k} \to \infty$, $\overline{\Gamma}_m(Ct) \leq \underline{\Gamma}_m(t)$ for some $C \geq 1$, and $\mathrm{mg}(M, M) < \infty$), then we immediately get a version for Denjoy–Carleman classes.

As another special case we obtain the following result for Braun–Meise–Taylor classes $\mathcal{E}^{[\omega]}$. For their definition and why it is a special case of Theorem 1.1 we refer the reader to [11, Section 4.5] and the references therein. By a *weight function* we mean a continuous increasing function $\omega : [0, \infty) \to [0, \infty)$ such that

- $\omega(2t) = O(\omega(t))$ as $t \to \infty$,
- $\omega(t) = o(t)$ as $t \to \infty$,
- $\log(t) = o(\omega(t))$ as $t \to \infty$,

- $t \mapsto \omega(e^t) =: \varphi_\omega(t)$ is convex on $[0, \infty)$.

Corollary 1.2. *Let ω be a concave weight function. Let $U \subseteq \mathbb{R}^d$ be an open subset. Then a function $f : U \to \mathbb{C}$ belongs to $\mathcal{E}^{[\omega]}(U)$ if $f^j, f^{j+1} \in \mathcal{E}^{[\omega]}(U)$ for some positive integer j.*

Actually, for a given weight function ω, the conclusion of Corollary 1.2 holds *if and only if* ω is equivalent to a concave weight function, as we shall see in Remark 4.3. Two weight functions ω_i, $i = 1, 2$, are said to be *equivalent* if $\omega_1(t) = O(\omega_2(t))$ and $\omega_2(t) = O(\omega_1(t))$ as $t \to \infty$ which holds if and only if $\mathcal{E}^{[\omega_1]} = \mathcal{E}^{[\omega_2]}$.

2. Proof

2.1. **Preliminaries.**

Lemma 2.1 ([11, Lemma 5.1]). *Let M, N be weight sequences satisfying $m_k^{1/k} \to \infty$, $n_k^{1/k} \to \infty$, and $C := \mathrm{mg}(M, N) < \infty$. Then*

$$h_m(t) \leq C^j n_j t^j h_n(Ct), \quad t > 0, \ j \in \mathbb{N}, \tag{4}$$

$$h_m(t) \leq h_n\left(\frac{eC}{2}t\right)^2, \quad t > 0. \tag{5}$$

For $\varepsilon > 0$ let Ω_ε be the interior of the ellipse in $\mathbb{C}$ with vertices $\pm \cosh(\varepsilon)$ and co-vertices $\pm i \sinh(\varepsilon)$. By $\mathcal{H}(\Omega_\varepsilon)$ we denote the space of holomorphic functions on Ω_ε. And $\|g\|_X := \sup_{z \in X} |g(z)|$ denotes the supremum norm.

The following lemma is a simple application of Hadamard's three lines theorem.

Lemma 2.2 ([11, Lemma 5.2] and [17, Lemma 3.2.4]). *Let M, N be two weight sequences satisfying $m_k^{1/k} \to \infty$, $n_k^{1/k} \to \infty$, and $C := \mathrm{mg}(M, N) < \infty$. Let $\varepsilon > 0$. Let $g \in \mathcal{H}(\Omega_\varepsilon) \cap C^0(\overline{\Omega}_\varepsilon)$ and assume that there are constants $L, a_1, a_2 > 0$ such that*

$$\|g\|_{\Omega_\varepsilon} \leq L, \quad \|g\|_{[-1,1]} \leq a_1 h_m(a_2 \varepsilon).$$

Then with $a_3 := \max\{a_1, L\}$ and $a_4 := eCa_2$ we have

$$\|g\|_{\Omega_{\varepsilon/2}} \leq a_3 h_n(a_4 \varepsilon).$$

2.2. **Uniform unidirectional holomorphic approximation.** Let $\mathbb{B} := \{x \in \mathbb{R}^d : \|x\| \leq 1\}$ be the closed unit ball in $\mathbb{R}^d$ and $\mathbb{S} := \{x \in \mathbb{R}^d : \|x\| = 1\}$ the unit sphere. For any open subset $U \subseteq \mathbb{R}^d$ consider $V_U := \bigcup_{x \in U}(x + \mathbb{B})$ which is again open. If $f : V_U \to \mathbb{C}$ is a smooth function, then the composites

$$f_{x,v}(t) := f(x + tv), \quad t \in [-1, 1],$$

are well-defined and smooth for all $x \in U$ and $v \in \mathbb{S}$. For notational convenience let Λ_U be the collection of all line segments $\lambda(t) = x + tv$, $t \in [-1, 1]$, where $x \in U$ and $v \in \mathbb{S}$, and $f_\lambda(t) := f_{x,v}(t) = f(x + tv)$.

The following theorem shows that ultradifferentiable bounds can be encoded by specific uniform unidirectional holomorphic approximation. It is closely related to [11, Theorem 5.3] (and [17, Proposition 3.3.2]), but special attention has to be paid to the uniformity (in λ) of all the estimates.

Theorem 2.3. *Let $U \subseteq \mathbb{R}^d$ be open and $f : V_U \to \mathbb{C}$ a smooth function.*

(i) *Let $M^{(i)}$, $1 \leq i \leq 3$, be weight sequences with $(m_k^{(i)})^{1/k} \to \infty$ and*

$$\exists B_1 \geq 1 \; \forall t > 0 : \overline{\Gamma}_{m^{(2)}}(B_1 t) \leq \underline{\Gamma}_{m^{(1)}}(t), \tag{6}$$

$$\exists B_2 \geq 1 \; \forall j \in \mathbb{N} : m_{j+1}^{(2)} \leq B_2^{j+1} m_j^{(3)}. \tag{7}$$

If $f \in \mathcal{B}_{B_0}^{M^{(1)}}(V_U)$, then there exist constants $K \geq 1$, $c_1, c_2 > 0$ and $0 < \varepsilon_0 \leq \frac{1}{2}$ and functions $f_{\lambda,\varepsilon} \in \mathcal{H}(\Omega_\varepsilon) \cap C^0(\overline{\Omega}_\varepsilon)$ such that for all $0 < \varepsilon \leq 2\varepsilon_0$ and all $\lambda \in \Lambda_U$,

$$\|f_{\lambda,\varepsilon}\|_{\Omega_\varepsilon} \leq K, \quad \|f_\lambda - f_{\lambda,\varepsilon}\|_{[-1,1]} \leq c_1 h_{m^{(3)}}(c_2 \varepsilon). \tag{8}$$

The constants $K, c_1, c_2, \varepsilon_0$ are independent of λ and ε, in particular, $c_2 = CB_0 B_1$, where C is an absolute constant.

(ii) *Let $N^{(i)}$, $1 \leq i \leq 3$, be weight sequences with $(n_k^{(i)})^{1/k} \to \infty$ and $\mathrm{mg}(N^{(i)}, N^{(i+1)}) = D^{(i)} < \infty$. Assume that there exist constants $K \geq 1$, $c_1, c_2 > 0$ and $0 < \varepsilon_0 \leq \frac{1}{2}$ and functions $f_{\lambda,\varepsilon} \in \mathcal{H}(\Omega_\varepsilon) \cap C^0(\overline{\Omega}_\varepsilon)$ such that for all $0 < \varepsilon \leq 2\varepsilon_0$ and all $\lambda \in \Lambda_U$,*

$$\|f_{\lambda,\varepsilon}\|_{\Omega_\varepsilon} \leq K, \quad \|f_\lambda - f_{\lambda,\varepsilon}\|_{[-1,1]} \leq c_1 h_{n^{(1)}}(c_2 \varepsilon), \tag{9}$$

where the constants $K, c_1, c_2, \varepsilon_0$ are independent of λ and ε. Then $f|_U \in \mathcal{B}_\sigma^{N^{(3)}}(U)$, where $\sigma := \frac{4e^2 D^{(1)} D^{(2)} c_2}{a}$ and $a > 0$ is an absolute constant.

Proof. (i) The assumption $f \in \mathcal{B}_{B_0}^{M^{(1)}}(V_U)$ implies

$$\sup_{\lambda \in \Lambda_U} \|f_\lambda^{(k)}\|_{[-1,1]} \leq C_0 B_0^k M_k^{(1)}, \quad k \in \mathbb{N}.$$

Then there are, by [3, Proposition 3.12] (thanks to (6) and (7)), constants $c_1, c_2 > 0$ and a function $F_\lambda \in C_c^1(\mathbb{C})$ extending f_λ such that

$$|\overline{\partial} F_\lambda(z)| \leq c_1 h_{m^{(3)}}(c_2 d(z, [-1,1])), \quad z \in \mathbb{C}. \tag{10}$$

Note that the constants $c_1 = c_1(C_0, B_0, B_1, B_2)$ and $c_2 = 12 B_0 B_1$ are independent of λ. By multiplication with a suitable cut-off function, we may assume that the support of F_λ is contained in the disk D centered at 0 with radius 2. Thus,

$$F_\lambda(z) = \frac{1}{2\pi i} \int_D \frac{\overline{\partial} F_\lambda(\zeta)}{\zeta - z} \, d\zeta \wedge d\overline{\zeta}, \quad z \in \mathbb{C},$$

and hence, in view of (10), $\|F_\lambda\|_{\mathbb{C}}$ is uniformly bounded in λ.

The function $w_{\lambda,\varepsilon} := \overline{\partial} F_\lambda \, \mathbf{1}_{\Omega_\varepsilon}$ satisfies

$$\|w_{\lambda,\varepsilon}\|_{\mathbb{C}} \leq c_1 h_{m^{(3)}}(C c_2 \varepsilon),$$

where $C > 0$ is an absolute constant such that $d(z, [-1,1]) \leq C\varepsilon$ for $z \in \Omega_\varepsilon$. Then the bounded continuous function $v_{\lambda,\varepsilon} = \mathcal{K} * w_{\lambda,\varepsilon}$, where $\mathcal{K}$ is the Cauchy kernel in $\mathbb{C}$, satisfies $\overline{\partial} v_{\lambda,\varepsilon} = w_{\lambda,\varepsilon}$ in the distributional sense in $\mathbb{C}$ and $\|v_{\lambda,\varepsilon}\|_{\mathbb{C}} \leq C \|w_{\lambda,\varepsilon}\|_{\mathbb{C}}$, where $C > 0$ again is an absolute constant. So $f_{\lambda,\varepsilon} := F_\lambda - v_{\lambda,\varepsilon}$ is holomorphic on Ω_ε, continuous on $\overline{\Omega}_\varepsilon$, and fulfills (8).

(ii) For $\lambda \in \Lambda_U$ and $0 < \varepsilon \leq \varepsilon_0$ consider $g_{\lambda,\varepsilon} := f_{\lambda,\varepsilon} - f_{\lambda,2\varepsilon} \in \mathcal{H}(\Omega_\varepsilon) \cap C^0(\overline{\Omega}_\varepsilon)$ which satisfy $\|g_{\lambda,\varepsilon}\|_{\Omega_\varepsilon} \leq 2K$ and $\|g_{\lambda,\varepsilon}\|_{[-1,1]} \leq 2c_1 h_{n^{(1)}}(2c_2\varepsilon)$, by (9). Then Lemma 2.2 implies, with $c_3 := 2\max\{c_1, K\}$,

$$\|g_{\lambda,\varepsilon}\|_{\Omega_{\varepsilon/2}} \leq c_3 \, h_{n^{(2)}}(2e D^{(1)} c_2 \varepsilon),$$

for all $\lambda \in \Lambda_U$ and $0 < \varepsilon \le \varepsilon_0$. There exists a (universal) constant $a > 0$ such that the closed disk with radius $a\varepsilon$ around any $t \in [-\frac{1}{2}, \frac{1}{2}]$ is contained in $\Omega_{\varepsilon/2}$. The Cauchy estimates and (4) yield

$$\|g_{\lambda,\varepsilon}^{(j)}\|_{[-\frac{1}{2},\frac{1}{2}]} \le \frac{c_3\, j!\, h_{n^{(2)}}(2eD^{(1)}c_2\varepsilon)}{(a\varepsilon)^j}$$

$$\le c_3\Big(\frac{2eD^{(1)}D^{(2)}c_2}{a}\Big)^j N_j^{(3)}\, h_{n^{(3)}}(2eD^{(1)}D^{(2)}c_2\varepsilon),$$

which means that

$$\|g_{\lambda,\varepsilon}\|_{[-\frac{1}{2},\frac{1}{2}],\rho}^{N^{(3)}} \le c_3\, h_{n^{(3)}}(2eD^{(1)}D^{(2)}c_2\varepsilon)$$

for all $\lambda \in \Lambda_U$ and $0 < \varepsilon \le \varepsilon_0$, where $\rho := \frac{2eD^{(1)}D^{(2)}c_2}{a}$. In an analogous way the first bound in (9) gives

$$\|f_{\lambda,\varepsilon_0}^{(j)}\|_{[-\frac{1}{2},\frac{1}{2}]} \le \frac{K}{(a\varepsilon_0)^j}j! \le AK\,\rho^j N_j^{(3)}.$$

Indeed, the assumption $(n_k^{(3)})^{1/k} \to \infty$ entails that for all $\tau > 0$ there exists A such that $j! \le A\,\tau^j N_j^{(3)}$ for all j; so take $\tau := 2eD^{(1)}D^{(2)}c_2\varepsilon_0$. It follows that

$$g_\lambda := f_{\lambda,\varepsilon_0} + \sum_{j=1}^{\infty} g_{\lambda,\varepsilon_0 2^{-j}} = f_{\lambda,\varepsilon_0} + \sum_{j=1}^{\infty}(f_{\lambda,\varepsilon_0 2^{-j}} - f_{\lambda,\varepsilon_0 2^{-j+1}})$$

converges absolutely in the Banach space $\mathcal{B}_\rho^{N^{(3)}}([-\frac{1}{2},\frac{1}{2}])$ for each $\lambda \in \Lambda_U$, and

$$\sup_{\lambda \in \Lambda_U} \|g_\lambda\|_{[-\frac{1}{2},\frac{1}{2}],\rho}^{N^{(3)}} < \infty. \tag{11}$$

We have $f_\lambda = g_\lambda$ on $[-\frac{1}{2}, \frac{1}{2}]$, since $g_\lambda = f_{\lambda,\varepsilon_0 2^{-k}} + \sum_{j=k+1}^{\infty}(f_{\lambda,\varepsilon_0 2^{-j}} - f_{\lambda,\varepsilon_0 2^{-j+1}})$ for every $k \in \mathbb{N}$, and so, for $t \in [-\frac{1}{2}, \frac{1}{2}]$,

$$|f_\lambda(t) - g_\lambda(t)| \le |f_\lambda(t) - f_{\lambda,\varepsilon_0 2^{-k}}(t)| + \Big|\sum_{j=k+1}^{\infty}(f_{\lambda,\varepsilon_0 2^{-j}}(t) - f_{\lambda,\varepsilon_0 2^{-j+1}}(t))\Big|$$

which tends to 0 as $k \to \infty$, by (9) and absolute convergence of the sum. Now if $\lambda(t) = x + tv$ then $d_v^k f(x) = f_\lambda^{(k)}(0) = g_\lambda^{(k)}(0)$. Thus, (11) and the polarization inequality (2), imply $f|_U \in \mathcal{B}_\sigma^{N^{(3)}}(U)$ with $\sigma := 2e\rho$.   □

2.3. **Proof of Theorem 1.1.** The next proposition extends [11, Lemma 6.1]; again uniformity in λ is crucial.

Proposition 2.4. *Let $U \subseteq \mathbb{R}^d$ be open and $f : V_U \to \mathbb{C}$ any function. Let j be a positive integer. Let $M^{(i)}$, $1 \le i \le k$ with $k := \lceil \log_2(j(j+1)) \rceil + 7$, be weight sequences satisfying $(m_\ell^{(i)})^{1/\ell} \to \infty$ and*

$$\exists B \ge 1\ \forall t > 0 : \overline{\Gamma}_{m^{(2)}}(Bt) \le \underline{\Gamma}_{m^{(1)}}(t),$$

$$\mathrm{mg}(M^{(i)}, M^{(i+1)}) < \infty, \quad for\ 2 \le i \le k-1.$$

There is a constant $D > 0$ such that $f^j, f^{j+1} \in \mathcal{B}_\rho^{M^{(1)}}(V_U)$ implies $f \in \mathcal{B}_{D\rho}^{M^{(k)}}(U)$. As a consequence $f^j, f^{j+1} \in \mathcal{B}^{[M^{(1)}]}(V_U)$ implies $f \in \mathcal{B}^{[M^{(k)}]}(U)$.

Proof. By the classical theorem of Joris, we may conclude that f is smooth.

Set $g := f^j$ and $h := f^{j+1}$ and consider g_λ and h_λ for $\lambda \in \Lambda_U$. (We may assume $j \geq 2$; otherwise the assertion is trivial.) By Theorem 2.3(i), there exist constants $K \geq 1$, $c_1, c_2 > 0$ and $0 < \varepsilon_0 \leq \frac{1}{2}$ and functions $g_{\lambda,\varepsilon}, h_{\lambda,\varepsilon} \in \mathcal{H}(\Omega_\varepsilon) \cap C^0(\overline{\Omega}_\varepsilon)$ such that for all $0 < \varepsilon \leq 2\varepsilon_0$ and all $\lambda \in \Lambda_U$,

$$\max\{\|g_{\lambda,\varepsilon}\|_{\Omega_\varepsilon}, \|h_{\lambda,\varepsilon}\|_{\Omega_\varepsilon}\} \leq K, \tag{12}$$

$$\max\{\|g_\lambda - g_{\lambda,\varepsilon}\|_{[-1,1]}, \|h_\lambda - h_{\lambda,\varepsilon}\|_{[-1,1]}\} \leq c_1 h_{m^{(3)}}(c_2\varepsilon). \tag{13}$$

Using $|a^\ell - b^\ell| \leq \ell \max\{|a|, |b|\}^{\ell-1}|a-b|$, we find that $g_{\lambda,\varepsilon}^{j+1} - h_{\lambda,\varepsilon}^j \in \mathcal{H}(\Omega_\varepsilon) \cap C^0(\overline{\Omega}_\varepsilon)$ satisfies

$$|g_{\lambda,\varepsilon}^{j+1} - h_{\lambda,\varepsilon}^j| \leq |g_{\lambda,\varepsilon}^{j+1} - g_\lambda^{j+1}| + |h_\lambda^j - h_{\lambda,\varepsilon}^j| \leq c_3 h_{m^{(3)}}(c_2\varepsilon), \quad \text{on } [-1,1].$$

The positive constant c_3 is again independent of λ and ε as are all further constants c_i appearing below. Thus Lemma 2.2 implies

$$\|h_{\lambda,\varepsilon}^j - g_{\lambda,\varepsilon}^{j+1}\|_{\Omega_{\varepsilon/2}} \leq c_4 h_{m^{(4)}}(Cec_2\varepsilon) =: \delta_\varepsilon, \tag{14}$$

where C fulfills $C \geq \mathrm{mg}(M^{(i)}, M^{(i+1)})$ for all $2 \leq i \leq k-1$. Set $r_\varepsilon := \delta_\varepsilon^{\frac{1}{j+1}}$. We may assume that $\varepsilon_0 > 0$ is chosen such that $\delta_\varepsilon \leq r_\varepsilon \leq \frac{1}{2}$ for all $0 < \varepsilon \leq 2\varepsilon_0$.

Consider the continuous function

$$u_{\lambda,\varepsilon} := \varphi_\varepsilon \frac{\overline{g}_{\lambda,\varepsilon} h_{\lambda,\varepsilon}}{\max\{|g_{\lambda,\varepsilon}|, r_\varepsilon\}^2},$$

where φ_ε is a smooth function compactly supported in Ω_ε and 1 on $\Omega_{\varepsilon/2}$. Note that $u_{\lambda,\varepsilon} = h_{\lambda,\varepsilon}/g_{\lambda,\varepsilon}$ in $\Omega_{\varepsilon/2} \cap \{|g_{\lambda,\varepsilon}| > r_\varepsilon\}$.

The uniformity of (12), (13), and (14) allows us to apply Lemmas 4.2.1 to 4.2.4 in [17] and conclude the following: for all $\lambda \in \Lambda_U$ and $0 < \varepsilon \leq 2\varepsilon_0$,

$$\|u_{\lambda,\varepsilon}\|_{\Omega_{\varepsilon/2}} \leq (2K)^{1/j}, \tag{15}$$

$$\|f_\lambda - u_{\lambda,\varepsilon}\|_{[-1,1]} \leq c_5 r_\varepsilon^{1/j}, \tag{16}$$

where $c_5 = c_5(K, j)$, cf. [17, Lemma 4.2.2]. The bounded continuous function $v_{\lambda,\varepsilon} = \mathcal{K} * (\overline{\partial} u_{\lambda,\varepsilon} \mathbf{1}_{\Omega_{\varepsilon/2}})$ satisfies $\overline{\partial} v_{\lambda,\varepsilon} = \overline{\partial} u_{\lambda,\varepsilon} \mathbf{1}_{\Omega_{\varepsilon/2}}$ in the distributional sense in $\mathbb{C}$ and

$$\|v_{\lambda,\varepsilon}\|_{\Omega_{\varepsilon/2}} \leq c_6 \delta_\varepsilon^{1/s} \tag{17}$$

where s is any real number with $s > j(j+1)$ and $c_6 = c_6(K, j, s)$, cf. [17, Lemma 4.2.4] and also [17, Lemma 3.1.1].

Then $f_{\lambda,\varepsilon} := u_{\lambda,2\varepsilon} - v_{\lambda,2\varepsilon}$ is holomorphic in Ω_ε and continuous on $\mathbb{C}$. By (15), (16), and (17), $\|f_{\lambda,\varepsilon}\|_{\Omega_\varepsilon}$ is uniformly bounded and

$$\|f_\lambda - f_{\lambda,\varepsilon}\|_{[-1,1]} \leq c_7 \delta_{2\varepsilon}^{1/s} = c_7\big(c_4 h_{m^{(4)}}(2eCc_2\varepsilon)\big)^{1/s},$$

for all $\lambda \in \Lambda_U$ and $0 < \varepsilon \leq \varepsilon_0$. Putting $s := 2^{k-6} =: 2^\ell$ and applying (5) repeatedly, we find

$$\|f_\lambda - f_{\lambda,\varepsilon}\|_{[-1,1]} \leq c_7 c_4^{1/s} h_{m^{(k-2)}}(2c_2(eC)^{\ell+1}\varepsilon).$$

So, by Theorem 2.3(ii) (paying attention to the dependence of the constants), we may conclude that there is a uniform constant $D > 0$ such that $f \in \mathcal{B}_{D\rho}^{M^{(k)}}(U)$ as claimed. □

Proof of Theorem 1.1. It suffices to show that f is locally of class $\mathcal{E}^{[\mathfrak{M}]}$. Up to an affine transformation, we may assume that $g := f^j$ and $h := f^{j+1}$ belong to $\mathcal{B}^{[\mathfrak{M}]}(V_U)$ and we will show that $f \in \mathcal{B}^{[\mathfrak{M}]}(U)$, where U is the open unit ball in $\mathbb{R}^d$.

In the Roumieu case there exists $M^{(1)} \in \mathfrak{M}$ such that $g, h \in \mathcal{B}^{\{M^{(1)}\}}(V_U)$. By R-admissibility of $\mathfrak{M}$, we find sequences $M^{(i)} \in \mathfrak{M}$ satisfying the assumptions of Proposition 2.4 which implies that $f \in \mathcal{B}^{\{M^{(k)}\}}(U)$.

In the Beurling case we fix any $M \in \mathfrak{M}$ and show that $f \in \mathcal{B}^{(M)}(U)$. Now B-admissibility of $\mathfrak{M}$ provides sequences $M^{(i)} \in \mathfrak{M}$ as required in Proposition 2.4 with $M^{(k)} = M$. As $g, h \in \mathcal{B}^{(M^{(1)})}(V_U)$, Proposition 2.4 gives $f \in \mathcal{B}^{(M)}(U)$. $\qquad\square$

Remark 2.5. Let j be some positive integer. The map $\varphi : \mathbb{C}^U \to \mathbb{C}^U \times \mathbb{C}^U$, $f \mapsto (f^j, f^{j+1})$ is injective. So there exists an inverse $\psi := \varphi^{-1}|_{\varphi(\mathbb{C}^U)}$. Under the assumptions of Theorem 1.1, the map ψ takes bounded sets in $\mathcal{E}^{[\mathfrak{M}]}(U) \times \mathcal{E}^{[\mathfrak{M}]}(U)$ to bounded sets in $\mathcal{E}^{[\mathfrak{M}]}(U)$, if $\mathcal{E}^{[\mathfrak{M}]}(U)$ is endowed with its natural locally convex topology; cf. [12, Sec. 4.2].

3. On Banach spaces and on convenient vector spaces

3.1. **Ultradifferentiable classes of functions on Banach spaces.** Let E be a Banach space and $U \subseteq E$ an open subset. Let $\mathfrak{M}$ be a weight matrix. For $M \in \mathfrak{M}$ and $\rho > 0$ we consider $\mathcal{B}_\rho^M(U) = \{f \in C^\infty(U, \mathbb{C}) : \|f\|_{U,\rho}^M < \infty\}$ as well as

$$\mathcal{B}^{\{\mathfrak{M}\}}(U) := \{f \in C^\infty(U, \mathbb{C}) : \exists M \in \mathfrak{M} \, \exists \rho > 0 : \|f\|_{U,\rho}^M < \infty\} \quad \text{and}$$

$$\mathcal{B}^{(\mathfrak{M})}(U) := \{f \in C^\infty(U, \mathbb{C}) : \forall M \in \mathfrak{M} \, \forall \rho > 0 : \|f\|_{U,\rho}^M < \infty\},$$

where $\|f\|_{U,\rho}^M$ is defined by (1) (with $\mathbb{R}^d$ replaced by E). Furthermore, we consider the local classes

$$\mathcal{E}^{\{\mathfrak{M}\}}(U) := \{f \in C^\infty(U, \mathbb{C}) : \forall K \subseteq_{cp} U \, \exists M \in \mathfrak{M} \, \exists \rho > 0 : \|f\|_{K,\rho}^M < \infty\} \quad \text{and}$$

$$\mathcal{E}^{(\mathfrak{M})}(U) := \{f \in C^\infty(U, \mathbb{C}) : \forall K \subseteq_{cp} U \, \forall M \in \mathfrak{M} \, \forall \rho > 0 : \|f\|_{K,\rho}^M < \infty\}.$$

Let $B(a, r) := \{x \in E : \|x - a\| < r\}$ be the open ball with center a and radius r.

Lemma 3.1. *Let $f : U \to \mathbb{C}$ be smooth. Then:*

(1) $f \in \mathcal{E}^{\{\mathfrak{M}\}}(U)$ *if and only if*

$$\forall a \in U \, \exists M \in \mathfrak{M} \, \exists \rho > 0 \, \exists r > 0 : f|_{B(a,r)} \in \mathcal{B}_\rho^M(B(a,r)). \tag{18}$$

(2) $f \in \mathcal{E}^{(\mathfrak{M})}(U)$ *if and only if*

$$\forall a \in U \, \forall M \in \mathfrak{M} \, \forall \rho > 0 \, \exists r > 0 : f|_{B(a,r)} \in \mathcal{B}_\rho^M(B(a,r)). \tag{19}$$

Proof. (1) That (18) is sufficient for $f \in \mathcal{E}^{\{\mathfrak{M}\}}(U)$ is easily seen. Fix a compact subset $K \subseteq U$. For each $a \in K$ we get $M_a \in \mathfrak{M}$, $\rho_a > 0$, and a small ball $B(a, r_a)$ such that $f|_{B(a,r_a)} \in \mathcal{B}_{\rho_a}^{M_a}(B(a, r_a))$. Now K is still covered by finitely many of the balls and we find $\|f\|_{K,\rho}^M < \infty$ with $M := \max M_a$ and $\rho := \max \rho_a$, where the maxima are taken over the corresponding a.

Let us show that $f \in \mathcal{E}^{\{\mathfrak{M}\}}(U)$ implies (18). If (18) does not hold, then there exists $a \in U$ such that

$$\forall M \in \mathfrak{M} \, \forall \rho > 0 \, \forall r > 0 \, \forall C > 0 \, \exists x \in B(a, r) \, \exists k \in \mathbb{N} : \|f^{(k)}(x)\|_{L^k(E,\mathbb{C})} \geq C\rho^k M_k.$$

By [3, Lemma 2.5], we may assume that $\mathfrak{M}$ is indexed by $\mathbb{N}$ in the following way: $\mathfrak{M} = \{M^{(\ell)} : \ell \in \mathbb{N}\}$ and $M^{(\ell_1)} \leq M^{(\ell_2)}$ if and only if $\ell_1 \leq \ell_2$. In particular, taking $C = \rho = n$, $r = \frac{1}{n}$ and $M = M^{(n)}$, we find

$$\forall n \in \mathbb{N} \,\exists x_n \text{ with } \|x_n - a\| < \tfrac{1}{n} \,\exists k_n \in \mathbb{N} : \|f^{(k_n)}(x_n)\|_{L^{k_n}(E, \mathbb{C})} \geq n^{k_n + 1} M^{(n)}_{k_n}.$$

For the compact set $K := \{x_n\}_{n \in \mathbb{N}} \cup \{a\}$ this means that $\|f\|^M_{K, \rho} = \infty$ for all choices of $M \in \mathfrak{M}$ and $\rho > 0$, contradicting $f \in \mathcal{E}^{\{\mathfrak{M}\}}(U)$.

(2) That (19) is sufficient for $f \in \mathcal{E}^{(\mathfrak{M})}(U)$ is again easy to check. For the necessity assume that (19) does not hold. Then there exist $a \in U$, $M \in \mathfrak{M}$, and $\rho > 0$ such that

$$\forall r > 0 \,\forall C > 0 \,\exists x \in B(a, r) \,\exists k \in \mathbb{N} : \|f^{(k)}(x)\|_{L^k(E, \mathbb{C})} \geq C\rho^k M_k.$$

Taking $C = n$ and $r = \frac{1}{n}$ we get

$$\forall n \in \mathbb{N} \,\exists x_n \text{ with } \|x_n - a\| < \tfrac{1}{n} \,\exists k_n \in \mathbb{N} : \|f^{(k_n)}(x_n)\|_{L^{k_n}(E, \mathbb{C})} \geq n\,\rho^{k_n} M_{k_n}.$$

Thus $\|f\|^M_{K, \rho} = \infty$ for $K := \{x_n\}_{n \in \mathbb{N}} \cup \{a\}$, contradicting $f \in \mathcal{E}^{(\mathfrak{M})}(U)$. $\qquad\square$

3.2. Theorem 1.1 holds on Banach spaces.

Let E be a Banach space and $U \subseteq E$ an open subset. Let $\mathbb{B} := \{x \in E : \|x\| \leq 1\}$ be the closed unit ball in E and $\mathbb{S} := \{x \in E : \|x\| = 1\}$ the unit sphere. Consider $V_U := \bigcup_{x \in U}(x + \mathbb{B})$.

Then Theorem 2.3 and Proposition 2.4 remain valid for functions $f : V_U \to \mathbb{C}$ (with literally the same proofs). Note that the classical Joris theorem is valid on Banach spaces by Boman's theorem.

Theorem 3.2. *Theorem 1.1 (and thus Corollary 1.2) hold if U is an open subset of a Banach space E.*

Proof. We treat Roumieu and Beurling case separately.

Roumieu case. Assume that $f^j, f^{j+1} \in \mathcal{E}^{\{\mathfrak{M}\}}(U)$. Fix $a \in U$; without loss of generality we may assume that a is the origin in E. By Lemma 3.1, there exist $M \in \mathfrak{M}$, $\rho > 0$, and $r > 0$ such that $g := f^j|_{B(0,r)}, h := f^{j+1}|_{B(0,r)} \in \mathcal{B}^M_\rho(B(0, r))$. Then $\tilde{g}(x) := g(\frac{r}{2}x)$ and $\tilde{h}(x) := h(\frac{r}{2}x)$ are elements of $\mathcal{B}^M_{\frac{r}{2}\rho}(B(0, 2))$; note that $B(0, 2) = V_{B(0,1)}$. By R-admissibility of $\mathfrak{M}$, we find sequences $M^{(i)} \in \mathfrak{M}$ with $M^{(1)} = M$ satisfying the assumptions of Proposition 2.4 which implies that the restriction of $\tilde{f}(x) := f(\frac{r}{2}x)$ to $B(0, 1)$ belongs to $\mathcal{B}^{M^{(k)}}_{D\frac{r}{2}\rho}(B(0, 1))$. Then $f|_{B(0, \frac{r}{2})} \in \mathcal{B}^{M^{(k)}}_{D\rho}(B(0, \frac{r}{2}))$ which ends the proof in view of Lemma 3.1.

Beurling case. Assume that $f^j, f^{j+1} \in \mathcal{E}^{(\mathfrak{M})}(U)$. Fix $a \in U$, $M \in \mathfrak{M}$, and $\rho > 0$; we may again assume that a is the origin. By B-admissibility of $\mathfrak{M}$, there are sequences $M^{(i)} \in \mathfrak{M}$ as required in Proposition 2.4 with $M^{(k)} = M$. Let $D > 0$ be the constant from Proposition 2.4. By Lemma 3.1, there exists $r > 0$ such that $g, h \in \mathcal{B}^{M^{(1)}}_{\frac{\rho}{D}}(B(0, r))$. Then $\tilde{g}, \tilde{h} \in \mathcal{B}^{M^{(1)}}_{\frac{r\rho}{2D}}(B(0, 2))$ (using the notation from the previous paragraph) and Proposition 2.4 implies $\tilde{f} \in \mathcal{B}^{M^{(k)}}_{\frac{r\rho}{2}}(B(0, 1))$ and, consequently, $f|_{B(0, \frac{r}{2})} \in \mathcal{B}^M_\rho(B(0, \frac{r}{2}))$. Invoking Lemma 3.1 we are done. $\qquad\square$

3.3. Theorem 1.1 holds on convenient vector spaces.

In [8, 9, 10] and [16] the calculus of $\mathcal{E}^{[\mathfrak{M}]}$-maps has been extended to maps between *convenient vector spaces*, i.e., locally convex spaces that are Mackey complete. We refer the reader to

[7] for a comprehensive treatment of convenient vector spaces and the c^∞-topology on them.

Let E be a convenient vector space and $U \subseteq E$ a c^∞-open subset. Let $\mathfrak{M}$ be a weight matrix. Then $\mathcal{E}^{[\mathfrak{M}]}(U)$ is by definition the set of all smooth functions $f : U \to \mathbb{C}$ (i.e. smooth on smooth curves in U) such that for each closed absolutely convex bounded subset B of E we have

$$f \circ i_B \in \mathcal{E}^{[\mathfrak{M}]}(U_B),$$

where $i_B : E_B \to E$ is the inclusion of the linear span E_B of B in E and $U_B := U \cap E_B$. Note that E_B equipped with the Minkowski functional $p_B(x) = \inf\{t > 0 : x \in tB\}$ is a Banach space, since E is convenient.

Theorem 3.3. *Theorem 1.1 (and thus Corollary 1.2) hold if U is an c^∞-open subset of a convenient vector space E.*

Proof. This is obvious by the definition of $\mathcal{E}^{[\mathfrak{M}]}(U)$ and Theorem 3.2. □

4. More general nonlinear conditions

As seen above the map $\Phi(t) = (t^j, t^{j+1})$, where j is some positive integer, has the property

$$\Phi \circ f \in \mathcal{E}^{[\mathfrak{M}]} \implies f \in \mathcal{E}^{[\mathfrak{M}]}, \tag{20}$$

provided that $\mathfrak{M}$ is an [admissible] weight matrix. Here t is a complex or a real variable. Equivalently, the map $\Phi(t) = (t^p, t^q)$, where p and q are positive coprime integers, has property (20); indeed, all integers $j \geq pq$ have the form $j = ap + bq$ with $a, b \in \mathbb{N}$.

The question arises which maps Φ have the property (20) for all continuous f. From now on we take continuity of f as a basic assumption. (The continuity of f was not an issue in Theorem 1.1, since one of the powers is necessarily odd and hence has a global continuous inverse.) This problem was investigated in [2] and in [5] in the C^∞-setting. In the latter article the smooth germs $\Phi : (\mathbb{R}, 0) \to (\mathbb{R}^n, 0)$ (and $\Phi : (\mathbb{C}, 0) \to (\mathbb{C}^n, 0)$ which possess a complex Taylor series, cf. [5, p.204]) with the property that $\Phi \circ f \in C^\infty$ implies $f \in C^\infty$ for all continuous germs f with $f(0) = 0$ were characterized in terms of a condition on the support of the formal Taylor series of f. They were called *pseudo-immersive* germs.

The proof is based (similar to most proofs in [2]) on a reduction to the case $\Phi(t) = (t^p, t^q)$ with $\gcd(p, q) = 1$ which is mostly algebraic and hence applies to our situation with marginal adjustments. We thus obtain a characterization of the *analytic* germs Φ satisfying (20) for all continuous germs f with $f(0) = 0$ by the same condition on the support of the Taylor series. Note that [admissibility] of the weight matrix $\mathfrak{M}$ implies that $\mathcal{E}^{[\mathfrak{M}]}$ contains all analytic maps and is stable under composition (by Corollaries 3.3 and 3.5 and Proposition 1.1 in [3]). See also Remark 4.4.

4.1. **A necessary condition.** Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$. Let $\Phi : (\mathbb{K}, 0) \to (\mathbb{K}^n, 0)$ be the germ of a non-zero analytic map. Then Φ is represented by its convergent Taylor series

$$\Phi(t) = \sum_{1 \leq k < N} a_k t^{n_k}, \quad \text{where } a_k \in \mathbb{K}^n \setminus \{0\} \text{ for all } k,$$

$1 \leq n_1 < n_2 < \cdots$, and $1 \leq N \leq \infty$. That means that $\{n_1, n_2, \ldots\}$ is the *support* of the power series $\Phi(t)$.

If (20) holds for all continuous germs f with $f(0) = 0$, then $\gcd(n_1, n_2, \ldots) = 1$. Indeed, if there are integers $p \geq 2$ and ℓ_k such that $n_k = p\ell_k$ for all k, then the power series $\sum_k a_k x^{\ell_k}$ is convergent, defines an analytic germ Ψ so that $\Phi(t) = \Psi(t^p)$, and it is easy to find a continuous germ $f \notin C^1$ with $f(0) = 0$ such that $\Phi \circ f$ is analytic; cf. [5, Theorem 2].

4.2. **A characterization.** We will see that the condition $\gcd(n_1, n_2, \ldots) = 1$ is also sufficient for (20) at least up to equivalence of analytic germs. Two analytic germs $\Phi, \Psi : (\mathbb{K}, 0) \to (\mathbb{K}^n, 0)$ are said to be *equivalent* if there exist germs of analytic diffeomorphisms $u : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$ and $v : (\mathbb{K}, 0) \to (\mathbb{K}, 0)$ such that $u \circ \Phi \circ v = \Psi$. Equivalent germs either both satisfy or do not satisfy (20). Any non-zero analytic germ $\Phi : (\mathbb{K}, 0) \to (\mathbb{K}^n, 0)$ is equivalent to a germ whose first component is a positive power. So it is no restriction to assume $\Phi_1(t) = t^p$.

Theorem 4.1. *Let $\Phi = (\Phi_1, \ldots, \Phi_n) : (\mathbb{K}, 0) \to (\mathbb{K}^n, 0)$ be an analytic germ such that $\Phi_1(t) = t^p$ and let $\{n_1, n_2, \ldots\}$ be the support of the power series $\Phi(t)$. Then the following conditions are equivalent:*

 (1) *Let $\mathfrak{M}$ be an [admissible] weight matrix. If $f : (\mathbb{R}^d, 0) \to (\mathbb{K}, 0)$ is a function germ and $\Phi \circ f$ is of class $\mathcal{E}^{[\mathfrak{M}]}$, then f is of class $\mathcal{E}^{[\mathfrak{M}]}$.*
 (2) *Let ω be a concave weight function. If $f : (\mathbb{R}^d, 0) \to (\mathbb{K}, 0)$ is a function germ and $\Phi \circ f$ is of class $\mathcal{E}^{[\omega]}$, then f is of class $\mathcal{E}^{[\omega]}$.*
 (3) $\gcd(n_1, n_2, \ldots) = 1$.

Proof. Since (2) is a special case of (1) (cf. [11, Section 4.5]) and the necessity of (3) for (2) follows from the discussion above, it remains to show that (3) implies (1). This is achieved by adjusting the arguments in the proofs of Theorems 3 and 3' in [5]; actually the proof simplifies a bit, since all series involved in the arguments are convergent.

We may assume that $p \geq 2$ and $n = 2$. In fact, if $p = 1$ there is nothing to prove and if $n > 2$ then one can find real constants γ_i such that the support of the power series $(t^p, \gamma_2\Phi_2(t) + \cdots + \gamma_n\Phi_n(t))$ satisfies (3).

Thus we may assume that $\Phi(t) = (t^p, \varphi(t))$, where $\varphi(t) = \sum_{1 \leq k < N} a_k t^{n_k}$ with $a_k \in \mathbb{K} \setminus \{0\}$ for all k. Then (3) takes the form $\gcd(p, n_1, n_2, \ldots) = 1$. The idea is to show that there exist a positive integer q and analytic germs α_j at 0 such that

$$t^{1+pq} = \sum_{j=0}^{p-1} \alpha_j(t^p)\varphi(t)^j. \tag{21}$$

Then (1) follows from Theorem 1.1 (by plugging f into (21)).

We will sketch how to establish (21) following the steps in [5]; the main difference is that in [5] all involved series are formal (power or Laurent) series so that at the end an application of Borel's lemma and some handling of the flat terms is required.

Let z be an indeterminate. Recall that $\mathbb{K}\{z\}$ denotes the ring of convergent power series in z with coefficients in $\mathbb{K}$. The convergent power series $\varphi(z) = \sum_{1 \leq k < N} a_k z^{n_k}$ has the form

$$\varphi(z) = \varphi_0(z^p) + z\varphi_1(z^p) + \cdots + z^{p-1}\varphi_{p-1}(z^p)$$

where $\varphi_j(z^p) \in \mathbb{K}\{z^p\}$ for all $j = 0, \ldots, p-1$. Introducing another independent indeterminate x, set

$$S(x) := \varphi_0(z^p) + x\varphi_1(z^p) + \cdots + x^{p-1}\varphi_{p-1}(z^p) \in \mathbb{K}\{z^p\}[x].$$

Furthermore, define

$$G_{kj}(x) := S(x)^j x^{kp}, \quad j = 0, 1, \ldots, p-1, \ k = 0, 1, \ldots, (p-1)^2.$$

Let $\mathbb{K}\{\{z\}\}$ denote the field of fractions of $\mathbb{K}\{z\}$. It is easy to see that $\mathbb{K}\{\{z\}\} = \{z^n g(z) : n \in \mathbb{Z}, \ g(z) \in \mathbb{K}\{z\}\}$. We look for nontrivial $a_{kj} \in \mathbb{K}\{\{z^p\}\}$ such that

$$\sum_{j=0}^{p-1} \sum_{k=0}^{(p-1)^2} a_{kj} G_{kj}(x) = x H(x^p)$$

with $H(x^p) \in \mathbb{K}\{\{z^p\}\}[x^p]$. This amounts to an underdetermined homogeneous system of linear equations over the field $\mathbb{K}\{\{z^p\}\}$; see [5, p.201] for details. So we find $a_{kj} \in \mathbb{K}\{\{z^p\}\}$ for $j = 0, 1, \ldots, p-1$, $k = 0, 1, \ldots, r$, and some $r \geq 0$, not all zero, and $H(x^p) \in \mathbb{K}\{\{z^p\}\}[x^p]$ such that

$$\sum_{j=0}^{p-1} \sum_{k=0}^{r} a_{kj} G_{kj}(x) = x H(x^p).$$

Regrouping the terms on the left-hand side, we obtain $A_j(x^p) \in \mathbb{K}\{\{z^p\}\}[x^p]$, $j = 0, 1, \ldots, p-1$, not all zero, such that

$$\sum_{j=0}^{p-1} A_j(x^p) S(x)^j = x H(x^p). \tag{22}$$

Taking r as small as possible, at least one of the series $A_0(z^p)$, ... , $A_{p-1}(z^p)$, which are obtained by replacing x^p with z^p, is non-zero and also $H(z^p)$ is non-zero. Indeed, if $H(z^p) = 0$ then (22) shows that $\varphi(z) = S(z)$ is algebraic over $\mathbb{K}\{\{z^p\}\} \subseteq \mathbb{K}((z^p)) \subseteq \mathbb{C}((z^p))$ of degree at most $p-1$; here $\mathbb{K}((z))$ is the field of formal Laurent series in z with coefficients in $\mathbb{K}$. The automorphism σ of $\mathbb{C}((z))$ induced by $z \mapsto \xi z$, where $\xi = e^{2\pi i/p}$, leaves invariant $\mathbb{C}((z^p))$ and generates the cyclic group $\langle \sigma \rangle$. Thus $\varphi(z)$ (as an element of $\mathbb{C}((z))$) is invariant under a non-trivial subgroup of $\langle \sigma \rangle$, say $\langle \sigma^b \rangle$ with $1 \leq b \leq p-1$ and $b|p$. Then

$$\sum_{1 \leq k < N} a_k z^{n_k} = \sum_{1 \leq k < N} a_k \xi^{bn_k} z^{n_k}$$

which implies $\xi^{bn_k} = 1$ for all k, since all $a_k \neq 0$. But that means that p/b divides all n_k, contradicting $\gcd(p, n_1, n_2, \ldots) = 1$. Cf. [5, p.202].

Multiplying (22) by a suitable power of z^p, we may suppose that the coefficients of A_j and H belong to $z^p \mathbb{K}\{z^p\}$. Substituting z for x, we find

$$\sum_{j=0}^{p-1} A_j(z^p) \varphi(z)^j = z H(z^p)$$

with $A_j(z^p)$ and $H(z^p)$ belonging to $z^p \mathbb{K}\{z^p\}$. Since $H(z^p) \neq 0$, there exist $q \in \mathbb{N}_{\geq 1}$, $c \in \mathbb{K} \setminus \{0\}$, and $H_1(z^p) \in \mathbb{K}\{z^p\}$ such that $H(z^p) = z^{pq}(c + z^p H_1(z^p))$ from which it is easy to conclude (21). $\square$

Remark 4.2. Let Φ be as in Theorem 4.1 with $\gcd(n_1, n_2, \ldots) = 1$ and assume that $\mathfrak{M}$ is an [admissible] weight matrix. Then the map $\mathcal{E}^{[\mathfrak{M}]} \ni \Phi \circ f \mapsto f \in \mathcal{E}^{[\mathfrak{M}]}$ in (1) takes bounded sets to bounded sets. This follows from Remark 2.5 and (21), since the superposition operator $g \mapsto h \circ g$ is bounded, even continuous, among

$\mathcal{E}^{[\mathfrak{M}]}$ (in finite dimensions), provided that $\mathcal{E}^{[\mathfrak{M}]}$ is stable under composition; see [12, Theorem 4.13].

Remark 4.3. Let g be a continuous germ at $0 \in \mathbb{R}$ such that $g(0) = 1$. Then $f := 1/g$ and $h := f - 1$ are continuous germs and $h(0) = 0$. Consider the analytic germ $\varphi(t) = \frac{1}{1+t} - 1 = \sum_{k \geq 1}(-1)^k t^k$. Then $\varphi \circ h = g - 1$. Any analytic germ $\Phi : (\mathbb{R}, 0) \to (\mathbb{R}^n, 0)$ that has φ as a component evidently satisfies the condition 4.1(3). Now if g is of class $\mathcal{E}^{[\mathfrak{M}]}$ and (20) holds, then h is of class $\mathcal{E}^{[\mathfrak{M}]}$ and consequently also $f = 1/g$. That means that (20) for analytic Φ implies that $\mathcal{E}^{[\mathfrak{M}]}$ is *inverse-closed.*

In view of [3, Theorem 4.8] we infer that concavity of the weight function ω (up to equivalence) is a necessary condition for the conclusions of Corollary 1.2 and Theorem 4.1(2).

Next we want to discuss implications of the conclusions of Theorem 1.1 and Theorem 4.1(1) and compare them with the assumption of admissibility for the weight matrix $\mathfrak{M}$; our goal is to convince the reader that admissibility is not only sufficient but also "close to" necessary. Suppose that

$$m_k^{1/k} \to \infty \quad \text{for all } M \in \mathfrak{M}; \tag{23}$$

this guarantees that $\mathcal{E}^{[\mathfrak{M}]}$ contains the real analytic class. Consider also the conditions

$$\forall M \in \mathfrak{M} \, \exists N \in \mathfrak{N} : \sup_k \left(\frac{M_{k+1}}{N_k}\right)^{1/k} < \infty \qquad \text{in the Roumieu case,}$$
$$\forall M \in \mathfrak{M} \, \exists N \in \mathfrak{N} : \sup_k \left(\frac{N_{k+1}}{M_k}\right)^{1/k} < \infty \qquad \text{in the Beurling case,} \tag{24}$$

which characterize stability under derivation of $\mathcal{E}^{[\mathfrak{M}]}$.

Fact i: Assuming (24), the conclusions of Theorem 1.1 and Theorem 4.1(1) imply that $\mathcal{E}^{[\mathfrak{M}]}$ is stable under composition.

Indeed, we saw that (20) for analytic Φ entails that $\mathcal{E}^{[\mathfrak{M}]}$ is inverse-closed which in turn implies (see [12, Theorem 4.9 and 4.11] and [13])

$$\forall M \in \mathfrak{M} \, \exists N \in \mathfrak{N} : \sup_{j \leq k} \frac{m_j^{1/j}}{n_k^{1/k}} < \infty \qquad \text{in the Roumieu case,}$$
$$\forall M \in \mathfrak{M} \, \exists N \in \mathfrak{N} : \sup_{j \leq k} \frac{n_j^{1/j}}{m_k^{1/k}} < \infty \qquad \text{in the Beurling case.} \tag{25}$$

Now (25), in conjunction with (24), yields that $\mathcal{E}^{[\mathfrak{M}]}$ is stable under composition (*loc. cit.*). Conversely, since $\mathcal{E}^{[\mathfrak{M}]}$ contains the real analytic class (thanks to (23)), stability under composition entails inverse-closedness and hence (25). Note that (24) follows from the moderate growth part in the definition of an [admissible] weight matrix.

Fact ii: Assuming (23) and (24), the other part in the definition of admissibility, i.e. the control of $\overline{\Gamma}_n$ by $\underline{\Gamma}_m$ (resp. $\overline{\Gamma}_m$ by $\underline{\Gamma}_n$), also implies stability under composition.

This follows from almost analytic extension (see [3, Remark 2.7]). But the property "control of $\overline{\Gamma}_n$ by $\underline{\Gamma}_m$" seems not to be *substantially* stronger than stability under composition. This is evidenced by Fact iii and Fact iv.

Fact iii: For any weight function ω, the class $\mathcal{E}^{[\omega]}$ being stable under composition implies the existence of a weight matrix $\mathfrak{M}$ with $\mathcal{E}^{[\omega]} = \mathcal{E}^{[\mathfrak{M}]}$ such that m is log-convex and thus $\overline{\Gamma}_m = \underline{\Gamma}_m$ for all $M \in \mathfrak{M}$.

See [3, Theorem 4.8] and [15, Proposition 3]); in this case (23) and moderate growth (hence (24)) are automatic.

Fact iv: If a general weight matrix $\mathfrak{M}$ fulfills (23), (25), and

$$
\begin{aligned}
&\forall M \in \mathfrak{M} \; \exists N \in \mathfrak{N} : \sup_k \frac{m_k}{m_{k-1}} n_k^{-1/k} < \infty && \text{in the Roumieu case,} \\
&\forall M \in \mathfrak{M} \; \exists N \in \mathfrak{N} : \sup_k \frac{n_k}{n_{k-1}} m_k^{-1/k} < \infty && \text{in the Beurling case,}
\end{aligned}
\tag{26}
$$

then $\mathfrak{M}$ admits the desired control of $\overline{\Gamma}_n$ by $\underline{\Gamma}_m$.

Indeed, in the Roumieu case, for given $M \in \mathfrak{M}$ there exist $L, N \in \mathfrak{M}$ and constants $C_i > 0$ such that for all $j \leq k$,

$$
\frac{m_j}{m_{j-1}} \overset{(26)}{\leq} C_1 \ell_j^{1/j} \overset{(25)}{\leq} C_2 n_k^{1/k} \leq C_3 \frac{n_k}{n_{k-1}};
$$

the last inequality it true for every weight sequence. It follows that $\overline{\Gamma}_n(C_3 t) \leq \underline{\Gamma}_m(t)$ for all $t > 0$ (see [14, Lemma 3.3]). The Beurling case is analogous.

Remark 4.4. In principle, one can apply the above method also to the more general case that Φ is the germ of a $\mathcal{E}^{[\mathfrak{M}]}$-mapping, but the ramification z^p causes a loss of regularity and, since the series are formal, an ultradifferentiable Borel lemma must be used which introduces flat terms whose handling again entails a loss of regularity.

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Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, A-1090 Wien, Austria.

Email address: `david.nicolas.nenning@univie.ac.at`

Email address: `armin.rainer@univie.ac.at`

Email address: `gerhard.schindl@univie.ac.at`