EQUIVALENCE OF STABILITY PROPERTIES FOR ULTRADIFFERENTIABLE FUNCTION CLASSES

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ABSTRACT. We characterize stability under composition, inversion, and solution of ordinary differential equations for ultradifferentiable classes, and prove that all these stability properties are equivalent.

1. INTRODUCTION

Let $\mathcal{F}$ denote some class of smooth mappings between non-empty open subsets of Euclidean spaces (of possibly different dimension). We say that

- $\mathcal{F}$ *is stable under composition* if the composite of any two $\mathcal{F}$-mappings $g : U \to V$ and $f : V \to W$ is an $\mathcal{F}$-mapping $f \circ g : U \to W$.
- $\mathcal{F}$ *is stable under solving ordinary differential equations (ODEs)* if for any $\mathcal{F}$-mapping $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ the solution of the initial value problem $x' = f(t, x)$, $x(0) = x_0 \in \mathbb{R}^n$ is of class $\mathcal{F}$ wherever it exists.
- $\mathcal{F}$ *is stable under inversion* if for any $\mathcal{F}$-mapping $f : \mathbb{R}^m \supseteq U \to V \subseteq \mathbb{R}^n$ so that $f'(x_0) \in L(\mathbb{R}^m, \mathbb{R}^n)$ is invertible at $x_0 \in U$ there exist neighborhoods $x_0 \in U_0 \subseteq U$ and $f(x_0) \in V_0 \subseteq V$ and an $\mathcal{F}$-mapping $g : V_0 \to U_0$ such that $f \circ g = \mathrm{id}_{V_0}$.
- $\mathcal{F}$ *is inverse closed* if $1/f \in \mathcal{F}(U)$ for each non-vanishing $f \in \mathcal{F}(U)$.

In this paper we shall prove that all these stability properties are equivalent for classes of ultradifferentiable mappings $\mathcal{F}$ satisfying some mild regularity conditions. We will treat

- the classical Denjoy–Carleman classes $\mathcal{E}^{[M]}$ determined by a weight sequence $M = (M_k)$,
- the classes $\mathcal{E}^{[\omega]}$ introduced by Braun, Meise, and Taylor [3] determined by a weight function ω,
- the classes $\mathcal{E}^{[\mathfrak{M}]}$ introduced in [13] determined by a weight matrix $\mathfrak{M}$.

The brackets [ ] stand for either { } in the Roumieu case or for ( ) in the Beurling case. For the precise definitions we refer to Section 2.

There are classes $\mathcal{E}^{[M]}$ that cannot be given in terms of a weight function ω and vice versa; see [2]. The classes $\mathcal{E}^{[\mathfrak{M}]}$ comprise all classes $\mathcal{E}^{[M]}$ and $\mathcal{E}^{[\omega]}$ and hence allow for a unified approach to the classes $\mathcal{E}^{[M]}$ and $\mathcal{E}^{[\omega]}$. Beyond that, they provide a convenient framework to describe unions and intersections of classical Denjoy–Carleman classes.

The characterization of the aforementioned stability properties for $\mathcal{E}^{[\mathfrak{M}]}$ was important for treating $\mathcal{E}^{[\mathfrak{M}]}$-differomorphism groups in [16].

1.1. **Stability properties of $\mathcal{E}^{[M]}$.** We assume from now on that any *weight sequence* $M = (M_k)$ is positive, $1 = M_0 \leq M_1$, and $k \mapsto k!M_k$ is log-convex (alias M is weakly log-convex).

Remark 1. For any weight sequence $M = (M_k)$ the sequence $(k!M_k)^{1/k}$ is increasing and $M_j M_k \leq \binom{j+k}{j} M_{j+k}$ for all $j, k \in \mathbb{N}$.

Theorem 1. *If* $\varliminf M_k^{1/k} > 0$ *and* $\sup(\frac{M_{k+1}}{M_k})^{1/k} < \infty$ *the following are equivalent:*

(1) $M_k^{1/k}$ *is almost increasing, i.e.,* $\exists C > 0 \; \forall j \leq k : M_j^{1/j} \leq CM_k^{1/k}$.
(2) M *has the (FdB)-property, i.e.,* $\exists C > 0 : M_k^\circ \leq C^k M_k$, *where*

$$M_k^\circ := \max\{M_j M_{\alpha_1} \ldots M_{\alpha_j} : \alpha_i \in \mathbb{N}_{>0}, \alpha_1 + \cdots + \alpha_j = k\}, \quad M_0^\circ := 1.$$

(3) $\mathcal{E}^{\{M\}}$ *is stable under composition.*
(4) $\mathcal{E}^{\{M\}}$ *is stable under solving ODEs.*
(5) $\mathcal{E}^{\{M\}}$ *is stable under inversion.*
(6) $\mathcal{E}^{\{M\}}$ *is inverse-closed.*

Note that $\varliminf M_k^{1/k} > 0$ iff $C^\omega \subseteq \mathcal{E}^{\{M\}}$, and $\sup(\frac{M_{k+1}}{M_k})^{1/k} < \infty$ iff $\mathcal{E}^{[M]}$ is stable under derivation; cf. [13]. If we replace the first condition by $\lim M_k^{1/k} = \infty$ which is equivalent to $C^\omega \subseteq \mathcal{E}^{(M)}$, we have the corresponding Beurling type result:

Theorem 2. *If* $\lim M_k^{1/k} = \infty$ *and* $\sup(\frac{M_{k+1}}{M_k})^{1/k} < \infty$ *the following are equivalent:*

(1) $M_k^{1/k}$ *is almost increasing.*
(2) M *has the (FdB)-property.*
(3) $\mathcal{E}^{(M)}$ *is stable under composition.*
(4) $\mathcal{E}^{(M)}$ *is stable under solving ODEs.*
(5) $\mathcal{E}^{(M)}$ *is stable under inversion.*
(6) $\mathcal{E}^{(M)}$ *is inverse-closed.*

Most implications of Theorems 1 and 2 are basically known, but scattered in the literature.

The equivalence of (1) and (6) is due to Rudin [15] in the Roumieu case and to Bruna [4] in the Beurling case; note that Rudin only considered non-quasianalytic classes and Hörmander dealt with the quasianalytic case (cf. [15, p. 799]). See also [17].

That (1) implies stability under inversion is due to Komatsu [10]; different proofs in the Banach space setting were given by Yamanaka [18] and Koike [7]. The sufficiency of (1) for stability under solving ODEs was obtained by Komatsu [11] and in Banach spaces by Yamanaka [19].

That the class $\mathcal{E}^{\{M\}}$ is stable under composition, provided that $M = (M_k)$ is log-convex (which implies (1)), is due to Roumieu [14]; other references are e.g. [8] and [1]. In [13] we proved the equivalence of (1), (2), and (3) (in both the Beurling and the Roumieu case).

It is worth mentioning that Dynkin [5] gave a characterization of the Roumieu classes $\mathcal{E}^{\{M\}}$ in terms of *almost holomorphic extensions*, provided that $M = (M_k)$ is

log-convex, which implies the stability properties (3), (4), (5), and (6) in a straight-forward manner.

That all the properties (1) – (6) are equivalent was, to our knowledge, not observed before.

1.2. Stability properties of $\mathcal{E}^{[\omega]}$.

The respective result in the weight function case, that is Theorems 3 and 4 below, was not known before, apart from a characterizaton for stability under composition obtained in [6] and in [13].

We henceforth assume that any *weight function* ω is a continuous increasing function $\omega : [0,\infty) \to [0,\infty)$ with $\omega|_{[0,1]} = 0$, $\lim_{t\to\infty} \omega(t) = \infty$, and so that:

(ω_1) $\qquad\qquad\qquad\qquad \omega(2t) = O(\omega(t))$ as $t \to \infty$.

(ω_2) $\qquad\qquad\qquad\qquad \log(t) = o(\omega(t))$ as $t \to \infty$.

(ω_3) $\qquad\qquad\qquad\qquad \varphi : t \mapsto \omega(e^t)$ is convex on $[0,\infty)$.

Note that $C^\omega \subseteq \mathcal{E}^{\{\omega\}}$ iff $\omega(t) = O(t)$, and $C^\omega \subseteq \mathcal{E}^{(\omega)}$ iff $\omega(t) = o(t)$, as $t \to \infty$.

Theorem 3. *If ω satisfies $\omega(t) = O(t)$ as $t \to \infty$ then the following are equivalent:*

(1) *ω satisfies $\exists C > 0 \ \exists t_0 > 0 \ \forall \lambda \geq 1 \ \forall t \geq t_0 : \omega(\lambda t) \leq C\lambda\omega(t)$.*
(2) *There exists a sub-additive weight function $\tilde{\omega}$ so that $\mathcal{E}^{[\omega]} = \mathcal{E}^{[\tilde{\omega}]}$.*
(3) *$\mathcal{E}^{\{\omega\}}$ is stable under composition.*
(4) *$\mathcal{E}^{\{\omega\}}$ is stable under solving ODEs.*
(5) *$\mathcal{E}^{\{\omega\}}$ is stable under inversion.*
(6) *$\mathcal{E}^{\{\omega\}}$ is inverse-closed.*

Theorem 4. *If ω satisfies $\omega(t) = o(t)$ as $t \to \infty$ then the following are equivalent:*

(1) *ω satisfies $\exists C > 0 \ \exists t_0 > 0 \ \forall \lambda \geq 1 \ \forall t \geq t_0 : \omega(\lambda t) \leq C\lambda\omega(t)$.*
(2) *There exists a sub-additive weight function $\tilde{\omega}$ so that $\mathcal{E}^{[\omega]} = \mathcal{E}^{[\tilde{\omega}]}$.*
(3) *$\mathcal{E}^{(\omega)}$ is stable under composition.*
(4) *$\mathcal{E}^{(\omega)}$ is stable under solving ODEs.*
(5) *$\mathcal{E}^{(\omega)}$ is stable under inversion.*
(6) *$\mathcal{E}^{(\omega)}$ is inverse-closed.*

1.3. Stability properties of $\mathcal{E}^{[\mathfrak{M}]}$.

Theorems 1, 2, 3, and 4 are corollaries of the corresponding result for the classes $\mathcal{E}^{[\mathfrak{M}]}$ defined in terms of weight matrices, namely Theorems 5 and 6 below.

A *weight matrix* $\mathfrak{M} = \{M^\lambda \in \mathbb{R}_{>0}^{\mathbb{N}} : \lambda \in \Lambda\}$ is a family of weight sequences $M^\lambda = (M_k^\lambda)$ indexed by an ordered subset Λ of $\mathbb{R}$ so that $\lim_k (k! M_k^\lambda)^{\frac{1}{k}} = \infty$ for each λ, and $M^\lambda \leq M^\mu$ if $\lambda \leq \mu$.

We shall again assume that the classes $\mathcal{E}^{[\mathfrak{M}]}$ contain the class of real analytic functions and are stable under derivation. Specifically we need the conditions

$(\mathfrak{M}_{\mathcal{H}})$ $\qquad\qquad \forall \lambda \in \Lambda : \underline{\lim}(M_k^\lambda)^{\frac{1}{k}} > 0$

$(\mathfrak{M}_{(C^\omega)})$ $\qquad\qquad \forall \lambda \in \Lambda : \lim(M_k^\lambda)^{\frac{1}{k}} = \infty$

$(\mathfrak{M}_{\{dc\}})$ $\qquad\qquad \forall \lambda \in \Lambda \ \exists \mu \in \Lambda \ \exists C > 0 \ \forall k \in \mathbb{N} : M_{k+1}^\lambda \leq C^k M_k^\mu$

$(\mathfrak{M}_{(dc)})$ $\qquad\qquad \forall \lambda \in \Lambda \ \exists \mu \in \Lambda \ \exists C > 0 \ \forall k \in \mathbb{N} : M_{k+1}^\mu \leq C^k M_k^\lambda$

We have

$$(\mathfrak{M}_{(C^\omega)}) \Longleftrightarrow C^\omega(U) \subseteq \mathcal{E}^{(\mathfrak{M})}(U)$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$(\mathfrak{M}_{\mathcal{H}}) \Longleftrightarrow \mathcal{H}(\mathbb{C}^n) \subseteq \mathcal{E}^{(\mathfrak{M})}(U)$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$(\mathfrak{M}_{\{C^\omega\}}) :\Leftrightarrow \exists \lambda \in \Lambda : \underline{\lim}(M_k^\lambda)^{\frac{1}{k}} > 0 \Longleftrightarrow C^\omega(U) \subseteq \mathcal{E}^{\{\mathfrak{M}\}}(U)$$

and $\mathcal{E}^{[\mathfrak{M}]}(U)$ is derivation closed iff $(\mathfrak{M}_{[dc]})$; see [13]. The conditions on the weight matrix $\mathfrak{M}$ that characterize the stability properties of $\mathcal{E}^{[\mathfrak{M}]}$ are natural generalizations of the condition of being almost increasing and of the (FdB)-property; clearly, the Roumieu and the Beurling version fall apart, see Remark 2 below:

$(\mathfrak{M}_{\{rai\}})$ $\qquad\qquad \forall \lambda \in \Lambda \, \exists \mu \in \Lambda \, \exists C > 0 \, \forall j \le k : (M_j^\lambda)^{\frac{1}{j}} \le C(M_k^\mu)^{\frac{1}{k}}$

$(\mathfrak{M}_{(rai)})$ $\qquad\qquad \forall \lambda \in \Lambda \, \exists \mu \in \Lambda \, \exists C > 0 \, \forall j \le k : (M_j^\mu)^{\frac{1}{j}} \le C(M_k^\lambda)^{\frac{1}{k}}$

$(\mathfrak{M}_{\{FdB\}})$ $\qquad\qquad \forall \lambda \in \Lambda \, \exists \mu \in \Lambda \, \exists C > 0 \, \forall k : (M^\lambda)_k^\circ \le C^k M_k^\mu$

$(\mathfrak{M}_{(FdB)})$ $\qquad\qquad \forall \lambda \in \Lambda \, \exists \mu \in \Lambda \, \exists C > 0 \, \forall k : (M^\mu)_k^\circ \le C^k M_k^\lambda$

Theorem 5. *For a weight matrix $\mathfrak{M}$ satisfying $(\mathfrak{M}_{\mathcal{H}})$ and $(\mathfrak{M}_{\{dc\}})$ the following are equivalent:*

- (1) $\mathfrak{M}$ *satisfies* $(\mathfrak{M}_{\{rai\}})$.
- (2) $\mathfrak{M}$ *satisfies* $(\mathfrak{M}_{\{FdB\}})$.
- (3) $\mathcal{E}^{\{\mathfrak{M}\}}$ *is stable under composition.*
- (4) $\mathcal{E}^{\{\mathfrak{M}\}}$ *is stable under solving ODEs.*
- (5) $\mathcal{E}^{\{\mathfrak{M}\}}$ *is stable under inversion.*
- (6) $\mathcal{E}^{\{\mathfrak{M}\}}$ *is inverse-closed.*

Theorem 6. *For a weight matrix $\mathfrak{M}$ satisfying $(\mathfrak{M}_{(C^\omega)})$ and $(\mathfrak{M}_{(dc)})$ the following are equivalent:*

- (1) $\mathfrak{M}$ *satisfies* $(\mathfrak{M}_{(rai)})$.
- (2) $\mathfrak{M}$ *satisfies* $(\mathfrak{M}_{(FdB)})$.
- (3) $\mathcal{E}^{(\mathfrak{M})}$ *is stable under composition.*
- (4) $\mathcal{E}^{(\mathfrak{M})}$ *is stable under solving ODEs.*
- (5) $\mathcal{E}^{(\mathfrak{M})}$ *is stable under inversion.*
- (6) $\mathcal{E}^{(\mathfrak{M})}$ *is inverse-closed.*

Remark 2. The weight matrix $\mathfrak{M}$ that consists of just two non-equivalent sequences $M^1 \le M^2$ satisfying

- $(M_k^i)^{1/k} \to \infty$ and $\sup_k (\frac{M_{k+1}^i}{M_k^i})^{1/k} < \infty$, $i = 1, 2$,
- $(M_k^1)^{1/k}$ almost increasing,
- $(M_k^2)^{1/k}$ not almost increasing,

satisfies $(\mathfrak{M}_{(rai)})$ but not $(\mathfrak{M}_{\{rai\}})$. Whereas, if $(M_k^2)^{1/k}$ is almost increasing and $(M_k^1)^{1/k}$ is not, $\mathfrak{M}$ satisfies $(\mathfrak{M}_{\{rai\}})$ but not $(\mathfrak{M}_{(rai)})$. We construct such sequences in Appendix A.

We shall prove Theorems 5 and 6 in Sections 3 and 4, respectively. In Section 5 we show that Theorems 1, 2, 3, and 4 are corollaries of Theorems 5 and 6.

Notation. The notation $\mathcal{E}^{[*]}$ for $* \in \{M, \omega, \mathfrak{M}\}$ stands for either $\mathcal{E}^{(*)}$ or $\mathcal{E}^{\{*\}}$ with the following restriction: Statements that involve more than one $\mathcal{E}^{[*]}$ symbol must not be interpreted by mixing $\mathcal{E}^{(*)}$ and $\mathcal{E}^{\{*\}}$.

2. Ultradifferentiable function classes

2.1. Ultradifferentiable functions defined by weight sequences. Let $M = (M_k)$ be a weight sequence. For non-empty open $U \subseteq \mathbb{R}^n$, define

$$\mathcal{E}^{(M)}(U, \mathbb{R}^m) := \left\{ f \in C^\infty(U, \mathbb{R}^m) : \forall K \subseteq U \text{ compact } \forall \rho > 0 : \|f\|_{K,\rho}^M < \infty \right\}$$

$$\mathcal{E}^{\{M\}}(U, \mathbb{R}^m) := \left\{ f \in C^\infty(U, \mathbb{R}^m) : \forall K \subseteq U \text{ compact } \exists \rho > 0 : \|f\|_{K,\rho}^M < \infty \right\}$$

$$\|f\|_{K,\rho}^M := \sup \left\{ \frac{\|f^{(k)}(x)\|_{L^k(\mathbb{R}^n, \mathbb{R}^m)}}{\rho^k k! M_k} : x \in K, k \in \mathbb{N} \right\}$$

and endow these spaces with their natural topologies:

$$\mathcal{E}^{(M)}(U, \mathbb{R}^m) = \varprojlim_{K \subseteq U} \varprojlim_{\ell \in \mathbb{N}_{>0}} \mathcal{E}_{\frac{1}{\ell}}^M(K, \mathbb{R}^m), \quad \mathcal{E}^{\{M\}}(U, \mathbb{R}^m) = \varprojlim_{K \subseteq U} \varinjlim_{\ell \in \mathbb{N}} \mathcal{E}_\ell^M(K, \mathbb{R}^m)$$

$$\text{where} \quad \mathcal{E}_\rho^M(K, \mathbb{R}^m) := \{ f \in C^\infty(K, \mathbb{R}^m) : \|f\|_{K,\rho}^M < \infty \}$$

We will need the following inclusion relations (cf. [13]):

$$\mathcal{E}^{[M]} \subseteq \mathcal{E}^{[N]} \quad \Leftrightarrow \quad M \preceq N \quad :\Leftrightarrow \quad \exists C, \rho > 0 \,\forall k : M_k \leq C \rho^k N_k$$

$$\mathcal{E}^{\{M\}} \subseteq \mathcal{E}^{(N)} \quad \Leftrightarrow \quad M \vartriangleleft N \quad :\Leftrightarrow \quad \forall \rho > 0 \,\exists C > 0 \,\forall k : M_k \leq C \rho^k N_k$$

In particular, $C^\omega(U) \subseteq \mathcal{E}^{\{M\}}(U) \Leftrightarrow \mathcal{H}(\mathbb{C}^n) \subseteq \mathcal{E}^{(M)}(U) \Leftrightarrow \varliminf M_k^{\frac{1}{k}} > 0$ and $C^\omega(U) \subseteq \mathcal{E}^{(M)}(U) \Leftrightarrow \lim M_k^{\frac{1}{k}} = \infty$.

2.2. Ultradifferentiable functions defined by weight functions. Let ω be a weight function (hence satisfying (ω_1), (ω_2), and (ω_3)). The *Young conjugate* of $\varphi(t) = \omega(e^t)$, given by

$$\varphi^*(t) := \sup\{st - \varphi(s) : s \geq 0\}, \quad t \geq 0,$$

is convex, increasing, and satisfies $\varphi^*(0) = 0$, $\varphi^{**} = \varphi$, and $\lim_{t \to \infty} t/\varphi^*(t) = 0$. Moreover, the functions $t \mapsto \varphi(t)/t$ and $t \mapsto \varphi^*(t)/t$ are increasing; see e.g. [3]. For non-empty open $U \subseteq \mathbb{R}^n$ define

$$\mathcal{E}^{(\omega)}(U, \mathbb{R}^m) := \left\{ f \in C^\infty(U, \mathbb{R}^m) : \forall K \subseteq U \text{ compact } \forall \rho > 0 : \|f\|_{K,\rho}^\omega < \infty \right\}$$

$$\mathcal{E}^{\{\omega\}}(U, \mathbb{R}^m) := \left\{ f \in C^\infty(U, \mathbb{R}^m) : \forall K \subseteq U \text{ compact } \exists \rho > 0 : \|f\|_{K,\rho}^\omega < \infty \right\}$$

$$\|f\|_{K,\rho}^\omega := \sup \left\{ \|f^{(k)}(x)\|_{L^k(\mathbb{R}^n, \mathbb{R}^m)} \exp(-\tfrac{1}{\rho}\varphi^*(\rho k)) : x \in K, k \in \mathbb{N} \right\}$$

and endow these spaces with their natural topologies:

$$\mathcal{E}^{(\omega)}(U, \mathbb{R}^m) = \varprojlim_{K \subseteq U} \varprojlim_{\ell \in \mathbb{N}_{>0}} \mathcal{E}_{\frac{1}{\ell}}^\omega(K, \mathbb{R}^m), \quad \mathcal{E}^{\{\omega\}}(U, \mathbb{R}^m) = \varprojlim_{K \subseteq U} \varinjlim_{\ell \in \mathbb{N}} \mathcal{E}_\ell^\omega(K, \mathbb{R}^m)$$

$$\text{where} \quad \mathcal{E}_\rho^\omega(K, \mathbb{R}^m) := \{ f \in C^\infty(K, \mathbb{R}^m) : \|f\|_{K,\rho}^\omega < \infty \}$$

We have $C^\omega(U) \subseteq \mathcal{E}^{\{\omega\}}(U)$ iff $\omega(t) = O(t)$ as $t \to \infty$, and $C^\omega(U) \subseteq \mathcal{E}^{(\omega)}(U)$ iff $\omega(t) = o(t)$ as $t \to \infty$; see e.g. [13].

2.3. **Ultradifferentiable functions defined by weight matrices.** Let $\mathfrak{M}$ be a weight matrix, let $U \subseteq \mathbb{R}^n$ be non-empty and open, and let $K \subseteq U$ be compact. We define

$$\mathcal{E}^{(\mathfrak{M})}(K, \mathbb{R}^m) := \bigcap_{\lambda \in \Lambda} \mathcal{E}^{(M^\lambda)}(K, \mathbb{R}^m), \quad \mathcal{E}^{\{\mathfrak{M}\}}(K, \mathbb{R}^m) := \bigcup_{\lambda \in \Lambda} \mathcal{E}^{\{M^\lambda\}}(K, \mathbb{R}^m),$$

$$\mathcal{E}^{(\mathfrak{M})}(U, \mathbb{R}^m) := \bigcap_{\lambda \in \Lambda} \mathcal{E}^{(M^\lambda)}(U, \mathbb{R}^m), \quad \mathcal{E}^{\{\mathfrak{M}\}}(U, \mathbb{R}^m) := \bigcap_{K \subseteq U} \bigcup_{\lambda \in \Lambda} \mathcal{E}^{\{M^\lambda\}}(K, \mathbb{R}^m),$$

and endow these spaces with their natural topologies:

$$\mathcal{E}^{(\mathfrak{M})}(U, \mathbb{R}^m) := \varprojlim_{\lambda \in \Lambda} \mathcal{E}^{(M^\lambda)}(U, \mathbb{R}^m), \quad \mathcal{E}^{\{\mathfrak{M}\}}(U, \mathbb{R}^m) := \varprojlim_{K \subseteq U} \varinjlim_{\lambda \in \Lambda} \mathcal{E}^{\{M^\lambda\}}(K, \mathbb{R}^m).$$

It is no loss of generality to assume that the limits are countable.

We have $\mathcal{H}(\mathbb{C}^n) \subseteq \mathcal{E}^{(\mathfrak{M})}(U)$ iff $(\mathfrak{M}_\mathcal{H})$, $C^\omega(U) \subseteq \mathcal{E}^{[\mathfrak{M}]}(U)$ iff $(\mathfrak{M}_{[C^\omega]})$, and $\mathcal{E}^{[\mathfrak{M}]}(U)$ is derivation closed iff $(\mathfrak{M}_{[dc]})$; see [13].

3. Proof of Theorem 5: the Roumieu case

The proof of the equivalence of the items (1)–(6) of Theorem 5 has the following structure:

(3.1)

where:

(6′) If $f \in \mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R})$ and $f(0) \neq 0$ then $1/f$ is $\mathcal{E}^{\{\mathfrak{M}\}}$ on its domain of definition.

We successively prove:

- the equivalences $(1) \Leftrightarrow (2) \Leftrightarrow (3)$
- the cycle $(1) \Rightarrow (4) \Rightarrow (6') \Rightarrow (1)$
- the cycle $(1) \Rightarrow (5) \Rightarrow (6) \Rightarrow (6') \Rightarrow (1)$

3.1. **The equivalences** $(1) \Leftrightarrow (2) \Leftrightarrow (3)$**.** The following lemma implies the equivalence of (1) and (2). The equivalence of (2) and (3) was shown in [13, 4.9].

Lemma 1. *For a weight matrix $\mathfrak{M}$ we have the following implications:*

(1) $(\mathfrak{M}_{[rai]})$ *and* $(\mathfrak{M}_{[dc]})$ *imply* $(\mathfrak{M}_{[FdB]})$.
(2) $(\mathfrak{M}_{[FdB]})$ *and* $(\mathfrak{M}_\mathcal{H})$ *imply* $(\mathfrak{M}_{[rai]})$

Proof. (1) was shown in [13, 4.9, 4.11]. To see (2) observe that $(M^\mu)^\circ \preceq M^\lambda$ implies $(M_j^\mu)^{\frac{1}{jk}}(M_k^\mu)^{\frac{1}{k}} \leq C(M_{jk}^\lambda)^{\frac{1}{jk}}$ for all j, k and some constant C. By $(\mathfrak{M}_\mathcal{H})$ we may conclude that $(M_k^\mu)^{\frac{1}{k}} \leq \tilde{C}(M_{jk}^\lambda)^{\frac{1}{jk}}$ for all j, k and some $\tilde{C}$. For $k \leq \ell$ choose $j \in \mathbb{N}$ such that $jk \leq \ell < (j+1)k$, then by Remark 1 and since $n! \leq n^n \leq e^n n!$,

$$\ell(M_\ell^\lambda)^{\frac{1}{\ell}} \geq (\ell! M_\ell^\lambda)^{\frac{1}{\ell}} \geq ((jk)! M_{jk}^\lambda)^{\frac{1}{jk}} \geq \frac{\tilde{C}jk}{e}(M_k^\mu)^{\frac{1}{k}} \geq \frac{\tilde{C}(j+1)k}{2e}(M_k^\mu)^{\frac{1}{k}} > \frac{\tilde{C}\ell}{2e}(M_k^\mu)^{\frac{1}{k}}$$

which implies the desired property. $\qquad\square$

3.2. **The cycle** $(1) \Rightarrow (4) \Rightarrow (6') \Rightarrow (1)$**.** The implication $(1) \Rightarrow (4)$ follows from the following proposition. We state and prove this result on ultradifferentiable solutions of ODEs (as well as the ultradifferentiable inverse mapping theorem below) for mappings between arbitrary Banach spaces, since we used such results in [12] and will need them in forthcoming work; cf. [16].

Remark 3. Observe the index shift in the estimates of (3.2) and (3.4). In order to deduce $(1) \Rightarrow (4)$ from Proposition 1 we use $(\mathfrak{M}_{\{dc\}})$ for the index shift in (3.2) and Remark 1 for the one in (3.4).

Henceforth we use the convention $(-1)!M_{-1} := 1$ for any weight sequence M.

Proposition 1. *Let $\mathfrak{M}$ be a weight matrix satisfying $(\mathfrak{M}_{\{rai\}})$. Let X be a Banach space and let $f : W \to X$ be a C^∞-mapping defined in an open subset $W \subseteq X \times \mathbb{R}$ and satisfying*

(3.2)
$$\exists \lambda \in \Lambda \; \exists C, \rho \geq 1 \; \forall (k, \ell) \in \mathbb{N}^2 \; \forall (x, t) \in W :$$
$$\|f^{(k,\ell)}(x,t)\|_{L^{k,\ell}(X,\mathbb{R};X)} \leq C\rho^{k+\ell}(k+\ell-1)!M^\lambda_{k+\ell-1}.$$

Then the solution $x : I \to X$ of the initial value problem

(3.3)
$$x'(t) = f(x(t), t), \quad x(0) = x_0,$$

satisfies

(3.4)
$$\exists \mu \in \Lambda \; \exists D, \sigma \geq 1 \; \forall k \in \mathbb{N} \; \forall t \in I : \|x^{(k)}(t)\| \leq D\sigma^k(k-1)!M^\mu_{k-1}.$$

Remark 4. A more general statement involving parameters u in a further Banach space Z is true: the solution of the initial value problem

$$x'(t) = f(x(t), t, u), \quad x(0) = x_0,$$

satisfies an estimate of the kind (3.4) in t, u, and x_0, given that f satisfies an estimate of the kind (3.2) in x, t, and u. For simplicity we prove only the result stated in the proposition; the general result is obtained by making obvious modifications in the proof of [19] the main ideas of which we follow here. Different arguments were given in [11] and [5].

In the proof of the proposition we will use Faà di Bruno's formula for Fréchet derivatives of mappings between Banach spaces. So let us recall this formula. For $k \geq 1$,

(3.5)
$$\frac{(f \circ g)^{(k)}(x)}{k!} = \mathrm{sym}\Big(\sum_{j \geq 1} \sum_{\substack{\alpha \in \mathbb{N}^j_{>0} \\ \sum_{i=1}^j \alpha_i = k}} \frac{f^{(j)}(g(x))}{j!} \circ \Big(\frac{g^{(\alpha_1)}(x)}{\alpha_1!} \times \cdots \times \frac{g^{(\alpha_j)}(x)}{\alpha_j!} \Big) \Big),$$

where sym denotes symmetrization of multilinear mappings.

Proof. We may reduce the initial value problem (3.3) to the problem

(3.6)
$$y' = g(y), \quad y(0) = y_0,$$

by setting $y = (x, t)$, $y_0 = (x_0, 0)$, and $g(y) = (f(y), 1)$. So we assume that Y is a Banach space, U is a neighborhood of 0 in Y, and $g \in C^\infty(U, Y)$ satisfies

(3.7) $\quad \exists \lambda \in \Lambda \; \exists C, \rho \geq 1 \; \forall k \in \mathbb{N} \; \forall y \in U : \|g^{(k)}(y)\|_{L^k(Y,Y)} \leq C\rho^k(k-1)!M^\lambda_{k-1}.$

Without loss of generality we assume $M_1^\lambda \geq 2$. By the classical existence and uniqueness result, there exists a unique C^∞ solution $y = y(t)$ of (3.6) for t in a neighborhood I of 0. We assume that $\sup_{t \in I} \|y(t)\| < \infty$.

By $(\mathfrak{M}_{\{\mathrm{rai}\}})$ there exists $\mu \in \Lambda$ and $H \geq 1$ such that for $2 \leq j \leq k$,

$$\left(\frac{M_{j-1}^\lambda}{j} \right)^{\frac{1}{j-1}} \leq H \left(\frac{M_{k-1}^\mu}{k} \right)^{\frac{1}{k-1}} =: p_k^\mu.$$

Then, since $1 \leq M_1^\lambda / 2 \leq p_k^\mu$,

$$(3.8) \qquad \frac{M_{j-1}^\lambda}{j} \leq (p_k^\mu)^j, \quad \text{for } 2 \leq j \leq k.$$

Let us choose constants A and η such that

$$(3.9) \qquad A \geq \max\{\sup_{t \in I} \|y(t)\|, C\} \quad \text{and} \quad \eta \geq \rho,$$

where C and ρ are the constants from (3.7). We define

$$G_k^\mu(s) := \frac{A}{1 - \eta p_k^\mu s},$$

for small $s \in \mathbb{R}$, and consider the initial value problem

$$(3.10) \qquad Y'(t) = G_k^\mu(Y(t) - A), \quad Y(0) = A.$$

We claim that the solution

$$Y_k^\mu(t) = A + \frac{1 - \sqrt{1 - 2A\eta p_k^\mu t}}{\eta p_k^\mu}$$

of (3.10) satisfies

$$(3.11) \qquad \sup_{t \in I} \|y^{(j)}(t)\| \leq (Y_k^\mu)^{(j)}(0) \quad \text{for } j \leq k.$$

This implies the statement of the proposition, since, for $j \geq 1$,

$$(Y_k^\mu)^{(j)}(0) = (2A)^j (\eta p_k^\mu)^{j-1} (2\sqrt{\pi})^{-1} \Gamma(j - \tfrac{1}{2})$$
$$\leq (2A)^j (\eta H)^{j-1} (j-1)! M_{j-1}^\mu \quad \text{if } k = j.$$

Let us prove (3.11). By the choice of the constant A, (3.11) is satisfied for $j = 0$. Suppose that (3.11) holds for all $j \leq \ell < k$. By (3.7), (3.8), and (3.9), we have

$$\sup_{y \in U} \|g^{(j)}(y)\|_{L^j(Y,Y)} \leq C\rho^j (j-1)! M_{j-1}^\lambda \leq C\rho^j j! (p_k^\mu)^j \leq A\eta^j j! (p_k^\mu)^j = (G_k^\mu)^{(j)}(0).$$

So, by applying Faà di Bruno's formula (3.5) twice, we may conclude that, for $j \leq \ell < k$,

$$\sup_{t \in I} \|(g \circ y)^{(j)}(t)\| \leq j! \sum_{h \geq 1} \frac{(G_k^\mu)^{(h)}(0)}{h!} \sum_{\substack{\alpha_1 + \cdots + \alpha_h = j \\ \alpha_i > 0}} \prod_{i=1}^h \frac{(Y_k^\mu)^{(\alpha_i)}(0)}{\alpha_i!}$$
$$= (G_k^\mu \circ (Y_k^\mu - A))^{(j)}(0).$$

But as y and Y_k^μ are solutions of (3.6) and (3.10), respectively, it follows that (3.11) holds for $j \leq \ell + 1$. By induction, (3.11) follows. $\qquad \square$

Let us check that (4) implies (6′). Let $f \in \mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R})$ satisfy $f(0) \neq 0$ and consider $g := 1/f$. Then g solves the initial value problem

$$x' = -f'(t)x^2, \quad x(0) = g(0).$$

By $(\mathfrak{M}_{\{dc\}})$, the mapping $(x, t) \mapsto f'(t)x^2$ is $\mathcal{E}^{\{\mathfrak{M}\}}$ and so g is $\mathcal{E}^{\{\mathfrak{M}\}}$, by (4).

The implication (6′) $\Rightarrow$ (1) was shown in the proof of [13, 4.9(2)$\Rightarrow$(3)], by following the argument of [17, Thm 3].

The next lemma, a variation of [7, Thm 1], is a preparation for Proposition 2 below. It gives, in particular, a direct proof of the implication (1) $\Rightarrow$ (6′).

Lemma 2. *Let $\mathfrak{M}$ be a weight matrix satisfying $(\mathfrak{M}_{\{rai\}})$. Let E, F, G be Banach spaces, $U \subseteq E$ be open, and let $T \in C^\infty(U, L(F, G))$ satisfy*

$$(3.12) \quad \exists \lambda \in \Lambda \; \exists C, \rho \geq 1 \; \forall k \in \mathbb{N} \; \forall x \in U : \|T^{(k)}(x)\|_{L^k(E, L(F,G))} \leq C\rho^k k! M_k^\lambda.$$

If $T(x_0) \in L(F, G)$ is invertible, then there is a neighborhood $x_0 \in U_0 \subseteq U$ such that $U_0 \ni x \mapsto S(x) := T(x)^{-1}$ satisfies

$$(3.13) \quad \exists \mu \in \Lambda \; \exists D, \sigma \geq 1 \; \forall k \in \mathbb{N} \; \forall x \in U_0 : \|S^{(k)}(x)\|_{L^k(E, L(G,F))} \leq D\sigma^k k! M_k^\mu.$$

Proof. There is an open neighborhood U_0 of x_0 so that for $x \in U_0$ we have $\|S(x)\|_{L(G,F)} \leq A$ for some constant $A > 0$ and $S(x)$ is given by the Neumann series

$$S(x) = T(x_0)^{-1} \sum_{j=0}^\infty \left((T(x_0) - T(x))T(x_0)^{-1}\right)^j.$$

For y near x we may consider

$$S(y) = S(x) \sum_{j=0}^\infty \left((T(x) - T(y))S(x)\right)^j = S(x)\left(\mathrm{id} - (T(x) - T(y))S(x)\right)^{-1}$$

and use Faà di Bruno's formula (3.5) and (3.12) to obtain, for $y = x$,

$$\frac{\|S^{(k)}(x)\|_{L^k(E, L(G,F))}}{k!} \leq A \sum_{j \geq 1} \sum_{\substack{\alpha_1 + \cdots + \alpha_j = k \\ \alpha_i > 0}} (AC)^j \rho^k \prod_{i=1}^j M_{\alpha_i}^\lambda$$

which implies (3.13), since $M_{\alpha_1}^\lambda \cdots M_{\alpha_j}^\lambda \leq H^k M_k^\mu$, by $(\mathfrak{M}_{\{rai\}})$. $\qquad \square$

3.3. The cycle $(1) \Rightarrow (5) \Rightarrow (6) \Rightarrow (6') \Rightarrow (1)$. The implications $(5) \Rightarrow (6) \Rightarrow (6')$ are obvious. And we already know from Subsection 3.2 that $(1) \Leftrightarrow (6')$. That (1) implies (5) follows from the next proposition, using the analogue of Remark 3.

Proposition 2. *Let $\mathfrak{M}$ be a weight matrix satisfying $(\mathfrak{M}_{\{rai\}})$. Let $f : U \to V$ be a C^∞-mapping between open subsets $U \subseteq E$ and $V \subseteq F$ of Banach spaces E, F satisfying*

$$(3.14) \quad \exists \lambda \in \Lambda \; \exists C, \rho \geq 1 \; \forall k \in \mathbb{N} \; \forall x \in U : \|f^{(k)}(x)\|_{L^k(E,F)} \leq C\rho^{k-1}(k-1)! M_{k-1}^\lambda$$

and so that $f'(x_0) \in L(E, F)$ is invertible. Then there exist neighborhoods $x_0 \in U_0 \subseteq U$ and $f(x_0) \in V_0 \subseteq V$ and a C^∞-mapping $g : V_0 \to U_0$ satisfying

$$(3.15)$$
$$\exists \mu \in \Lambda \; \exists D, \sigma \geq 1 \; \forall k \in \mathbb{N} \; \forall y \in V_0 : \|g^{(k)}(y)\|_{L^k(F,E)} \leq D\sigma^{k-1}(k-1)! M_{k-1}^\mu$$

and such that $f \circ g = \mathrm{id}_{V_0}$.

Proof. We adapt the proof of [7]. Different arguments were given in [10] and [18], under more restrictive assumptions also in [5] and [1]. By the classical C^∞ inverse mapping theorem there exist neighborhoods $x_0 \in U_0 \subseteq U$ and $f(x_0) \in V_0 \subseteq V$ and a C^∞-mapping $g : V_0 \to U_0$ such that $f \circ g = \mathrm{id}_{V_0}$. We can assume that $(f'(x))^{-1}$ is bounded for $x \in U_0$. We shall show that g satisfies (3.15).

Let $S_k \in C^\infty(U_0, L(F, E))$, $k \geq 1$, be given and define $R_k(x)$, $k \geq 0$, for $x \in U_0$ recursively by setting

$$(3.16) \qquad R_0(x) := \mathrm{id}_E, \quad R_k(x) := (R_{k-1}(x)S_k(x))'.$$

Thus $R_{k-1} \in C^\infty(U_0, L(E, L^{k-1}(F, E)))$ and $R_{k-1}(x)S_k(x) \in L^k(F, E)$. It follows that

$$\|R_k(x)\|_{L(E, L^k(F, E))} \leq \sum_{\substack{\beta_1 + \cdots + \beta_k = k \\ \beta_i \geq 0}} N(\beta_1, \dots, \beta_k) \prod_{i=1}^{k} \|S_i^{(\beta_i)}(x)\|_{L^{\beta_i}(E, L(F, E))},$$

where the nonnegative integers $N(\beta_1, \dots, \beta_k)$ are given by the identity

$$\sum_{\substack{\beta_1 + \cdots + \beta_k = k \\ \beta_i \geq 0}} N(\beta_1, \dots, \beta_k) \prod_{i=1}^{k} t_i^{\beta_i} = \prod_{j=1}^{k} \sum_{\ell=1}^{j} t_\ell.$$

Since $\prod_{j=1}^{k} \sum_{\ell=1}^{j} t_\ell \leq (\sum_{\ell=1}^{k} t_\ell)^k = \sum k! \prod_{i=1}^{k} \frac{t_i^{\beta_i}}{\beta_i!}$, where the sum is taken over all $\beta_i \geq 0$ so that $\beta_1 + \cdots + \beta_k = k$, we obtain

$$(3.17) \qquad \|R_k(x)\|_{L(E, L^k(F, E))} \leq \sum_{\substack{\beta_1 + \cdots + \beta_k = k \\ \beta_i \geq 0}} k! \prod_{i=1}^{k} \frac{\|S_i^{(\beta_i)}(x)\|_{L^{\beta_i}(E, L(F, E))}}{\beta_i!}.$$

If we set $S(x) = S_k(x) := (f'(x))^{-1}$ for all $k \geq 1$, then for $n \geq 1$

$$g^{(n)}(y) = R_{n-1}(x)S(x), \quad (x = g(y)),$$

where the sequence R_n is defined by (3.16). Applying Lemma 2 to $T = f'$ and using (3.17), we find that there exist $\mu, \nu \in \Lambda$ and $D, H, \sigma, \tau \geq 1$ so that

$$\|R_k(x)\|_{L(E, L^k(F, E))} \leq \sum_{\substack{\beta_1 + \cdots + \beta_k = k \\ \beta_i \geq 0}} k!(D\sigma)^k \prod_{i=1}^{k} M_{\beta_i}^\mu \leq H\tau^k k! M_k^\nu,$$

by $(\mathfrak{M}_{\{\mathrm{rai}\}})$. This implies (3.15). $\qquad\square$

4. Proof of Theorem 6: the Beurling case

The structure of the proof of the equivalence of the six items in Theorem 6 is again represented by the diagram in (3.1), where now:

(6′) If $f \in \mathcal{E}^{(\mathfrak{M})}(\mathbb{R})$ and $f(0) \neq 0$ then $1/f$ is $\mathcal{E}^{(\mathfrak{M})}$ on its domain of definition.

4.1. **The equivalences** $(1) \Leftrightarrow (2) \Leftrightarrow (3)$**.** Lemma 1 implies $(1) \Leftrightarrow (2)$, since $(\mathfrak{M}_\mathcal{H})$ follows from $(\mathfrak{M}_{(C^\omega)})$. The equivalence of (2) and (3) was shown in [13, 4.11].

4.2. **The cycle** $(1) \Rightarrow (4) \Rightarrow (6') \Rightarrow (1)$**.** That (4) implies $(6')$ follows in the same way as in the Roumieu case, see Subsection 3.2. The implication $(6') \Rightarrow (1)$ is a consequence of the following lemma.

Lemma 3. *If $\mathcal{E}^{(\mathfrak{M})}(\mathbb{R})$ is inverse closed, then $\mathfrak{M}$ satisfies $(\mathfrak{M}_{(\mathrm{rai})})$.*

Proof. We follow an argument of [4]. Consider the algebra $\mathcal{A} := \mathcal{E}^{(\mathfrak{M})}(\mathbb{R})$ and its subalgebra $\mathcal{B} := \{f \in \mathcal{A} : \|f\|_\infty := \|f\|_{L^\infty(\mathbb{R})} < \infty\}$. Endow $\mathcal{B}$ with the topology generated by all seminorms $Q := \{\| \ \|_{K,\rho}^{M^\lambda}\}_{\lambda,K,\rho} \cup \{\| \ \|_\infty\}$. Then $\mathcal{B}$ is a Fréchet algebra. If $f, g \in \mathcal{B}$ are such that $|f(x)| \geq c > 0$ for all $x \in \mathbb{R}$ and $\|f - g\|_\infty \leq c/2$, then $|g(x)| \geq c/2 > 0$, i.e., the set $\{f \in \mathcal{B} : 1/f \in \mathcal{B}\}$ is open in $\mathcal{B}$. By [20, Thm 13.17], see also [4, Prop 5.2], we may conclude that the algebra $\mathcal{B}$ is locally m-convex, i.e., $\mathcal{B}$ has an equivalent seminorm system $P = \{p\}$ such that $p(fg) \leq p(f)p(g)$ for all $f, g \in \mathcal{B}$. So for each $\lambda \in \Lambda$, compact $K \subseteq \mathbb{R}$, and $\rho > 0$ there exist $p \in P$, $q \in Q$, and constants $C, D > 0$ such that

$$\|f^m\|_{K,\rho}^{M^\lambda} \leq Cp(f^m) \leq Cp(f)^m \leq CD^m q(f)^m, \quad f \in \mathcal{B}, m \in \mathbb{N}.$$

We shall use this inequality for the functions $f_t(x) = e^{itx}$, and, since $\|f_t\|_\infty = 1 \leq \|f_t\|_{[-a,a],\sigma}^{M^\mu}$ for each $\mu \in \Lambda$ and $a, \sigma > 0$, we can replace q in the very same inequality by some seminorm $\| \ \|_{[-a,a],\sigma}^{M^\mu}$. Then the proof of [13, 4.11(2) $\Rightarrow$ (3)] yields $(\mathfrak{M}_{(\mathrm{rai})})$. $\qquad\square$

The remaining implication $(1) \Rightarrow (4)$ (as well as $(1) \Rightarrow (5)$ below) we shall deduce from the corresponding result in the Roumieu case by means of the following lemma, which is a variation of [9, Lemma 6].

Lemma 4. *Let $L \in \mathbb{R}_{\geq 0}^{\mathbb{N}}$ and $M^1, M^2, M^3 \in \mathbb{R}_{>0}^{\mathbb{N}}$ be sequences satisfying $L \lhd M^1 \leq M^2 \leq M^3$ and $(M_k^i)^{1/k} \to \infty$ for $i = 1, 2, 3$, and assume that there exist $1 \leq H_1 \leq H_2$ so that*

(4.1) $$(M_j^1)^{1/j} \leq H_1(M_k^2)^{1/k} \leq H_2(M_\ell^3)^{1/\ell} \quad \text{for } j \leq k \leq \ell.$$

Then there exist sequences $N^1, N^2 \in \mathbb{R}_{>0}^{\mathbb{N}}$ with $L \leq N^1 \leq N^2 \lhd M^3$ satisfying $(N_k^i)^{1/k} \to \infty$ for $i = 1, 2$ and so that

(4.2) $$(N_j^1)^{1/j} \leq \sqrt{H_1}(N_k^2)^{1/k} \quad \text{for } j \leq k.$$

Proof. Without loss of generality we may assume that $L_k > 0$ for all k; otherwise replace L by $\bar{L}$ where $\bar{L}_k = L_k$ if $L_k > 0$ and $\bar{L}_k = 1$ if $L_k = 0$ (we still have $\bar{L} \lhd M^1$ since $(M_k^1)^{1/k} \to \infty$). The sequences, for $i = 1, 2, 3$,

$$c_k^i := \left(\frac{M_k^i}{L_k}\right)^{\frac{1}{k}}$$

satisfy $c_k^i \to \infty$, since $L \lhd M^i$. We define, for $i = 1, 2$,

(4.3) $$(N_k^i)^{\frac{1}{k}} := \max\left\{\sqrt{(M_k^i)^{\frac{1}{k}}}, \max_{j \leq k} \frac{(M_j^i)^{\frac{1}{j}}}{c_j^i}\right\} = \max\left\{\sqrt{(M_k^i)^{\frac{1}{k}}}, \max_{j \leq k} L_j^{\frac{1}{j}}\right\}.$$

Then clearly $(N_k^i)^{1/k} \to \infty$ and $L \leq N^1 \leq N^2$. For each $\epsilon > 0$ there exists $j_{\epsilon,i}$ so that $1/c_j^i \leq \epsilon$ for $j > j_{\epsilon,i}$. Thus, by (4.1),

$$\left(\frac{N_k^i}{M_k^{i+1}} \right)^{\frac{1}{k}} \leq \max \left\{ (M_k^{i+1})^{-\frac{1}{2k}}, (M_k^{i+1})^{-\frac{1}{k}} \max_{j \leq k} \frac{(M_j^i)^{\frac{1}{j}}}{c_j^i} \right\}$$

$$\leq \max \left\{ (M_k^{i+1})^{-\frac{1}{2k}}, (M_k^{i+1})^{-\frac{1}{k}} \max_{j \leq j_{\epsilon,i}} \frac{(M_j^i)^{\frac{1}{j}}}{c_j^i}, H_i \epsilon \right\} \leq H_i \epsilon,$$

for k sufficiently large, i.e., $N^1 \lhd M^2$ and $N^2 \lhd M^3$. It remains to show (4.2). But this is immediate from (4.1) and (4.3), indeed for $j \leq k$,

$$(N_j^1)^{\frac{1}{j}} = \max \left\{ \sqrt{(M_j^1)^{\frac{1}{j}}}, \max_{h \leq j} L_h^{\frac{1}{h}} \right\} \leq \max \left\{ \sqrt{H_1 (M_k^2)^{\frac{1}{k}}}, \max_{h \leq k} L_h^{\frac{1}{h}} \right\} \leq \sqrt{H_1} (N_k^2)^{\frac{1}{k}}$$

as required. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

The following proposition implies (1) $\Rightarrow$ (4), by the analogue of Remark 3.

Proposition 3. *Let $\mathfrak{M}$ be a weight matrix satisfying $(\mathfrak{M}_{(\mathrm{rai})})$ and $(\mathfrak{M}_{(C^\omega)})$. Let X be a Banach space and let $f : W \to X$ be an C^∞-mapping defined in an open subset $W \subseteq X \times \mathbb{R}$ and satisfying*

(4.4)
$$\forall \nu \in \Lambda, \rho > 0 \ \exists C \geq 1 \ \forall (k, \ell) \in \mathbb{N}^2 \ \forall (x, t) \in W :$$
$$\|f^{(k,\ell)}(x,t)\|_{L^{k,\ell}(X, \mathbb{R}; X)} \leq C \rho^{k+\ell} (k + \ell - 1)! M_{k+\ell-1}^\nu.$$

Then the solution $x : I \to X$ of the initial value problem

(4.5)
$$x'(t) = f(x(t), t), \quad x(0) = x_0,$$

satisfies

(4.6) $\forall \lambda \in \Lambda, \sigma > 0 \ \exists D \geq 1 \ \forall k \in \mathbb{N} \ \forall t \in I : \|x^{(k)}(t)\| \leq D \sigma^k (k-1)! M_{k-1}^\lambda.$

Remark 5. The analogue of Remark 4 applies.

Proof. In the same way as in the Roumieu case (Proposition 1) we may reduce to the initial value problem (3.6), where now $g \in C^\infty(U, Y)$ satisfies

(4.7)
$$\forall \nu \in \Lambda, \rho > 0 \ \exists C \geq 1 \ \forall k \in \mathbb{N} \ \forall y \in U : \|g^{(k)}(y)\|_{L^k(Y,Y)} \leq C \rho^k (k-1)! M_{k-1}^\nu.$$

There is a unique solution $y \in C^\infty(I, Y)$ defined on some interval I. Let $\lambda \in \Lambda$ be fixed. By $(\mathfrak{M}_{(\mathrm{rai})})$ there exist $\mu, \nu \in \Lambda$ and $H, J \geq 1$ such that

(4.8) $(M_j^\nu)^{1/j} \leq H(M_k^\mu)^{1/k} \leq J(M_\ell^\lambda)^{1/\ell}$ for $j \leq k \leq \ell,$

and, by $(\mathfrak{M}_{(C^\omega)})$,

(4.9) $\lim (M_k^\nu)^{1/k} = \lim (M_k^\mu)^{1/k} = \lim (M_k^\lambda)^{1/k} = \infty.$

We may assume without loss of generality that $\nu \leq \mu \leq \lambda$ and thus $M^\nu \leq M^\mu \leq M^\lambda$. If we set

$$L_{k-1} := \sup_{y \in U} \tfrac{1}{(k-1)!} \|g^{(k)}(y)\|_{L^k(Y,Y)},$$

then (4.7) implies $L \lhd M^\nu$. Thus, by applying Lemma 4, we find sequences N^1 and N^2 with $L \leq N^1 \leq N^2 \lhd M^\lambda$ satisfying $(N_k^i)^{1/k} \to \infty$ for $i = 1, 2$ and so that

$$(N_j^1)^{1/j} \leq \sqrt{H}(N_k^2)^{1/k}   \text{ for } j \leq k.$$

Repeating the proof of Proposition 1 (with N^1 in place of M^λ and N^2 in place of M^μ) we may conclude that

$$\exists D, \sigma \geq 1 \; \forall k \in \mathbb{N} \; \forall t \in I : \|y^{(k)}(t)\| \leq D\sigma^k (k-1)! N_{k-1}^2$$

which implies

$$\forall \tau > 0 \; \exists E \geq 1 \; \forall k \in \mathbb{N} \; \forall t \in I : \|y^{(k)}(t)\| \leq E\tau^k (k-1)! M_{k-1}^\lambda,$$

since $N^2 \lhd M^\lambda$. As λ was arbitrary, the result follows. $\qquad\square$

4.3. **The cycle** $(1) \Rightarrow (5) \Rightarrow (6) \Rightarrow (6') \Rightarrow (1)$. Obviously, $(5) \Rightarrow (6) \Rightarrow (6')$, and $(1) \Leftrightarrow (6')$, by Subsection 4.2. Finally, $(1) \Rightarrow (5)$ follows from the following proposition.

Proposition 4. *Let* $\mathfrak{M}$ *be a weight matrix satisfying* $(\mathfrak{M}_{(\mathrm{rai})})$ *and* $(\mathfrak{M}_{(C^\omega)})$. *Let* $f : U \to V$ *be a* C^∞-*mapping between open subsets* $U \subseteq E$ *and* $V \subseteq F$ *of Banach spaces* E, F *satisfying*

(4.10)
$$\forall \nu \in \Lambda, \rho > 0 \; \exists C \geq 1 \; \forall k \in \mathbb{N} \; \forall x \in U : \|f^{(k)}(x)\|_{L^k(E,F)} \leq C\rho^{k-1}(k-1)! M_{k-1}^\nu$$

and so that $f'(x_0) \in L(E, F)$ *is invertible. Then there exist neighborhoods* $x_0 \in U_0 \subseteq U$ *and* $f(x_0) \in V_0 \subseteq V$ *and a* C^∞-*mapping* $g : V_0 \to U_0$ *satisfying*

(4.11)
$$\forall \mu \in \Lambda, \sigma > 0 \; \exists D \geq 1 \; \forall k \in \mathbb{N} \; \forall y \in V_0 : \|g^{(k)}(y)\|_{L^k(F,E)} \leq D\sigma^{k-1}(k-1)! M_{k-1}^\mu$$

and such that $f \circ g = \mathrm{id}_{V_0}$.

Proof. Let $\lambda \in \Lambda$ be fixed. By $(\mathfrak{M}_{(\mathrm{rai})})$ and $(\mathfrak{M}_{(C^\omega)})$, there exist $\mu, \nu \in \Lambda$ satisfying (4.8) and (4.9). If we set

$$L_{k-1} := \sup_{x \in U} \tfrac{1}{(k-1)!} \|f^{(k)}(x)\|_{L^k(E,F)},$$

then (4.10) implies $L \lhd M^\nu$. In analogy to the proof of Proposition 3, we repeat the proof of Proposition 2 (and that of Lemma 2) with the sequences N^1 and N^2 provided by Lemma 4. $\qquad\square$

5. Proof of Theorems 1, 2, 3, and 4

5.1. **Proof of Theorems 1 and 2.** Apply Theorems 5 and 6 to the constant weight matrix $\mathfrak{M} = \{M\}$.

5.2. **Proof of Theorems 3 and 4.** For a weight function ω and each $\rho > 0$ consider the sequence $\Omega^\rho \in \mathbb{R}_{>0}^{\mathbb{N}}$ defined by

$$\Omega_k^\rho := \tfrac{1}{k!} \exp(\tfrac{1}{\rho} \varphi^*(\rho k)).$$

By the properties of φ^*, the collection $\mathfrak{W} := \{\Omega^\rho : \rho > 0\}$ forms a weight matrix, and we have $\mathcal{E}^{[\omega]}(U) = \mathcal{E}^{[\mathfrak{W}]}(U)$ as locally convex spaces, by [13]. Moreover, $\mathfrak{W}$ satisfies $(\mathfrak{M}_{(\mathrm{dc})})$ as well as $(\mathfrak{M}_{\{\mathrm{dc}\}})$. If $\omega(t) = O(t)$ as $t \to \infty$, then $\mathfrak{W}$ satisfies $(\mathfrak{M}_{\mathcal{H}})$, and if $\omega(t) = o(t)$ as $t \to \infty$, then $\mathfrak{W}$ satisfies $(\mathfrak{M}_{(C^\omega)})$. Thus Theorems 3 and 4 are immediate consequences of Theorems 5, 6, and [13, 6.3, 6.5].

Appendix A. Weight sequences as required in Remark 2

Let us now find explicit sequences that satisfy the requirements of Remark 2. To this end we construct a weight sequence $M = (M_k)$ such that $(M_{k+1}/M_k)^{1/k}$ is bounded, $M_k^{1/k}$ tends to ∞ but is not almost increasing, and $k!^s \leq M_k \leq k!^t$ for suitable $s, t > 0$ and sufficiently large k. Since for every Gevrey sequence $G^s = (k!^s)_k$ (where $s \geq 0$), $(G_k^s)^{1/k}$ is increasing (and tends to ∞ if $s > 0$), the pair of sequences (G^s, M), or (M, G^t), will fulfill the requirements of Remark 2 (after adjusting finitely many terms of one sequence).

Let $k_j := 2 \uparrow\uparrow j = 2^{2^{\cdot^{\cdot^{2}}}}$ (j times) for $j \geq 1$ and $k_0 := 0$. Let $\varphi : [0, \infty) \to [0, \infty)$ be the function whose graph is the polygon with vertices $\{v_j = (k_j, \varphi(k_j)) : j \in \mathbb{N}\}$ defined by

$$\varphi(0) := 0, \quad \varphi(2) := 8 \log 2, \quad \varphi(k_j) := \begin{cases} k_j \log k_{j+1} & j \text{ even} \\ k_j \log(k_j k_{j-2}) & j \text{ odd} \end{cases}, \quad (j \geq 2).$$

We claim that the sequence $M = (M_k)$ defined by $M_k := \exp(\varphi(k))/k!$ satisfies:

(1) M is a weight sequence, i.e., M is weakly log-convex,
(2) $\sup_k (\frac{M_{k+1}}{M_k})^{1/k} < \infty$,
(3) $M_k^{1/k} \to \infty$,
(4) $M_k^{1/k}$ is not almost increasing,
(5) $k!^s \leq M_k \leq k!^t$ for all $0 \leq s \leq 1/4$, $t > 3$, and all $k \geq k_0(t)$.

To see that M is weakly log-convex it suffices to show that the slopes of the line segments in the graph of φ are increasing. Let a_j denote the slope of the line segment left of the vertex v_j. Then, for $i \geq 2$,

$$\begin{aligned}
a_{2i-1} &= \frac{k_{2i-1} \log(k_{2i-1} k_{2i-3}) - k_{2i-2} \log k_{2i-1}}{k_{2i-1} - k_{2i-2}} = \frac{\frac{5}{4} k_{2i-2} - 1}{k_{2i-2} - 1} \log k_{2i-1} \\
a_{2i} &= \frac{k_{2i} \log k_{2i+1} - k_{2i-1} \log(k_{2i-1} k_{2i-3})}{k_{2i} - k_{2i-1}} = \frac{4 k_{2i-1} - \frac{5}{4}}{k_{2i-1} - 1} \log k_{2i-1} \\
a_{2i+1} &= \frac{k_{2i+1} \log(k_{2i+1} k_{2i-1}) - k_{2i} \log k_{2i+1}}{k_{2i+1} - k_{2i}} = \frac{5 k_{2i} - 4}{k_{2i} - 1} \log k_{2i-1}
\end{aligned}$$

(A.1)

and $a_{2i-1} \leq a_{2i} \leq a_{2i+1}$. This proves (1). Let us check (2). Since

$$\left(\frac{M_{k+1}}{M_k} \right)^{1/k} = \frac{\exp(\frac{\varphi(k+1)}{k} - \frac{\varphi(k)}{k})}{(k+1)^{1/k}} \leq \exp\left(\frac{\varphi(k+1)}{k} - \frac{\varphi(k)}{k} \right),$$

it suffices to show that $\frac{\varphi(k+1)}{k} - \frac{\varphi(k)}{k}$ is bounded or equivalently that the slope of the line segments of φ increases at most linearly in k. This is obvious from (A.1). Thanks to $k! \leq k^k \leq e^k k!$ we have

$$\frac{\exp(\frac{\varphi(k)}{k})}{k} \leq M_k^{1/k} \leq \frac{\exp(\frac{\varphi(k)}{k} + 1)}{k}.$$

This implies (3). To show (4) let j be even. Then

$$\log \frac{M_{k_j}^{1/k_j}}{M_{k_{j+1}}^{1/k_{j+1}}} \geq \log k_j + \log k_{j+1} - \log(k_{j+1} k_{j-1}) - 1 = \log k_{j-1} - 1 \to \infty,$$

as required. Finally, $k!^s \le M_k \le k!^t$ is equivalent to $1 + s \le \frac{\varphi(k)}{\log k!} \le 1 + t$. We have

$$\frac{\varphi(k_j)}{k_j \log k_j} = \begin{cases} 2 & j \text{ even} \\ 1 + \frac{1}{4} & j \text{ odd} \end{cases}$$

and $\frac{\varphi(k)}{k \log k} \le \frac{\varphi(k)}{\log k!} \le 2 \frac{\varphi(k)}{k \log k}$. So the first inequality in (5) follows thanks to the fact that $k!^s$ is log-convex for each $s \ge 0$. For the second inequality we observe that in view of (A.1) the slope a_i is dominated by the increment $\log((k+1)!^{1+t}) - \log(k!^{1+t}) = (1 + t) \log(k + 1)$ for all $k_{i-1} \le k < k_i$ provided that $t > 3$ and that i is sufficiently large depending on t.

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