INTERPOLATION OF DERIVATIVES AND ULTRADIFFERENTIABLE REGULARITY

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Abstract. We review some interpolation inequalities for derivatives and results on ultradifferentiable regularity by lacunary estimates.

1. Introduction

Ultradifferentiable classes are classes of C^∞ functions defined by prescribed growth behavior of the infinite sequence of derivatives. The most classical among them are the Denjoy–Carleman classes, where the growth of the derivatives is dominated by a weight sequence $M = (M_j)_{j\ge0}$. The well-known Gevrey classes are special cases thereof.

In applications, sometimes the question arises whether the ultradifferentiable regularity can be concluded if only lacunary information on the growth of the derivatives is available. More precisely, suppose that we know that a smooth function satisfies certain ultradifferentiable bounds for the derivatives of order k_j, $j \ge 0$, can we deduce that its derivatives of all orders satisfy the ultradifferentiable bounds? An affirmative answer clearly depends on conditions for the base sequence (k_j), the weight $M = (M_j)$, and on suitable interpolation inequalities.

In the recent paper [1], Albano and Mughetti gave sufficient conditions for Denjoy–Carleman classes of Roumieu type on a compact interval of $\mathbb{R}$, based on the Cartan–Gorny inequality (cf. Proposition 2.6). The base sequence (k_j) is required to be such that k_{j+1}/k_j is bounded. For the weight sequence, the authors assume a rather strong condition (which we recall and discuss in Remark 6.4). In fact, we prove a version under a weaker and more natural condition in Theorem 6.1 (see also Theorem 4.1), namely, boundedness of $m_{k_{j+1}}/m_{k_j}$, where $m_j := M_j^{1/j}$.

The main purpose of this paper is to treat the interpolation problem in a broad and comprehensive way, striving for minimal assumptions on the weights. To this end, we review various interpolation inequalities for L^p-norms, where $1 \le p \le \infty$, and work in arbitrary (finite) dimensions; the growth of the involved constants is crucial in the ultradifferentiable setting. We consider ultradifferentiable classes of local and global type and allow very general weight systems (so that also Braun–Meise–Taylor classes are covered). All the ingredients seem to be well-known, even though somewhat scattered in the literature, but we think a unified treatment can be useful.

Date: December 12, 2023.

2020 *Mathematics Subject Classification.* 26D10, 26E05, 26E10, 41A17, 46E10.

Key words and phrases. Interpolation inequalities, Denjoy–Carleman classes, ultradifferentiable regularity, lacunary estimates.

Supported by the Austrian Science Fund (FWF), Grants DOI 10.55776/P32905 and DOI 10.55776/P33417.

For local Denjoy–Carleman classes of Roumieu type, the paper of Liess [14] must be mentioned which itself refers to the work of Bolley, Camus, and Métivier [2] in the analytic case. Note that Liess in particular showed that, under certain assumptions on M, boundedness of $m_{k_{j+1}}/m_{k_j}$ is also a necessary condition for the interpolation problem (see Remark 4.4). The results on ultradifferentiable regularity by lacunary estimates have natural applications to the case, where one has bounds for the sequence $(P^j f)_{j\ge 0}$ and P is an elliptic linear partial differential operator. See for instance [17], where this approach was used to prove Chevalley type results.

2. Interpolation inequalities

2.1. **The global setting.** The Landau–Kolmogorov inequality states that a C^m function $f : \mathbb{R} \to \mathbb{R}$ with finite $\|f\|_{L^\infty(\mathbb{R})}$ and $\|f^{(m)}\|_{L^\infty(\mathbb{R})}$ satisfies

$$\|f^{(j)}\|_{L^\infty(\mathbb{R})} \le K_{m,j}\, \|f\|_{L^\infty(\mathbb{R})}^{1-j/m} \|f^{(m)}\|_{L^\infty(\mathbb{R})}^{j/m}, \quad j = 1, \dots, m-1.$$

Due to Kolmogorov [9], the optimal constants $K_{m,j}$ are given by

$$K_{m,j} = \frac{k_{m-j}}{k_m^{1-j/m}},$$

where $k_r := \frac{4}{\pi}\sum_{i=0}^\infty [\frac{(-1)^i}{2i+1}]^{r+1}$ are the Favard constants. Note that $1 \le k_r \le 2$ so that $K_{m,j} \le 2$. By a simple functional-analytic argument, Certain and Kurtz [5] inferred that, if $(E, \|\cdot\|)$ is a real Banach space and A is the generator of a strongly continuous group of isometries, then

$$\|A^j x\| \le K_{m,j}\, \|x\|^{1-j/m} \|A^m x\|^{j/m}, \quad j = 1, \dots, m-1,$$

for x in the domain of A^m. In particular (see also [13, Section 4.4]), we have the following lemma.

Lemma 2.1. *Let $1 \le p \le \infty$ and $m \in \mathbb{N}_{\ge 2}$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^m function. Then, for all $v \in \mathbb{S}^{n-1}$,*

$$\|d_v^j f\|_{L^p(\mathbb{R}^n)} \le 2\, \|f\|_{L^p(\mathbb{R}^n)}^{1-j/m} \|d_v^m f\|_{L^p(\mathbb{R}^n)}^{j/m}, \quad j = 1, \dots, m-1,$$

if the right-hand side is finite, where $d_v^j f(x) := \partial_t^j f(x + tv)|_{t=0}$.

Proof. We may assume $v = (1, 0, \dots, 0)$ by choosing a suitable orthonormal system of coordinates. Let us first consider $p = \infty$. We have

$$\begin{aligned}\sup_{x\in\mathbb{R}^n} |\partial_1^j f(x)| &\le \sup_{\substack{x_i\in\mathbb{R}\\ 2\le i\le n}} \Big[2 \Big(\sup_{x_1\in\mathbb{R}} |f(x)|\Big)^{1-j/m} \Big(\sup_{x_1\in\mathbb{R}} |\partial_1^m f(x)|\Big)^{j/m}\Big] \\ &\le 2 \Big(\sup_{x\in\mathbb{R}^n} |f(x)|\Big)^{1-j/m} \Big(\sup_{x\in\mathbb{R}^n} |\partial_1^m f(x)|\Big)^{j/m}.\end{aligned}$$

For $p < \infty$, we apply the above remarks to $A = \partial_1$ and $E = L^p(\mathbb{R}^n)$. It generates the group of translations $T(s)f(x_1, \dots, x_n) = f(x_1+s, x_2, \dots, x_n)$ which is strongly continuous, by the dominated convergence theorem. □

Remark 2.2. For $p = 2$, the factor 2 can be omitted as follows from an application of the Fourier transform:

$$\int |\xi_1^j \widehat{f}(\xi)|^2\, d\xi = \int (|\xi_1|^m |\widehat{f}(\xi)|)^{\frac{2j}{m}} |\widehat{f}(\xi)|^{2(1-\frac{j}{m})}\, d\xi \le \|\xi_1^m \widehat{f}(\xi)\|_{L^2(\mathbb{R}^n)}^{2j/m} \|\widehat{f}(\xi)\|_{L^2(\mathbb{R}^n)}^{2(1-j/m)},$$

by Hölder's inequality.

2.2. **The local setting.** The next two lemmas follow from an easy adaptation of the proof of [2, Lemmas 2.3-2.5]. We sketch the argument for the convenience of the reader.

Lemma 2.3. *There is a constant $C > 0$ such that the following holds. Let $1 \le p \le \infty$, $a > 0$, $m \in \mathbb{N}$, $0 \le j \le m$, and $f \in C^m([-2a, 2a])$. Then*

$$\frac{a^j}{j!}\|f^{(j)}\|_{L^p([-a,a])} \le C^m\Big(\frac{a^m}{m!}\|f^{(m)}\|_{L^p([-2a,2a])} + \|f\|_{L^p([-2a,2a])}\Big).$$

Proof. By rescaling, it suffices to assume $a = 1$. Fix m. Let φ be a C^∞ function with support in $(-2, 2)$ that equals 1 near $[-1, 1]$ and satisfies $\|\varphi^{(k)}\|_{L^\infty(\mathbb{R})} \le (C_0 m)^k$ for $0 \le k \le m$ for a universal constant C_0 (cf. [8, Theorem 1.3.5]). By Taylor's theorem, for $t \in [-2, 2]$ and $0 \le j \le m - 1$,

$$(\varphi f)^{(j)}(t) = \int_{-2}^{t} \frac{(t-s)^{m-j-1}}{(m-j-1)!}(\varphi f)^{(m)}(s)\, ds = \sum_{i=0}^{m}\binom{m}{i}A_i(t),$$

where

$$A_i(t) := \frac{1}{(m-j-1)!}\int_{-2}^{t}(t-s)^{m-j-1}\varphi^{(m-i)}(s)f^{(i)}(s)\, ds.$$

For $i = m$, we have

$$|A_m(t)| \le \frac{1}{(m-j-1)!}\int_{-2}^{2}|t-s|^{m-j-1}|\varphi(s)f^{(m)}(s)|\, ds,$$

and thus, by Young's inequality,

$$\begin{aligned}\|A_m\|_{L^p([-1,1])} &\le \|A_m\|_{L^p([-2,2])}\\ &\le \frac{1}{(m-j-1)!}\|\mathbf{1}_{[-2,2]}t^{m-j-1}\|_{L^1(\mathbb{R})}\|\varphi f^{(m)}\|_{L^p(\mathbb{R})}\\ &\le \frac{2^{m-j+1}}{(m-j)!}\|f^{(m)}\|_{L^p([-2,2])}.\end{aligned}$$

For $0 \le i \le m-1$ and $t \in [-1, 1]$, integration by parts gives

$$\begin{aligned}A_i(t) &= \frac{(-1)^i}{(m-j-1)!}\int_{-2}^{-1}\partial_s^i[(t-s)^{m-j-1}\varphi^{(m-i)}(s)]f(s)\, ds\\ &= \sum_{\ell=0}^{k}\binom{i}{\ell}\frac{(-1)^{i+\ell}}{(m-j-1-\ell)!}\int_{-2}^{-1}(t-s)^{m-j-1-\ell}\varphi^{(m-\ell)}(s)f(s)\, ds,\end{aligned}$$

where $k := \min\{i, m-j-1\}$. Using Young's inequality as before, we get

$$\|A_i\|_{L^p([-1,1])} \le \sum_{\ell=0}^{k}\binom{i}{\ell}\frac{2^{m-j-\ell+1}}{(m-j-\ell)!}(C_0 m)^{m-\ell}\|f\|_{L^p([-2,2])}.$$

Consequently,

$$\sum_{i=0}^{m-1}\binom{m}{i}\|A_i\|_{L^p([-1,1])} \le \sum_{i=0}^{m-1}\sum_{\ell=0}^{k}\binom{m}{i}\binom{i}{\ell}\frac{2^{m-j-\ell+1}}{(m-j-\ell)!}(C_0 m)^{m-\ell}\|f\|_{L^p([-2,2])}$$

and the lemma follows if we prove that

$$\sum_{i=0}^{m-1}\sum_{\ell=0}^{k}\binom{m}{i}\binom{i}{\ell}\frac{2^{m-j-\ell+1}}{(m-j-\ell)!}(C_0 m)^{m-\ell} \le C^m j!$$

for a universal constant C. The left-hand side equals

$$\sum_{\ell=0}^{m-j-1}\sum_{i=\ell}^{m-1}\frac{(m-\ell)!}{(m-i)!(i-\ell)!}\frac{m!}{\ell!(m-\ell)!}\frac{2^{m-j-\ell+1}}{(m-j-\ell)!}(C_0m)^{m-\ell}$$

which is bounded by

$$\sum_{\ell=0}^{m-j-1} 2^{m-\ell}2^m\frac{2^{m-j-\ell+1}}{(m-j-\ell)!}C_0^{m-\ell}e^m\frac{m!}{\ell!}$$
$$\le e^m 2^{2m}2^{j+1}C_0^j\, j!\sum_{\ell=0}^{m-j}\frac{(m-j)!}{\ell!(m-j-\ell)!}(2^2C_0)^{m-j-\ell}$$

which is of the required form. □

Lemma 2.4. *There exist constants $C_0, C>0$, depending only on the dimension n, such that the following holds. Let $1\le p\le\infty$. Let U,V bounded open subsets of $\mathbb{R}^n$ such that $\overline{U}\subseteq V$. Let $f\in C^m(\overline{V})$. For all $0<a\le C_0\operatorname{dist}(U,\partial V)$, all $0\le j\le m$, and all $v\in\mathbb{S}^{n-1}$,*

$$\frac{a^j}{j!}\|d_v^jf\|_{L^p(U)}\le C^m\Big(\frac{a^m}{m!}\|d_v^mf\|_{L^p(V)}+\|f\|_{L^p(V)}\Big).$$

Proof. We may assume that $v=(1,0,\dots,0)$. Let us first show the assertion for $U=(-a,a)^n$ and $V=(-2a,2a)^n$. For $x':=(x_2,\dots,x_n)\in(-a,a)^{n-1}$, Lemma 2.3 gives

$$\frac{a^j}{j!}\|\partial_1^jf(\cdot,x')\|_{L^p((-a,a))}\le C^m\Big(\frac{a^m}{m!}\|\partial_1^mf(\cdot,x')\|_{L^p((-2a,2a))}+\|f(\cdot,x')\|_{L^p((-2a,2a))}\Big).$$

For $p=\infty$, the statement follows, by taking the supremum over all $x'\in(-a,a)^{n-1}$. For $p<\infty$, take the p-th power an integrate over $x'\in(-a,a)^{n-1}$.

In general, there is a constant $C_0>0$ depending only on n such that, if $a\le C_0\operatorname{dist}(U,\partial V)$, then U can be covered by a family $\mathcal{Q}$ of cubes $Q=x+[-a,a]^n$ such that $2Q=x+[-2a,2a]^n\subseteq V$ and such that any two cubes in $\mathcal{Q}$ have disjoint interior. Then, for each $Q\in\mathcal{Q}$,

$$\frac{a^j}{j!}\|\partial_1^jf\|_{L^p(Q)}\le C^m\Big(\frac{a^m}{m!}\|\partial_1^mf\|_{L^p(2Q)}+\|f\|_{L^p(2Q)}\Big).$$

For $p=\infty$, take the maximum over all $Q\in\mathcal{Q}$, for $p<\infty$, take the p-power and the sum over all $Q\in\mathcal{Q}$. □

Now we combine Lemma 2.4 with Lemma 2.1 (following [14, p. 193]).

Corollary 2.5. *Let $1\le p\le\infty$. Let U,V bounded open subsets of $\mathbb{R}^n$ such that $\overline{U}\subseteq V$. Let $f\in C^\infty(\overline{V})$. Then, for each integer $m\ge2$, $j=1,\dots,m-1$, and all $v\in\mathbb{S}^{n-1}$,*

$$\|d_v^jf\|_{L^p(U)}\le C^m\|f\|_{L^p(V)}^{1-j/m}\Big(\|d_v^mf\|_{L^p(V)}^{j/m}+m^j\|f\|_{L^p(V)}^{j/m}\Big),$$

where $C>0$ is a constant depending only on U, V, and n.

Proof. Let W be the open δ-neighborhood of $\overline{U}$, where $\delta:=\frac{1}{2}\operatorname{dist}(U,\partial V)$. Let φ be a C^∞ function with support in W that equals 1 near $\overline{U}$ and satisfies $\|\partial^\alpha\varphi\|_{L^\infty(\mathbb{R}^n)}\le C_0^{|\alpha|+1}m^{|\alpha|}$ for $|\alpha|\le m$, where C_0 only depends on U, V, and n (cf. [8, Theorem

1.4.2]). Then φf has a C^∞ extension by zero outside W to all of $\mathbb{R}^n$ which we also denote by φf. By Lemma 2.1,

$$\|d_v^j f\|_{L^p(U)} \le \|d_v^j(\varphi f)\|_{L^p(\mathbb{R}^n)} \le 2\,\|(\varphi f)\|_{L^p(W)}^{1-j/m}\|d_v^m(\varphi f)\|_{L^p(W)}^{j/m}$$

and, by Lemma 2.4 with $a = C_1\delta$, where $C_1 = C_1(n)$,

$$\begin{aligned}
\|d_v^m(\varphi f)\|_{L^p(W)} &\le \sum_{i=0}^m \binom{m}{i} \|d_v^{m-i}\varphi\|_{L^\infty(\mathbb{R}^n)}\|d_v^i f\|_{L^p(W)} \\
&\le \sum_{i=0}^m \binom{m}{i} C_0(C_0 m)^{m-i} C^m \frac{i!}{(C_1\delta)^i}\Big(\frac{(C_1\delta)^m}{m!}\|d_v^m f\|_{L^p(V)} + \|f\|_{L^p(V)}\Big) \\
&\le \sum_{i=0}^m \binom{m}{i} \frac{C_0(eC_0C\max\{C_1\delta,1\})^m}{(C_0C_1\delta)^i}\big(\|d_v^m f\|_{L^p(V)} + m!\,\|f\|_{L^p(V)}\big) \\
&\le C_2^m\Big(\|d_v^m f\|_{L^p(V)} + m^m\|f\|_{L^p(V)}\Big).
\end{aligned}$$

This implies the assertion. □

2.3. **The Cartan–Gorny inequality.** The following result is due to Gorny [7] and independently to Cartan [4].

Proposition 2.6. *Let $I \subseteq \mathbb{R}$ be a compact interval and $f \in C^m(I)$. Then, for $j = 1, \ldots, m-1$,*

$$\|f^{(j)}\|_{L^\infty(I)} \le 4e^{2j}\Big(\frac{m}{j}\Big)^j \|f\|_{L^\infty(I)}^{1-j/m} \max\Big\{\|f^{(m)}\|_{L^\infty(I)}, \frac{m!}{|I|^m}\|f\|_{L^\infty(I)}\Big\}^{j/m}.$$

The factor $(\frac{m}{j})^j$ is bounded by e^m.

3. Ultradifferentiable regularity by lacunary estimates

In this section, we prove three technical propositions on which most of the subsequent results are based.

Let $0 =: k_0 < k_1 < k_2 < \cdots$ be a strictly increasing infinite sequence of positive integers. A sequence (k_j) with these properties is called an *base sequence*. If additionally k_{j+1}/k_j, $j \ge 1$, is bounded we say that (k_j) is a *special base sequence*.

Proposition 3.1. *Let (k_j) be a base sequence. Let (m_j) and (m'_j) be positive increasing sequences of reals satisfying $m_j \le m'_j$ for all j. Assume that $m_{k_{j+1}}/m'_{k_j}$ is bounded. Let $1 \le p \le \infty$ and $f \in C^\infty(\mathbb{R}^n)$.*

If there are $C, \rho > 0$ such that

$$\max_{|\alpha|=k_j}\|\partial^\alpha f\|_{L^p(\mathbb{R}^n)} \le C(\rho m_{k_j})^{k_j}, \quad j \ge 0, \tag{1}$$

then there exist $C', \rho' > 0$ such that

$$\max_{|\alpha|=\ell}\|\partial^\alpha f\|_{L^p(\mathbb{R}^n)} \le C'(\rho' m'_\ell)^\ell, \quad \ell \ge 0. \tag{2}$$

We have $\rho' = O(\rho)$ as $\rho \to 0$.

Proof. By (1), there are constants $C_1, \rho_1 > 0$ such that

$$\|d_v^{k_j} f\|_{L^p(\mathbb{R}^n)} \le C_1(\rho_1 m_{k_j})^{k_j}, \quad j \ge 0,\ v \in \mathbb{S}^{n-1}. \tag{3}$$

Let $\ell \ge 0$ and let $j \ge 0$ be such that $k_j \le \ell < k_{j+1}$. By Lemma 2.1 and the assumptions on (k_j), (m_j), and (m'_j),

$$
\begin{aligned}
\|d_v^\ell f\|_{L^p(\mathbb{R}^n)} &\le 2\, \|d_v^{k_j} f\|_{L^p(\mathbb{R}^n)}^{1-\frac{\ell-k_j}{k_{j+1}-k_j}} \|d_v^{k_{j+1}} f\|_{L^p(\mathbb{R}^n)}^{\frac{\ell-k_j}{k_{j+1}-k_j}} \\
&\le 2C_1 \,(\rho_1 m_{k_j})^{k_j\left(1-\frac{\ell-k_j}{k_{j+1}-k_j}\right)} (\rho_1 m_{k_{j+1}})^{k_{j+1}\frac{\ell-k_j}{k_{j+1}-k_j}} \\
&\le 2C_1 \,(\rho_1 m'_{k_j})^{k_j\left(1-\frac{\ell-k_j}{k_{j+1}-k_j}\right)} (C_2\rho_1 m'_{k_j})^{k_{j+1}\frac{\ell-k_j}{k_{j+1}-k_j}} \\
&\le 2C_1 \,(C_2\rho_1 m'_\ell)^\ell.
\end{aligned} \tag{4}
$$

By polarization [10, Lemma 7.13], for $|\alpha| = \ell$ and $v_1, \ldots, v_\ell$ the list of standard unit vectors

$$\underbrace{e_1, \ldots, e_1}_{\alpha_1}, \underbrace{e_2, \ldots, e_2}_{\alpha_2}, \ldots, \underbrace{e_n, \ldots, e_n}_{\alpha_n},$$

we have

$$\partial^\alpha f = \frac{1}{\ell!} \sum_{\epsilon_1,\ldots,\epsilon_\ell=0}^{1} (-1)^{\ell-\sum \epsilon_\ell} \Big(\sum \epsilon_i\Big)^\ell d^\ell_{\sum \epsilon_i v_i/\sum \epsilon_i} f. \tag{5}$$

Therefore,

$$\|\partial^\alpha f\|_{L^p(\mathbb{R}^n)} \le 2C_1 (C_2\rho_1 m'_\ell)^\ell \frac{1}{\ell!} \sum_{k=0}^{\ell} \binom{\ell}{k} k^\ell \le 2C_1 (2eC_2\rho_1 m'_\ell)^\ell.$$

Thus (2) is proved. $\square$

Proposition 3.2. *Let (k_j) be an special base sequence. Let (m_j) and (m'_j) be positive increasing sequences of reals satisfying $m_j \le m'_j$ for all j and such that j/m_j is bounded. Assume that $m_{k_{j+1}}/m'_{k_j}$ is bounded. Let $1 \le p \le \infty$. Let U, V bounded open subsets of $\mathbb{R}^n$ such that $\overline{U} \subseteq V$. Let $f \in C^\infty(\overline{V})$.*

If there are $C, \rho > 0$ such that

$$\max_{|\alpha|=k_j} \|\partial^\alpha f\|_{L^p(V)} \le C(\rho m_{k_j})^{k_j}, \quad j \ge 0, \tag{6}$$

then there exist $C', \rho' > 0$ such that

$$\max_{|\alpha|=\ell} \|\partial^\alpha f\|_{L^p(U)} \le C'(\rho' m'_\ell)^\ell, \quad \ell \ge 0. \tag{7}$$

Under the additional assumption that $j/m_j \to 0$, we have that $\rho' = O(\rho)$ as $\rho \to 0$ and C' depends on ρ'.

Proof. By (6), there are constants $C_1, \rho_1 > 0$ such that

$$\|d_v^{k_j} f\|_{L^p(V)} \le C_1(\rho_1 m_{k_j})^{k_j}, \quad j \ge 0,\ v \in \mathbb{S}^{n-1}.$$

Let $\ell \ge 0$ and let $j \ge 0$ be such that $k_j \le \ell < k_{j+1}$. By Corollary 2.5 and the assumptions on (k_j), (m_j), and (m'_j),

$$
\begin{aligned}
&\|d_v^\ell f\|_{L^p(U)} \\
&\le C^{k_{j+1}-k_j} \Big(\|d_v^{k_j} f\|_{L^p(V)}^{1-\frac{\ell-k_j}{k_{j+1}-k_j}} \|d_v^{k_{j+1}} f\|_{L^p(V)}^{\frac{\ell-k_j}{k_{j+1}-k_j}} + (k_{j+1}-k_j)^{\ell-k_j} \|d_v^{k_j} f\|_{L^p(V)} \Big) \\
&\le C_1 C_2^{k_j} \Big((\rho_1 m_{k_j})^{k_j\left(1-\frac{\ell-k_j}{k_{j+1}-k_j}\right)} (\rho_1 m_{k_{j+1}})^{k_{j+1}\frac{\ell-k_j}{k_{j+1}-k_j}} + (C_3 k_j)^{\ell-k_j} (\rho_1 m_{k_j})^{k_j} \Big)
\end{aligned}
$$

$$\le C_1C_2^{k_j}\Big((\rho_1 m'_{k_j})^{k_j\left(1-\frac{\ell-k_j}{k_{j+1}-k_j}\right)}(C_4\rho_1 m'_{k_j})^{k_{j+1}\frac{\ell-k_j}{k_{j+1}-k_j}} + (C_5 m'_{k_j})^{\ell-k_j}(\rho_1 m'_{k_j})^{k_j}\Big)$$
$$\le C_1C_2^{k_j}\Big((C_4\rho_1 m'_{k_j})^{\ell} + ((C_5+\rho_1)m'_{k_j})^{\ell}\Big)$$
$$\le C_1(C_6 m'_\ell)^\ell.$$

If we assume that $j/m_j \to 0$ (actually $k_j/m_{k_j} \to 0$ is enough), then for each $\tau > 0$ there is j_τ such that $k_j/m_{k_j} \le \tau$ for all $j \ge j_\tau$. Thus $C_5 = C_3\rho_1$ provided that $j \ge j_{\rho_1}$ and so

$$\|d_v^\ell f\|_{L^p(U)} \le C_1(C_7\rho_1 m'_\ell)^\ell, \quad \ell \ge k_{j_{\rho_1}}.$$

For $\ell < k_{j_{\rho_1}}$,

$$\|d_v^\ell f\|_{L^p(U)} \le C_1(C_6 m'_\ell)^\ell \le C_1 \max\{\rho_1^{-\ell} : \ell < k_{j_{\rho_1}}\}\cdot (C_6\rho_1 m'_\ell)^\ell.$$

Since $\rho_1 = O(\rho)$, we get

$$\|d_v^\ell f\|_{L^p(U)} \le C'(\rho' m'_\ell)^\ell, \quad \ell \ge 0,$$

where $\rho' = O(\rho)$ and $C' = C'(\rho')$.

To end the proof, it suffices to apply $\|\cdot\|_{L^p(U)}$ to (5) and use the estimate for $\|d_v^\ell f\|_{L^p(U)}$. □

Remark 3.3. The requirement $k_0 = 0$ for the base sequence is important in Proposition 3.1. Without this assumption there is no reason why a function satisfying (1) should also fulfill (2) for $\ell = 0$. In the local setting of Proposition 3.2, the assumption $k_0 = 0$ can be made without loss of generality, by adjusting the constant C.

Proposition 3.4. *Let (k_j) be a base sequence. Let (m_j) and (m'_j) be positive increasing sequences of reals satisfying $m_j \le m'_j$ for all j. Assume that $m_{k_{j+1}}/m'_{k_j}$ is bounded. Let $f \in C^\infty(\mathbb{R}^n)$.*

If there are $C, \sigma > 0$ such that

$$\max_{|\alpha|=k_j} \|x^\alpha f\|_{L^\infty(\mathbb{R}^n)} \le C(\sigma m_{k_j})^{k_j}, \quad j \ge 0, \tag{8}$$

then there exist $C', \sigma' > 0$ such that

$$\max_{|\alpha|=\ell} \|x^\alpha f\|_{L^\infty(\mathbb{R}^n)} \le C'(\sigma' m'_\ell)^\ell, \quad \ell \ge 0. \tag{9}$$

We have $\sigma' = O(\sigma)$.

Proof. If $|\alpha| = k$, then, for a constant $D > 0$,

$$|x^\alpha| \le |x|^k \le D\sum_{i=1}^n |x_i|^k \le D\sum_{|\alpha|=k}|x^\alpha| \le D\, n^k \max_{|\alpha|=k}|x^\alpha|.$$

(Indeed, $\sum_{i=1}^n |x_i|^k$ has a positive minimum d on the sphere $\{|x| = 1\}$, thus $\sum_{i=1}^n |x_i|^k \ge d|x|^k$, by homogeneity.) The assumption (8) gives

$$|x|^{k_j}|f(x)| \le C_1(n\sigma\, m_{k_j})^{k_j}, \quad j \ge 0,\ x \in \mathbb{R}^n.$$

Then, for $k_j \le \ell < k_{j+1}$,

$$|x|^\ell|f(x)| = (|x|^{k_j}|f(x)|)^{1-\frac{\ell-k_j}{k_{j+1}-k_j}}(|x|^{k_{j+1}}|f(x)|)^{\frac{\ell-k_j}{k_{j+1}-k_j}}.$$

So (8) implies (9), by a computation analogous to (4). □

4. Denjoy–Carleman classes

Let $M = (M_k)_{k\ge 0}$ be a positive sequence of real numbers. We say that M is a *weight sequence* if $M_0 = 1 \le M_1 \le \mu_2 \le \mu_3 \le \cdots$, where $\mu_k := M_k/M_{k-1}$. Then also the sequence $m_k := M_k^{1/k}$ is increasing.

Let U be an open subset of $\mathbb{R}^n$ and $1 \le p \le \infty$. The *L^p-based local Denjoy–Carleman class of Roumieu type* on U is the set $\mathcal{E}^{\{M\}}_{L^p}(U)$ of all $f \in C^\infty(U)$ such that for all open relatively compact subsets $\Omega \Subset U$ there exist constants $C, \rho > 0$ such that

$$\max_{|\alpha|=k} \|\partial^\alpha f\|_{L^p(\Omega)} \le C(\rho m_k)^k, \quad k \in \mathbb{N}. \tag{10}$$

Analogously, the *L^p-based local Denjoy–Carleman class of Beurling type* is the set $\mathcal{E}^{(M)}_{L^p}(U)$ of all $f \in C^\infty(U)$ that satisfy (10) for all Ω and all ρ with $C = C(\Omega, \rho)$. We use the placeholder $[\cdot]$ for either the Roumieu case $\{\cdot\}$ or the Beurling case $(\cdot)$.

Replacing (10) with various global requirements we obtain natural *global Denjoy–Carleman classes*,

$$\mathcal{B}^{\{M\}}_{L^p}(\mathbb{R}^n) := \{f \in C^\infty(\mathbb{R}^n) : \exists C, \rho > 0\ \forall \alpha : \|\partial^\alpha f\|_{L^p(\mathbb{R}^n)} \le C(\rho m_{|\alpha|})^{|\alpha|}\},$$
$$\mathcal{B}^{(M)}_{L^p}(\mathbb{R}^n) := \{f \in C^\infty(\mathbb{R}^n) : \forall \rho > 0\ \exists C > 0\ \forall \alpha : \|\partial^\alpha f\|_{L^p(\mathbb{R}^n)} \le C(\rho m_{|\alpha|})^{|\alpha|}\},$$

Gelfand–Shilov classes $\mathcal{S}^{\{M\}}(\mathbb{R}^n)$ (resp. $\mathcal{S}^{(M)}(\mathbb{R}^n)$) consisting of all $f \in C^\infty(\mathbb{R}^n)$ such that there exist $\rho, \sigma > 0$ such that (resp. for all $\rho, \sigma > 0$)

$$\sup_{\alpha,\beta\in\mathbb{N}^n} \sup_{x\in\mathbb{R}^n} \frac{|x^\alpha \partial^\beta f(x)|}{(\rho m_{|\alpha|})^{|\alpha|}(\sigma m_{|\beta|})^{|\beta|}} < \infty,$$

and the classes $\mathcal{D}^{[M]}(\mathbb{R}^n) := \mathcal{B}^{[M]}_{L^\infty}(\mathbb{R}^n) \cap C^\infty_c(\mathbb{R}^n)$ of all $\mathcal{B}^{[M]}_{L^\infty}$-functions with compact support. (Cleary, $\mathcal{D}^{[M]}(\mathbb{R}^n)$ is nontrivial only in the non-quasianalytic setting.) Note that various aspects of these global classes have recently been studied in [11, 12, 16]. We have continuous (with respect to the natural locally convex topologies) inclusions

$$\begin{array}{ccccccc}
\mathcal{D}^{\{M\}}(\mathbb{R}^n) & \longrightarrow & \mathcal{S}^{\{M\}}(\mathbb{R}^n) & \longrightarrow & \mathcal{B}^{\{M\}}_{L^p}(\mathbb{R}^n) & \longrightarrow & \mathcal{E}^{\{M\}}_{L^p}(\mathbb{R}^n) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathcal{D}^{(M)}(\mathbb{R}^n) & \longrightarrow & \mathcal{S}^{(M)}(\mathbb{R}^n) & \longrightarrow & \mathcal{B}^{(M)}_{L^p}(\mathbb{R}^n) & \longrightarrow & \mathcal{E}^{(M)}_{L^p}(\mathbb{R}^n)
\end{array}$$

and, for $1 \le p \le q \le \infty$,

$$\mathcal{B}^{[M]}_{L^p}(\mathbb{R}^n) \longrightarrow \mathcal{B}^{[M]}_{L^q}(\mathbb{R}^n) \qquad\qquad \mathcal{E}^{[M]}_{L^p}(U) = \mathcal{E}^{[M]}_{L^q}(U), \tag{11}$$

provided that M is *derivation closed*, i.e.,

$$\sup_{k\ge 0} \Big(\frac{M_{k+1}}{M_k}\Big)^{1/(k+1)} < \infty. \tag{12}$$

This follows from the Sobolev inequality, since the latter condition guarantees stability under taking derivatives. For the local classes, we have equality if (12) holds, because $\|\cdot\|_{L^p(\Omega)} \le |\Omega|^{1/p-1/q}\|\cdot\|_{L^q(\Omega)}$.

If (k_j) is a base sequence, we consider the sets $\mathcal{E}^{[M]}_{L^p,(k_j)}(U)$, $\mathcal{B}^{[M]}_{L^p,(k_j)}(\mathbb{R}^n)$, $\mathcal{D}^{[M]}_{(k_j)}(\mathbb{R}^n)$ of C^∞ functions that satisfy the defining estimates for multiindices α with $|\alpha| = k_j$.

Theorem 4.1. *Let $1 \le p \le \infty$. Let (k_j) be a base sequence and $M = (M_j)$ a weight sequence such that $m_{k_{j+1}}/m_{k_j}$ is bounded. We have:*

(a) $\mathcal{B}^{[M]}_{L^p,(k_j)}(\mathbb{R}^n) = \mathcal{B}^{[M]}_{L^p}(\mathbb{R}^n)$.
(b) $\mathcal{D}^{[M]}_{(k_j)}(\mathbb{R}^n) = \mathcal{D}^{[M]}(\mathbb{R}^n)$.

Let (k_j) be a special base sequence and $U \subseteq \mathbb{R}^n$ open. Then:

(c) $\mathcal{E}^{\{M\}}_{L^p,(k_j)}(U) = \mathcal{E}^{\{M\}}_{L^p}(U)$ *provided that j/m_j is bounded.*
(d) $\mathcal{E}^{(M)}_{L^p,(k_j)}(U) = \mathcal{E}^{(M)}_{L^p}(U)$ *provided that $j/m_j \to 0$.*

Proof. (a) follows from Proposition 3.1 and (b) is a consequence of (a) by intersecting with $C^\infty_c(\mathbb{R}^n)$. (c) and (d) follow from Proposition 3.2. □

In (c) and (d), the assumption that the base sequence (k_j) is *special* cannot be omitted; this follows from Remark 6.5 below.

That j/m_j is bounded (resp. tends to zero) is equivalent to the fact that $\mathcal{E}^{\{M\}}_{L^\infty}(U)$ (resp. $\mathcal{E}^{(M)}_{L^\infty}(U)$) contains $C^\omega(U)$. Note that $\mathcal{E}^{[M]}_{L^p}(U) \subseteq \mathcal{E}^{[M]}_{L^q}(U)$ if $1 \le q \le p \le \infty$.

Before we come to a similar result in the Gelfand–Shilov case, let us recall a result due to [6]. (In that paper only the Roumieu case is treated, but the Beurling case can be proved analogously, see [17, Lemma 4].) To this end we need some further conditions on the weight sequence. We say that a weight sequence $M = (M_j)$ has *moderate growth* if

$$\sup_{j,k\ge 1} \Big(\frac{M_{j+k}}{M_j M_k}\Big)^{1/(j+k)} < \infty. \tag{13}$$

Note that (13) implies (12).

Proposition 4.2 ([6])**.** *Let $M = (M_j)$ be a weight sequence with moderate growth. Assume that j/m_j is bounded in the Roumieu case and tends to zero in the Beurling case. Then the following are equivalent:*

(a) $f \in \mathcal{S}^{[M]}(\mathbb{R}^n)$.
(b) *There are constants $C, \rho, \sigma > 0$ (resp. for all $\rho, \sigma > 0$ there is $C > 0$) such that*
$$\sup_x |x^\alpha f(x)| \le C(\rho m_{|\alpha|})^{|\alpha|} \quad \text{and} \quad \sup_x |\partial^\beta f(x)| \le C(\sigma m_{|\beta|})^{|\beta|}$$
for all $\alpha, \beta \in \mathbb{N}^n$.
(c) *There are constants $C, \rho, \sigma > 0$ (resp. for all $\rho, \sigma > 0$ there is $C > 0$) such that*
$$\sup_x |x^\alpha f(x)| \le C(\rho m_{|\alpha|})^{|\alpha|} \quad \text{and} \quad \sup_x |x^\beta \widehat{f}(x)| \le C(\sigma m_{|\beta|})^{|\beta|}$$
for all $\alpha, \beta \in \mathbb{N}^n$.

In view of this proposition, we allow a second base sequence (ℓ_j) and define $\mathcal{S}^{[M]}_{(k_j),(\ell_j)}(\mathbb{R}^n)$ to be the set of $f \in C^\infty(\mathbb{R}^n)$ such that there exist $C, \rho, \sigma > 0$ (resp.

for all $\rho, \sigma > 0$ there is $C > 0$) such that

$$\max_{|\alpha|=k_j} \sup_x |x^\alpha f(x)| \le C(\rho m_{|\alpha|})^{|\alpha|} \quad \text{and} \quad \max_{|\beta|=\ell_j} \sup_x |\partial^\beta f(x)| \le C(\sigma m_{\ell_j})^{\ell_j}$$

for all $j \ge 0$.

Theorem 4.3. *Let (k_j) and (ℓ_j) be base sequences. Let $M = (M_j)$ be a weight sequence with moderate growth. Assume that $m_{k_{j+1}}/m_{k_j}$ and $m_{\ell_{j+1}}/m_{\ell_j}$ are bounded. Then:*

(a) $\mathcal{S}^{\{M\}}_{(k_j),(\ell_j)}(\mathbb{R}^n) = \mathcal{S}^{\{M\}}(\mathbb{R}^n)$ *provided that j/m_j is bounded.*
(b) $\mathcal{S}^{(M)}_{(k_j),(\ell_j)}(\mathbb{R}^n) = \mathcal{S}^{(M)}(\mathbb{R}^n)$ *provided that $j/m_j \to 0$.*

Proof. This follows from Proposition 3.1, Proposition 3.4, and Proposition 4.2. □

Remark 4.4. Liess [14] proved that, under a number of assumptions on the sequence $M = (M_j)$, the equality $\mathcal{E}^{\{M\}}_{L^2,(k_j)}(U) = \mathcal{E}^{\{M\}}_{L^2}(U)$ implies that $m_{k_{j+1}}/m_{k_j}$ is bounded. His assumptions are the following:

(a) $m_j \le m_{j+1}$ for all j and j/m_j is bounded.
(b) m_{j+1}/m_j is bounded.
(c) There exists $g : \mathbb{N} \to (0, \infty)$ such that $m_{kj} \le g(k)m_j$ for all k and j.
(d) For every $c > 0$ there is $c' > 0$ such that $m_j \le c\, m_k$ implies $j \le c'k$.

Let us assume that M is a weight sequence of moderate growth such that j/m_j is bounded. Then it is easily seen that the conditions (a)–(c) are satisfied. On the other hand, for a weight sequence, (c) is equivalent to (13) (take $k = 2$ and use [15, Theorem 1]).

Observe that, assuming $m_j \le m_{j+1}$, (d) is equivalent to the existence of an integer $n \ge 2$ such that $\liminf_{k\to\infty} m_{nk}/m_k =: a > 1$. Indeed, for all $\ell \ge 1$ and large k, we have $a^\ell m_k \le m_{n^\ell k}$. If $a > 1$ and $m_j \le c\, m_k$, then $c/a^{-\ell} < 1$, provided that ℓ is large enough, and thus $m_j < m_{n^\ell k}$. As m_j is increasing, we conclude that $j \le n^\ell k$. It is clear that $a = 1$ violates (d).

5. Other ultradifferentiable classes

Classically, besides Denjoy–Carleman classes a lot of attention was devoted to ultradifferentiable classes, sometimes called Braun–Meise–Taylor classes due to the foundational article [3], defined in terms of a weight function; they have their origin in work of Beurling.

A *weight function* is a continuous increasing function $\omega : [0, \infty) \to [0, \infty)$ satisfying $\omega(0) = 0$, $\omega(2t) = O(\omega(t))$, $\log t = o(\omega(t))$ as $t \to \infty$ and such that $\varphi(t) := \omega(e^t)$ is convex. It is no restriction to assume $\omega(t) = 0$ for $0 \le t \le 1$ (cf. [18, Section 11.1]).

The local classes $\mathcal{E}^{\{\omega\}}_{L^p}(U)$ and $\mathcal{E}^{(\omega)}_{L^p}(U)$ are defined in analogy to Denjoy–Carleman classes, where now

$$\max_{|\alpha|=k} \|\partial^\alpha f\|_{L^p(\Omega)} \le C \exp\big(\tfrac{1}{\rho}\varphi^*(\rho k)\big), \quad k \ge 0, \tag{14}$$

with $\varphi^*(s) := \sup_{t\ge0}(st - \varphi(t))$, plays the role of (10). More precisely, we have

$$\mathcal{E}^{\{\omega\}}_{L^p}(U) := \bigcap_{\Omega \Subset U} \bigcup_{\rho>0} \bigcup_{C>0} \{f \in C^\infty(U) : f \text{ satisfies (14)}\},$$

$$\mathcal{E}^{(\omega)}_{L^p}(U) := \bigcap_{\Omega \Subset U} \bigcap_{\rho>0} \bigcup_{C>0} \{f \in C^\infty(U) : f \text{ satisfies (14)}\}.$$

The global classes $\mathcal{B}^{[\omega]}_{L^p}(\mathbb{R}^n)$, $\mathcal{S}^{[\omega]}(\mathbb{R}^n)$, and $\mathcal{D}^{[\omega]}(\mathbb{R}^n)$ are defined in a straightforward way. Furthermore, we consider $\mathcal{E}^{[\omega]}_{L^p,(k_j)}(U)$, $\mathcal{B}^{[\omega]}_{L^p,(k_j)}(\mathbb{R}^n)$, $\mathcal{S}^{[\omega]}_{(k_j),(\ell_j)}(\mathbb{R}^n)$, and $\mathcal{D}^{[\omega]}_{(k_j)}(\mathbb{R}^n)$, all of them defined in the obvious manner. (Note that $\mathcal{D}^{[\omega]}(\mathbb{R}^n)$ is nontrivial if and only if $\int_1^\infty t^{-2}\omega(t)\,dt < \infty$; cf. [3].)

There is an overarching framework for ultradifferentiable classes introduced in [19] which goes beyond Denjoy–Carleman and Braun–Meise–Taylor classes and, on a technical level, reduces the proofs to handling certain families of weight sequences. See [18] for a comprehensive survey of the theory.

From now on, let $\mathfrak{M}$ be a totally ordered family of weight sequences, i.e., if $M, M' \in \mathfrak{M}$, then either $M_j \le M'_j$ or $M'_j \le M_j$ for all j.

We define

$$\mathcal{E}^{\{\mathfrak{M}\}}_{L^p}(U) := \bigcap_{\Omega \Subset U} \bigcup_{M\in\mathfrak{M}} \bigcup_{\rho>0} \bigcup_{C>0} \{f \in C^\infty(U) : f \text{ satisfies (10)}\},$$
$$\mathcal{E}^{(\mathfrak{M})}_{L^p}(U) := \bigcap_{\Omega \Subset U} \bigcap_{M\in\mathfrak{M}} \bigcap_{\rho>0} \bigcup_{C>0} \{f \in C^\infty(U) : f \text{ satisfies (10)}\},$$

and in the evident analogous way also the global classes $\mathcal{B}^{[\mathfrak{M}]}_{L^p}(\mathbb{R}^n)$, $\mathcal{S}^{[\mathfrak{M}]}(\mathbb{R}^n)$, and $\mathcal{D}^{[\mathfrak{M}]}(\mathbb{R}^n)$ as well as $\mathcal{E}^{[\mathfrak{M}]}_{L^p,(k_j)}(U)$, $\mathcal{B}^{[\mathfrak{M}]}_{L^p,(k_j)}(\mathbb{R}^n)$, $\mathcal{S}^{[\mathfrak{M}]}_{(k_j),(\ell_j)}(\mathbb{R}^n)$, and $\mathcal{D}^{[\mathfrak{M}]}_{(k_j)}(\mathbb{R}^n)$.

Theorem 5.1. *Let $1 \le p \le \infty$. Let ω be a weight function. For $\mathfrak{M} := \{M^{(\rho)}\}_{\rho>0}$, where $M^{(\rho)}_k := \exp(\frac{1}{\rho}\varphi^*(\rho k))$, we have the identities:*

$$\mathcal{E}^{[\omega]}_{L^p}(U) = \mathcal{E}^{[\mathfrak{M}]}_{L^p}(U), \quad \mathcal{E}^{[\omega]}_{L^p,(k_j)}(U) = \mathcal{E}^{[\mathfrak{M}]}_{L^p,(k_j)}(U),$$
$$\mathcal{B}^{[\omega]}_{L^p}(\mathbb{R}^n) = \mathcal{B}^{[\mathfrak{M}]}_{L^p}(\mathbb{R}^n), \quad \mathcal{B}^{[\omega]}_{L^p,(k_j)}(\mathbb{R}^n) = \mathcal{B}^{[\mathfrak{M}]}_{L^p,(k_j)}(\mathbb{R}^n),$$
$$\mathcal{S}^{[\omega]}(\mathbb{R}^n) = \mathcal{S}^{[\mathfrak{M}]}(\mathbb{R}^n), \quad \mathcal{S}^{[\omega]}_{(k_j),(\ell_j)}(\mathbb{R}^n) = \mathcal{S}^{[\mathfrak{M}]}_{(k_j),(\ell_j)}(\mathbb{R}^n),$$
$$\mathcal{D}^{[\omega]}(\mathbb{R}^n) = \mathcal{D}^{[\mathfrak{M}]}(\mathbb{R}^n), \quad \mathcal{D}^{[\omega]}_{(k_j)}(\mathbb{R}^n) = \mathcal{D}^{[\mathfrak{M}]}_{(k_j)}(\mathbb{R}^n),$$

where (k_j) and (ℓ_j) are base sequences.

Proof. This is shown by the proof of [19, Lemma 5.14] which is based on the following property of $\mathfrak{M}$, see [19, (5.10)],

$$\forall \sigma > 0\ \exists H \ge 1\ \forall \rho > 0\ \exists C \ge 1\ \forall k \in \mathbb{N} : \sigma^k M^{(\rho)}_k \le C\, M^{(H\rho)}_k. \tag{15}$$

In fact, for given $\rho, \sigma > 0$ there exist $C, H \ge 1$ such that

$$\sup_{j\ge 0} \frac{\max_{|\alpha|=k_j} \|\partial^\alpha f\|_{L^p(\Omega)}}{M^{(H\rho)}_{k_j}} \le C \sup_{j\ge 0} \frac{\max_{|\alpha|=k_j} \|\partial^\alpha f\|_{L^p(\Omega)}}{\sigma^{k_j} M^{(\rho)}_{k_j}}$$

and $M^{(H\rho)}_{k_j} = \exp(\frac{1}{H\rho}\varphi^*(H\rho k_j))$. In the Gelfand–Shilov case, we observe that additionally

$$\sup_{j\ge 0} \frac{\max_{|\alpha|=\ell_j} \sup_x |x^\alpha f(x)|}{M^{(H\rho)}_{\ell_j}} \le C \sup_{j\ge 0} \frac{\max_{|\alpha|=\ell_j} \sup_x |x^\alpha f(x)|}{\sigma^{\ell_j} M^{(\rho)}_{\ell_j}}.$$

The case $\mathcal{S}^{[\omega]}(\mathbb{R}^n) = \mathcal{S}^{[\mathfrak{M}]}(\mathbb{R}^n)$ follows from

$$\sup_{\alpha,\beta} \frac{\sup_x |x^\alpha \partial^\beta f(x)|}{M^{(H\rho)}_{|\alpha|} M^{(H\rho)}_{|\beta|}} \le C \sup_{\alpha,\beta} \frac{\sup_x |x^\alpha \partial^\beta f(x)|}{\tau^{|\alpha|}\sigma^{|\beta|} M^{(\rho)}_{|\alpha|} M^{(\rho)}_{|\beta|}}.$$

taking constants C, H which work for both σ and τ. □

The family $\mathfrak{M}$ from Theorem 5.1 has an inherent moderate growth property (see [19, (5.6)]):

$$M^{(\rho)}_{j+k} \le M^{(2\rho)}_j M^{(2\rho)}_k, \quad j,k \ge 0. \tag{16}$$

In general, we say that $\mathfrak{M}$ has [*moderate growth*] if, in the Roumieu case (i.e., $[\cdot] = \{\cdot\}$),

$$\forall M \in \mathfrak{M}\ \exists M' \in \mathfrak{M}\ \exists C > 0\ \forall j,k \ge 0 : M_{j+k} \le C^{j+k} M'_j M'_k,$$

and, in the Beurling case (i.e., $[\cdot] = (\cdot)$),

$$\forall M \in \mathfrak{M}\ \exists M' \in \mathfrak{M}\ \exists C > 0\ \forall j,k \ge 0 : M'_{j+k} \le C^{j+k} M_j M_k.$$

Lemma 5.2. *If (k_j) is a special base sequence and $\mathfrak{M}$ has [moderate growth], then, in the Roumieu case (i.e., $[\cdot] = \{\cdot\}$),*

$$\forall M \in \mathfrak{M}\ \exists M' \in \mathfrak{M} : m_{k_{j+1}}/m'_{k_j} \text{ is bounded,} \tag{17}$$

and, in the Beurling case (i.e., $[\cdot] = (\cdot)$),

$$\forall M \in \mathfrak{M}\ \exists M' \in \mathfrak{M} : m'_{k_{j+1}}/m_{k_j} \text{ is bounded.} \tag{18}$$

Proof. By assumption, there exists a positive integer a such that $k_{j+1} \le ak_j$ for all $j \ge 1$. Moderate growth in the Roumieu case guarantees that, for given $M \in \mathfrak{M}$ there exist $C > 0$ and $M' \in \mathfrak{M}$ (depending on a) such that

$$M_{ak_j} \le C^{ak_j} (M'_{k_j})^a.$$

This implies $m_{k_{j+1}} \le m_{ak_j} \le Cm'_{k_j}$. The Beurling case is similar. □

Theorem 5.3. *Let $1 \le p \le \infty$. Let $(k_j), (\ell_j)$ be base sequences. Let $\mathfrak{M}$ satisfy (17) in the Roumieu case and (18) in the Beurling case. We have:*

(a) $\mathcal{B}^{[\mathfrak{M}]}_{L^p,(k_j)}(\mathbb{R}^n) = \mathcal{B}^{[\mathfrak{M}]}_{L^p}(\mathbb{R}^n)$.
(b) $\mathcal{D}^{[\mathfrak{M}]}_{(k_j)}(\mathbb{R}^n) = \mathcal{D}^{[\mathfrak{M}]}(\mathbb{R}^n)$.

Assume additionally that, for all $M \in \mathfrak{M}$, j/m_j is bounded, in the Roumieu case, and tends to zero, in the Beurling case. Then:

(c) $\mathcal{E}^{[\mathfrak{M}]}_{L^p,(k_j)}(U) = \mathcal{E}^{[\mathfrak{M}]}_{L^p}(U)$ *for all open $U \subseteq \mathbb{R}^n$, provided that (k_j) is a special base sequence.*
(d) $\mathcal{S}^{[\mathfrak{M}]}_{(k_j),(\ell_j)}(\mathbb{R}^n) = \mathcal{S}^{[\mathfrak{M}]}(\mathbb{R}^n)$, *provided that $\mathfrak{M}$ has [moderate growth].*

Proof. Since the sequences in $\mathfrak{M}$ are totally ordered, we may assume that the sequences provided by (17) and (18) also satisfy $m_j \le m'_j$ or $m'_j \le m_j$ for all j, respectively. Then (a), (b), and (c) follow easily from Proposition 3.1 and Proposition 3.2. Finally, (d) follows from Proposition 3.1, Proposition 3.4, and the generalization [17, Lemma 4] of Proposition 4.2. □

To deduce a version for Braun–Meise–Taylor classes, it suffices to note that for a weight function ω and $\mathfrak{M} = \{M^{(\rho)}\}_{\rho>0}$ the associated family from Theorem 5.1,

- $\omega(t) = O(t)$ as $t \to \infty$ if and only if $j/m_j^{(\rho)}$ is bounded for some $\rho > 0$,
- $\omega(t) = o(t)$ as $t \to \infty$ if and only if $j/m_j^{(\rho)} \to 0$ for all $\rho > 0$;

cf. [19, Lemma 5.7 and Corollary 5.15].

Theorem 5.4. *Let $1 \le p \le \infty$. Let (k_j), (ℓ_j) be special base sequences. Let ω be a weight function. We have:*

(a) $\mathcal{B}^{[\omega]}_{L^p,(k_j)}(\mathbb{R}^n) = \mathcal{B}^{[\omega]}_{L^p}(\mathbb{R}^n)$.
(b) $\mathcal{D}^{[\omega]}_{(k_j)}(\mathbb{R}^n) = \mathcal{D}^{[\omega]}(\mathbb{R}^n)$.

Assume additionally that $\omega(t) = O(t)$ as $t \to \infty$, in the Roumieu case, and $\omega(t) = o(t)$, in the Beurling case. Then:

(c) $\mathcal{E}^{[\omega]}_{L^p,(k_j)}(U) = \mathcal{E}^{[\omega]}_{L^p}(U)$ *for all open* $U \subseteq \mathbb{R}^n$.
(d) $\mathcal{S}^{[\omega]}_{(k_j),(\ell_j)}(\mathbb{R}^n) = \mathcal{S}^{[\omega]}(\mathbb{R}^n)$.

Proof. This is a corollary of Theorem 5.1, Lemma 5.2, and Theorem 5.3. □

Remark 5.5. The family $\mathfrak{M} = \{M^{(\rho)}\}_{\rho>0}$ associated with a weight function in Theorem 5.1 has the following property:

(a) Let $0 < \sigma \le \rho$. If there is $C_1 > 0$ such that $m_j^{(\sigma)} \le C_1\, m_k^{(\rho)}$, then there is $C_2 = C_2(\sigma, \rho, C_1) > 0$ such that $j \le C_2\, k$.

So, in this case, validity of (17) or (18) entails that (k_j) is a special base sequence; the converse holds by (16) and Lemma 5.2.

To see (a), note first that for any positive integer N we have

$$m_k^{(N\rho)} = \exp(\tfrac{1}{N\rho k}\varphi^*(N\rho k)) = m_{Nk}^{(\rho)}.$$

By (15), there exist $C = C(\rho) \ge 1$ and an integer $H \ge 2$ such that

$$4m_k^{(\rho)} \le C^{1/k} m_k^{(H\rho)} = C^{1/k} m_{Hk}^{(\rho)}, \quad k \ge 0.$$

For $k \ge k_0$, we have $C^{1/k} < 2$ and hence $2m_k^{(\rho)} < m_{Hk}^{(\rho)}$ and, by iteration,

$$2^\ell m_k^{(\rho)} < m_{H^\ell k}^{(\rho)}.$$

If $m_j^{(\sigma)} \le C_1\, m_k^{(\rho)}$, then, choosing ℓ such that $C_1 2^{-\ell} < 1$, we find

$$m_j^{(\sigma)} < m_{H^\ell k}^{(\rho)}.$$

Then the assumption $j \ge \lceil \frac{\rho}{\sigma} \rceil H^\ell k$ leads to

$$m_{H^\ell k}^{(\rho)} \le m_{H^\ell k}^{(\lceil \frac{\rho}{\sigma} \rceil \sigma)} = m_{\lceil \frac{\rho}{\sigma} \rceil H^\ell k}^{(\sigma)} < m_{H^\ell k}^{(\rho)},$$

a contradiction. Thus $j < C_2\, k$, where $C_2 := \lceil \frac{\rho}{\sigma} \rceil H^\ell$. For $k < k_0$, there are only finitely many j satisfying $m_j^{(\sigma)} \le C_1\, m_{k_0}^{(\rho)}$ (since $\lim_{j\to\infty} m_j^{(\sigma)} = \infty$; see [3, Remark 1.3]). So adding this bound to C_2 gives the assertion for all k.

Remark 5.6. Under the [moderate growth] assumption, we have, for $1 \le p \le q \le \infty$, the continuous inclusions $\mathcal{B}^{[\mathfrak{M}]}_{L^p}(\mathbb{R}^n) \subseteq \mathcal{B}^{[\mathfrak{M}]}_{L^q}(\mathbb{R}^n)$ and $\mathcal{E}^{[\mathfrak{M}]}_{L^p}(U) = \mathcal{E}^{[\mathfrak{M}]}_{L^q}(U)$, in particular, $\mathcal{B}^{[\omega]}_{L^p}(\mathbb{R}^n) \subseteq \mathcal{B}^{[\omega]}_{L^q}(\mathbb{R}^n)$ and $\mathcal{E}^{[\omega]}_{L^p}(U) = \mathcal{E}^{[\omega]}_{L^q}(U)$.

6. Complements

In this section, we discuss the setting considered in [1].

Let $M = (M_j)$ be a weight sequence. Let $I \subseteq \mathbb{R}$ be a compact interval. We define

$$\mathcal{B}^{\{M\}}_{L^\infty}(I) := \{f \in C^\infty(I) :\ \exists C, \rho > 0\ \forall j \ge 0 : \|f^{(j)}\|_{L^\infty(I)} \le C(\rho m_j)^j\},$$
$$\mathcal{B}^{(M)}_{L^\infty}(I) := \{f \in C^\infty(I) :\ \forall \rho > 0\ \exists C > 0\ \forall j \ge 0 : \|f^{(j)}\|_{L^\infty(I)} \le C(\rho m_j)^j\},$$

as well as $\mathcal{B}^{[M]}_{L^\infty,(k_j)}(I)$ in the obvious way.

Theorem 6.1. *Let (k_j) be a special base sequence. Let $M = (M_j)$ be a weight sequence such that $m_{k_{j+1}}/m_{k_j}$ is bounded. Then:*

(a) $\mathcal{B}^{\{M\}}_{L^\infty,(k_j)}(I) = \mathcal{B}^{\{M\}}_{L^\infty}(I)$ *provided that j/m_j is bounded.*
(b) $\mathcal{B}^{(M)}_{L^\infty,(k_j)}(I) = \mathcal{B}^{(M)}_{L^\infty}(I)$ *provided that $j/m_j \to 0$.*

Proof. Assume that $f \in \mathcal{B}^{\{M\}}_{L^\infty,(k_j)}(I)$. Let $\ell \ge 0$ and let $j \ge 0$ be such that $k_j \le \ell < k_{j+1}$. By Proposition 2.6 and the properties of (k_j) and (m_j),

$$\begin{aligned}\|f^{(\ell)}\|_{L^\infty(I)} &\le 4e^{2(\ell-k_j)} e^{k_{j+1}-k_j}\, \|f^{(k_j)}\|_{L^\infty(I)}^{1-\frac{\ell-k_j}{k_{j+1}-k_j}} \\ &\qquad \cdot \max\Big\{\|f^{(k_{j+1})}\|_{L^\infty(I)}^{\frac{\ell-k_j}{k_{j+1}-k_j}}, \Big(\frac{k_{j+1}-k_j}{|I|}\Big)^{\ell-k_j} \|f^{(k_j)}\|_{L^\infty(I)}^{\frac{\ell-k_j}{k_{j+1}-k_j}}\Big\} \\ &\le 4C\, C_1^\ell\, (\rho m_{k_j})^{k_j\left(1-\frac{\ell-k_j}{k_{j+1}-k_j}\right)} \Big((\rho m_{k_{j+1}})^{\frac{k_{j+1}(\ell-k_j)}{k_{j+1}-k_j}} + \Big(\frac{C_2 k_j}{|I|}\Big)^{\ell-k_j} (\rho m_{k_j})^{\frac{k_j(\ell-k_j)}{k_{j+1}-k_j}}\Big) \\ &\le 4C\, C_1^\ell \Big((C_3\rho m_{k_j})^\ell + \Big(\frac{C_4 m_{k_j}}{|I|}\Big)^{\ell-k_j} (\rho m_{k_j})^{k_j}\Big) \\ &\le 8C\, (C_5 \max\{\rho, C_4|I|^{-1}\})^\ell\, m_\ell^\ell.\end{aligned}$$

Hence $f \in \mathcal{B}^{\{M\}}_{L^\infty}(I)$.

Assume that $f \in \mathcal{B}^{(M)}_{L^\infty,(k_j)}(I)$. Let $\sigma > 0$ be given. If we assume that $j/m_j \to 0$, then we may proceed as in the proof of Proposition 3.2 to see that choosing $\rho > 0$ sufficiently small we may find

$$\|f^{(\ell)}\|_{L^\infty(I)} \le D(\sigma m_\ell)^\ell, \quad \ell \ge 0,$$

where $D = D(\sigma)$. □

It is now an easy exercise to deduce the following extensions.

Theorem 6.2. *Let (k_j) be a special base sequence. Let $\mathfrak{M}$ have [moderate growth] and assume that, for all $M \in \mathfrak{M}$, j/m_j is bounded, in the Roumieu case, and tends to zero, in the Beurling case. Then $\mathcal{B}^{[\mathfrak{M}]}_{L^\infty,(k_j)}(I) = \mathcal{B}^{[\mathfrak{M}]}_{L^\infty}(I)$.*

Theorem 6.3. *Let (k_j) be a special base sequence. Let ω be a weight function such that $\omega(t) = O(t)$ as $t \to \infty$, in the Roumieu case, and $\omega(t) = o(t)$, in the Beurling case. Then $\mathcal{B}^{[\omega]}_{L^\infty,(k_j)}(I) = \mathcal{B}^{[\omega]}_{L^\infty}(I)$.*

Remark 6.4. In [1], instead of assuming $m_{k_{j+1}}/m_{k_j}$ bounded, the authors require that

$$\exists i_0 \geq 0 \; \forall i, j > i_0 \text{ and } i < j < ik : m_j \leq m_i m_k. \tag{19}$$

If there is $\epsilon > 0$ such that $m_j > \epsilon$ for all j, then it is not hard to check that (19) implies

$$\exists C \geq 1 \; \forall i, j \geq 1 \text{ and } i < j < ik : m_j \leq C \, m_i m_k. \tag{20}$$

But (20) (for $k = 3$ and $j = 2i$) implies that M has moderate growth (cf. [15, Theorem 1]). And, if (k_j) is a special base sequence, then Lemma 5.2 implies that $m_{k_{j+1}}/m_{k_j}$ is bounded.

Remark 6.5. In Theorem 6.1, the assumption that (k_j) is a *special* base sequence cannot be omitted. In fact, by [1, Theorem 1.6], for each positive sequence $M = (M_j)$ such that $\limsup_{j\to\infty} m_j = \infty$ there is a base sequence (k_j) and $f \in \mathcal{B}^{\{M\}}_{L^\infty,(k_j)}(I)$ but $f \notin \mathcal{B}^{\{M\}}(I)$. More precisely, a weight sequence $N = (N_j)$ and sequences of positive integers (k_j) and (ℓ_j) with the following properties are constructed:

- $\cdots < \ell_j < k_j < \ell_{j+1} < k_{j+1} < \cdots,$
- $N_{k_j} = M_{k_j}$, and
- $N_{\ell_j} = 2^{2^{\ell_j}} M_{\ell_j}$.

It is well-known that for each weight sequence N there exists $f \in \mathcal{B}^{\{N\}}_{L^\infty}(\mathbb{R})$ with $|f^{(j)}(0)| \geq N_j$ for all $j \geq 0$, from which the assertion follows easily.

Note that this result does not contradict Theorem 4.1: if also $M = (M_j)$ is a weight sequence, then the above conditions imply

$$m_{k_{j+1}} = n_{k_{j+1}} \geq n_{\ell_{j+1}} = 2^{2^{\ell_{j+1}}/\ell_{j+1}} m_{\ell_{j+1}} \geq 2^{2^{\ell_{j+1}}/\ell_{j+1}} m_{k_j}$$

so that $m_{k_{j+1}}/m_{k_j}$ is unbounded.

It is possible to adapt the proof of this result to the Beurling case as well as to the general setting of a weight structure $\mathfrak{M}$; we omit the details.

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