ON THE EXTENSION OF WHITNEY ULTRAJETS, II

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Abstract. We characterize the validity of the Whitney extension theorem in the ultradifferentiable Roumieu setting with controlled loss of regularity. Specifically, we show that in the main Theorem 1.3 of [16] condition (1.3) can be dropped. Moreover, we clarify some questions that remained open in [16].

1. Introduction

The main goal of this paper is to prove:

Theorem 1. *Let ω be a non-quasianalytic concave weight function. Let σ be a weight function satisfying $\sigma(t) = o(t)$ as $t \to \infty$. Then the following conditions are equivalent:*

(i) *For every compact $E \subseteq \mathbb{R}^n$ we have $j^\infty_E(\mathcal{B}^{\{\omega\}}(\mathbb{R}^n)) \supseteq \mathcal{B}^{\{\sigma\}}(E)$, where j^∞_E assigns to each $f \in C^\infty(\mathbb{R}^n)$ its infinite jet $(f^{(\alpha)}|_E)_{\alpha\in\mathbb{N}^n}$ on E.*

(ii) *There is $C > 0$ such that $\int_1^\infty \frac{\omega(tu)}{u^2}\, du \le C\sigma(t) + C$ for all $t > 0$.*

(Here $\mathcal{B}^{\{\omega\}}$ denotes the Roumieu class defined by the weight function ω; we use the symbol $\mathcal{B}$ to emphasize that the defining estimates are global, cf. [16, 2.2 and 2.6].) It means that Theorem 1.3 of [16] holds without the assumption (1.3) that the associated weight matrix $\mathfrak{S}$ of σ satisfies

$$\forall S \in \mathfrak{S}\ \exists T \in \mathfrak{S}\ \exists C \ge 1\ \forall 1 \le j \le k : \frac{S_j}{jS_{j-1}} \le C\, \frac{T_k}{kT_{k-1}}. \tag{1}$$

Theorem 1 is proved in Section 2. In Section 3 we clarify some questions that remained open in [16] and obtain several characterizations of concave weight functions. For an overview of the background of Theorem 1 we refer to the introduction in [16]. We use the notation and the definitions of said paper; the concept of *weight matrices* is recalled in the appendix at the end of this paper.

Note that in the special case that ω and σ coincide we recover the result of [1]:

Corollary 2. *Let ω be a weight function. The following conditions are equivalent:*

(i') *For every compact $E \subseteq \mathbb{R}^n$ we have $j^\infty_E(\mathcal{B}^{\{\omega\}}(\mathbb{R}^n)) = \mathcal{B}^{\{\omega\}}(E)$.*

(ii') *There is $C > 0$ such that $\int_1^\infty \frac{\omega(tu)}{u^2}\, du \le C\omega(t) + C$ for all $t > 0$.*

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Indeed, if ω satisfies (ii') then it is non-quasianalytic, equivalent to a concave weight function [9, Proposition 1.3], and $\omega(t) = o(t)$ as $t \to \infty$ [2, Remark 3.20]. That (ii') is a necessary condition for (i') is well-known. Note that also (i') implies that ω is non-quasianalytic. Indeed, if ω is quasianalytic, then the Borel map $j^\infty_{\{0\}} : \mathcal{B}^{\{\omega\}}(\mathbb{R}^n) \to \mathcal{B}^{\{\omega\}}(\{0\})$ is never surjective. For $t \ne O(\omega(t))$ as $t \to \infty$ this follows from [15], for $t = O(\omega(t))$ as $t \to \infty$ consider e.g. the formal series $\sum_{k=0}^\infty x^k$ which converges to the unbounded real analytic function $1/(1-x)$ function for $|x| < 1$.

2. Proof of Theorem 1

Preparations. First we recall a few definitions and facts. Let $m = (m_k)$ be a positive sequence satisfying $m_0 = 1$ and $m_k^{1/k} \to \infty$. The *log-convex minorant* of m is given by

$$\underline{m}_k := \sup_{t>0} \frac{t^k}{\exp(\omega_m(t))}, \quad k \in \mathbb{N},$$

where

$$\omega_m(t) := \sup_{k\in\mathbb{N}} \log\Big(\frac{t^k}{m_k}\Big), \quad t > 0.$$

The function ω_m is increasing, convex in $\log t$, and zero for sufficiently small $t > 0$. Related is the function $h_m(t) := \inf_{k\in\mathbb{N}} m_k t^k$, for $t > 0$, and $h_m(0) := 0$. It is increasing, continuous, positive for $t > 0$, and equals 1 for large t.

Let $m = (m_k)$ be a positive *log-convex* sequence (i.e., $m = \underline{m}$) such that $m_0 = 1$ and $m_k^{1/k} \to \infty$. Then the functions $\overline{\Gamma}_m$ and $\underline{\Gamma}_m$ defined in [16, Definition 3.1] coincide, we simply write Γ_m in this case; note that log-convexity and $m_k^{1/k} \to \infty$ imply $m_k/m_{k-1} \to \infty$. Thus

$$\Gamma_m(t) = \min\{k : h_m(t) = m_k t^k\} = \min\Big\{k : \frac{m_{k+1}}{m_k} \ge \frac{1}{t}\Big\}, \quad t > 0. \tag{2}$$

By [16, Lemma 3.2], Γ_m is decreasing, tending to ∞ as $t \to 0$, and

$$k \mapsto m_k t^k \text{ is decreasing for } k \le \Gamma_m(t). \tag{3}$$

Recall that with every weight function σ (always understood as defined in [16, Section 2.1]) is associated a weight matrix $\mathfrak{S} = \{S^\xi\}_{\xi>0}$, where

$$S^\xi_k := \exp\big(\tfrac{1}{\xi}\varphi^*(\xi k)\big), \qquad \text{(here } \varphi = \sigma \circ \exp \text{ and } \varphi^* \text{ is its Young conjugate)},$$

such that $\mathcal{B}^{\{\sigma\}} = \mathcal{B}^{\{\mathfrak{S}\}}$ and $\mathcal{B}^{(\sigma)} = \mathcal{B}^{(\mathfrak{S})}$ algebraically and topologically; cf. [16, 2.5] and [12]. In the following we set $s^\xi_k := S^\xi_k/k!$.

The next proposition shows that for a weight function σ which is equivalent to a concave weight function and satisfies $\sigma(t) = o(t)$ as $t \to \infty$ we additionally have $\mathcal{B}^{\{\sigma\}} = \mathcal{B}^{\{\underline{\mathfrak{S}}\}}$ and $\mathcal{B}^{(\sigma)} = \mathcal{B}^{(\underline{\mathfrak{S}})}$, where $\underline{\mathfrak{S}} = \{\underline{S}^\xi\}_{\xi>0}$ and

$$\underline{S}^\xi_k := k!\, \underline{s}^\xi_k.$$

In particular, $\underline{\mathfrak{S}}$ satisfies (1). We say that $\underline{S}^\xi$ is *strongly log-convex* meaning that $\underline{s}^\xi_k = \underline{S}^\xi_k/k!$ is log-convex. (Note the abuse of notation: $\underline{S}^\xi$ is *not* necessarily the log-convex minorant of S^ξ; this will cause no confusion.) Recall that two weight functions ω and σ are called *equivalent* if $\omega(t) = O(\sigma(t))$ and $\sigma(t) = O(\omega(t))$ as $t \to \infty$; this means that they define the same ultradifferentiable class.

Proposition 3. *Let σ be a weight function satisfying $\sigma(t) = o(t)$ as $t \to \infty$ which is equivalent to a concave weight function. For each $\xi > 0$ there exist constants $A, B, C > 0$ such that*

$$A^{-1} s_k^{\xi/B} \le \underline{s}_k^\xi \le s_k^\xi \le C^k \underline{s}_k^{B\xi} \quad \text{for all } k \in \mathbb{N}. \tag{4}$$

Moreover, there is a constant $H \ge 1$ such that $\underline{s}_{j+k}^\xi \le H^{j+k} \underline{s}_j^{2\xi} \underline{s}_k^{2\xi}$, for all $\xi > 0$ and all $j, k \in \mathbb{N}$, and thus $h_{\underline{s}^\xi}(t) \le h_{\underline{s}^{2\xi}}(Ht)^2$, for all $\xi > 0$ and all $t > 0$.

Proof. Clearly, $\underline{s}^\xi \le s^\xi$. Let $\underline{S}_k^\xi := k!\, \underline{s}_k^\xi$. By [6, Lemma 3.6], ω_{S^ξ} and $\omega_{\underline{S}^\xi}$ are equivalent, in particular, there exists $C \ge 1$ such that

$$\omega_{\underline{S}^\xi} \le C\omega_{S^\xi} + C. \tag{5}$$

By [16, Lemma 2.4(3)] and [14, Remark 2.5], we have

$$2\omega_{S^{2\xi}} \le \omega_{S^\xi}, \quad \text{for all } \xi > 0.$$

If n is an integer such that $B := 2^n \ge C$, then $\omega_{\underline{S}^\xi} \le \omega_{S^{\xi/B}} + C$ and hence

$$\underline{S}_k^\xi = \sup_{t>0} \frac{t^k}{\exp(\omega_{\underline{S}^\xi}(t))} \ge e^{-C} \sup_{t>0} \frac{t^k}{\exp(\omega_{S^{\xi/B}}(t))} = e^{-C} S_k^{\xi/B}.$$

This shows the first inequality in (4).

By [16, Lemma 3.13], there exists $D \ge 1$ such that for all $\xi > 0$,

$$2\omega_{s^{2\xi}}(t) \le \omega_{s^\xi}(Dt), \quad \text{for } t > 0$$

and therefore

$$\underline{s}_{2k}^\xi = \sup_{t>0} \frac{(Dt)^{2k}}{\exp(\omega_{s^\xi}(Dt))} \le D^{2k} \sup_{t>0} \frac{t^{2k}}{\exp(2\omega_{s^{2\xi}}(t))} = D^{2k} (\underline{s}_k^{2\xi})^2.$$

Thus, by [17, Theorem 9.5.1] (which is a generalization of [8]), there exists a constant $H \ge 1$ such that $\underline{s}_{j+k}^\xi \le H^{j+k} \underline{s}_j^{2\xi} \underline{s}_k^{2\xi}$, for all j, k. That $h_{\underline{s}^\xi}(t) \le h_{\underline{s}^{2\xi}}(Ht)^2$, for all $\xi > 0$ and all $t > 0$, follows from [16, Lemma 3.12]. By [18, Proposition 3.6],

$$2\omega_{\underline{S}^{2\xi}}(t) \le \omega_{\underline{S}^\xi}(Ht), \quad \text{for } t > 0,$$

for some (possibly different) $H \ge 1$. As above, using (5), we find $\omega_{\underline{S}^{B\xi}}(bt) \le \omega_{S^\xi}(t) + 1$ for some constant $0 < b \le 1$. Then

$$\underline{S}_k^{B\xi} = \sup_{t>0} \frac{(bt)^k}{\exp(\omega_{\underline{S}^{B\xi}}(bt))} \ge e^{-1} b^k \sup_{t>0} \frac{t^k}{\exp(\omega_{S^\xi}(t))} = e^{-1} b^k S_k^\xi.$$

The last inequality of (4) follows. □

Proposition 3 alone is not enough to get rid of the assumption (1). It is not clear that $\underline{\mathfrak{S}}$ has the property that for all $S \in \underline{\mathfrak{S}}$ there is a $T \in \underline{\mathfrak{S}}$ such that $S_{2k}/S_{2k-1} \lesssim T_k/T_{k-1}$. Note that $\mathfrak{S}$ has this property (see [16, Lemma 2.4(4)]) and it enters crucially in Lemma 3.4 and Proposition 3.7 of [16].

We deal with this problem by introducing another intimately related weight matrix $\mathfrak{V} := \{V^\xi\}_{\xi>0}$. For each $\xi > 0$ we define $V_k^\xi := k!\, v_k^\xi$ by setting

$$v_k^\xi := \min_{0 \le j \le k} \underline{s}_j^{2\xi} \underline{s}_{k-j}^{2\xi}, \quad k \in \mathbb{N}. \tag{6}$$

That means that for the sequence of quotients v^ξ_k/v^ξ_{k-1} we have (cf. [7, Lemma 3.5])

$$\Big(\frac{v^\xi_1}{v^\xi_0},\frac{v^\xi_2}{v^\xi_1},\frac{v^\xi_3}{v^\xi_2},\frac{v^\xi_4}{v^\xi_3},\dots\Big)=\Big(\frac{\underline{s}^{2\xi}_1}{\underline{s}^{2\xi}_0},\frac{\underline{s}^{2\xi}_1}{\underline{s}^{2\xi}_0},\frac{\underline{s}^{2\xi}_2}{\underline{s}^{2\xi}_1},\frac{\underline{s}^{2\xi}_2}{\underline{s}^{2\xi}_1},\frac{\underline{s}^{2\xi}_3}{\underline{s}^{2\xi}_2},\frac{\underline{s}^{2\xi}_3}{\underline{s}^{2\xi}_2},\dots\Big).$$

Thus the sequence $v^\xi=(v^\xi_k)$ is log-convex and satisfies

$$\frac{v^\xi_{2k-1}}{v^\xi_{2k-2}}=\frac{v^\xi_{2k}}{v^\xi_{2k-1}}=\frac{\underline{s}^{2\xi}_k}{\underline{s}^{2\xi}_{k-1}},\quad\text{ for all } k\ge 1. \tag{7}$$

So, in view of (2),

$$2\Gamma_{\underline{s}^{2\xi}}(t)=\Gamma_{v^\xi}(t),\quad\text{ for all } t>0. \tag{8}$$

By Proposition 3, there is $H\ge 1$ such that for all $\xi>0$

$$\underline{s}^\xi_k\le H^k v^\xi_k\le H^k\underline{s}^{2\xi}_k,\quad\text{ for all } k\in\mathbb{N}. \tag{9}$$

Thus, we also have $\mathcal{B}^{\{\sigma\}}=\mathcal{B}^{\{\mathfrak{V}\}}$ and $\mathcal{B}^{(\sigma)}=\mathcal{B}^{(\mathfrak{V})}$ algebraically and topologically.

Proof of Theorem 1. The implication (i) $\Rightarrow$ (ii) follows from [2]. So we only prove the converse implication. Condition (ii) means that the weight function

$$\kappa(t):=\int_1^\infty\frac{\omega(tu)}{u^2}\,du \tag{10}$$

satisfies $\kappa(t)=O(\sigma(t))$ as $t\to\infty$, i.e., $\mathcal{B}^{\{\sigma\}}\subseteq\mathcal{B}^{\{\kappa\}}$. Now κ is concave and $\kappa(t)=o(t)$ as $t\to\infty$, see [9, Proposition 1.3]. We will show that Whitney ultrajets of class $\mathcal{B}^{\{\kappa\}}$ admit extensions of class $\mathcal{B}^{\{\omega\}}$. Thus from now on we assume without loss of generality that $\sigma=\kappa$ is concave. Since ω is increasing we have $\sigma=\kappa\ge\omega$ and hence, if $\mathfrak{W}=\{W^\xi\}_{\xi>0}$ denotes the weight matrix associated with ω,

$$\underline{S}^\xi\le S^\xi\le W^\xi,\quad\text{ for all } \xi>0. \tag{11}$$

Moreover, Proposition 3 as well as (8) and (9) apply. Let us now indicate the necessary changes in the proof of [16, Theorem 1.3]. The changes also lead to some simplifications. We provide details in the hope that this contributes to a better understanding.

- Every Whitney ultrajet $F=(F^\alpha)$ of class $\mathcal{B}^{\{\sigma\}}$ on the compact set $E\subseteq\mathbb{R}^n$ is an element of $\mathcal{B}^{\{V^\xi\}}(E)$ for some $\xi>0$, i.e., there exist $C>0$ and $\rho\ge 1$ such that

$$|F^\alpha(a)|\le C\rho^{|\alpha|}\,V^\xi_{|\alpha|},\quad \alpha\in\mathbb{N}^n,\ a\in E, \tag{12}$$

$$|(R^k_aF)^\alpha(b)|\le C\rho^{k+1}\,|\alpha|!\,v^\xi_{k+1}\,|b-a|^{k+1-|\alpha|},\quad k\in\mathbb{N},\,|\alpha|\le k,\ a,b\in E. \tag{13}$$

Let $p\in\mathbb{N}$ be fixed (and to be specified later). Let $\{\varphi_{i,p}\}_{i\in\mathbb{N}}$ be the partition of unity provided by [16, Proposition 4.9], relative to the family of cubes $\{Q_i\}_{i\in\mathbb{N}}$ from [16, Lemma 4.7], and let $r_0=r_0(p)$ be the constant appearing in this proposition. The center of Q_i is denoted by x_i. We claim that an extension of class $\mathcal{B}^{\{\omega\}}$ of F to a suitable neighborhood of E in $\mathbb{R}^n$ is provided by

$$f(x):=\begin{cases}\sum_{i\in\mathbb{N}}\varphi_{i,p}(x)\,T^{p(x_i)}_{\hat{x}_i}F(x), & \text{if } x\in\mathbb{R}^n\setminus E,\\ F^0(x), & \text{if } x\in E,\end{cases}$$

where, given $x\in\mathbb{R}^n\setminus E$, $\hat{x}$ is any point in E with $d(x):=d(x,E)=|x-\hat{x}|$ and

$$p(x):=\max\{2\Gamma_{\underline{s}^{2\xi}}(Ld(x))-1,0\}.$$

Here L is a positive constant to be specified below. Recall that Q_i^* is the closed cube with the same center as Q_i expanded by the factor $9/8$. By [16, Corollary 4.8],

$$\frac{1}{2}d(x) \le d(x_i) \le 3d(x), \qquad \text{for all } x \in Q_i^*. \tag{14}$$

Then $d(x) < 1/(3L\underline{s}_1^{2\xi})$ guarantees that both $\Gamma_{\underline{s}^{2\xi}}(Ld(x_i))$ and $\Gamma_{\underline{s}^{2\xi}}(Ld(x))$ are ≥ 1, by (2), thus $p(x_i) = 2\Gamma_{\underline{s}^{2\xi}}(Ld(x_i)) - 1$ and $p(x) = 2\Gamma_{\underline{s}^{2\xi}}(Ld(x)) - 1$.

• Replace [16, Lemma 5.2] by the following lemma. The only difference in the proof is that one uses (8) instead of [16, (5.4)].

Lemma 4. *There is a constant $C_0 = C_0(n) > 1$ such that, for all Whitney ultrajets $F = (F^\alpha)_\alpha$ of class $\mathcal{B}^{\{V^\xi\}}$ that satisfy* (12) *and* (13)*, all $L \ge C_0\rho$, all $x \in \mathbb{R}^n$, and all $\alpha \in \mathbb{N}^n$,*

$$|(T_{\hat{x}}^{p(x)}F)^{(\alpha)}(x)| \le C(2L)^{|\alpha|+1}V^\xi_{|\alpha|}, \tag{15}$$

and, if $|\alpha| < p(x)$,

$$|(T_{\hat{x}}^{p(x)}F)^{(\alpha)}(x) - F^\alpha(\hat{x})| \le C(2L)^{|\alpha|+1}|\alpha|!\, v^\xi_{|\alpha|+1}d(x). \tag{16}$$

We remark that (here and below) by $(T_{\hat{x}}^{p(x)}F)^{(\alpha)}(x)$ we mean the α-th partial derivative of the polynomial $y \mapsto T_{\hat{x}}^{p(x)}F(y)$ evaluated at $y = x$.

• Replace [16, Lemma 5.3] by:

Lemma 5. *There is a constant $C_1 = C_1(n) > 0$ such that for all $L > C_1\rho$, all $\beta \in \mathbb{N}^n$, and all $x \in Q_i^*$ with $d(x) < 1/(3L\underline{s}_1^{2\xi})$,*

$$|\partial^\beta(T_{\hat{x}_i}^{p(x_i)}F - T_{\hat{x}}^{p(x_i))}F)(x)| \le CL^{|\beta|+1}\underline{S}^{2\xi}_{|\beta|}\, h_{\underline{s}^{2\xi}}(Ld(x_i)). \tag{17}$$

Proof. It suffices to consider $|\beta| \le p(x_i) = 2\Gamma_{\underline{s}^{2\xi}}(Ld(x_i)) - 1 =: 2q - 1$. Let H_1 denote the left-hand side of (17). By [16, Lemma 5.1 and Corollary 4.8] and (6),

$$H_1 \le C(2n^2\rho)^{2q}|\beta|!\, v^\xi_{2q}(6d(x_i))^{2q-|\beta|} \le C(2n^2\rho)^{2q}|\beta|!\, (\underline{s}_q^{2\xi})^2(6d(x_i))^{2q-|\beta|}.$$

By the definition of q, $h_{\underline{s}^{2\xi}}(Ld(x_i)) = \underline{s}_q^{2\xi}(Ld(x_i))^q \le \underline{s}_k^{2\xi}(Ld(x_i))^k$ for all k. Thus

$$H_1 \le C\Big(\frac{12n^2\rho}{L}\Big)^{2q} L^{|\beta|}\, |\beta|!\, \underline{s}^{2\xi}_{|\beta|}\, h_{\underline{s}^{2\xi}}(Ld(x_i)).$$

If $L > 12n^2\,\rho$, then (17) follows. □

• Replace [16, Lemma 5.4] by:

Lemma 6. *There is a constant $C_2 = C_2(n) > 0$ such that for all $L > C_2\rho$, all $\beta \in \mathbb{N}^n$, and all $x \in Q_i^*$ with $d(x) < 1/(3L\underline{s}_1^{2\xi})$,*

$$|\partial^\beta(T_{\hat{x}}^{p(x_i)}F - T_{\hat{x}}^{p(x)}F)(x)| \le C\Big(\frac{3L}{n}\Big)^{|\beta|+1}\underline{S}^{2\xi}_{|\beta|}h_{\underline{s}^{2\xi}}(3Ld(x)). \tag{18}$$

Proof. Both $p(x_i)$ and $p(x)$ are majorized by $\Gamma_{v^\xi}(Ld(x)/2)$, indeed, by (8), (14), and since Γ_{v^ξ} is decreasing,

$$p(x_i) = 2\Gamma_{\underline{s}^{2\xi}}(Ld(x_i)) - 1 \le 2\Gamma_{\underline{s}^{2\xi}}(Ld(x_i)) = \Gamma_{v^\xi}(Ld(x_i)) \le \Gamma_{v^\xi}(Ld(x)/2).$$

So the degree of the polynomial $T_{\hat{x}}^{p(x_i)}F - T_{\hat{x}}^{p(x)}F$ is at most $\Gamma_{v^\xi}(Ld(x)/2)$. The valuation of the polynomial is equal to $\min\{p(x_i), p(x)\} + 1$ (unless $p(x_i) = p(x)$ in which case (18) is trivial) and so at least $2\Gamma_{\underline{s}^{2\xi}}(3Ld(x)) =: 2q$, by (14). So if H_2 denotes the left-hand side of (18), then (see the calculation in [16, (5.7)])

$$H_2 \le \frac{C|\beta|!}{(nd(x))^{|\beta|}} \sum_{j=2q}^{\Gamma_{v^\xi}(Ld(x)/2)} (2n^2\rho d(x))^j v_j^\xi.$$

By (3), $v_j^\xi(Ld(x)/2)^j \le v_{2q}^\xi(Ld(x)/2)^{2q}$ for $2q \le j \le \Gamma_{v^\xi}(Ld(x)/2)$. By the definition of q, $h_{\underline{s}^{2\xi}}(3Ld(x)) = \underline{s}_q^{2\xi}(3Ld(x))^q \le \underline{s}_k^{2\xi}(3Ld(x))^k$ for all k. With (6) this leads to

$$\begin{aligned} H_2 &\le \frac{C|\beta|!}{(nd(x))^{|\beta|}} \sum_{j=2q}^{\Gamma_{v^\xi}(Ld(x)/2)} \Big(\frac{4n^2\rho}{L}\Big)^j v_{2q}^\xi \Big(\frac{Ld(x)}{2}\Big)^{2q} \\ &\le \frac{C|\beta|!}{(nd(x))^{|\beta|}} \sum_{j=2q}^{\Gamma_{v^\xi}(Ld(x)/2)} \Big(\frac{4n^2\rho}{L}\Big)^j (\underline{s}_q^{2\xi})^2 \Big(\frac{Ld(x)}{2}\Big)^{2q} \\ &\le C\Big(\frac{3L}{n}\Big)^{|\beta|} |\beta|!\, \underline{s}_{|\beta|}^{2\xi} h_{\underline{s}^{2\xi}}(3Ld(x)) \sum_{j=2q}^{\Gamma_{v^\xi}(Ld(x)/2)} \Big(\frac{4n^2\rho}{L}\Big)^j. \end{aligned}$$

If we choose $L \ge 8n^2\rho$, then the sum is bounded by 2, and (18) follows. ☐

- Assume that L is chosen such that

$$L > \max\{C_0, C_1, C_2\}\, \rho \tag{19}$$

so that (15), (16), (17), and (18) are valid. Recall that $\mathfrak{W}$ denotes the weight matrix associated with ω. The next lemma is a substitute for the claim in the proof of Theorem 5.5 in [16].

Lemma 7. *There exist constants $K_j = K_j(n,\omega)$, $j = 1,2,3$, such that the following holds. If $p = K_1L$ and $L > K_2\rho$, then there exist a weight sequence $W \in \mathfrak{W}$ and a constant $M_1 = M_1(n,\omega,L) > 0$ such that for all $x \in \mathbb{R}^n \setminus E$ with $d(x) < \min\{r_0/(3B_1), 1/(3L\underline{s}_1^{2\xi})\}$ and all $\alpha \in \mathbb{N}^n$,*

$$|\partial^\alpha(f - T_{\hat{x}}^{p(x)}F)(x)| \le CM_1^{|\alpha|+1} W_{|\alpha|} h_{\underline{s}^{4\xi}}(K_3Ld(x)), \tag{20}$$

where C and ρ are the constants from (12) *and* (13) *(and B_1 is the universal constant from* [16, Lemma 4.7]*).*

Proof. By the Leibniz rule,

$$\partial^\alpha(f - T_{\hat{x}}^{p(x)}F)(x) = \sum_{\beta\le\alpha} \binom{\alpha}{\beta} \sum_i \varphi_{i,p}^{(\alpha-\beta)}(x)\, \partial^\beta(T_{\hat{x}_i}^{p(x_i)}F - T_{\hat{x}}^{p(x)}F)(x). \tag{21}$$

Now (17) and (18) imply, that for $x \in Q_i^*$ with $d(x) < 1/(3L\underline{s}_1^{2\xi})$,

$$|\partial^\beta(T_{\hat{x}_i}^{p(x_i)}F - T_{\hat{x}}^{p(x)}F)(x)| \le C(6L)^{|\beta|+1} \underline{S}_{|\beta|}^{2\xi}\, h_{\underline{s}^{2\xi}}(3Ld(x)). \tag{22}$$

As in [16] we conclude (using [16, Proposition 4.9]) that there exist $W = W(p) \in \mathfrak{W}$ and $M = M(p) > 0$ such that for all $i \in \mathbb{N}$, all $x \in \mathbb{R}^n \setminus E$ with $d(x) < r_0/(3B_1)$, and all $\beta \in \mathbb{N}^n$,

$$|\varphi_{i,p}^{(\beta)}(x)| \le MW_{|\beta|}\, \Pi(p,x) \tag{23}$$

where, by [16, Corollary 3.11],

$$\begin{aligned}\Pi(p,x) &= \exp\Big(\frac{A_1(n)}{p}\sigma^\star\Big(\frac{b_1 p}{9A_2(n)}d(x)\Big)\Big)\\ &\le \Big(\frac{e}{h_{\underline{s}^\eta}(\frac{b_1 p d(x)}{9A_2(n)B})}\Big)^{\frac{A_1(n)B}{p}}, \quad \text{for some } B \ge 1 \text{ and all } \eta > 0. \end{aligned} \tag{24}$$

(b_1 is the universal constant from [16, Lemma 4.7] and $A_1(n) \le A_2(n)$ are constants depending only on n.) By (11), we may assume that $\underline{S}^{2\xi} \le W$. Then, by (21), (22), (23), and [16, Lemma 4.7], for $x \in \mathbb{R}^n \setminus E$ with $d(x) < \min\{r_0/(3B_1), 1/(3L\underline{s}_1^{2\xi})\}$,

$$\begin{aligned}&|\partial^\alpha(f - T_{\hat{x}}^{p(x)}F)(x)|\\ &\le \sum_{\beta\le\alpha}\frac{\alpha!}{\beta!(\alpha-\beta)!}\cdot 12^{2n}\cdot MW_{|\alpha|-|\beta|}\Pi(p,x)\cdot C(6L)^{|\beta|+1}\underline{S}^{2\xi}_{|\beta|}\, h_{\underline{s}^{2\xi}}(3Ld(x))\\ &\le 12^{2n}CM\Big(\sum_{j=0}^{|\alpha|}\frac{|\alpha|!\,n^{|\alpha|+j}}{j!(|\alpha|-j)!}(6L)^{j+1}W_{|\alpha|-j}\underline{S}^{2\xi}_j\Big)\,\Pi(p,x)\,h_{\underline{s}^{2\xi}}(3Ld(x))\\ &\le 6\cdot 12^{2n}LCMn^{|\alpha|}W_{|\alpha|}\Big(\sum_{j=0}^{|\alpha|}\frac{|\alpha|!}{j!(|\alpha|-j)!}(6Ln)^j\Big)\,\Pi(p,x)\,h_{\underline{s}^{2\xi}}(3Ld(x))\\ &= 6\cdot 12^{2n}LCM(n(1+6Ln))^{|\alpha|}W_{|\alpha|}\Pi(p,x)\,h_{\underline{s}^{2\xi}}(3Ld(x)).\end{aligned}$$

By Proposition 3, there is $H \ge 1$ (independent of ξ) such that $h_{\underline{s}^{2\xi}}(t) \le h_{\underline{s}^{4\xi}}(Ht)^2$ for $t > 0$. Let us choose L according to (19) and such that $p := 27\,A_2(n)BHL/b_1 \ge A_1(n)B$ is an integer. Then, by (24) and since $h_{\underline{s}^{4\xi}} \le 1$,

$$\Pi(p,x)\,h_{\underline{s}^{2\xi}}(3Ld(x)) \le \frac{e\,h_{\underline{s}^{2\xi}}(3Ld(x))}{h_{\underline{s}^{4\xi}}(3HLd(x))} \le e\,h_{\underline{s}^{4\xi}}(3HLd(x))$$

and we obtain (20). (Note that M depends on p, and hence on L, which results in the non-explicit dependence of M_1.) □

• Let us finish the proof of Theorem 1. By (15) and (20), for all $x \in \mathbb{R}^n \setminus E$ with $d(x) < \min\{r_0/(3B_1), 1/(3L\underline{s}_1^{2\xi})\}$ and all $\alpha \in \mathbb{N}^n$,

$$|f^{(\alpha)}(x)| \le |(T_{\hat{x}}^{p(x)}F)^{(\alpha)}(x)| + |\partial^\alpha(f - T_{\hat{x}}^{p(x)}F)(x)| \le CM^{|\alpha|+1}W_{|\alpha|} \tag{25}$$

for a suitable constant $M = M(n,\omega,L)$.

Let us fix a point $a \in E$ and $\alpha \in \mathbb{N}^n$. Since $\Gamma_{\underline{s}^{2\xi}}(t) \to \infty$ as $t \to 0$, we have $|\alpha| < p(x)$ if $x \in \mathbb{R}^n \setminus E$ is sufficiently close to a. Thus, as $x \to a$,

$$\begin{aligned}&|f^{(\alpha)}(x) - F^\alpha(a)|\\ &\le |\partial^\alpha(f - T_{\hat{x}}^{p(x)}F)(x)| + |(T_{\hat{x}}^{p(x)}F)^{(\alpha)}(x) - F^\alpha(\hat{x})| + |F^\alpha(\hat{x}) - F^\alpha(a)|\\ &= O(h_{\underline{s}^{4\xi}}(K_3Ld(x))) + O(d(x)) + O(|\hat{x} - a|),\end{aligned}$$

by (13), (16), and (20). Hence $f^{(\alpha)}(x) \to F^\alpha(a)$ as $x \to a$. We may conclude that $f \in C^\infty(\mathbb{R}^n)$ and extends F. After multiplication with a suitable cut-off function of class $\mathcal{B}^{\{\omega\}}$ with support in $\{x : d(x) < \min\{r_0/(3B_1), 1/(3L\underline{s}_1^{2\xi})\}\}$, we find that $f \in \mathcal{B}^{\{\omega\}}(\mathbb{R}^n)$ thanks to (12), (25), and [16, Lemma 2.4(5)]. The proof of Theorem 1 is complete.

3. Concave, good, and strong weight functions

In [16, Definition 3.5] we called a weight function σ *good* if its associated weight matrix $\mathfrak{S}$ satisfies (1). A non-quasianalytic weight function ω is called *strong* if there is a constant $C > 0$ such that

$$\int_1^\infty \frac{\omega(tu)}{u^2}\, du \le C\omega(t) + C, \qquad \text{for all } t > 0.$$

Otherwise put, ω is strong if and only if it is equivalent to the concave weight function $\kappa = \kappa(\omega)$ defined in (10). In [16] we asked the following questions:

Question 3.21: *Is every concave weight function equivalent to a good one?*
Question 5.11: *Is every strong weight function equivalent to a good one?*

We will give partial answers to these questions and reveal some related connections in Theorem 11 below.

In [16] it was important that the associated weight matrix *itself* satisfies (1) as explained after the proof of Proposition 3. Since we could overcome this problem (by introducing $\mathfrak{V} = \{V^\xi\}$), it is more natural to allow for a wider concept of goodness. For completeness we will also treat the Beurling case. A weight function ω is called *R-good* if there exists a weight matrix $\mathfrak{M}$ satisfying

$$\forall M \in \mathfrak{M}\ \exists N \in \mathfrak{M}\ \exists C \ge 1\ \forall 1 \le j \le k : \frac{\mu_j}{j} \le C \frac{\nu_k}{k} \tag{26}$$

such that $\mathcal{B}^{\{\omega\}} = \mathcal{B}^{\{\mathfrak{M}\}}$. Recall that $\mu_k := M_k/M_{k-1}$ and $\nu_k := N_k/N_{k-1}$. Similarly, ω is called *B-good* if there exists a weight matrix $\mathfrak{M}$ satisfying

$$\forall N \in \mathfrak{M}\ \exists M \in \mathfrak{M}\ \exists C \ge 1\ \forall 1 \le j \le k : \frac{\mu_j}{j} \le C \frac{\nu_k}{k} \tag{27}$$

such that $\mathcal{B}^{(\omega)} = \mathcal{B}^{(\mathfrak{M})}$.

The next lemma, which is inspired by [5, Proposition 4.15], implies that for any weight matrix $\mathfrak{M}$ satisfying (26) (resp. (27)) there is a weight matrix $\mathfrak{S}$ consisting of strongly log-convex weight sequences such that $\mathcal{B}^{\{\mathfrak{M}\}} = \mathcal{B}^{\{\mathfrak{S}\}}$ (resp. $\mathcal{B}^{(\mathfrak{M})} = \mathcal{B}^{(\mathfrak{S})}$).

Lemma 8. *Assume that $1 = \mu_0 \le \mu_1 \le \cdots$ and $1 = \nu_0 \le \nu_1 \le \cdots$ satisfy*

$$\exists C > 0 : \frac{\mu_j}{j} \le C \frac{\nu_k}{k}, \qquad \textit{for all } j \le k.$$

Then the sequence $\tilde\nu$ defined by

$$\frac{\tilde\nu_k}{k} := \inf_{\ell \ge k} \frac{\nu_\ell}{\ell}, \quad \tilde\nu_0 := 1,$$

is such that $\tilde\nu_k/k$ is increasing and $C^{-1}\mu \le \tilde\nu \le \nu$. □

The next two corollaries are immediate from Lemma 8 and results of [12], [13], and [14].

Corollary 9. *Let $\mathfrak{M}$ be a weight matrix with the property that for all $M \in \mathfrak{M}$ there is $N \in \mathfrak{M}$ such that $(M_{k+1}/N_k)^{1/k}$ is bounded. Consider the following conditions:*

(a) $\mathfrak{M}$ *satisfies* (26).
(b) *There is a weight matrix $\mathfrak{S}$ consisting of strongly log-convex weight sequences such that $\mathcal{B}^{\{\mathfrak{M}\}} = \mathcal{B}^{\{\mathfrak{S}\}}$.*
(c) $\mathcal{B}^{\{\mathfrak{M}\}}$ *is stable under composition.*
(d) $\forall M \in \mathfrak{M}\ \exists N \in \mathfrak{M}\ \exists C > 0\ \forall j \le k : m_j^{1/j} \le C\, n_k^{1/k}$.

Then (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c) $\Leftrightarrow$ (d). *If additionally* $\mathfrak{M}$ *satisfies*

$$\forall M \in \mathfrak{M}\ \exists N \in \mathfrak{M} : \mu_k \lesssim N_k^{1/k}, \tag{28}$$

then all four conditions are equivalent.

Corollary 10. *Let* $\mathfrak{M}$ *be a weight matrix with the property that for all* $N \in \mathfrak{M}$ *there is* $M \in \mathfrak{M}$ *such that* $(M_{k+1}/N_k)^{1/k}$ *is bounded. Consider the following conditions:*

(a) $\mathfrak{M}$ *satisfies* (27).
(b) *There is a weight matrix* $\mathfrak{S}$ *consisting of strongly log-convex weight sequences such that* $\mathcal{B}^{(\mathfrak{M})} = \mathcal{B}^{(\mathfrak{S})}$.
(c) $\mathcal{B}^{(\mathfrak{M})}$ *is stable under composition.*
(d) $\forall N \in \mathfrak{M}\ \exists M \in \mathfrak{M}\ \exists C > 0\ \forall j \le k : m_j^{1/j} \le C\, n_k^{1/k}$.

Then (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c) $\Leftrightarrow$ (d). *If additionally* $\mathfrak{M}$ *satisfies*

$$\forall N \in \mathfrak{M}\ \exists M \in \mathfrak{M} : \mu_k \lesssim N_k^{1/k}, \tag{29}$$

then all four conditions are equivalent.

In general, (c) $\not\Rightarrow$ (b) in neither of the corollaries which follows from [12, Example 3.6]. Note that if $M = N$ then (28) and (29) reduce to a condition which is usually called *moderate growth* or M.

For weight functions ω we get a full characterization.

Theorem 11. *Let* ω *be a weight function satisfying* $\omega(t) = o(t)$ *as* $t \to \infty$. *Then the following are equivalent.*

(a) ω *is equivalent to a concave weight function.*
(b) $\exists C > 0\ \exists t_0 > 0\ \forall \lambda \ge 1\ \forall t \ge t_0 : \omega(\lambda t) \le C\lambda\, \omega(t)$.
(c) $\mathcal{B}^{\{\omega\}}$ *is stable under composition.*
(d) $\mathcal{B}^{(\omega)}$ *is stable under composition.*
(e) *There is a weight matrix* $\mathfrak{S}$ *consisting of strongly log-convex weight sequences such that* $\mathcal{B}^{\{\omega\}} = \mathcal{B}^{\{\mathfrak{S}\}}$.
(f) *There is a weight matrix* $\mathfrak{S}$ *consisting of strongly log-convex weight sequences such that* $\mathcal{B}^{(\omega)} = \mathcal{B}^{(\mathfrak{S})}$.
(g) ω *is R-good.*
(h) ω *is B-good.*

Notice that the conditions in the theorem are furthermore equivalent to the classes $\mathcal{B}^{\{\omega\}}$ and $\mathcal{B}^{(\omega)}$ to be stable under inverse/implicit functions and solving ODEs, and, in terms of the associated weight matrix $\mathfrak{W} = \{W^\xi\}_{\xi>0}$, to

$$\forall \xi > 0\ \exists \eta > 0 : (w_j^\xi)^{1/j} \le C\, (w_k^\eta)^{1/k} \quad \text{for } j \le k,$$

as well as

$$\forall \eta > 0\ \exists \xi > 0 : (w_j^\xi)^{1/j} \le C\, (w_k^\eta)^{1/k} \quad \text{for } j \le k,$$

see [13]. In the forthcoming paper [4] we shall see that they are also equivalent to the property that $\mathcal{B}^{\{\omega\}}$, resp. $\mathcal{B}^{(\omega)}$, can be described by almost analytic extensions; see also [11].

Proof. The equivalence of the first four conditions (a)–(d) is well-known, see e.g. [13], which is based on [10, Lemma 1] and [3]. That (a) implies (e) and (f) follows from Proposition 3. (e) $\Rightarrow$ (c) and (f) $\Rightarrow$ (d) are clear; cf. [12]. The equivalences (e) $\Leftrightarrow$ (g) and (f) $\Leftrightarrow$ (h) follow from Lemma 8. ☐

Appendix A. Weight matrices

By a *weight matrix* we mean a family $\mathfrak{M}$ of weight sequences $M \ge (k!)_k$ which is totally ordered with respect to the pointwise order relation on sequences, i.e.,

- $\mathfrak{M} \subseteq \mathbb{R}^{\mathbb{N}}$,
- each $M \in \mathfrak{M}$ is a weight sequence, which means that $M_0 = 1$, $M_k^{1/k} \to \infty$, and M is log-convex,
- each $M \in \mathfrak{M}$ satisfies $k! \le M_k$ for all k,
- for all $M, N \in \mathfrak{M}$ we have $M \le N$ or $M \ge N$.

For a weight matrix $\mathfrak{M}$ and an open $U \subseteq \mathbb{R}^n$ we consider the Roumieu class

$$\mathcal{B}^{\{\mathfrak{M}\}}(U) := \operatorname{ind}_{M \in \mathfrak{M}} \mathcal{B}^{\{M\}}(U),$$

and the Beurling class

$$\mathcal{B}^{(\mathfrak{M})}(U) := \operatorname{proj}_{M \in \mathfrak{M}} \mathcal{B}^{(M)}(U).$$

For weight matrices $\mathfrak{M}$, $\mathfrak{N}$ we have (cf. [12])

$$\mathcal{B}^{\{\mathfrak{M}\}} \subseteq \mathcal{B}^{\{\mathfrak{N}\}} \quad \Leftrightarrow \quad \forall M \in \mathfrak{M}\, \exists N \in \mathfrak{N} : (M_k/N_k)^{1/k} \text{ is bounded},$$

$$\mathcal{B}^{(\mathfrak{M})} \subseteq \mathcal{B}^{(\mathfrak{N})} \quad \Leftrightarrow \quad \forall N \in \mathfrak{N}\, \exists M \in \mathfrak{M} : (M_k/N_k)^{1/k} \text{ is bounded}.$$

Analogous equivalences hold for the *local* classes

$$\mathcal{E}^{\{\mathfrak{M}\}}(U) := \operatorname{proj}_{V \Subset U} \mathcal{B}^{\{\mathfrak{M}\}}(V) \quad \text{and} \quad \mathcal{E}^{(\mathfrak{M})}(U) := \operatorname{proj}_{V \Subset U} \mathcal{B}^{(\mathfrak{M})}(V).$$

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