

ON OPTIMAL SOLUTIONS OF THE BOREL PROBLEM IN THE ROUMIEU CASE

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ABSTRACT. The Borel problem for Denjoy–Carleman and Braun–Meise–Taylor classes has well-known optimal solutions. The unified treatment of these ultradifferentiable classes by means of one-parameter families of weight sequences allows to compare these optimal solutions. We determine the relations among them and give conditions for their equivalence in the Roumieu case.

1. INTRODUCTION

The Borel map $j^\infty : C^\infty(\mathbb{R}) \rightarrow \mathbb{C}^{\mathbb{N}}$ takes a smooth function f to its infinite jet $(f^{(n)}(0))_{n \in \mathbb{N}}$ at 0. We will be concerned with the restriction of j^∞ to Denjoy–Carleman classes $\mathcal{E}^{\{M\}}(\mathbb{R})$, Braun–Meise–Taylor classes $\mathcal{E}^{\{\omega\}}(\mathbb{R})$, and, most generally, classes $\mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R})$, where \mathfrak{M} is a one-parameter family of weight sequences. We will only treat the Roumieu case in this paper; the Beurling case will be discussed in a separate paper. The j^∞ -image of any of these ultradifferentiable classes sits in a sequence space $\Lambda^{\{M\}}$, $\Lambda^{\{\omega\}}$, and $\Lambda^{\{\mathfrak{M}\}}$ defined by analogous bounds. The (mixed) Borel problem asks for conditions for the validity of the inclusion

$$(1.1) \quad \Lambda^{\{\text{weight}'\}} \subseteq j^\infty \mathcal{E}^{\{\text{weight}\}}(\mathbb{R})$$

in any of the above cases, where different weights may appear on the left and on the right.

For Denjoy–Carleman and Braun–Meise–Taylor classes the *optimal* solution to this problem is well-known:

- (1) Let M be a non-quasianalytic weight sequence. If M' is a suitable other weight sequence, then $\Lambda^{\{M'\}} \subseteq j^\infty \mathcal{E}^{\{M\}}(\mathbb{R})$ is equivalent to a condition (namely (4.1)) purely in terms of M', M which we denote by $M' \prec_{SV} M$; see [26]. There is an explicit positive sequence $L = L(M)$ such that $L \prec_{SV} M$, that is $\Lambda^{\{L\}} \subseteq j^\infty \mathcal{E}^{\{M\}}(\mathbb{R})$, and L is optimal with respect to this property; see [24].
- (2) The condition $M' \prec_{\gamma_1} M$ (see (4.4)) is generally stronger than $M' \prec_{SV} M$; it plays a crucial role in the more general Whitney problem [5, 7, 13, 20, 18]. In many important cases the conditions are equivalent. There is an explicit weight sequence $S = S(M)$ such that $S \prec_{\gamma_1} M$ and S is optimal with respect to this property; see [20].

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- (3) Let ω be a non-quasianalytic weight function. If ω' is a suitable other weight function, then $\Lambda^{\{\omega'\}} \subseteq j^\infty \mathcal{E}^{\{\omega\}}(\mathbb{R})$ is equivalent to a condition (namely (2.2)) purely in terms of ω', ω which we denote by $\omega' \prec_{st} \omega$. There is an explicit weight function $\kappa = \kappa(\omega)$ such that $\kappa \prec_{st} \omega$, that is $\Lambda^{\{\kappa\}} \subseteq j^\infty \mathcal{E}^{\{\omega\}}(\mathbb{R})$, and κ is optimal with respect to this property; see [3]. The condition $\omega' \prec_{st} \omega$ is also crucial in the respective Whitney problem [1, 22, 23, 17, 18].
- (4) Beside these optimal solutions there is Carleson's solution [6] based on a universal moment problem. It gives a sequence $Q = Q(M)$ (but is also intimately related with (3)) such that $\Lambda^{\{Q\}} \subseteq j^\infty \mathcal{E}^{\{M\}}(\mathbb{R})$.

The assumption that the weight on the right-hand side of (1.1) is non-quasianalytic is no restriction: it is a necessary consequence of the inclusion (1.1) (for germs at 0) if the involved classes strictly contain the real analytic class; see [21] and [18, Section 5].

The use of one-parameter families \mathfrak{M} of weight sequences admits an efficient unified treatment of Denjoy–Carleman and Braun–Meise–Taylor classes alike. In fact, for a weight function ω there is a canonical well-behaved family $\mathfrak{M} = \mathfrak{M}_\omega$ such that $\Lambda^{\{\omega\}} = \Lambda^{\{\mathfrak{M}\}}$ and $\mathcal{E}^{\{\omega\}}(\mathbb{R}) = \mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R})$ as locally convex spaces; cf. [19]. It also abolishes the borders between the four listed solutions and begs the question about the relationships among them.

In this note we answer this question. Given a suitable family \mathfrak{M} we lift the derived weights L, S, κ , and Q to the level of families of sequences $\mathfrak{L}, \mathfrak{S}, \mathfrak{K}$, and \mathfrak{Q} and clarify the relations among them. While the construction of \mathfrak{L} and \mathfrak{S} is straightforward, \mathfrak{K} and \mathfrak{Q} are obtained by a more convoluted procedure involving the interplay between weight sequences $M \in \mathfrak{M}$ and their associated functions $\omega_M, \kappa_{\omega_M}$, and P_{ω_M} . In summary we find, provided that \mathfrak{M} and \mathfrak{M}' satisfy some standard conditions,

$$(1.2) \quad \begin{aligned} \Lambda^{\{\mathfrak{S}\}} \subseteq \Lambda^{\{\mathfrak{K}\}} = \Lambda^{\{\mathfrak{Q}\}} \subseteq \Lambda^{\{\mathfrak{L}\}} \subseteq \Lambda^{\{\mathfrak{S}\}} \subseteq j^\infty \mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R}), \\ \Lambda^{\{\mathfrak{M}'\}} \subseteq j^\infty \mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R}) \implies \Lambda^{\{\mathfrak{M}'\}} \subseteq \Lambda^{\{\mathfrak{L}\}}, \end{aligned}$$

underlining the overall optimality of \mathfrak{L} in regard of the mixed Borel problem; here $\underline{\mathfrak{L}}$ is obtained from \mathfrak{L} by passing to the log-convex minorants of the sequences in \mathfrak{L} .

In the most important Braun–Meise–Taylor case we prove that, for each non-quasianalytic weight function ω ,

$$(1.3) \quad \Lambda^{\{\kappa\}} = \Lambda^{\{\mathfrak{K}\}} = \Lambda^{\{\mathfrak{Q}\}} = \Lambda^{\{\mathfrak{L}\}},$$

where the families $\mathfrak{L}, \mathfrak{K}$, and \mathfrak{Q} are derived from the canonical associated family \mathfrak{M}_ω . So in this case optimality in the mixed Borel problem is achieved by all these different approaches. In view of the inherent regularity properties of κ it is not too surprising that we have to pass from \mathfrak{L} which might be quite irregular to $\underline{\mathfrak{L}}$.

After recalling background on weights and function and sequence spaces in Section 2 we review Carleson's solution to the Borel problem in Section 3 and adapt it to our setting. In Section 4 we prove (1.2) and give sufficient conditions for equality everywhere in (1.2). The consequences for the classical cases of Braun–Meise–Taylor and Denjoy–Carleman classes are discussed in Section 5. The proof of (1.3) relies on a variant of a result of [3] which we prove in Appendix A.

2. WEIGHTS

2.1. Weight functions. Let $\omega : [0, \infty) \rightarrow [0, \infty)$ be a continuous increasing function satisfying $\omega(0) = 0$ and $\lim_{t \rightarrow \infty} \omega(t) = \infty$. We call ω a *pre-weight function* if additionally

- $\log(t) = o(\omega(t))$ as $t \rightarrow \infty$,
- $\varphi_\omega : t \mapsto \omega(e^t)$ is convex.

A pre-weight function ω is a *weight function* if it also fulfills

- $\omega(2t) = O(\omega(t))$ as $t \rightarrow \infty$.

Another important condition is

$$(2.1) \quad \exists H \geq 1 \forall t \geq 0 : 2\omega(t) \leq \omega(Ht) + H.$$

Given two pre-weight functions ω, σ we write $\sigma \preceq \omega$ if $\omega(t) = O(\sigma(t))$ as $t \rightarrow \infty$ and call ω and σ *equivalent* if $\sigma \preceq \omega$ and $\omega \preceq \sigma$. Note that \preceq induces a partial order on the set of equivalence classes. Any equivalence class $[\omega]$ contains a pre-weight function $\tilde{\omega}$ with $\tilde{\omega}|_{[0,1]} = 0$; we say that $\tilde{\omega}$ is *normalized*. We will often tacitly assume that this property is satisfied.

Also note that if one representative in $[\omega]$ is a weight function then all representatives are weight functions.

A pre-weight function ω is called *non-quasianalytic* if

$$\int_0^\infty \frac{\omega(t)}{1+t^2} dt < \infty$$

and *quasianalytic* otherwise. If ω is a non-quasianalytic pre-weight function, then we may consider

$$\kappa_\omega(t) := \int_1^\infty \frac{\omega(ts)}{s^2} ds = t \int_t^\infty \frac{\omega(s)}{s^2} ds, \quad t \geq 0,$$

which turns out to be a concave weight function satisfying $\kappa_\omega(t) = o(t)$ as $t \rightarrow \infty$ and $\omega \leq \kappa_\omega$ (which entails $\kappa_\omega \preceq \omega$) since ω is increasing; it might be quasianalytic. Note that if σ and ω are equivalent, then so are κ_σ and κ_ω .

The importance of κ_ω relies on its *optimality* with respect to the Borel problem [3]: Given that ω', ω are weight functions, ω non-quasianalytic, $\omega'(t) = o(t)$ as $t \rightarrow \infty$, then $\Lambda^{\{\omega'\}} \subseteq j^\infty \mathcal{E}^{\{\omega\}}(\mathbb{R})$ if and only if $\omega' \preceq \kappa_\omega$, i.e.,

$$(2.2) \quad \exists C > 0 \forall t \geq 0 : \int_1^\infty \frac{\omega'(ts)}{s^2} ds \leq C\omega'(t) + C;$$

we will write $\omega' \prec_{st} \omega$ for (2.2). This induces a relation on the equivalence classes of pre-weight functions which is antisymmetric, since $\omega' \prec_{st} \omega$ implies $\omega' \preceq \omega$, and transitive (indeed, $\omega_1 \prec_{st} \omega_0$ and $\omega_2 \prec_{st} \omega_1$ yield $\omega_2 \preceq \kappa_{\omega_1} \preceq \omega_1 \preceq \kappa_{\omega_0}$ so that $\omega_2 \prec_{st} \omega_0$). The pre-weight functions ω that satisfy $\omega \prec_{st} \omega$, that is $\Lambda^{\{\omega\}} = j^\infty \mathcal{E}^{\{\omega\}}(\mathbb{R})$, are often called *strong weight functions*; being equivalent to κ_ω they actually are weight functions.

For any pre-weight function ω we consider the *Young conjugate* of φ_ω ,

$$\varphi_\omega^*(x) := \sup\{xy - \varphi_\omega(y) : y \geq 0\}, \quad x \geq 0.$$

It is convex, increasing, and satisfies $(\varphi_\omega^*)^* = \varphi_\omega$, $\varphi_\omega^*(t)/t \nearrow \infty$ as $t \rightarrow \infty$, and $\varphi_\omega^*(0) = 0$ (provided that ω is normalized).

2.2. Weight sequences. A positive sequence $M = (M_k)_{k \geq 0}$ is called a *weight sequence* if $M_k = \mu_0 \mu_1 \cdots \mu_k$, where $1 = \mu_0 \leq \mu_1 \leq \cdots \leq \mu_{k-1} \leq \mu_k \nearrow \infty$. That μ is increasing amounts to log-convexity of M . We have $M_k^{1/k} \nearrow \infty$. If even μ_k/k is increasing, we say that M is *strongly log-convex*.

For two positive sequences M, N we write $M \preceq N$ if

$$\sup_{k \geq 1} \left(\frac{M_k}{N_k} \right)^{1/k} < \infty,$$

and we call M and N *equivalent*, if $M \preceq N$ and $N \preceq M$. The relation $M \preceq N$ induces a partial order on the set of equivalence classes.

Note that if M is log-convex and $\mu_k \nearrow \infty$ then the equivalence class $[M]$ contains a weight sequence \widetilde{M} . That means \widetilde{M} also satisfies $1 = \widetilde{M}_0 \leq \widetilde{M}_1$.

A weight sequence M is called *non-quasianalytic*, if

$$\sum_{k=1}^{\infty} \frac{1}{\mu_k} < \infty$$

and *quasianalytic* otherwise. We say that M has *moderate growth* if

$$\exists C \geq 1 \forall j, k \in \mathbb{N} : M_{j+k} \leq C^{j+k} M_j M_k.$$

2.3. Associated function. With a positive sequence M satisfying $M_k^{1/k} \rightarrow \infty$ we associate (cf. [14, Chapitre I] and [12, Definition 3.1]) the function $\omega_M : [0, \infty) \rightarrow [0, \infty)$ defined by

$$\omega_M(t) := \sup_{k \in \mathbb{N}} \log \left(\frac{t^k M_0}{M_k} \right), \quad \text{for } t > 0, \quad \omega_M(0) := 0.$$

Lemma 2.1 (Cf. [8, Lemma 2.4] and [9, Lemma 3.1]). *For a weight sequence M :*

- (1) ω_M is a pre-weight function.
- (2) M has moderate growth if and only if ω_M satisfies (2.1).
- (3) $(M_k/k!)^{1/k} \rightarrow \infty$ if and only if $\omega_M(t) = o(t)$ as $t \rightarrow \infty$.

The *log-convex minorant* \underline{M} of a positive sequence M satisfying $M_k^{1/k} \rightarrow \infty$ is given by

$$(2.3) \quad \underline{M}_k := M_0 \sup_{t \geq 0} \frac{t^k}{e^{\omega_M(t)}}, \quad k \in \mathbb{N}.$$

We have $\underline{M}_k^{1/k} \rightarrow \infty$ and if $L \leq M$ is log-convex then $L \leq \underline{M} \leq M$.

2.4. Weight matrices. Cf. [19, Section 4]. A *weight matrix* $\mathfrak{M} = \{M^{(x)} : x \in \mathbb{R}_{>0}\}$ is a one parameter family of weight sequences $M^{(x)}$ such that $M^{(x)} \leq M^{(y)}$ if $x \leq y$. Weight matrices are a convenient technical tool for working with weight functions:

Lemma 2.2 ([19, Section 5]). *With every normalized pre-weight function ω one can associate a weight matrix $\mathfrak{M} = \mathfrak{M}_\omega := \{M^{(x)} : x > 0\}$ by setting*

$$(2.4) \quad M_k^{(x)} := \exp \left(\frac{1}{x} \varphi_\omega^*(xk) \right).$$

ω is non-quasianalytic if and only if some/each $M^{(x)}$ is non-quasianalytic. All weight sequences $M^{(x)}$ are equivalent if and only if ω satisfies (2.1) which in turn is equivalent to some/each $M^{(x)}$ having moderate growth.

If ω is even a weight function, then

$$(2.5) \quad \forall h \geq 1 \exists A \geq 1 \forall x > 0 \exists D \geq 1 \forall k \in \mathbb{N} : h^k M_k^{(x)} \leq DM_k^{(Ax)}.$$

If ω is not normalized, we still define \mathfrak{M}_ω by (2.4), but $1 = M_0^{(x)} \leq M_1^{(x)}$ might fail.

A weight matrix \mathfrak{M} is called *non-quasianalytic* if all $M \in \mathfrak{M}$ are non-quasianalytic.

Let \mathfrak{M} and \mathfrak{N} be weight matrices. We write $\mathfrak{M}\{\preceq\}\mathfrak{N}$ if

$$\forall M \in \mathfrak{M} \exists N \in \mathfrak{N} : M \preceq N$$

and say that \mathfrak{M} and \mathfrak{N} are *R-equivalent* if $\mathfrak{M}\{\preceq\}\mathfrak{N}$ and $\mathfrak{N}\{\preceq\}\mathfrak{M}$. Note that if there exists a non-quasianalytic $M \in \mathfrak{M}$, then there is a non-quasianalytic weight matrix R-equivalent to \mathfrak{M} .

A weight matrix \mathfrak{M} is said to have *R-moderate growth* if

$$\forall M \in \mathfrak{M} \exists N \in \mathfrak{M} \exists C \geq 1 \forall j, k \in \mathbb{N} : M_{j+k} \leq C^{j+k} N_j N_k.$$

Note that \mathfrak{M}_ω has R-moderate growth; see [19].

We remark that the prefix ‘‘R-’’ and the brackets ‘‘{.}’’ indicate that the notions are tied to the Roumieu case; they have Beurling-relatives which will not be discussed in this paper.

2.5. Function and sequence spaces. Let M be a positive sequence. For $\sigma > 0$ and $n \in \mathbb{N}$ we define the Banach space

$$\mathcal{E}_\sigma^M([-n, n]) := \left\{ f \in C^\infty([-n, n]) : \sup_{x \in [-n, n], k \in \mathbb{N}} \frac{|f^{(k)}(x)|}{\sigma^k M_k} < \infty \right\}$$

and the *Denjoy–Carleman classes of Roumieu type*

$$\mathcal{E}^{\{M\}}(\mathbb{R}) := \text{proj}_{n \in \mathbb{N}} \text{ind}_{\sigma \in \mathbb{N}} \mathcal{E}_\sigma^M([-n, n]).$$

Let \mathfrak{M} be a directed family of positive sequences. We set

$$\mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R}) := \text{proj}_{n \in \mathbb{N}} \text{ind}_{\sigma \in \mathbb{N}} \text{ind}_{M \in \mathfrak{M}} \mathcal{E}_\sigma^M([-n, n]).$$

Let ω be a normalized pre-weight function. We define

$$\mathcal{E}_\sigma^\omega([-n, n]) := \left\{ f \in C^\infty([-n, n]) : \sup_{x \in [-n, n], k \in \mathbb{N}} \frac{|f^{(k)}(x)|}{e^{\frac{1}{\sigma} \varphi_\omega^*(\sigma k)}} < \infty \right\}$$

and the *Braun–Meise–Taylor class of Roumieu type*

$$\mathcal{E}^{\{\omega\}}(\mathbb{R}) := \text{proj}_{n \in \mathbb{N}} \text{ind}_{\sigma \in \mathbb{N}} \mathcal{E}_\sigma^\omega([-n, n]).$$

The corresponding sequence spaces are defined as follows:

$$\Lambda_\sigma^M := \left\{ a \in \mathbb{C}^\mathbb{N} : \sup_{k \in \mathbb{N}} \frac{|a_k|}{\sigma^k M_k} < \infty \right\}, \quad \Lambda_\sigma^\omega := \left\{ a \in \mathbb{C}^\mathbb{N} : \sup_{k \in \mathbb{N}} \frac{|a_k|}{e^{\frac{1}{\sigma} \varphi_\omega^*(\sigma k)}} < \infty \right\},$$

$$\Lambda^{\{M\}} := \text{ind}_{\sigma \in \mathbb{N}} \Lambda_\sigma^M, \quad \Lambda^{\{\mathfrak{M}\}} := \text{ind}_{M \in \mathfrak{M}} \Lambda^{\{M\}}, \quad \Lambda^{\{\omega\}} := \text{ind}_{\sigma \in \mathbb{N}} \Lambda_\sigma^\omega.$$

By Lemma 2.2 (and [19]), we always have $\mathcal{E}^{\{\omega\}}(\mathbb{R}) \subseteq \mathcal{E}^{\{\mathfrak{M}_\omega\}}(\mathbb{R})$ and $\Lambda^{\{\omega\}} \subseteq \Lambda^{\{\mathfrak{M}_\omega\}}$ and topological isomorphisms $\mathcal{E}^{\{\omega\}}(\mathbb{R}) = \mathcal{E}^{\{\mathfrak{M}_\omega\}}(\mathbb{R})$ and $\Lambda^{\{\omega\}} = \Lambda^{\{\mathfrak{M}_\omega\}}$ if ω is even a weight function.

The order relations \preceq and $\{\preceq\}$ on the weights reflect inclusion relations of the corresponding function and sequence spaces. In particular, for positive sequences M', M we have

$$\begin{aligned} M' \preceq M &\iff \Lambda^{\{M'\}} \subseteq \Lambda^{\{M\}}, \\ M' \preceq M &\implies \mathcal{E}^{\{M'\}}(\mathbb{R}) \subseteq \mathcal{E}^{\{M\}}(\mathbb{R}), \end{aligned}$$

and provided that M' is a weight sequence

$$M' \preceq M \iff \mathcal{E}^{\{M'\}}(\mathbb{R}) \subseteq \mathcal{E}^{\{M\}}(\mathbb{R}).$$

So equivalent weights determine the same function and sequence space. For details we refer to [19] and [18].

3. ON CARLESON'S SOLUTION OF THE BOREL PROBLEM

3.1. A moment problem. Let ω be a non-quasianalytic pre-weight function which we extend to \mathbb{R} by setting $\omega(t) := \omega(|t|)$. (It is easy to check that $W := \exp \circ \omega$ satisfies the assumptions in [6].)

The harmonic extension

$$P_\omega(x + iy) := \begin{cases} \frac{|y|}{\pi} \int_{-\infty}^{\infty} \frac{\omega(t)}{(x-t)^2 + y^2} dt & \text{if } y \neq 0 \\ \omega(x) & \text{if } y = 0 \end{cases}$$

is continuous on \mathbb{C} and harmonic in the upper and lower halfplane. By [6, (2.3)] (see also [3, Lemma 3.3]),

$$\begin{aligned} \max_{x^2 + y^2 \leq r^2} P_\omega(x + iy) &\leq P_\omega(ir) = \frac{r}{\pi} \int_{-\infty}^{\infty} \frac{\omega(t)}{t^2 + r^2} dt = \frac{2}{\pi} \int_0^{\infty} \frac{\omega(rs)}{1 + s^2} ds \\ &\leq \frac{2}{\pi} \left(\omega(r) + \int_1^{\infty} \frac{\omega(rs)}{s^2} ds \right) = \frac{2}{\pi} (\omega(r) + \kappa_\omega(r)) \leq \frac{4}{\pi} \kappa_\omega(r). \end{aligned}$$

On the other hand,

$$\kappa_\omega(r) = \int_1^{\infty} \frac{\omega(rs)}{s^2} ds \leq 2 \int_1^{\infty} \frac{\omega(rs)}{1 + s^2} ds \leq 2 \int_0^{\infty} \frac{\omega(rs)}{1 + s^2} ds = \pi P_\omega(ir).$$

We see that

$$(3.1) \quad P_\omega(ir) \leq \frac{4}{\pi} \kappa_\omega(r) \leq 4P_\omega(ir), \quad r > 0.$$

We define

$$(3.2) \quad Q_k := \sup_{r > 0} \frac{r^{k + \frac{1}{2}}}{e^{\frac{1}{2} P_\omega(ir)}}, \quad k \in \mathbb{N}.$$

Let \mathcal{F}_ω be the space of measurable functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\|f\|_\omega^2 := \int_{-\infty}^{\infty} |f(t)|^2 e^{-\omega(t)} dt < \infty.$$

Fix a positive real sequence $\lambda = (\lambda_k)_{k \in \mathbb{N}}$ and let \mathcal{S}_λ be the space of sequences $s = (s_k)_{k \in \mathbb{N}}$ such that

$$\|s\|_\lambda^2 := \sum_{k=0}^{\infty} \frac{|s_k|^2}{\lambda_k^2} < \infty.$$

Theorem 3.1 ([6, Theorem 1]). *For each $s \in \mathcal{S}_\lambda$ there exists $f \in \mathcal{F}_\omega$ with*

$$L_k(f) := \int_{-\infty}^{\infty} f(t)t^k e^{-\omega(t)} dt = s_k, \quad k \in \mathbb{N},$$

provided that

$$\sum_{k=0}^{\infty} \left(\frac{\lambda_k}{Q_k} \right)^2 < \infty.$$

3.2. A solution of the Borel problem. Let M be a non-quasianalytic weight sequence and ω_M the associated function. Then

$$\tilde{\omega}_M(t) := \omega_M(t) + \log(1 + t^2)$$

is a non-quasianalytic pre-weight function that is equivalent to ω_M .

Let $\mathfrak{M} = \{M^{(\alpha)} : \alpha > 0\}$ be a non-quasianalytic weight matrix. For $\alpha > 0$ and $j \in \mathbb{N}$ set

$$\kappa_\alpha := \kappa_{\tilde{\omega}_{M^{(\alpha)}}} \quad \text{and} \quad K_j^{(\alpha)} := \exp(\varphi_{\kappa_\alpha}^*(j)).$$

Then $K^{(\alpha)} = (K_j^{(\alpha)})$ is a weight sequence by the properties of the Young conjugate and $\mathfrak{K} = \mathfrak{K}(\mathfrak{M}) := \{K^{(\alpha)} : \alpha > 0\}$ is a weight matrix. (Strictly speaking, we take a normalized representative in the equivalence class of κ_α in order to have $1 = K_0^{(\alpha)} \leq K_1^{(\alpha)}$.) On the other hand we have the collection $\mathfrak{Q} = \mathfrak{Q}(\mathfrak{M}) := \{Q^{(\alpha)} : \alpha > 0\}$, where $Q^{(\alpha)}$ is the sequence defined in (3.2) with P_ω replaced by

$$P_\alpha := P_{\tilde{\omega}_{M^{(\alpha)}}}.$$

Lemma 3.2. *Let \mathfrak{M} be a non-quasianalytic weight matrix. Then $\mathfrak{K} = \mathfrak{K}(\mathfrak{M})$ is a weight matrix such that $(K_j^{(\alpha)}/j!)^{1/j} \rightarrow \infty$ and $K_j^{(\alpha)}/M_j^{(\alpha)}$ is bounded for all $\alpha > 0$.*

Proof. That \mathfrak{K} is a weight matrix is an easy consequence of the definitions. Now $\omega_{K^{(\alpha)}}$ is equivalent to κ_α , by [19, Lemma 5.7], and $\kappa_\alpha(t) = o(t)$ as $t \rightarrow \infty$ which shows $(K_j^{(\alpha)}/j!)^{1/j} \rightarrow \infty$, by Lemma 2.1. Finally, $\omega_{M^{(\alpha)}} \leq \tilde{\omega}_{M^{(\alpha)}} \leq \kappa_\alpha$ implies $K^{(\alpha)} \leq M^{(\alpha)}$ in view of (2.3). \square

Lemma 3.3. *Let \mathfrak{M} be a non-quasianalytic weight matrix of R-moderate growth. Then $\mathfrak{K} = \mathfrak{K}(\mathfrak{M})$ has R-moderate growth which is equivalent to*

$$(3.3) \quad \forall \alpha > 0 \exists \beta > 0 \exists H \geq 1 \forall t \geq 0 : 2\kappa_\beta(t) \leq \kappa_\alpha(Ht) + H.$$

Proof. That \mathfrak{M} has R-moderate growth is, by [25, Proposition 3.6], equivalent to

$$(3.4) \quad \forall \alpha > 0 \exists \beta > 0 \exists H \geq 1 \forall t \geq 0 : 2\omega_{M^{(\beta)}}(t) \leq \omega_{M^{(\alpha)}}(Ht) + H.$$

Since $\tilde{\omega}_{M^{(\alpha)}}$ is equivalent to $\omega_{M^{(\alpha)}}$, this implies by iteration that

$$\forall \alpha > 0 \exists \beta > 0 \exists H \geq 1 \forall t \geq 0 : 2\tilde{\omega}_{M^{(\beta)}}(t) \leq \tilde{\omega}_{M^{(\alpha)}}(Ht) + H.$$

Thus (3.3) holds:

$$2\kappa_\beta(r) = 2 \int_1^\infty \frac{\tilde{\omega}_{M^{(\beta)}}(rt)}{t^2} dt \leq \int_1^\infty \frac{\tilde{\omega}_{M^{(\alpha)}}(Hrt)}{t^2} dt + H \int_1^\infty \frac{1}{t^2} dt = \kappa_\alpha(Hr) + H.$$

To see that (3.3) is equivalent to \mathfrak{K} having R-moderate growth observe that, for each $\alpha > 0$, κ_α is equivalent to $\omega_{K^{(\alpha)}}$, by [19, Lemma 5.7], and use the remark at the beginning of the proof. \square

Proposition 3.4. *Let \mathfrak{M} be a non-quasianalytic weight matrix of R -moderate growth. Then $\mathfrak{K} = \mathfrak{K}(\mathfrak{M})$ and $\mathfrak{Q} = \mathfrak{Q}(\mathfrak{M})$ are R -equivalent.*

Proof. Let $\alpha > 0$ be fixed. By (3.1) and (3.3), there are $\beta > 0$ and $H \geq 1$ such that

$$\frac{1}{2}P_\beta(it) \leq 2\kappa_\beta(t) \leq \kappa_\alpha(Ht) + H, \quad t > 0.$$

Then, using without loss of generality $\kappa_\alpha|_{[0,1]} = 0$,

$$\begin{aligned} Q_n^{(\beta)} &= \sup_{r>0} \frac{r^{n+\frac{1}{2}}}{e^{\frac{1}{2}P_\beta(ir)}} \geq \frac{1}{e^H} \sup_{r>0} \frac{r^{n+\frac{1}{2}}}{e^{\kappa_\alpha(Hr)}} = \frac{1}{H^{n+\frac{1}{2}}e^H} \sup_{s>0} \frac{s^{n+\frac{1}{2}}}{e^{\kappa_\alpha(s)}} \\ &= \frac{1}{H^{n+\frac{1}{2}}e^H} \sup_{s \geq 1} \frac{s^{n+\frac{1}{2}}}{e^{\kappa_\alpha(s)}} \geq \frac{1}{H^{n+\frac{1}{2}}e^H} \sup_{s \geq 1} \frac{s^n}{e^{\kappa_\alpha(s)}} = \frac{1}{H^{n+\frac{1}{2}}e^H} K_n^{(\alpha)}, \end{aligned}$$

that is $K^{(\alpha)} \preceq Q^{(\beta)}$. Again by (3.1) and (3.3), there are $\beta > 0$ and $H \geq 1$ such that

$$\kappa_\beta(t) \leq \frac{1}{8}\kappa_\alpha(Ht) + \frac{H}{8} \leq \frac{1}{2}P_\alpha(iHt) + \frac{H}{8}, \quad t > 0.$$

Thus, using $\kappa_\beta|_{[0,1]} = 0$,

$$\begin{aligned} Q_n^{(\alpha)} &= H^{n+\frac{1}{2}} \sup_{r>0} \frac{r^{n+\frac{1}{2}}}{e^{\frac{1}{2}P_\alpha(iHr)}} \leq H^{n+\frac{1}{2}} e^{\frac{H}{8}} \sup_{r>0} \frac{r^{n+\frac{1}{2}}}{e^{\kappa_\beta(r)}} = H^{n+\frac{1}{2}} e^{\frac{H}{8}} \sup_{r \geq 1} \frac{r^{n+\frac{1}{2}}}{e^{\kappa_\beta(r)}} \\ &\leq H^{n+\frac{1}{2}} e^{\frac{H}{8}} \sup_{r \geq 1} \frac{r^{n+1}}{e^{\kappa_\beta(r)}} = H^{n+\frac{1}{2}} e^{\frac{H}{8}} K_{n+1}^{(\beta)}. \end{aligned}$$

Since \mathfrak{K} has R -moderate growth we are done. \square

The following theorem is a simple generalization of [6, Theorem 2].

Theorem 3.5. *Let \mathfrak{M} be a non-quasianalytic weight matrix of R -moderate growth and consider $\mathfrak{K} = \mathfrak{K}(\mathfrak{M})$ and $\mathfrak{Q} = \mathfrak{Q}(\mathfrak{M})$. Then $\Lambda^{\{\mathfrak{K}\}} = \Lambda^{\{\mathfrak{Q}\}} \subseteq j^\infty \mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R})$.*

Proof. The identity $\Lambda^{\{\mathfrak{K}\}} = \Lambda^{\{\mathfrak{Q}\}}$ is a consequence of Proposition 3.4. Let $a \in \Lambda^{\{\mathfrak{Q}\}}$, i.e., there exist $\alpha, C, \rho > 0$ such that $|a_n| \leq C\rho^n Q_n^{(\alpha)}$ for all n . Set $s_n := (3\rho)^{-n} a_n$ and $\lambda_n := 2^{-n} Q_n^{(\alpha)}$. By Theorem 3.1, there exists $f \in \mathcal{F}_{\tilde{\omega}_{M(\alpha)}}$ such that $L_n(f) = s_n$ for all n . The function

$$g(t) := \int_{-\infty}^{\infty} e^{3\rho ixt} f(x) e^{-\tilde{\omega}_{M(\alpha)}(x)} dx$$

fulfills $j^\infty g = a$ and belongs to $\mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R})$. Indeed,

$$\begin{aligned} |g^{(n)}(t)| &\leq (3\rho)^n \int_{-\infty}^{\infty} |x|^n |f(x)| e^{-\tilde{\omega}_{M(\alpha)}(x)} dx \\ &\leq (3\rho)^n \|f\|_{\tilde{\omega}_{M(\alpha)}} \left(\int_{-\infty}^{\infty} x^{2n} e^{-\tilde{\omega}_{M(\alpha)}(x)} dx \right)^{1/2} \end{aligned}$$

and

$$x^{2n} e^{-\tilde{\omega}_{M(\alpha)}(x)} = \frac{x^{2n}}{\exp(\omega_{M(\alpha)}(x))} \frac{1}{1+x^2} \leq \frac{1}{1+x^2} \sup_{x \geq 0} \frac{x^{2n}}{\exp(\omega_{M(\alpha)}(x))} = \frac{M_{2n}^{(\alpha)}}{1+x^2}.$$

Since \mathfrak{M} has R-moderate growth, there exist $\beta > 0$ and $H \geq 1$ such that $M_{2n}^{(\alpha)} \leq H^n (M_n^{(\beta)})^2$ for all n . Thus

$$|g^{(n)}(t)| \leq (3\rho\sqrt{H})^n \|f\|_{\tilde{\omega}_{M^{(\alpha)}}} M_n^{(\beta)} \left(\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx \right)^{1/2},$$

that is $g \in \mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R})$. \square

4. OTHER DERIVED WEIGHTS RELATED TO THE BOREL PROBLEM

4.1. Optimal solution of the Borel problem. Let M be a non-quasianalytic weight sequence and M' a positive sequence. The condition that gives the optimal solution in the Borel problem is

$$(4.1) \quad \exists s \in \mathbb{N}_{\geq 1} : \sup_{j \in \mathbb{N}_{\geq 1}} \frac{\sup_{0 \leq i < j} \left(\frac{M'_j}{s^j M_i} \right)^{1/(j-i)}}{j} \sum_{k \geq j} \frac{1}{\mu_k} < \infty$$

which we abbreviate by $M' \prec_{SV} M$:

Theorem 4.1 ([26], [10, Theorem 3.2], [24, Theorem 2.2]). *Let M be a non-quasianalytic weight sequence and M' a positive sequence such that $\liminf_{k \rightarrow \infty} (M'_k/k!)^{1/k} > 0$. Then $\Lambda^{\{M'\}} \subseteq j^\infty \mathcal{E}^{\{M\}}(\mathbb{R})$ if and only if $M' \prec_{SV} M$.*

In the listed references also the assumption $M' \preceq M$ (or even a stronger assumption) is made. But note that, by [24, Lemma 3.2],

$$(4.2) \quad M' \prec_{SV} M \implies M' \preceq M,$$

and clearly also $\Lambda^{\{M'\}} \subseteq j^\infty \mathcal{E}^{\{M\}}(\mathbb{R})$ implies $M' \preceq M$ so that Theorem 4.1 holds as stated without the additional assumption $M' \preceq M$.

We remark that \prec_{SV} induces a relation on the set of equivalence classes of weight sequences M such that $\liminf_{k \rightarrow \infty} (M_k/k!)^{1/k} > 0$ which is antisymmetric and transitive (indeed, $M_1 \prec_{SV} M_0$ and $M_2 \prec_{SV} M_1$ imply $\Lambda^{\{M_2\}} \subseteq j^\infty \mathcal{E}^{\{M_1\}}(\mathbb{R}) \subseteq j^\infty \mathcal{E}^{\{M_0\}}(\mathbb{R})$ by (4.2) and so $M_2 \prec_{SV} M_0$ by Theorem 4.1).

Theorem 4.2. *Let \mathfrak{M} be a non-quasianalytic weight matrix and \mathfrak{M}' a one-parameter family of positive sequences such that $\liminf_{k \rightarrow \infty} (M'_k/k!)^{1/k} > 0$ for all $M' \in \mathfrak{M}'$. Then the following conditions are equivalent:*

- (1) $\Lambda^{\{\mathfrak{M}'\}} \subseteq j^\infty \mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R})$.
- (2) $\forall M' \in \mathfrak{M}' \exists M \in \mathfrak{M} : M' \prec_{SV} M$.

Proof. Theorem 4.1 yields that (2) is equivalent to

$$(3) \quad \forall M' \in \mathfrak{M}' \exists M \in \mathfrak{M} : \Lambda^{\{M'\}} \subseteq j^\infty \mathcal{E}^{\{M\}}(\mathbb{R}).$$

That (3) implies (1) is clear. To see that (1) implies (3) let $M' \in \mathfrak{M}'$ and note that we may assume that $\mathfrak{M} = \{M^{(n)} : n \in \mathbb{N}_{\geq 1}\}$. By (1), $\Lambda_1^{M'} \subseteq j^\infty \mathcal{D}^{\{\mathfrak{M}\}}([-1, 1])$ (by multiplication with a suitable cutoff function), where

$$\mathcal{D}^{\{\mathfrak{M}\}}([-1, 1]) := \text{ind}_{n \in \mathbb{N}_{\geq 1}} \mathcal{D}_n^{M^{(n)}}([-1, 1])$$

and $\mathcal{D}_n^{M^{(n)}}([-1, 1]) := \{f \in \mathcal{E}_n^{M^{(n)}}([-1, 1]) : \text{supp}(f) \subseteq [-1, 1]\}$. By Grothendieck's factorization theorem [15, 24.33], $\Lambda_1^{M'} \subseteq j^\infty \mathcal{D}_n^{M^{(n)}}([-1, 1])$ for some n . This inclusion implies $\Lambda^{\{M'\}} \subseteq j^\infty \mathcal{E}^{\{M^{(n)}\}}(\mathbb{R})$ and hence (3) is proved. Indeed, if $a = (a_k) \in \Lambda^{\{M'\}}$ then $a \in \Lambda_\rho^{M'}$ for some $\rho > 0$ and hence $b := (\rho^{-k} a_k) \in \Lambda_1^{M'}$.

There exists $g \in \mathcal{D}_n^{M^{(n)}}([-1, 1])$ with $b = j^\infty g$ so that $f(x) := g(\rho x) \in \mathcal{E}^{\{M^{(n)}\}}(\mathbb{R})$ satisfies $a = j^\infty f$. \square

Corollary 4.3. *Let \mathfrak{M} be a non-quasianalytic weight matrix of R -moderate growth and consider $\mathfrak{K} = \mathfrak{K}(\mathfrak{M})$. Then $\forall \alpha > 0 \exists \beta > 0 : K^{(\alpha)} \prec_{SV} M^{(\beta)}$.*

Proof. This follows from Lemma 3.2, Theorem 3.5, and Theorem 4.2. \square

4.2. The derived sequence L . Let M be a non-quasianalytic weight sequence. We define the sequence $L = L(M)$ by setting

$$(4.3) \quad L_k := \min_{0 \leq j < k} \left(\left(\frac{k}{\sum_{\ell \geq k} \mu_\ell^{-1}} \right)^{k-j} M_j \right), \quad k \geq 1, \quad L_0 := 1.$$

The importance of L relies on its *optimality* with respect to $\cdot \prec_{SV} M$ (cf. [24, Theorem 3.3]): We have $L \prec_{SV} M$ and if M' is a positive sequence with $\liminf_{k \rightarrow \infty} (M'_k/k!)^{1/k} > 0$ and $M' \prec_{SV} M$ then $M' \preceq L \preceq M$. If M' is log-convex we also have $M' \preceq \underline{L} \preceq M$, where \underline{L} is the log-convex minorant of L . We remark that $(L_k/k!)^{1/k} \rightarrow \infty$; see [24, Lemma 3.2].

Lemma 4.4. *If $M \preceq N$ are two non-quasianalytic weight sequences, then $L(M) \preceq L(N)$ and $\underline{L}(M) \preceq \underline{L}(N)$.*

Proof. By Theorem 4.1, $L(M) \prec_{SV} M$ implies $L(M) \prec_{SV} N$ and so $L(M) \preceq L(N)$ by the optimality. Passing to the log-convex minorant preserves the order relation, see [19, Lemma 2.6]. \square

4.3. The derived sequence S . Let M be a non-quasianalytic weight sequence and M' a positive sequence. We consider the relation $M' \prec_{\gamma_1} M$ defined by

$$(4.4) \quad \sup_{j \in \mathbb{N}_{\geq 1}} \frac{\mu'_j}{j} \sum_{k \geq j} \frac{1}{\mu_k} < \infty.$$

Let us recall a construction (see [20, Section 4.1], [11, Remark 9], and also [16]) which yields a weight sequence that is *optimal* with respect to $\cdot \prec_{\gamma_1} M$ in the following sense. The strongly log-convex weight sequence $S = S(M)$ defined by $S_k = \sigma_0 \sigma_1 \cdots \sigma_k$, where $\sigma_0 := 1$ and

$$\sigma_k := \tau_1 \frac{k}{\tau_k}, \quad \tau_k := \frac{k}{\mu_k} + \sum_{\ell \geq k} \frac{1}{\mu_\ell}, \quad k \geq 1,$$

satisfies $\sigma \lesssim \mu$, $S \prec_{\gamma_1} M$, and if M' is another weight sequence satisfying $\mu' \lesssim \mu$ and $M' \prec_{\gamma_1} M$ then $\mu' \lesssim \sigma$; see [20, Lemma 4.2]. Note that $\sigma \lesssim \mu$ means that σ/μ is bounded and implies $S \preceq M$.

We have $S(M) \preceq L(M)$, since $M' \prec_{\gamma_1} M$ implies $M' \prec_{SV} M$; cf. [10, Lemma 2.4] and [26].

Suppose that also M' is a weight sequence and $\mu' \leq \mu$. Then $M' \prec_{\gamma_1} M$ implies $\omega_{M'} \prec_{st} \omega_M$, see [22, Lemma 5.7]. If M' additionally has moderate growth, then

$$M' \prec_{SV} M \iff M' \prec_{\gamma_1} M \iff \omega_{M'} \prec_{st} \omega_M;$$

cf. [10, Lemma 5.8] and [24, Remark 2.1]. So \prec_{γ_1} induces a relation on equivalence classes of positive sequences which is always antisymmetric and becomes transitive if we restrict to weight sequences of moderate growth.

In light of this it is important to know under which circumstances the derived sequence S has moderate growth. This is the case if M has moderate growth and also under the weaker condition

$$(4.5) \quad \liminf_{j \rightarrow \infty} \frac{\mu_j}{j} \sum_{k \geq 2j} \frac{1}{\mu_k} > 0;$$

see [11, Lemma 6]. In that case S and L are equivalent, see [24, Theorem 3.11].

4.4. Relations among the derived sequences. Let $\mathfrak{M} = \{M^{(\alpha)} : \alpha > 0\}$ be a non-quasianalytic weight matrix and $\mathfrak{K} = \{K^{(\alpha)} : \alpha > 0\}$ the derived weight matrix from Section 3.2. For each $M^{(\alpha)}$ we consider the derived sequences $S^{(\alpha)}$, $L^{(\alpha)}$, and $\underline{L}^{(\alpha)}$ (the log-convex minorant of $L^{(\alpha)}$) and the families

$$\mathfrak{S} = \{S^{(\alpha)} : \alpha > 0\}, \quad \mathfrak{L} = \{L^{(\alpha)} : \alpha > 0\}, \quad \underline{\mathfrak{L}} = \{\underline{L}^{(\alpha)} : \alpha > 0\}.$$

Formally, these collections are not weight matrices as defined above, but their deficiencies are minor and carry no weight. We have $S^{(\alpha)} \preceq \underline{L}^{(\alpha)} \leq L^{(\alpha)}$ for each $\alpha > 0$.

Theorem 4.5. *Let \mathfrak{M} be a non-quasianalytic weight matrix of R-moderate growth. Then the derived families satisfy $\mathfrak{S}\{\preceq\}\mathfrak{K}\{\preceq\}\underline{\mathfrak{L}}\{\preceq\}\mathfrak{L}$.*

Proof. Let us first show $\mathfrak{K}\{\preceq\}\underline{\mathfrak{L}}$. By Corollary 4.3, for each $\alpha > 0$ there is $\beta > 0$ such that $K^{(\alpha)} \prec_{SV} M^{(\beta)}$ and thus $K^{(\alpha)} \preceq L^{(\beta)}$ by optimality. Since $K^{(\alpha)}$ is log-convex, we also have $K^{(\alpha)} \preceq \underline{L}^{(\beta)}$.

For $\mathfrak{S}\{\preceq\}\mathfrak{K}$ observe that for each $\alpha > 0$ we may assume that $\sigma^{(\alpha)} \leq \mu^{(\alpha)}$ by dividing $\sigma^{(\alpha)}$ by a suitable constant. Then $\omega_{S^{(\alpha)}} \prec_{st} \omega_{M^{(\alpha)}}$ holds (cf. Section 4.3), i.e., $\omega_{S^{(\alpha)}} \preceq \kappa_\alpha$, since κ_α and $\kappa_{\omega_{M^{(\alpha)}}}$ are equivalent. By [19, Lemma 5.7], $\omega_{K^{(\alpha)}}$ is equivalent to κ_α so that $\omega_{K^{(\alpha)}} \leq C\omega_{S^{(\alpha)}} + C$ for some positive integer C . Then

$$\begin{aligned} S_n^{(\alpha)} &= \sup_{t \geq 0} \frac{t^n}{\exp(\omega_{S^{(\alpha)}}(t))} \leq e \sup_{t \geq 0} \frac{t^n}{\exp(C^{-1}\omega_{K^{(\alpha)}}(t))} \\ &= e \left(\sup_{t \geq 0} \frac{t^{Cn}}{\exp(\omega_{K^{(\alpha)}}(t))} \right)^{1/C} = e(K_{Cn}^{(\alpha)})^{1/C}. \end{aligned}$$

There exists $\beta > 0$ such that $S^{(\alpha)} \preceq K^{(\beta)}$, since \mathfrak{K} has R-moderate growth, by Lemma 3.3. Since $\underline{\mathfrak{L}}\{\preceq\}\mathfrak{L}$ is obvious, we are done. \square

We may now complete the proof of (1.2): Let \mathfrak{M} be a non-quasianalytic weight matrix of R-moderate growth. Then Proposition 3.4, Theorem 4.1, Theorem 4.2, and Theorem 4.5 yield the sequence of inclusions in the first line of (1.2).

Suppose that \mathfrak{M}' is a one-parameter family of positive sequences such that $\liminf_{k \rightarrow \infty} (M'_k/k!)^{1/k} > 0$ for all $M' \in \mathfrak{M}'$ and $\Lambda^{\{\mathfrak{M}'\}} \subseteq j^\infty \mathcal{E}^{\{\mathfrak{M}'\}}(\mathbb{R})$. Then for each $M' \in \mathfrak{M}'$ there is $M \in \mathfrak{M}$ such that $M' \prec_{SV} M$, by Theorem 4.2, and hence $M' \preceq L(M)$, by optimality. That means $\mathfrak{M}'\{\preceq\}\mathfrak{L}$ and the second line in (1.2) is proved.

4.5. Sufficient conditions for R-equivalence of the derived families \mathfrak{S} , \mathfrak{K} and \mathfrak{L} .

Theorem 4.6. *Let $\mathfrak{M} = \{M^{(\alpha)} : \alpha > 0\}$ be a non-quasianalytic weight matrix of R -moderate growth such that $\mu^{(\alpha)} \leq \mu^{(\beta)}$ if $\alpha \leq \beta$ and*

$$(4.6) \quad \forall \alpha > 0 \exists \beta > 0 : \liminf_{k \rightarrow \infty} \frac{\mu_k^{(\beta)}}{k} \sum_{j \geq k} \frac{1}{\mu_j^{(\alpha)}} > 0.$$

Assume that the derived family \mathfrak{S} has the property

$$(4.7) \quad \forall \alpha > 0 \exists \beta > 0 \exists A \geq 1 \forall j \in \mathbb{N}_{\geq 1} : \sigma_j^{(\alpha)} \leq A(S_j^{(\beta)})^{1/j}.$$

Then \mathfrak{S} , \mathfrak{R} , $\underline{\mathfrak{L}}$, and \mathfrak{L} are R -equivalent.

Proof. By Theorem 4.5, it suffices to show $\mathfrak{L}\{\preceq\}\mathfrak{S}$. Let $\alpha > 0$ be fixed. By (4.6), there exist $\beta > 0$ and $C \geq 1$ such that

$$\frac{k}{\mu_k^{(\beta)}} \leq C \sum_{j \geq k} \frac{1}{\mu_j^{(\alpha)}}.$$

We may assume that $\alpha \leq \beta$ and hence $\mu^{(\alpha)} \leq \mu^{(\beta)}$. Thus we get

$$\tau_k^{(\beta)} = \frac{k}{\mu_k^{(\beta)}} + \sum_{j \geq k} \frac{1}{\mu_j^{(\beta)}} \leq (C+1) \sum_{\ell \geq k} \frac{1}{\mu_\ell^{(\alpha)}}$$

and consequently,

$$\frac{k}{\sigma_k^{(\beta)}} \leq D \sum_{\ell \geq k} \frac{1}{\mu_\ell^{(\alpha)}}.$$

Then, by (4.7), there exist $\gamma > 0$ and $A \geq 1$ such that

$$(L_k^{(\alpha)})^{1/k} \leq \frac{k}{\sum_{\ell \geq k} (\mu_\ell^{(\alpha)})^{-1}} \leq D \sigma_k^{(\beta)} \leq AD(S_k^{(\gamma)})^{1/k}$$

and we are done. \square

The conclusion of Theorem 4.6 (invoking Theorem 3.5) means that

$$\Lambda\{\mathfrak{S}\} = \Lambda\{\mathfrak{R}\} = \Lambda\{\Omega\} = \Lambda\{\underline{\mathfrak{L}}\} = \Lambda\{\mathfrak{L}\}.$$

So under the assumption of the theorem all presented solutions to the mixed Borel problem coincide with the optimal one; cf. Section 4.2. We shall see in Theorem 5.5 that for non-quasianalytic Braun–Meise–Taylor classes similarly optimality is achieved by the different approaches leading to $\Lambda\{\kappa\}$, $\Lambda\{\mathfrak{R}\} = \Lambda\{\Omega\}$, and $\Lambda\{\underline{\mathfrak{L}}\}$.

Remark 4.7. Let us discuss what R -equivalence of \mathfrak{S} , \mathfrak{R} , $\underline{\mathfrak{L}}$, and \mathfrak{L} implies for \mathfrak{M} . In view of Theorem 4.5, this means that for each $\alpha > 0$ there exists $\beta > 0$ such that $(L_k^{(\alpha)})^{1/k} \lesssim (S_k^{(\beta)})^{1/k}$. Since $(S_k^{(\beta)})^{1/k} \leq \sigma_k^{(\beta)} = \tau_1^{(\beta)} \frac{k}{\tau_k^{(\beta)}}$, we may infer

$$\left(\left(\frac{k}{\sum_{\ell \geq k} (\mu_\ell^{(\alpha)})^{-1}} \right)^{k-j_k} M_{j_k}^{(\alpha)} \right)^{1/k} \lesssim \frac{k}{\tau_k^{(\beta)}},$$

where j_k is an integer $0 \leq j < k$ where the minimum in the definition of $L_k^{(\alpha)}$ is attained. A simple conversion of terms gives

$$\left(\frac{1}{k} \sum_{\ell \geq k} \frac{1}{\mu_\ell^{(\alpha)}} \right)^{j_k/k} (M_{j_k}^{(\alpha)})^{1/k} \lesssim \frac{1}{\tau_k^{(\beta)}} \sum_{\ell \geq k} \frac{1}{\mu_\ell^{(\alpha)}}$$

and so, by the definition of $\tau_k^{(\beta)}$,

$$\left(\frac{1}{k} \sum_{\ell \geq k} \frac{1}{\mu_\ell^{(\alpha)}}\right)^{j_{k/k}} (M_{j_k}^{(\alpha)})^{1/k} \lesssim \frac{\mu_k^{(\beta)}}{k} \sum_{\ell \geq k} \frac{1}{\mu_\ell^{(\alpha)}}.$$

We see that (4.6) would follow if the left-hand side is bounded away from zero.

In the following lemma we give conditions *purely for* \mathfrak{M} which imply the assumptions of Theorem 4.6. Notice however that (4.9) seems to be quite restrictive.

Lemma 4.8. *Let $\mathfrak{M} = \{M^{(\alpha)} : \alpha > 0\}$ be a non-quasianalytic weight matrix such that $\mu^{(\alpha)} \leq \mu^{(\beta)}$ if $\alpha \leq \beta$,*

$$(4.8) \quad \forall \alpha > 0 \exists \beta > 0 : \liminf_{k \rightarrow \infty} \frac{\mu_k^{(\beta)}}{k} \sum_{j \geq 2k} \frac{1}{\mu_j^{(\alpha)}} > 0,$$

and

$$(4.9) \quad \forall \alpha > 0 \exists \beta > 0 \exists A \geq 1 \forall j \in \mathbb{N} : (\mu_j^{(\alpha)})^2 \leq A \mu_{2j}^{(\beta)}.$$

Then (4.6) and (4.7) hold.

Proof. Obviously, (4.8) implies (4.6). Next we claim that (4.8) implies

$$(4.10) \quad \forall \alpha > 0 \exists \beta > 0 \exists B \geq 1 \forall j \in \mathbb{N} : \tau_j^{(\beta)} \leq B \tau_{2j}^{(\alpha)}.$$

Indeed,

$$\begin{aligned} \tau_j^{(\beta)} &= \frac{j}{\mu_j^{(\beta)}} + \sum_{k \geq 2j} \frac{1}{\mu_k^{(\beta)}} + \sum_{j \leq k < 2j} \frac{1}{\mu_k^{(\beta)}} \leq \frac{2j}{\mu_j^{(\beta)}} + \sum_{k \geq 2j} \frac{1}{\mu_k^{(\beta)}} \\ &\leq C \sum_{k \geq 2j} \frac{1}{\mu_k^{(\alpha)}} + \sum_{k \geq 2j} \frac{1}{\mu_k^{(\beta)}} \leq (C+1) \sum_{k \geq 2j} \frac{1}{\mu_k^{(\alpha)}} \leq (C+1) \tau_{2j}^{(\alpha)}. \end{aligned}$$

Consequently, since $\tau_j^{(\alpha)}$ is decreasing,

$$\begin{aligned} B^{2j} (\tau_1^{(\alpha)})^j (\tau_j^{(\alpha)})^j &\geq B^{2j} \tau_1^{(\alpha)} \dots \tau_{2j}^{(\alpha)} \\ &\geq B^{2j} (\tau_2^{(\alpha)})^2 (\tau_4^{(\alpha)})^2 \dots (\tau_{2j}^{(\alpha)})^2 \geq (\tau_1^{(\beta)} \dots \tau_j^{(\beta)})^2. \end{aligned}$$

Now (4.9) implies that there exist $\gamma > 0$ and $A \geq 1$ such that

$$\begin{aligned} (\tau_j^{(\beta)})^2 &\geq \frac{j^2}{(\mu_j^{(\beta)})^2} + \sum_{k \geq j} \frac{1}{(\mu_k^{(\beta)})^2} \geq \frac{1}{2A} \frac{2j}{\mu_{2j}^{(\gamma)}} + \frac{1}{A} \sum_{k \geq j} \frac{1}{\mu_{2k}^{(\gamma)}} \\ &\geq \frac{1}{2A} \left(\frac{2j}{\mu_{2j}^{(\gamma)}} + \sum_{k \geq 2j} \frac{1}{\mu_k^{(\gamma)}} \right) = \frac{1}{2A} \tau_{2j}^{(\gamma)}. \end{aligned}$$

In view of (4.10) there exist $\delta > 0$ and $D \geq 1$ such that $\tau_{2j}^{(\gamma)} \geq \frac{1}{D} \tau_j^{(\delta)}$. Thus

$$B^{2j} (\tau_1^{(\alpha)})^j (\tau_j^{(\alpha)})^j \geq (\tau_1^{(\beta)} \dots \tau_j^{(\beta)})^2 \geq (2AD)^{-j} \tau_1^{(\delta)} \dots \tau_j^{(\delta)}.$$

It is easy to see that $\sigma_j^{(\alpha)} \leq A(S_j^{(\delta)})^{1/j}$ is equivalent to

$$\tau_1^{(\delta)} \dots \tau_j^{(\delta)} \leq \tilde{A}^j (\tau_j^{(\alpha)})^j$$

and hence the statement is proved. \square

5. CLASSICAL CASES

5.1. Braun–Meise–Taylor classes. The goal of this section is to show that, for any non-quasianalytic weight function ω ,

$$\Lambda^{\{\kappa\}} = \Lambda^{\{\mathfrak{K}\}} = \Lambda^{\{\Omega\}} = \Lambda^{\{\mathfrak{L}\}},$$

where $\kappa = \kappa_\omega$ and the families \mathfrak{K} , Ω , and \mathfrak{L} are derived from $\mathfrak{M} = \mathfrak{M}_\omega$.

Proposition 5.1. *Let ω be a non-quasianalytic weight function, $\kappa = \kappa_\omega$, and $\mathfrak{K} = \mathfrak{K}(\mathfrak{M}_\omega)$. Then $\Lambda^{\{\kappa\}} = \Lambda^{\{\mathfrak{K}\}}$.*

Proof. Now κ and κ_α are equivalent for all $\alpha > 0$, since ω and $\omega_{M(\alpha)}$ are, by [19, Lemma 5.7]. So there is $C \in \mathbb{N}_{\geq 1}$ such that $\kappa_\alpha \leq C\kappa + C$. Let $\mathfrak{H} = \mathfrak{H}_\kappa = \{H^{(\alpha)} : \alpha > 0\}$ be the weight matrix associated with κ (cf. Lemma 2.2). Then, for $\alpha = 1$, assuming that κ and κ_1 are normalized,

$$H_j^{(1)} = \exp(\varphi_\kappa^*(j)) = \sup_{t \geq 0} \frac{t^j}{\exp(\kappa(t))} \leq e \sup_{t \geq 0} \frac{t^j}{\exp(\frac{1}{C}\kappa_1(t))} = e(K_{Cj}^{(1)})^{1/C}.$$

For $x \in \mathbb{N}_{\geq 1}$, we conclude $H_j^{(x)} = (H_{xj}^{(1)})^{1/x} \leq e^{1/x} (K_{Cxj}^{(1)})^{1/(Cx)}$ and thus $\mathfrak{H}\{\preceq\}\mathfrak{K}$, since \mathfrak{K} has R-moderate growth.

We also have $\kappa \leq C\kappa_\alpha + C$ for some $C \in \mathbb{N}_{\geq 1}$ so that an analogous computation gives $K_j^{(\alpha)} \leq e(H_{Cj}^{(1)})^{1/C}$ and hence $\mathfrak{K}\{\preceq\}\mathfrak{H}$, since \mathfrak{H} has R-moderate growth. \square

As a technical tool we will associate with a weight sequence M and a positive integer n the weight sequence $M^{[n]}$ defined by $M_j^{[n]} := M_{nj}^{1/n}$. Note that

$$\mu_j^{[n]} := \frac{M_j^{[n]}}{M_{j-1}^{[n]}} = (\mu_{n(j-1)+1} \cdots \mu_{nj})^{1/n}$$

satisfies $\mu_{2j} \leq A\mu_j^{[4]}$ for some constant $A \geq 1$, which follows easily from the fact that μ is increasing. We always have $M \leq M^{[n]}$, and $M^{[n]} \preceq M$ holds provided that M has moderate growth. If M is non-quasianalytic, so is $M^{[n]}$.

Lemma 5.2. *Let M be a non-quasianalytic weight sequence and M' a positive sequence. Then $M' \prec_{SV} M$ implies $M'^{[n]} \prec_{SV} M^{[4n]}$ for all positive integers n .*

Proof. For all $0 \leq i < j$,

$$\left(\frac{M_j^{[n]}}{s^j M_i^{[4n]}} \right)^{\frac{1}{j-i}} = \left(\frac{M'_{nj}}{s^{nj} M_{ni}^{[4]}} \right)^{\frac{1}{n(j-i)}} \leq \left(\frac{M'_{nj}}{s^{nj} M_{ni}} \right)^{\frac{1}{n(j-i)}} \leq \sup_{0 \leq i < nj} \left(\frac{M'_{nj}}{s^{nj} M_i} \right)^{\frac{1}{nj-i}}.$$

Moreover,

$$\mu_k^{[4n]} = (\mu_{n(k-1)+1}^{[4]} \cdots \mu_{nk}^{[4]})^{1/n} \geq \mu_{n(k-1)}^{[4]} \geq A^{-1} \mu_{2n(k-1)}$$

so that, for $j \geq 2$,

$$\sum_{k \geq j} \frac{1}{\mu_k^{[4n]}} \leq A \sum_{k \geq nj} \frac{1}{\mu_k}.$$

The assertion follows. \square

Proposition 5.3. *Let \mathfrak{M} be a non-quasianalytic weight matrix of R -moderate growth and \mathfrak{M}' a one-parameter family of positive sequences such that $\liminf_{k \rightarrow \infty} (M'_k/k!)^{1/k} > 0$ for all $M' \in \mathfrak{M}'$. Assume that*

$$(5.1) \quad \exists \widehat{M}' \in \mathfrak{M}' \forall M' \in \mathfrak{M}' \exists n \in \mathbb{N}_{\geq 1} : M' \preceq (\widehat{M}')^{[n]}.$$

If there is $\widehat{M} \in \mathfrak{M}$ such that $\widehat{M}' \prec_{SV} \widehat{M}$, then $\Lambda^{\{\mathfrak{M}'\}} \subseteq j^\infty \mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R})$.

Proof. Let $a \in \Lambda^{\{\mathfrak{M}'\}}$. Then $a \in \Lambda^{\{M'\}}$ for some $M' \in \mathfrak{M}'$. By (5.1), we find $n \geq 1$ such that $M' \preceq (\widehat{M}')^{[n]}$. Consequently, $a \in \Lambda^{\{(\widehat{M}')^{[n]}\}}$. By Lemma 5.2, we have $(\widehat{M}')^{[n]} \prec_{SV} \widehat{M}^{[4n]}$. We may conclude that $a \in j^\infty \mathcal{E}^{\{\widehat{M}^{[4n]}\}}(\mathbb{R})$; cf. Theorem 4.1. Since \mathfrak{M} has R -moderate growth, $\mathcal{E}^{\{\widehat{M}^{[4n]}\}}(\mathbb{R}) \subseteq \mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R})$. \square

Lemma 5.4. *If ω is a pre-weight function, then $\mathfrak{M} = \mathfrak{M}_\omega$ satisfies (5.1).*

Proof. We have $M^{(n)} = (M^{(1)})^{[n]}$ for all positive integers n ; cf. (2.4). \square

Theorem 5.5. *Let ω be a non-quasianalytic weight function and $\mathfrak{M} = \mathfrak{M}_\omega$. Then the derived families \mathfrak{K} and \mathfrak{L} are R -equivalent.*

Proof. By Theorem 4.5, it remains to show $\mathfrak{L}\{\preceq\}\mathfrak{K}$. Fix $\underline{L} \in \mathfrak{L}$ and $M \in \mathfrak{M}$ such that $L = L(M)$. We have $\underline{L} \preceq M$ and thus there is $H \geq 1$ such that $\omega_M(t) \leq \omega_{\underline{L}}(Ht)$ for all $t \geq 0$. Now ω_M , being equivalent to ω by [19, Lemma 5.7], is a weight function so that, for some $A \geq 1$,

$$\omega_M(t) \leq A\omega_M(tH^{-1}) + A \leq A\omega_{\underline{L}}(t) + A.$$

That means $\omega_{\underline{L}} \preceq \omega$ and thus $\mathfrak{B}\{\preceq\}\mathfrak{M}$, where $\mathfrak{B} = \{B^{(\alpha)} : \alpha > 0\} = \mathfrak{M}_{\omega_{\underline{L}}}$ is the weight matrix associated with $\omega_{\underline{L}}$ (cf. Lemma 2.2; we do not assume that $\omega_{\underline{L}}$ is normalized). In general, $\omega_{\underline{L}}$ might just be a pre-weight function (not a weight function) so that only the inclusion $\Lambda^{\{\omega_{\underline{L}}\}} \subseteq \Lambda^{\{\mathfrak{B}\}}$ is available. Observe that $B^{(1)}$ and \underline{L} are closely related: by (2.3) and (2.4),

$$\underline{L}_k = \sup_{t \geq 0} \frac{t^k}{e^{\omega_{\underline{L}}(t)}}, \quad B_k^{(1)} = e^{\varphi_{\omega_{\underline{L}}}^*(k)} = \sup_{t \geq 1} \frac{t^k}{e^{\omega_{\underline{L}}(t)}},$$

and consequently

$$\begin{aligned} B_k^{(1)} &\leq \underline{L}_k = \max \left\{ \sup_{0 \leq t \leq 1} \frac{t^k}{e^{\omega_{\underline{L}}(t)}}, \sup_{t \geq 1} \frac{t^k}{e^{\omega_{\underline{L}}(t)}} \right\} \\ &\leq \max \left\{ 1, \sup_{t \geq 1} \frac{t^k}{e^{\omega_{\underline{L}}(t)}} \right\} = \max\{1, B_k^{(1)}\}. \end{aligned}$$

Since $B_k^{(1)} \rightarrow \infty$, we see that the sequences $B^{(1)}$ and \underline{L} are equivalent.

By Lemma 5.4, we have $B^{(n)} = (B^{(1)})^{[n]}$. Since $L \prec_{SV} M$ and thus $\underline{L} \prec_{SV} M$ and the sequences $B^{(1)}$ and \underline{L} are equivalent, we find $\Lambda^{\{\mathfrak{B}\}} \subseteq j^\infty \mathcal{E}^{\{\mathfrak{M}\}}(\mathbb{R})$, by Proposition 5.3. (We have $\omega_{\underline{L}}(t) = o(t)$ as $t \rightarrow \infty$ and consequently $(B_j^{(n)}/j!)^{1/j} \rightarrow \infty$ for all n , by Lemma 2.1.) Thus $\Lambda^{\{\omega_{\underline{L}}\}} \subseteq j^\infty \mathcal{E}^{\{\omega\}}(\mathbb{R})$ which implies $\omega_{\underline{L}} \preceq \kappa$, by Proposition A.1.

It follows that $B^{(1)} \in \Lambda_1^{B^{(1)}} = \Lambda_1^{\omega_{\underline{L}}} \subseteq \Lambda^{\{\omega_{\underline{L}}\}} \subseteq \Lambda^{\{\kappa\}} = \Lambda^{\{\mathfrak{K}\}}$, by Proposition 5.1, and consequently there exists $K \in \mathfrak{K}$ such that $\underline{L} \preceq K$, since $B^{(1)}$ and \underline{L} are equivalent. \square

Now (1.3) follows from Proposition 3.4, Proposition 5.1, and Theorem 5.5.

Note that a necessary and sufficient condition for ω being a strong weight function is that \mathfrak{R} , \mathfrak{Q} , \mathfrak{L} , and \mathfrak{L} are all R-equivalent to \mathfrak{M}_ω .

5.2. Denjoy–Carleman classes.

Theorem 5.6. *Let M be a non-quasianalytic weight sequence of moderate growth. Then the derived sequences S , K , and L are equivalent.*

Proof. We have $S \preceq K \preceq L$ by Theorem 4.5. That $L \preceq S$ was shown in [24, Theorem 3.10]. \square

Remark 5.7. The assumption that M has moderate growth can be replaced by the weaker condition

$$\liminf_{k \rightarrow \infty} \frac{\mu_k}{k} \sum_{j \geq 2k} \frac{1}{\mu_j} > 0,$$

which guarantees that S has moderate growth and $L \preceq S$; cf. [24, Theorem 3.10].

APPENDIX A.

The goal of this section is to prove the following proposition. It is due to [3] if σ is a weight function. We will show that it is valid if σ is just a pre-weight function by slightly modifying the proof of [3]. The proposition is used in this more general form in the proof of Theorem 5.5.

Proposition A.1. *Let ω be a non-quasianalytic weight function. Let σ be a pre-weight function such that $\sigma(t) = o(t)$ as $t \rightarrow \infty$. Then $\Lambda^{\{\sigma\}} \subseteq j^\infty \mathcal{E}^{\{\omega\}}(\mathbb{R})$ implies $\sigma \preceq \kappa_\omega$.*

For a pre-weight function σ we have $\Lambda^{\{\sigma\}} \subseteq \Lambda^{\{\mathfrak{M}_\sigma\}}$ (and in general not equality). But since ω is a weight function, $\Lambda^{\{\sigma\}} \subseteq j^\infty \mathcal{E}^{\{\omega\}}(\mathbb{R})$ entails

$$(A.1) \quad \Lambda^{\{\mathfrak{M}_\sigma\}} \subseteq j^\infty \mathcal{E}^{\{\omega\}}(\mathbb{R}).$$

Indeed, let $a \in \Lambda^{\{\mathfrak{M}_\sigma\}}$, i.e., $|a_k| \leq CH^k e^{\frac{1}{\alpha} \varphi_\sigma^*(\alpha k)}$ for some $C, H \geq 1$ and $\alpha > 0$. Then $b = (H^{-k} a_k)_k$ belongs to $\Lambda^{\{\sigma\}}$. If $g \in \mathcal{E}^{\{\omega\}}(\mathbb{R})$ satisfies $j^\infty g = b$, then $f(x) := g(Hx)$ fulfills $j^\infty f = a$ and belongs to $\mathcal{E}^{\{\omega\}}(\mathbb{R})$ since ω is a weight function (cf. (2.5)).

Our goal is now to show that (A.1) implies $\sigma \preceq \kappa_\omega$. The one crucial task is to identify the dual of $\Lambda^{\{\mathfrak{M}_\sigma\}}$. Let us write $\mathfrak{S} = \{S^{(\alpha)} : \alpha > 0\} := \mathfrak{M}_\sigma$ throughout this section.

Lemma A.2. *Let $\Lambda^k := \Lambda_k^{S^{(k)}} = \{a \in \mathbb{C}^{\mathbb{N}} : \sup_{j \in \mathbb{N}} \frac{|a_j|}{k^j S_j^{(k)}} < \infty\}$ for $k \in \mathbb{N}_{\geq 1}$. Then the inclusion*

$$\Lambda^k \hookrightarrow \Lambda^{k+1}$$

is compact. In particular $\Lambda^{\{\mathfrak{M}_\sigma\}} (= \text{ind}_{k \in \mathbb{N}} \Lambda^k)$ is a (DFS)-space (cf. [15, 25.20]).

Proof. Let $(a^{(n)})_n$ be a bounded sequence in Λ^k . Then $\{b_j^{(n)} := \frac{a_j^{(n)}}{k^j S_j^{(k)}} : n, j \in \mathbb{N}\}$ is bounded. Thus the sequence $(b_1^{(n)})$ is bounded and after passing to a subsequence we may assume that it is convergent. Passing to a subsequence again we may assume that $(b_2^{(n)})$ converges. Iterating this procedure and taking the diagonal sequence, we end up with a subsequence $b^{(n_j)}$ such that $b_i^{(n_j)} \rightarrow c_i \in \mathbb{C}$ for all $i \in \mathbb{N}$. To finish

the proof we show that $a^{(n_j)} \rightarrow c$ in Λ^{k+1} , where $c = (c_i)$. It is clear that for fixed $\varepsilon > 0$ there exists i_0 such that for all $i \geq i_0$ and all j , we have

$$\frac{|a_i^{(n_j)}|}{(k+1)^i S_i^{(k+1)}} = |b_i^{(n_j)}| \frac{k^i S_i^{(k)}}{(k+1)^i S_i^{(k+1)}} \leq \varepsilon \quad \text{and} \quad \frac{|c_i|}{(k+1)^i S_i^{(k+1)}} \leq \varepsilon.$$

In addition there exists j_0 such that $|a_i^{(n_j)} - c_i| \leq \varepsilon$ for $j \geq j_0$ and $i \leq i_0$. This yields that $a^{(n_j)} \rightarrow c$ in Λ^{k+1} . \square

Lemma A.3. *Let σ be a pre-weight function. Then*

$$(\Lambda^{\{\mathfrak{M}_\sigma\}})' \cong \{f \in \mathcal{H}(\mathbb{C}) : \forall n \in \mathbb{N} \exists A > 0 \forall z \in \mathbb{C} : |f(z)| \leq A e^{\omega_{S(n)}(\frac{|z|}{n})}\} =: \mathcal{A}^0.$$

The isomorphism is explicitly given by

$$T \mapsto \Phi(T) := \left(z \mapsto \sum_j T(e_j) z^j \right),$$

where e_j denotes the j -th unit vector.

Proof. As a consequence of the compactness of the connecting mappings (Lemma A.2) the collection $\{B_n\}_n$ of closed unit balls $B_n \subseteq \Lambda^n$ forms a fundamental system of bounded sets in $\Lambda^{\{\mathfrak{M}_\sigma\}}$ (cf. [15, 25.19]). So a set B is bounded in $\Lambda^{\{\mathfrak{M}_\sigma\}}$ if and only if there exist $n \in \mathbb{N}$ and $\lambda > 0$ such that $B \subseteq \lambda B_n$. Therefore a 0-neighborhood base in the (strong) dual is given by the collection of the polars $(nB_n)^\circ$. Let $T \in (2nB_{2n})^\circ$. Then,

$$|T(e_j)| \leq \frac{1}{2n} \frac{1}{(2n)^j S_j^{(2n)}}$$

and hence

$$\begin{aligned} |\Phi(T)(z)| &\leq \sum_j |T(e_j)| |z|^j \leq \frac{1}{2n} \sum_j \frac{|z|^j}{(2n)^j S_j^{(2n)}} \\ &\leq \frac{1}{2n} \sup_{k \in \mathbb{N}} \frac{|z|^k}{n^k S_k^{(n)}} \sum_{j \geq 0} 2^{-j} = \frac{1}{n} e^{\omega_{S(n)}(\frac{|z|}{n})}. \end{aligned}$$

Thus $\Phi((2nB_{2n})^\circ)$ is contained in the ball of radius $\frac{1}{n}$ with respect to the weight $e^{\omega_{S(n)}(\frac{|z|}{n})}$, whence Φ is continuous.

Conversely, let $f(z) = \sum_j c_j z^j \in \mathcal{A}^0$ be such that

$$|f(z)| \leq \frac{1}{2n} e^{\omega_{S(2n)}(\frac{|z|}{2n})}.$$

Then, by the Cauchy estimates,

$$|c_j| \leq \frac{1}{2n} \inf_{r>0} \frac{e^{\omega_{S(2n)}(\frac{r}{2n})}}{r^j} = \frac{1}{2n} \left(\sup_{r>0} \frac{r^j}{e^{\omega_{S(2n)}(\frac{r}{2n})}} \right)^{-1} = \frac{1}{(2n)^{j+1} S_j^{(2n)}}.$$

Therefore, if $a \in nB_n$ then

$$|\Phi^{-1}(f)(a)| \leq \sum_{j \geq 0} \frac{n^{j+1} S_j^{(n)}}{(2n)^{j+1} S_j^{(2n)}} \leq 1,$$

and so $\Phi^{-1}(f) \in (nB_n)^\circ$, which shows continuity of Φ^{-1} . \square

Now we are ready to show that (A.1) implies $\sigma \preceq \kappa_\omega$. If $\sigma \not\preceq \kappa_\omega$, then in view of (3.1) we can find a sequence of positive real numbers $a_j \rightarrow \infty$ such that

$$(A.2) \quad 6j\sigma(a_j) \leq P_\omega(ia_j),$$

$$(A.3) \quad \log(1 + |z|^2) \leq \frac{1}{j}\sigma(|z|) \quad \text{for } |z| \geq a_j.$$

By the proof of Proposition 2.4 in [2], there exists of a sequence of polynomials h_j and constants C and m such that

$$|h_j(z)| \leq C(1 + a_j^2)^2 e^{\frac{m}{j}P_\omega(z)}, \quad z \in \mathbb{C}, \quad \text{and} \quad h_j(ia_j) = e^{\frac{1}{j}P_\omega(ia_j)}.$$

Now set

$$f_j(z) := e^{(\sigma(a_j) - \frac{1}{j}P_\omega(ia_j))} h_j(z).$$

Using (A.3) and (A.2), we get

$$|f_j(z)| \leq C e^{(1 + \frac{2}{j})\sigma(a_j) - \frac{1}{j}P_\omega(ia_j)} e^{\frac{m}{j}P_\omega(z)} \leq C e^{-\frac{1}{2j}P_\omega(ia_j)} e^{\frac{m}{j}P_\omega(z)}.$$

Applying the estimate $P_\omega(x + iy) \leq |y| + A(\omega(x) + 1)$ (see [4, Lemma 2.2]), where A is some absolute constant, gives

$$|f_j(z)| \leq C e^{Am} e^{-\frac{1}{2j}P_\omega(ia_j)} e^{m|y| + \frac{Am}{j}\omega(|z|)}.$$

It follows that a subsequence of (f_j) tends to 0 in $(\mathcal{E}^{\{\omega\}}(\mathbb{R}))'$ by means of the identification in [3, Remark 1.4]. On the other hand,

$$f_j(ia_j) = e^{\sigma(a_j)} \geq e^{\omega_{S(n)}(\frac{a_j}{n})},$$

since $\sigma(t) \geq \omega_{S(1)}(t) \geq \omega_{S(n)}(t) \geq \omega_{S(n)}(t/n)$. Thus, no subsequence of (f_j) converges to 0 in $(\Lambda^{\{\mathfrak{M}_\sigma\}})' \cong \mathcal{A}^0$; see Lemma A.3. Now [3, Corollary 2.2] gives that $\Lambda^{\{\mathfrak{M}_\sigma\}}$ is not contained in $j^\infty \mathcal{E}^{\{\omega\}}(\mathbb{R})$ (for this we need the (DFS)-property, i.e., Lemma A.2). So the assumption $\sigma \not\preceq \kappa_\omega$ contradicts (A.1) and the proof of Proposition A.1 is complete.

Remark A.4. The Beurling version of Proposition A.1 is valid, too. We omit the proof.

REFERENCES

- [1] J. Bonet, R. W. Braun, R. Meise, and B. A. Taylor, *Whitney's extension theorem for non-quasianalytic classes of ultradifferentiable functions*, *Studia Math.* **99** (1991), no. 2, 155–184.
- [2] J. Bonet, R. Meise, and B. A. Taylor, *Whitney's extension theorem for ultradifferentiable functions of Roumieu type*, *Proc. Roy. Irish Acad. Sect. A* **89** (1989), no. 1, 53–66.
- [3] ———, *On the range of the Borel map for classes of non-quasianalytic functions*, *Progress in functional analysis* (Peñíscola, 1990), *North-Holland Math. Stud.*, vol. 170, North-Holland, Amsterdam, 1992, pp. 97–111.
- [4] R. W. Braun, R. Meise, and B. A. Taylor, *Ultradifferentiable functions and Fourier analysis*, *Results Math.* **17** (1990), no. 3-4, 206–237.
- [5] J. Bruna, *An extension theorem of Whitney type for non-quasi-analytic classes of functions*, *J. London Math. Soc. (2)* **22** (1980), no. 3, 495–505.
- [6] L. Carleson, *On universal moment problems*, *Math. Scand.* **9** (1961), 197–206.
- [7] J. Chaumat and A.-M. Chollet, *Surjectivité de l'application restriction à un compact dans des classes de fonctions ultradifférentiables*, *Math. Ann.* **298** (1994), no. 1, 7–40.
- [8] J. Jiménez-Garrido, J. Sanz, and G. Schindl, *Sectorial extensions, via Laplace transforms, in ultraholomorphic classes defined by weight functions*, *Results Math.* **74** (2019), no. 27.
- [9] ———, *Sectorial extensions for ultraholomorphic classes defined by weight functions*, *Math. Nachr.* **293** (2020), no. 11, 2140–2174.

- [10] ———, *The surjectivity of the Borel mapping in the mixed setting for ultradifferentiable ramification spaces*, *Monatsh. Math.* **191** (2020), no. 3, 537–576.
- [11] ———, *Ultraholomorphic extension theorems in the mixed setting*, *Banach J. Math. Anal.* **14** (2020), no. 4, 1630–1669.
- [12] H. Komatsu, *Ultradistributions. I. Structure theorems and a characterization*, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **20** (1973), 25–105.
- [13] M. Langenbruch, *Extension of ultradifferentiable functions*, *Manuscripta Math.* **83** (1994), no. 2, 123–143.
- [14] S. Mandelbrojt, *Séries adhérentes, régularisation des suites, applications*, Gauthier-Villars, Paris, 1952.
- [15] R. Meise and D. Vogt, *Introduction to functional analysis*, Oxford Graduate Texts in Mathematics, vol. 2, The Clarendon Press Oxford University Press, New York, 1997. MR 1483073 (98g:46001)
- [16] H.-J. Petzsche, *On E. Borel's theorem*, *Math. Ann.* **282** (1988), no. 2, 299–313.
- [17] A. Rainer, *On the extension of Whitney ultrajets of Beurling type*, *Results Math.* **76** (2021), Article Number 36, <https://doi.org/10.1007/s00025-021-01347-z>.
- [18] ———, *Ultradifferentiable extension theorems: a survey*, *Expositiones Mathematicae* **40** (2022), no. 3, 679–757. <https://doi.org/10.1016/j.exmath.2021.12.001>.
- [19] A. Rainer and G. Schindl, *Composition in ultradifferentiable classes*, *Studia Math.* **224** (2014), no. 2, 97–131.
- [20] ———, *Extension of Whitney jets of controlled growth*, *Math. Nachr.* **290** (2017), no. 14-15, 2356–2374, doi:10.1002/mana.201600321.
- [21] ———, *On the Borel mapping in the quasianalytic setting*, *Math. Scand.* **121** (2017), 293–310.
- [22] ———, *On the extension of Whitney ultrajets*, *Studia Math.* **245** (2019), no. 3, 255–287.
- [23] ———, *On the extension of Whitney ultrajets, II*, *Studia Math.* **250** (2020), no. 3, 283–295.
- [24] G. Schindl, *On the maximal extension in the mixed ultradifferentiable weight sequence setting*, *Studia Math.* **263** (2022), no. 2, 209–240.
- [25] ———, *Characterization of ultradifferentiable test functions defined by weight matrices in terms of their Fourier transform*, *Note di Matematica* **36** (2016), no. 2, 1–35.
- [26] J. Schmets and M. Valdivia, *On certain extension theorems in the mixed Borel setting*, *J. Math. Anal. Appl.* **297** (2003), 384–403.

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