LIFTING SMOOTH CURVES OVER INVARIANTS FOR REPRESENTATIONS OF COMPACT LIE GROUPS, II

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ABSTRACT. Any sufficiently often differentiable curve in the orbit space of a compact Lie group representation can be lifted to a once differentiable curve into the representation space.

1. INTRODUCTION

In [2] the following problem was investigated. Consider an orthogonal representation of a compact Lie group G on a real finite dimensional Euclidean vector space V. Let $\sigma_1, \ldots, \sigma_n$ be a system of homogeneous generators for the algebra $\mathbb{R}[V]^G$ of invariant polynomials on V. Then the mapping $\sigma = (\sigma_1, \ldots, \sigma_n) : V \to \mathbb{R}^n$ induces a homeomorphism between the orbit space V/G and the semialgebraic set $\sigma(V)$. Suppose a smooth curve $c : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ in the orbit space is given (smooth as curve in $\mathbb{R}^n$), does there exist a smooth lift to V, i.e., a smooth curve $\bar{c} : \mathbb{R} \to V$ with $c = \sigma \circ \bar{c}$?

It was shown in [2] that a real analytic curve in V/G admits a local real analytic lift to V, and that a smooth curve in V/G admits a global smooth lift, if certain genericity conditions are satisfied. In both cases the lifts may be chosen orthogonal to each orbit they meet and then they are unique up to a transformation in G, whenever the representation of G on V is polar, i.e., admits sections.

In this paper we treat the same problem under weaker differentiability conditions for $c : \mathbb{R} \to V/G$ and without the mentioned genericity conditions. In section 3 we show that a continuous curve in the orbit space V/G allows a global continuous lift to V. As a consequence we can prove in section 4 that a sufficiently often differentiable curve in V/G can be lifted to a once differentiable curve in V. What we mean by sufficiently often differentiable will be specified there.

In the special case that the symmetric group S_n is acting on $\mathbb{R}^n$, in other words (see [2]), if smooth parameterizations of the roots of smooth curves of polynomials with all roots real are looked for, the following results were proved in [5]: Any differentiable lift of a C^{2n}-curve (of polynomials) $c : \mathbb{R} \to \mathbb{R}^n/S_n$ is actually C^1, and there always exists a twice differentiable but in general not better lift of c, if it is of class C^{3n}. Note that here the differentiability assumptions on c are not the weakest possible which is shown by the case $n = 2$, elaborated in [1] 2.1. The proof there is based on the fact that the roots of a C^n-curve of polynomials $c : \mathbb{R} \to \mathbb{R}^n/S_n$ may be chosen differentiable with locally bounded derivative; this is due to Bronshtein [4] and Wakabayashi [12]. Therefore, our long-term objective is to prove the existence of a twice differentiable lift also in the general setting. The key is the generalization of Bronshtein's and Wakabayashi's result which seems to be difficult.

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The polynomial results have applications in the theory of partial differential equations and perturbation theory, see [6].

2. Preliminaries

2.1. **The setting.** Let G be a compact Lie group and let $\rho : G \to O(V)$ be an orthogonal representation in a real finite dimensional Euclidean vector space V with inner product $\langle \quad | \quad \rangle$. By a classical theorem of Hilbert and Nagata, the algebra $\mathbb{R}[V]^G$ of invariant polynomials on V is finitely generated. So let $\sigma_1, \ldots, \sigma_n$ be a system of homogeneous generators of $\mathbb{R}[V]^G$ of positive degrees $d_1, \ldots, d_n$. We may assume that $\sigma_1 : v \mapsto \langle v|v\rangle$ is the inner product. Consider the *orbit map* $\sigma = (\sigma_1, \ldots, \sigma_n) : V \to \mathbb{R}^n$. Note that, if $(y_1, \ldots, y_n) = \sigma(v)$ for $v \in V$, then $(t^{d_1}y_1, \ldots, t^{d_n}y_n) = \sigma(tv)$ for $t \in \mathbb{R}$, and that $\sigma^{-1}(0) = \{0\}$. The image $\sigma(V)$ is a semialgebraic set in the categorical quotient $V/\!\!/G := \{y \in \mathbb{R}^n : P(y) = 0 \text{ for all } P \in I\}$ where I is the ideal of relations between $\sigma_1, \ldots, \sigma_n$. Since G is compact, σ is proper and separates orbits of G, it thus induces a homeomorphism between V/G and $\sigma(V)$.

2.2. **The slice theorem.** For a point $v \in V$ we denote by G_v its isotropy group and by $N_v = T_v(G.v)^\perp$ the normal subspace of the orbit $G.v$ at v. It is well known that there exists a G-invariant open neighborhood U of v which is real analytically G-isomorphic to the crossed product (or associated bundle) $G \times_{G_v} S_v = (G \times S_v)/G_v$, where S_v is a ball in N_v with center at the origin. The quotient U/G is homeomorphic to S_v/G_v. It follows that the problem of local lifting curves in V/G passing through $\sigma(v)$ reduces to the same problem for curves in N_v/G_v passing through 0. For more details see [2], [8] and [10], theorem 1.1.

A point $v \in V$ (and its orbit $G.v$ in V/G) is called regular if the isotropy representation $G_v \to O(N_v)$ is trivial. Hence a neighborhood of this point is analytically G-isomorphic to $G/G_v \times S_v \cong G.v \times S_v$. The set V_{reg} of regular points is open and dense in V, and the projection $V_{\text{reg}} \to V_{\text{reg}}/G$ is a locally trivial fiber bundle. A non regular orbit or point is called singular.

2.3. **Removing fixed points.** Let V^G be the space of G-invariant vectors in V, and let V' be its orthogonal complement in V. Then we have $V = V^G \oplus V'$, $\mathbb{R}[V]^G = \mathbb{R}[V^G] \otimes \mathbb{R}[V']^G$ and $V/G = V^G \times V'/G$.

Lemma. *Any lift $\bar{c}$ of a curve $c = (c_0, c_1)$ of class C^k ($k = 0, 1, \ldots, \infty, \omega$) in $V^G \times V'/G$ has the form $\bar{c} = (c_0, \bar{c}_1)$, where $\bar{c}_1$ is a lift of c_1 to V' of class C^k ($k = 0, 1, \ldots, \infty, \omega$). The lift $\bar{c}$ is orthogonal if and only if the lift $\bar{c}_1$ is orthogonal.* □

2.4. **Multiplicity.** For a continuous function f defined near 0 in $\mathbb{R}$, let the *multiplicity* or *order of flatness* $m(f)$ at 0 be the supremum of all integers p such that $f(t) = t^p g(t)$ near 0 for a continuous function g. If f is C^n and $m(f) < n$, then $f(t) = t^{m(f)} g(t)$, where now g is $C^{n-m(f)}$ and $g(0) \neq 0$. Similarly, one can define multiplicity of a function at any $t \in \mathbb{R}$.

Lemma. *Let $c = (c_1, \ldots, c_n)$ be a curve in $\sigma(V) \subseteq \mathbb{R}^n$, where c_i is C^{d_i}, for $1 \le i \le n$, and $c(0) = 0$. Then the following two conditions are equivalent:*

(1) *$c_1(t) = t^2 c_{1,1}(t)$ near 0 for a continuous function $c_{1,1}$;*
(2) *$c_i(t) = t^{d_i} c_{i,i}(t)$ near 0 for a continuous function $c_{i,i}$, for all $1 \le i \le n$.*

Proof. The proof of the nontrivial implication (1) $\Rightarrow$ (2) is the same as in the smooth case with $r = 1$, see [2] 3.3. for details. □

3. Lifting continuous curves over invariants

Proposition 3.1. *Let $c = (c_1, \dots, c_n) : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ be continuous. Then there exists a global continuous lift $\bar{c} : \mathbb{R} \to V$ of c.*

This result is due to Montgomery and Yang [7] see also [3]. We present a short proof adapted to our setting:

Proof. We will make induction on the size of G. More precisely, for two compact Lie groups G' and G we denote $G' < G$, if

- $\dim G' < \dim G$ or
- if $\dim G' = \dim G$, then G' has less connected components than G has.

In the simplest case, when $G = \{e\}$ is trivial, we find $\sigma(V) = V/G = V$, whence we can put $\bar{c} := c$.

Let us assume that for any $G' < G$ and any continuous $c : \mathbb{R} \to V/G'$ there exists a global continuous lift $\bar{c} : \mathbb{R} \to V$ of c, where $G' \to O(V)$ is an orthogonal representation on an arbitrary real finite dimensional Euclidean vector space V.

We shall prove that then the same is true for G. Let $c : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ be continuous. By lemma 2.3, we may remove the nontrivial fixed points of the G-action on V and suppose that $V^G = \{0\}$. The set $c^{-1}(0)$ is closed in $\mathbb{R}$ and, consequently, $c^{-1}(\sigma(V)\backslash\{0\}) = \mathbb{R}\backslash c^{-1}(0)$ is open in $\mathbb{R}$. Thus, we can write $c^{-1}(\sigma(V)\backslash\{0\}) = \bigcup_{i\in I}(a_i, b_i)$, a disjoint union, where $a_i, b_i \in \mathbb{R} \cup \{\pm\infty\}$ with $a_i < b_i$ such that each (a_i, b_i) is maximal with respect to not containing zeros of c, and I is an at most countable set of indices. In particular, we have $c(a_i) = c(b_i) = 0$ for all $a_i, b_i \in \mathbb{R}$ appearing in the above presentation.

We assert that on each (a_i, b_i) there exists a continuous lift $\bar{c} : (a_i, b_i) \to V\backslash\{0\}$ of the restriction $c|_{(a_i,b_i)} : (a_i, b_i) \to \sigma(V)\backslash\{0\}$. In fact, since $V^G = \{0\}$, for all $v \in V\backslash\{0\}$ the isotropy groups G_v, acting orthogonally on N_v, satisfy $G_v < G$. Therefore, by induction hypothesis and by 2.2, we find local continuous lifts of $c|_{(a_i,b_i)}$ near any $t \in (a_i, b_i)$ and through all $v \in \sigma^{-1}(c(t))$. Suppose $\bar{c}_1 : (a_i, b_i) \supseteq (a, b) \to V\backslash\{0\}$ is a local continuous lift of $c|_{(a_i,b_i)}$ with maximal domain (a, b), where, say, $b < b_i$. Then there exists a local continuous lift $\bar{c}_2$ of $c|_{(a_i,b_i)}$ near b, and there is a $t_0 < b$ such that both $\bar{c}_1$ and $\bar{c}_2$ are defined near t_0. Since $\bar{c}_1(t_0)$ and $\bar{c}_2(t_0)$ lie in the same orbit, there must exist a $g \in G$ such that $\bar{c}_1(t_0) = g.\bar{c}_2(t_0)$. But then,

$$\bar{c}_{12}(t) := \begin{cases} \bar{c}_1(t) & \text{for} \quad t \le t_0 \\ g.\bar{c}_2(t) & \text{for} \quad t \ge t_0 \end{cases}$$

is a local continuous lift of $c|_{(a_i,b_i)}$ defined on a larger interval than $\bar{c}_1$. Thus we have shown that each local continuous lift of $c|_{(a_i,b_i)}$ defined on an open interval $(a, b) \subseteq (a_i, b_i)$ can be extended to a larger interval whenever $(a, b) \subsetneq (a_i, b_i)$. This proves the assertion.

We put $\bar{c}|_{c^{-1}(0)} := 0$, since, by $\sigma^{-1}(0) = \{0\}$, this is the only choice. Then $\bar{c}$ is also continuous at points $t_0 \in c^{-1}(0)$ since $\langle\bar{c}(t)|\bar{c}(t)\rangle = \sigma_1(\bar{c}(t)) = c_1(t)$ converges to 0 as $t \to t_0$. □

4. Lifting differentiably

Throughout the whole section we let $d \ge 2$ be the maximum of all degrees of systems of minimal generators of invariant polynomials of all slice representations of ρ. Of these there are only finitely many isomorphism types.

Lemma 4.1. *A curve $c : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ of class C^d admits an orthogonal C^d-lift $\bar{c}$ in a neighborhood of a regular point $c(t_0) \in V_{\text{reg}}/G$. It is unique up to a transformation from G.*

Proof. The proof works analogously as in the smooth case, see [2] 3.1. □

Theorem 4.2. *Let $c = (c_1, \dots, c_n) : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ be a curve of class C^d. Then for any $t_0 \in \mathbb{R}$ there exists a local lift $\bar{c}$ of c near t_0 which is differentiable at t_0.*

Proof. We follow partially the algorithm given in [2] 3.4. Without loss of generality we may assume that $t_0 = 0$. We show the existence of local lifts of c which are differentiable at 0 through any $v \in \sigma^{-1}(c(0))$. By lemma 2.3 we can assume $V^G = \{0\}$.

If $c(0) \neq 0$ corresponds to a regular orbit, then unique orthogonal C^d-lifts defined near 0 exist through all $v \in \sigma^{-1}(c(0))$, by lemma 4.1.

If $c(0) = 0$, then c_1 must vanish of at least second order at 0, since $c_1(t) \geq 0$ for all $t \in \mathbb{R}$. That means $c_1(t) = t^2 c_{1,1}(t)$ near 0 for a continuous function $c_{1,1}$ since c_1 is C^2. By the multiplicity lemma 2.4 we find that $c_i(t) = t^{d_i} c_{i,i}(t)$ near 0 for $1 \leq i \leq n$, where $c_{1,1}, c_{2,2}, \dots, c_{n,n}$ are continuous functions. We consider the following curve in $\sigma(V)$ which is continuous since $\sigma(V)$ is closed in $\mathbb{R}^n$, see [9]:

$$\begin{aligned} c_{(1)}(t) : &= (c_{1,1}(t), c_{2,2}(t), \dots, c_{n,n}(t)) \\ &= (t^{-2} c_1(t), t^{-d_2} c_2(t), \dots, t^{-d_n} c_n(t)). \end{aligned}$$

By proposition 3.1, there exists a continuous lift $\bar{c}_{(1)}$ of $c_{(1)}$. Thus, $\bar{c}(t) := t \cdot \bar{c}_{(1)}(t)$ is a local lift of c near 0 which is differentiable at 0:

$$\sigma(\bar{c}(t)) = \sigma(t \cdot \bar{c}_{(1)}(t)) = (t^2 c_{1,1}(t), \dots, t^{d_n} c_{n,n}(t)) = c(t),$$

and

$$\lim_{t \to 0} \frac{t \cdot \bar{c}_{(1)}(t)}{t} = \lim_{t \to 0} \bar{c}_{(1)}(t) = \bar{c}_{(1)}(0).$$

Note that $\sigma^{-1}(0) = \{0\}$, therefore we are done in this case.

If $c(0) \neq 0$ corresponds to a singular orbit, let v be in $\sigma^{-1}(c(0))$ and consider the isotropy representation $G_v \to O(N_v)$. By 2.2, the lifting problem reduces to the same problem for curves in N_v/G_v now passing through 0. □

Lemma 4.3. *Consider a continuous curve $c : (a, b) \to X$ in a compact metric space X. Then the set A of all accumulation points of $c(t)$ as $t \searrow a$ is connected.*

Proof. On the contrary suppose that $A = A_1 \cup A_2$, where A_1 and A_2 are disjoint open and closed subsets of A. Since A is closed in X, also A_1 and A_2 are closed in X. There exist disjoint open subsets $A_1', A_2' \subseteq X$ with $A_1 \subseteq A_1'$ and $A_2 \subseteq A_2'$. Consider $F := X \backslash (A_1' \cup A_2')$ which is closed in X and hence compact. Since c visits A_1' and A_2' infinitely often and $c^{-1}(A_1')$ and $c^{-1}(A_2')$ are disjoint and open in $\mathbb{R}$, there exists a sequence $t_m \to a$ and $c(t_m) \in F$ for all m. By compactness of F, this sequence has a cluster point y in F. Hence y is in A by definition, which contradicts $F \cap A = \emptyset$. □

Theorem 4.4. *Let $c = (c_1, \dots, c_n) : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ be a curve of class C^d. Then there exists a global differentiable lift $\bar{c} : \mathbb{R} \to V$ of c.*

Proof. The proof, as the one of proposition 3.1, will be carried out by induction on the size of G.

If $G = \{e\}$ is trivial, then $\bar{c} := c$ is a global differentiable lift.

So let us assume that for any $G' < G$ and any $c : \mathbb{R} \to V/G'$ satisfying the differentiability conditions of the theorem there exists a global differentiable lift $\bar{c} : \mathbb{R} \to V$ of c, where $G' \to O(V)$ is an orthogonal representation on an arbitrary real finite dimensional Euclidean vector space V.

We shall prove that the same is true for G. Let $c = (c_1, \dots, c_n) : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ be of class C^d. We may assume that $V^G = \{0\}$, by lemma 2.3. As in the proof of proposition 3.1 we can write $c^{-1}(\sigma(V)\backslash\{0\}) = \bigcup_i (a_i, b_i)$, a disjoint union, where $a_i, b_i \in \mathbb{R} \cup \{\pm\infty\}$ with $a_i < b_i$. In particular, we have $c(a_i) = c(b_i) = 0$ for all $a_i, b_i \in \mathbb{R}$ appearing in the above presentation.

Claim: On each (a_i, b_i) there exists a differentiable lift $\bar{c} : (a_i, b_i) \to V\backslash\{0\}$ of the restriction $c|_{(a_i,b_i)} : (a_i, b_i) \to \sigma(V)\backslash\{0\}$. The lack of nontrivial fixed points guarantees that for all $v \in V\backslash\{0\}$ the isotropy groups G_v acting on N_v satisfy $G_v < G$. Therefore, by induction hypothesis and by 2.2, we find local differentiable lifts of $c|_{(a_i,b_i)}$ near any $t \in (a_i, b_i)$ and through all $v \in \sigma^{-1}(c(t))$. Suppose that $\bar{c}_1 : (a_i, b_i) \supseteq (a, b) \to V\backslash\{0\}$ is a local differentiable lift of $c|_{(a_i,b_i)}$ with maximal domain (a, b), where, say, $b < b_i$. Then there exists a local differentiable lift $\bar{c}_2$ of $c|_{(a_i,b_i)}$ near b, and there exists a $t_0 < b$ such that both $\bar{c}_1$ and $\bar{c}_2$ are defined near t_0. We may assume without loss that $\bar{c}_1(t_0) = \bar{c}_2(t_0) =: v_0$, by applying a transformation $g \in G$ to $\bar{c}_2$, say. We want to show that we can arrange the lift $\bar{c}_2$ in such a way that its derivative at t_0 matches with the derivative of $\bar{c}_1$ at t_0. We decompose $\bar{c}_i'(t_0) = \bar{c}_i'(t_0)^\top + \bar{c}_i'(t_0)^\perp$ into the parts tangent to the orbit $G.v_0$ and normal to it.

First we deal with the normal parts $\bar{c}_i'(t_0)^\perp \in V$. We consider the projection $p : G.S_{v_0} \cong G \times_{G_{v_0}} S_{v_0} \to G/G_{v_0} \cong G.v_0$ of the fiber bundle associated to the principal bundle $\pi : G \to G/G_{v_0}$. Then, for t close to t_0, $\bar{c}_1$ and $\bar{c}_2$ are differentiable curves in $G.S_{v_0}$, whence $p \circ \bar{c}_i$ $(i = 1, 2)$ are differentiable curves in G/G_{v_0} which admit differentiable lifts g_i into G with $g_i(t_0) = e$ (via the horizontal lift of a principal connection, say). Consequently, $t \mapsto g_i(t)^{-1}.\bar{c}_i(t) =: \tilde{c}_i(t)$ are differentiable lifts of $c|_{(a_i,b_i)}$ near t_0 which lie in S_{v_0}, whence $\tilde{c}_i'(t_0) = \frac{d}{dt}\big|_{t=t_0}(g_i(t)^{-1}.\bar{c}_i(t)) = -g_i'(t_0).v_0 + \bar{c}_i'(t_0) \in N_{v_0}$. So, $\bar{c}_i'(t_0)^\top = (g_i'(t_0).v_0)^\top = g_i'(t_0).v_0$, and so for the normal part we get $\bar{c}_i'(t_0)^\perp = \tilde{c}_i'(t_0)$.

Since $\tilde{c}_i$ lie in S_{v_0} we can change to the isotropy representation $G_{v_0} \to O(N_{v_0})$ (using the same letters σ_i for the generators of $\mathbb{R}[N_{v_0}]^{G_{v_0}}$). We can suppose that $v_0 = 0$, i.e., $c(t_0) = 0$.

Recall the continuous curve in $\sigma(V)$ defined in the proof of theorem 4.2 which depends on the point t_0:

$$c_{(1,t_0)}(t) := ((t-t_0)^{-2}c_1(t), (t-t_0)^{-d_2}c_2(t), \ldots, (t-t_0)^{-d_n}c_n(t)).$$

We find that for $i = 1, 2$:

$$\sigma(\tilde{c}_i'(t_0)) = \sigma\left(\lim_{t\to t_0}\frac{\tilde{c}_i(t) - \tilde{c}_i(t_0)}{t - t_0}\right) = \lim_{t\to t_0}\sigma\left(\frac{\tilde{c}_i(t)}{t - t_0}\right) = c_{(1,t_0)}(t_0).$$

So $\tilde{c}_1'(t_0)$ and $\tilde{c}_2'(t_0)$ are lying in the same orbit. This shows also that

- ♦ for any two lifts of c near $t_0 \in c^{-1}(0)$ which are one-sided differentiable at t_0 the derivatives at t_0 lie in the same G-orbit.

Thus, there must exist a $g_0 \in G_{v_0}$ such that $\bar{c}_1'(t_0)^\perp = \tilde{c}_1'(t_0) = g_0.\tilde{c}_2'(t_0) = g_0.\bar{c}_2'(t_0)^\perp = (g_0.\bar{c}_2)'(t_0)^\perp$.

Now we deal with the tangential parts. We search for a differentiable curve $t \mapsto g(t)$ in G with $g(t_0) = g_0$ and

$$\bar{c}_1'(t_0)^\top = \left(\tfrac{d}{dt}|_{t=t_0}(g(t).\bar{c}_2(t))\right)^\top = g'(t_0).v_0 + g_0.\bar{c}_2'(t_0)^\top.$$

But this linear equation can be solved for $g'(t_0)$, and, hence, the required curve $t \mapsto g(t)$ exists. Note that the normal parts still fit since

$$\left(\tfrac{d}{dt}|_{t=t_0}(g(t).\bar{c}_2(t))\right)^\perp = \left(g'(t_0).v_0 + g_0.\bar{c}_2'(t_0)\right)^\perp = 0 + g_0.\bar{c}_2'(t_0)^\perp = \bar{c}_1'(t_0)^\perp.$$

The two lifts $\bar{c}_1$ for $t \le t_0$ and $g.\bar{c}_2$ for $t \ge t_0$ fit together differentiably at t_0. This proves the claim.

Now let $\bar{c} : (a_i, b_i) \to V\backslash\{0\}$ be the differentiable lift of $c|_{(a_i,b_i)}$ constructed above. For $a_i \ne -\infty$, we put $\bar{c}(a_i) := 0$, the only choice. Consider the expression

$\gamma(t) := \frac{\bar{c}(t)}{t-a_i}$ which is a differentiable curve in $V\backslash\{0\}$ for $t \in (a_i, b_i)$. We want to show that $\lim_{t\searrow a_i} \gamma(t)$ exists. For t sufficiently close to a_i we have

$$\sigma(\gamma(t)) = \sigma\left(\frac{\bar{c}(t)}{t-a_i}\right) = c_{(1,a_i)}(t) \to c_{(1,a_i)}(a_i) \qquad \text{as } t \searrow a_i,$$

where now $c_{(1,a_i)}(t) := ((t-a_i)^{-2}c_1(t), (t-a_i)^{-d_2}c_2(t), \dots, (t-a_i)^{-d_n}c_n(t))$. Let $\bar{c}_{(1,a_i)}$ be a corresponding continuous lift of $c_{(1,a_i)}$ which exists by proposition 3.1. This shows that the set A of all accumulation points of $(\gamma(t))_{t\searrow a_i}$ lies in the orbit $G.\bar{c}_{(1,a_i)}(a_i)$ through $\bar{c}_{(1,a_i)}(a_i)$. By lemma 4.3, A is connected. In particular, the limit $\lim_{t\searrow a_i} \gamma(t)$ must exist, if G is a finite group. In general let us consider the projection $p : G.S_{v_1} \cong G \times_{G_{v_1}} S_{v_1} \to G/G_{v_1} \cong G.v_1$ of a fiber bundle associated to the principal bundle $\pi : G \to G/G_{v_1}$, where we choose some $v_1 \in A$. For t close to a_i the curve $t \mapsto \gamma(t)$ is differentiable in $G.S_{v_1}$, whence $t \mapsto p(\gamma(t))$ defines a differentiable curve in G/G_{v_1} which admits a differentiable lift $t \mapsto g(t)$ into G. Now, $t \mapsto g(t)^{-1}.\gamma(t)$ is a differentiable curve in S_{v_1} whose accumulation points for $t \searrow a_i$ have to lie in $G.v_1 \cap S_{v_1} = \{v_1\}$, since $\sigma(g(t)^{-1}.\gamma(t)) = \sigma(\gamma(t))$. That means that $t \mapsto g(t)^{-1}.\bar{c}(t)$ defines a differentiable lift of $c|_{(a_i,b_i)}$, for $t > a_i$ close to a_i, whose one-sided derivative at a_i exists:

$$\lim_{t\searrow a_i} \frac{g(t)^{-1}.\bar{c}(t)}{t-a_i} = \lim_{t\searrow a_i} g(t)^{-1}.\gamma(t) = v_1.$$

Let $t \mapsto g(t)$ be extended smoothly to (a_i, b_i) so that near b_i it is constant and replace $t \mapsto \bar{c}(t)$ by $t \mapsto g(t)^{-1}\bar{c}(t)$. Thus

$$\bar{c}'(a_i) := \lim_{t\searrow a_i} \frac{\bar{c}(t)}{t-a_i} = v_1.$$

The same reasoning is true for $b_i \neq +\infty$. Thus we have extended $\bar{c}$ differentiably to the closure of (a_i, b_i).

Let us now construct a global differentiable lift of c defined on the whole of $\mathbb{R}$. For isolated points $t_0 \in c^{-1}(0)$ the two differentiable lifts on the neighboring intervals can be made to match differentiably, by applying a fixed $g \in G$ to one of them by ♦. Let E be the set of accumulation points of $c^{-1}(0)$. For connected components of $\mathbb{R} \setminus E$ we can proceed inductively to obtain differentiable lifts on them.

We extend the lift by 0 on the set E of accumulation points of $c^{-1}(0)$. Note that every lift $\tilde{c}$ of c has to vanish on E and is continuous there since $\langle \tilde{c}(t)|\tilde{c}(t)\rangle = \sigma_1(\tilde{c}(t)) = c_1(t)$. We also claim that any lift $\tilde{c}$ of c is differentiable at any point $t' \in E$ with derivative 0. Namely, the difference quotient $t \mapsto \frac{\tilde{c}(t)}{t-t'}$ is a lift of the curve $c_{(1,t')}$ which vanishes at t' by the following argument: Consider the local lift $\bar{c}$ of c near t' which is differentiable at t', provided by theorem 4.2. Let $(t_m)_{m\in\mathbb{N}} \subseteq c^{-1}(0)$ be a sequence with $t' \neq t_m \to t'$, consisting exclusively of zeros of c. Such a sequence always exists since $t' \in E$. Then we have

$$\bar{c}'(t') = \lim_{t\to t'} \frac{\bar{c}(t) - \bar{c}(t')}{t-t'} = \lim_{m\to\infty} \frac{\bar{c}(t_m)}{t_m - t'} = 0.$$

Thus $c_{(1,t')}(t') = \lim_{t\to t'} \sigma(\frac{\tilde{c}(t)}{t-t'}) = \sigma(\bar{c}'(t')) = 0$. □

Remark 4.5. Note that the differentiability conditions of the curve c in the current section are best possible: In the case when the symmetric group S_n is acting in $\mathbb{R}^n$ by permuting the coordinates, and $\sigma_1, \dots, \sigma_n$ are the elementary symmetric polynomials with degrees $1, \dots, n$, there need not exist a differentiable lift if the differentiability assumptions made on c are weakened, see [1] 2.3. first example.

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