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Binet-Legendge ellipsoid in conformal finsler geometry

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Based on the paper arXiv:1104.1647 joint with Marc Troyanov

Abstract: I show a simple construction from convex geometry that solves many named problems in Finsler geometry

Definition of finsler metrics: Finsler metric ist a continuous function $F : TM \to R$ such that for every $x \in M$ the restriction $F_{|T_xM}$ is a Minkowski norm, that is $\forall u, v \in T_xM, \ \forall \lambda > 0$
(a) $F(\lambda \cdot v) = \lambda \cdot F(v)$, (b) $F(u + v) \leq F(u) + F(v)$,
(c) $F(v) = 0 \iff v = 0$.

Euclidean norm:
$E : R^n \to R$ of the form
$E(v) = \sqrt{\sum_{i,j} a_{ij} v^i v^j}$,
where (a_{ij}) is a positively definite symmetric matrix

$\longrightarrow$

(Minkowski) norm:
$B : R^n \to R_{\geq 0}$ with

(a) $B(\lambda \cdot v) = \lambda \cdot B(v)$,

(b) $B(u + v) \leq B(u) + B(v)$,

(c) $B(v) = 0 \iff v = 0$

$\downarrow$ $\downarrow$

(Local) Riemannian metric:
$g : \underbrace{R^n}_{x} \times \underbrace{R^n}_{v} \to R_{\geq 0}$
of the form
$g_x(v, u) = \sum_{i,j} a_{ij}(x) v^i u^j$,
where for every x
$(a_{ij}(x))$ is a positively definite symmetric matrix

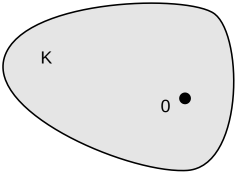
$\longrightarrow$

(LOCAL) FINSLER METRIC:
$F : \underbrace{R^n}_{x} \times \underbrace{R^n}_{v} \to R_{\geq 0}$ such that for every x
$F(x, \underbrace{\cdot}_{v}) : R^n \to R$ is a norm, i.e., satisfies
(a), (b), (c).

How to visualize finsler metrics

It is known (Minkowski) that the unit ball determines the norm uniquely:

for a given convex body $K \in R^n$ such that $0 \in int(K)$ there exists an unique norm B such that $K = \{x \in R^n \mid B(x) \leq 1\}$.

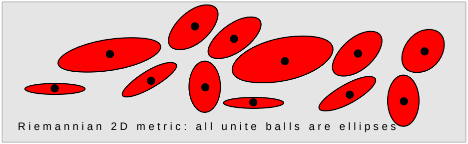


There exists a unique norm such that (the convex body) K is the unite ball in this norm

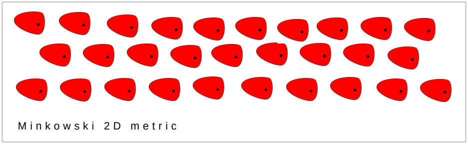
Thus, in order to make a picture of a finsler metric it is sufficient to draw unit balls at the tangent spaces.

Examples:

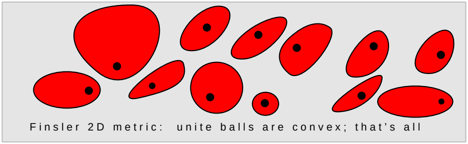
Riemannian metric: every unit ball is an ellipsoid symmetric w.r.t. 0.



Minkowski metric on R^n: $F(x, v) = B(v)$ for a certain norm B, i.e., the metric is invariant w.r.t. the standard translations of R^n.



Arbitrary finsler metric on R^n:



Main Trick

$$F \quad \xmapsto{\textit{canonically construct}} \quad g_F$$

Given a (smooth) finsler metrics F we construct a (smooth) RIEMANNIAN metric on g_F such that

- The Riemannian metric g_F has the same (or better) regularity as the finsler metric F
- If F is Riemannian, i.e. if $F(x,\xi) = \sqrt{g_x(\xi,\xi)}$ for a some Riemannian metric g, then $g_F = g$
- If two finsler metrics F_1 and F_2 are conformally equivalent, i.e., if $F_1(x,\xi) = \lambda(x)F_2(x,\xi)$ for some function $\lambda : M \to R$, then the corresponding Riemannian metrics are also conformally equivalent with essentially the same conformal factor: $g_{F_1} = \lambda^2 g_{F_2}$
- If F_1 and F_2 are C^0-close, then so are g_{F_1} and g_{F_2}.
- If F_1 and F_2 are bilipschitzly equivalent, then so are g_{F_1} and g_{F_2}.

This allows to use the results and methods from (much better developed) Riemannian geometry to finsler geometry. I will show many application

Construction of the (Binet-Legendre) Euclidean structure in every tangent space

For every convex body $K \subseteq V$ such that $0 \in \mathrm{int}(K)$, let us now construct an Euclidean structure in V.

We take an arbitrary linear volume form Ω in V and construct contravariant bilinear form $g^* : V^* \times V^* \to R$ (where V^* is the dual vector space to V) by

$$g^*(\xi, \nu) := \frac{1}{Vol_\Omega(K)} \int_K \xi(k)\nu(k) d\Omega$$

(i.e., the function we integrate takes on $k \in K \subset V$ the value $\xi(k)\nu(k)$; ξ and ν are elements of V^*, i.e., are functions on V.)

Equivalent definition: $g^*(\xi, \nu) = \langle \xi_{|K}, \nu_{|K} \rangle_{L_2}$ where we fixed the linear volume form Ω on V by requiring $Vol_\Omega(K) = 1$.

g^* allows to identify canonically V and V^* and gives therefore an Euclidean structure on V, which we denote by g.

$$g^*(\xi, \nu) := \frac{1}{Vol_\Omega(K)} \int_K \xi(k)\nu(k) d\Omega$$

Evidently, g is a well-defined Euclidean structure

- it does not depend on Ω (because the only freedom is choosing Ω, multiplication by a constant, does not influence the result),
- It is bilinear and positive definite

  Moreover,
- g' constructed by $K' := \frac{1}{\lambda} \cdot K$ is given by $g' = \lambda^2 \cdot g$

Remark 1. The construction is too easy to be new – our motivation came from classical mechanics, and our construction is close to one of the inertia ellipsoid (Poinsot, Binet, Legendre). In the convex geometry, Milman et al 1990 had a similar construction in an Euclidean space

Remark 2. There exist other constructions for example Vincze 2005 and M$\sim$, Rademacher, Troyanov, Zeghib 2009. The present construction has better properties.

Thus, by a finsler metric F, we canonically constructed a Euclidean structure on every tangent space, i.e., a Riemannian metric g_F. If the finsler metric is smooth, then the Riemannian metric is also smooth.

This metric has the following property: $g_{\lambda \cdot F} = \lambda^2 \cdot g_F$.

In particular, if ϕ is isometry, similarity, or conformal transformation of F, it is an isometry, similarity, or conformal transformation of g_F.

First application: Wang's Theorem for all dimensions.

Theorem. *Let (M^n, F) be a C^2-smooth connected Finlser manifold. If the dimension of the space of Killing vector fields of (M, F) is greater than $\frac{n(n-1)}{2} + 1$, then F is actually a Riemannian metric.*

History: For $n \neq 2, 4$ Theorem was proved 1947 by H.C. Wang. This theorem answers a question of S. Deng and Z. Hou (2007).

Proof. I will use: if ϕ is an isometry of F, then it is an isometry of g_F.
Let $r > \frac{n(n-1)}{2} + 1$ be the dimension of the space of Killing vector fields. Take a point x and choose $r - n$ linearly independent Killing vector fields $K_1, \ldots, K_{r-n}$ vanishing at x. The point x is then a fixed point of the corresponding local flows $\phi_t^{K_1}$, ... , $\phi_t^{K_{r-n}}$.
Then, for every t, the differentials of $\phi_t^{K_1}$, ... , $\phi_t^{K_{r-n}}$ at x are linear isometries of (T_xM, g_F).
Thus, the subgroup of $SO(T_xM, g_F)$ preserving the function $F_{|T_xM}$ is at least $r - n$ dimensional.

Now, it is well-known that every subgroup of $SO(T_xM, g_F)$ of dimension $r - n > \frac{n(n-1)}{2} + 1 - n = \frac{(n-2)(n-1)}{2}$ acts transitively on the g_F-sphere $S^{n+1} \subset T_xM$. Then, the ratio $F(\xi)^2/g(\xi, \xi)$ is constant for all $\xi \in T_xM$ and the metric F is actually a Riemannian metric $\square$

The Liouville Theorem for Minkowski spaces and the solution to a problem by Matsumoto.

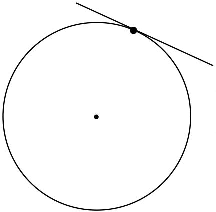
Theorem. *Let (V, F) be an non-euclidean Minkowski space. If $\phi : U_1 \to U_2$ is a conformal map between two domains $U_1 \subset V$ and $U_2 \subset V$, then ϕ is (the restriction of) a similarity, that is the composition of an isometry and a homothety $x \mapsto \mathrm{const} \cdot x$.*

Remark. Theorem generalizes classical result of Liouville for Minkowski metrics: Liouville has shown 1850 that every conformal transformation of the standard $(R^{n \geq 3}, g_{\mathrm{euclidean}})$ is a similarity or a Möbius transformation, i.e., a composition of a similarity and an inversion. We see that for noneuclidean finsler metrics only similarities are allowed.

Theorem answers the question of Matsumoto 2001 and will be uses below.

Proof of: Every conformal mapping of a Minkowski space is a similarity

Proof for $dim(M) > 2$. I will use: if ϕ is a conformal transformation of F, then it is a conformal transformation of g_F. Moreover, if ϕ is a conformal transformation of F and similarity of g_F, then it is a similarity of F.
We consider the metric g_F. It is Euclidean; w.l.o.g. we think that $g_F = dx_1^2 + ... + dx_n^2$.
Then, by the classical Liouville Theorem 1850, ϕ is as we want or a Möbius transformation, i.e., a composition of of a similarity and an inversion. We thus only need to prove that *a composition of of a similarity an inversion cannot be a conformal map of some non euclidean Minkowski norm on R^n,* which is an easy exercise.



The differential of the inversion at every point of the sphere is the reflection with respect to the tangent line to the sphere. The only convex body invariant with respect to all such reflection is the standard ball

Conformally flat compact Finsler Manifolds

Def. A metric F is conformally flat, if locally, in a neighborhood of every point, it is conformally Minkowski.

Corollary. *Any smooth connected compact conformally flat non Riemannian Finsler manifold is either a Bieberbach manifolds or a Hopf manifolds. In particular, it is finitely covered either by a torus T^n or by $S^{n-1} \times S^1$.*

Proof. Assuming M to be non Riemannian, it follows from Theorem from the previous slide that these changes of coordinates are euclidean similarities.

The manifold M carries therefore a similarity structure.

Compact manifolds with a similarity structure have been topologically classified by N. H. Kuiper (1950) and D. Fried (1980): they are either Bieberbach manifolds (i.e. R^n/Γ, where Γ is some crystallographic group of R^n), or they are Hopf-manifolds i.e. compact quotients of $R^n \setminus \{0\} = S^{n-1} \times R_+$ by a group G which is a semi-direct product of an infinite cyclic group with a finite subgroup of $O(n+1)$.

Finsler spaces with a non trivial self-similarity

Def. A C^1-map $f : (M, F) \to (M', F')$ is a *similarity* if there exists a constant $a > 0$, $a \neq 1$ (called the *dilation constant*) such that $F(f(x), df_x(\xi)) = a \cdot F(x, \xi)$ for all $(x, \xi) \in TM$.

Theorem. Let (M, F) be a forward complete connected C^0-Finsler manifold (the manifold M is of class C^1, the metric F is C^0). If there exists a non isometric self-similarity $f : M \to M$ of class C^1, then (M, F) is a Minkowski space.

Remark. In the case of smooth Finsler manifolds, Theorem is known. A first proof was given by Heil and Laugwitz in 1974, however R. L. Lovas, and J. Szilasi found a gap in the argument and gave a new proof in 2009.

In the proof, I will use:

(Fact 1.) if f is similarity for F, then it is a similarity for g_F;

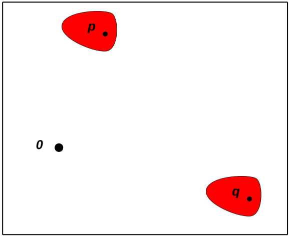
(Fact 2.) A similarity of a forward-complete manifold always has a fixed point, i.e. x such that $f(x) = x$ (since for every x the sequence $x, f(x), f(f(x)), f(f(f(x))), ...$ is forward Cauchy and its limit is a fixed point.

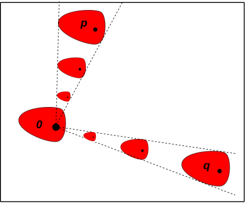
(Fact 3.) A Riemannian metric admitting similarity with a fixed point is flat. Indeed, for smooth metrics this statement reduces to a classical Riemannian argument, since the existence of a non trivial self-similarity in a C^2-Riemannian manifold easily implies that the sectional curvature of that manifold vanishes because otherwise it goes to infinity at the sequence of points $y, f(y), f(f(y)), f(f(f(y)))... \to x$. For nonsmooth metrics, the proof is slightly more tricky and is given in our paper; though it is known to experts in metric geometry.

Proof. By Fact 3, g_F is the standard Euclidean metric, and the similarity f is a similarity of R^n.

We consider two points $p, q \in R^n$. Our goal is to show that the unit ball in q is the parallel translation of the unit ball in p.

Let us first assume for simplicity that f is already a homothety $x \mapsto C \cdot x$ for a constant $1 > C > 0$ (we known that actually it is $\psi \circ \phi$, where ψ is an isometry and ϕ a homothety; I will explain on the next slide that w.l.o.g. $\psi = Id$)





We consider the points $p, f(p) = C \cdot p, f \circ f(p) = C^2 \cdot p$, ... , $\stackrel{\text{converge}}{\longrightarrow} 0.$

The unit ball of the push-forward $f_*^k(F)$ of the metric at the point $f^k(p)$ are as on the picture; therefore, the unit ball of $\frac{1}{C^k} f_*^k(F)$ at the point $f^k(p)$ is the parallel translation of the unit ball at the unit ball at the point p. But the unit ball of $\frac{1}{C^k} f_*^k(F)$ at $f^k(p)$ is the unit ball of F!

Thus, for every k the unit ball of F at $f_k(p)$ is the parallel translation of the unit ball of F at p.

Sending $k \to \infty$, we obtain that the unit ball at $0 = \lim_{k \to \infty} f^k(p)$ is the parallel translation of the unit ball at p. The same is true for q. Then, the unit ball at q is the parallel translation of the unit ball at p

Why we can think that the similarity f is a homothety, and not the composition $\psi \circ \phi$, where $\psi \in O(n)$ is an isometry and ϕ is a homothety

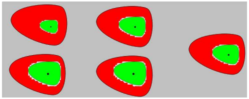
Because the group $O(n)$ is compact. Hence, any sequence of the form $\psi, \psi^2, \psi^3, \ldots$, has a subsequence converging to Id.
Thus, in the arguments on the previous slide we can take the subsequence $k \to \infty$ such that

$$(\psi \circ \phi)^k \stackrel{\phi\circ\psi=\psi\circ\phi}{=} \underbrace{\psi^k}_{\sim Id} \circ \phi^k$$

is "almost" ϕ^k, and the proof works.

Examples of conformal transformations and Theorem

(i) If $\phi : M \to M$ is an isometry for F, and $\lambda : M \to R_{>0}$ is a function, then ϕ is a conformal transformation of $F_1 := \lambda \cdot F$.



(ii) Let F_m be a Minkowski metric on R^n. Then, the mapping $x \mapsto \text{const} \cdot x$ (for $\text{const} \neq 0$) is a conformal transformation. Moreover, it is also a conformal transformation of $F := \lambda \cdot F_m$. Moreover, if ψ is an isometry of F_m, then $\psi \circ \phi$ is a conformal transformation of every $F := \lambda \cdot F_m$.

(iii) Let g be the standard (Riemannian) metric on the standard sphere S^n. Then, the standard Möbius transformations of S^n are conformal transformations of every metric $F := \lambda \cdot g$.

Theorem (finsler verion of conformal Lichnerowicz conjecture).
That's all: Let ϕ be a conformal transformation of a connected (smooth) finsler manifold $(M^{n\geq 2}, F)$. Then (M, F) and ϕ are as in Examples (i, ii, iii) above.

Even in the Riemannian case, Theorem above is nontrivial

Corollary (proved before by Alekseevsky 1971, Schoen 1995, (Lelong)-Ferrand 1996) Let ϕ be a conformal transformation of a connected RIEMANNIAN manifold $(M^{n\geq 2}, g)$. Then for a certain $\lambda : M \to R$ one of the following conditions holds

(a) ϕ is an isometry of $\lambda \cdot g$, or

(b) $(M, \lambda \cdot g)$ is $(R^n, g_{\rm flat})$,

(c) or $(S^n, g_{\rm round})$.

The story: This statement is known as *conformal Lichnerowicz conjecture* $\sim$ 1960
1970: Obata proved it under the assumption that M is closed.
1971: Alekseevsky proved it for all manifolds; later many mathematicians (for example Yoshimatsu 1976 amd Gutschera 1995 (basing on example of Ziller)) claimed the existence of flaws in the proof
1974–1996: (Lelong)-Ferrand gave another proof using her theory of quasiconformal mappings
1995: Schoen: New proof using completely new ideas

Remark. *In the pseudo-Riemannian case, the analog of Theorem is wrong (a counterexample in signature* $(2, n-1)$ *of Frances). In the lorenz signature, the question is still open.*

Proof

Let ϕ is a conformal transformation of F. Then, it is a conformal transformation of g_F. By the Riemannian version of Main Theorem, the following cases are possible:

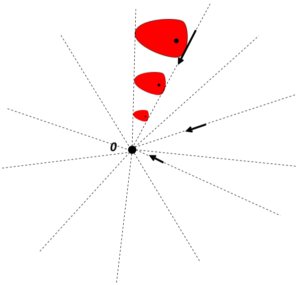
(Trivial case): ϕ is an isometry of a certain $\lambda \cdot g_F$.
Then, it is an isometry of $\lambda^2 \cdot F$.

(Case R^n): After the multiplication of F by an appropriate function, g_F is the standard Euclidean metric, and ϕ is a similarity of g_F. Then, as we have shown above, F is Minkowski.

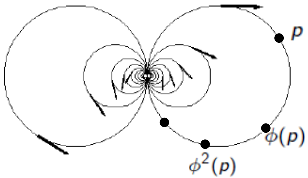
(Case S^n): After the multiplication F by an appropriate function, g_F is the standard metric on the sphere, and ϕ is a möbius transformation of the sphere.

(Case S^n): After the multiplication F be an appropriate function, g_F is the standard "round" (Riemannian) metric on the sphere

Conformal transformation of S^n were described by J. Liouville 1850 in dim n = 2, and by S. Lie 1872. For the sphere, the analog of the picture (a) for the conformal transformation (which are homotheties) of R^n is the picture (b).



Picture (a)



Picture (b)

We will generalize our proof for R^n to the case S^n (the principal observation that sequence of the points $p, \phi(p), \phi^2(p), \ldots$ converges to a fixed point is also true on the sphere; the analysis is slightly more complicated).

Facts: J. Liouville 1850, S. Lie 1872

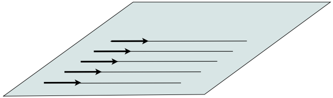
Fact 1. Let ϕ be a conformal nonisometric orientation-preserving transformation of the round sphere (S, g_{round}). Then, there exists a one parameter subgroup $(R, +) \subset \mathrm{Conf}(S, g_{\mathrm{round}})$ containing ϕ.

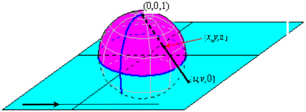
Fakt 2. Any one-parametric subgroup of $(R, +) \subset \mathrm{Conf}(S, g_{\mathrm{round}})$ which is not a subgroup of $Iso(S, g_{\mathrm{round}})$ can be constructed by one of the following ways:

- Way 1. (General case)

  (i) One takes the sliding rotation $\Phi_t : x \to \exp(tA) + tv$, where A is a skew-symmetric matrix such that the vector v is its eigenvector

  (ii) and then pullback this transformation to the sphere with the help of stereographic projection

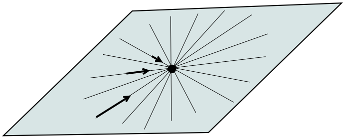




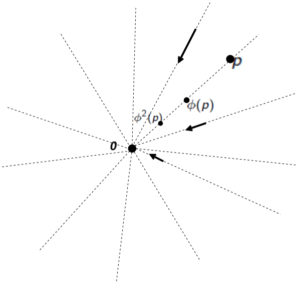
- Way 2. (Special case)

  (i) One takes $\Psi \circ \Phi$, where Φ is a homothety on the plane and Ψ is a rotation on the plane

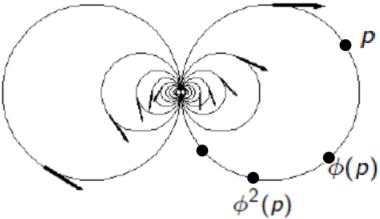
  (ii) and then pullback this transformation to the sphere with the help of stereographic projection



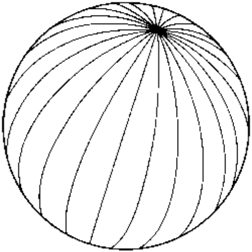
A neighborhood of the pole on the sphere is as on the picture:



Special case



General case



in the special case two points of the sphere have such neighborhood (south and north pols), in the general case only one

The proof for the special case

In this case, the metric is conformally Minkowski, as every metric admitting a similarity transformations. If it is not Riemannian, the manifold is finitely covered either by a torus T^n or by $S^{n-1} \times S^1$ which is not the case.

Thus, it is Riemannian as we want.

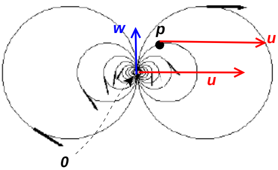
The proof for the general case

We have: the finsler metric F is invariant with respect to ϕ. We consider the following two functions:

$$M(q) := \max_{\eta\in T_qS^n,\ \eta\neq 0} \frac{F(q,\eta)}{\sqrt{g_{(q)}(\eta,\eta)}} - \min_{\eta\in T_qS^n,\ \eta\neq 0} \frac{F(q,\eta)}{\sqrt{g_{(q)}(\eta,\eta)}}.$$

$M(q)=0 \iff F(q,\cdot)$ **is proportional to** $\sqrt{g_{(q)}(\cdot,\cdot)}$**.**

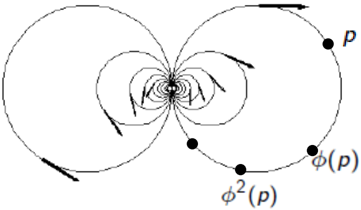
$m(q) := \frac{F(q,v(q))}{g_{(q)}(v(q),v(q))}$, where v is the generator of the 1-parameter group of the conformal transformations containing ϕ. **Both functions are continuous and invariant with respect to ϕ.** We need to show that the function M is identically zero; we first do it at the point 0.



We will show that for every vector u at 0 we have $\frac{F(0,u)}{\sqrt{g_{(0)}(u,u)}} = \frac{F(0,w)}{\sqrt{g_{(0)}(w,w)}}$, where w is as on the picture. We take a point p very close to 0 such that at this point u is proportional to v with a positive coefficient. Such points exist in arbitrary small neighborhood of 0. We have:

$$\frac{F(p,u)}{\sqrt{g_{(p)}(u,u)}} = \frac{F(p,v)}{\sqrt{g_{(p)}(v,v)}} := m(p) \overset{m(p) \text{ is invariant w.r.t. } \phi}{=} m(0) = \frac{F(0,w)}{\sqrt{g_{(0)}(w,w)}}.$$

Replacing p by a sequence of the points converging to 0 (such that at these points u is proportional to v) we obtain that $\frac{F(0,u)}{\sqrt{g_{(0)}(u,u)}} = \frac{F(0,w)}{\sqrt{g_{(0)}(w,w)}}$ implying $M(q)=0$ implying $F(0,\cdot)=\lambda\cdot\sqrt{g_{(0)}(\cdot,\cdot)}$.



We have:

- $M(0) = 0$,
- M is invariant w.r.t. ϕ and continuous,
- For every point p the sequence $p, \phi(p), \phi^2(p), \ldots$ converges to 0.

Then, $M \equiv 0$ implying the metric F is actually a Riemannian metric, $\square$

Solution of Deng-Hou conjecture

Def. The Finsler manifold (M, F) is called *locally symmetric*, if for every point $x \in M$ there exists $r = r(x) > 0$ (called the symmetry radius) and an isometry $\tilde{I}_x : B_r(x) \to B_r(x)$ (called the *reflexion* at x) such that $\tilde{I}_x(x) = x$ and $d_x(\tilde{I}_x) = -\mathrm{id} : T_xM \to T_xM$.

Def. A Finsler metric is *Berwald*, if there exists a symmetric affine connection $\Gamma = (\Gamma^i_{jk})$ such that the parallel transport with respect to this connection preserves the function F.

Theorem. Let (M, F) be a C^2-smooth Finsler manifold. If (M, F) is locally symmetric, then F is Berwald.

Remark. This theorem answers positively a conjecture of Deng-Hou 2009, where it has been proved for globally symmetric spaces.

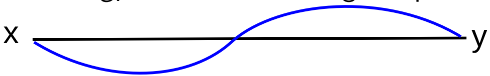
Remark. Locally symmetric Berwald metrics are easy to construct — take the Levi-Civita connection ∇ of a locally symmetric Riemannian manifolds, choose a reversible norm at one T_xM invariant with respect to the holonomy group, and extend the norm to all points $y \in M$ with the help of parallel transport. The obtained finsler metric is then automatically invariant w.r.t. the reflections.

Corollary. Every locally symmetric C^2-smooth Finsler manifold is locally isometric to a globally symmetric Finsler space.

Proof under the additional assumption that the symmetry radius is locally bounded from zero.

The Binet-Legendge metric is a locally symmetric metric. Let us now show that the metrics g_F and F are affinely equivalent, that is, for every arclength parameterised F-geodesic $\tilde\gamma$ there exists a nonzero constant c such that $\tilde\gamma(c \cdot t)$ is an arclength parameterised g_F-geodesic.

It is sufficient to show that for every sufficiently close points $x, y \in M$ the midpoints of the geodesic segments γ and $\tilde\gamma$ in the metrics g_F and F connecting the points x and y coincide.



Indeed, if it is true, then the geodesics γ and $\tilde\gamma$ coincide on its dense subset implying they coincide.

Take a short F-geodesic $\tilde{\gamma} : [-\tilde{\varepsilon}, \tilde{\varepsilon}] \to M$. Let $\gamma : [-\varepsilon, \varepsilon] \to W$ be the unique shortest g_F-geodesic such that $\gamma(-\varepsilon) = \tilde{\gamma}(-\tilde{\varepsilon})$ and $\gamma(\varepsilon) = \tilde{\gamma}(\tilde{\varepsilon})$. Let $x = \tilde{\gamma}(0)$ be the midpoint of $\tilde{\gamma}$ and let I_x be the g_F reflexion centered at x. Then, $I_x(\gamma(-\varepsilon)) = I_x(\tilde{\gamma}(-\tilde{\varepsilon})) = \tilde{\gamma}(\tilde{\varepsilon}) = \gamma(\varepsilon)$ implying $I_x(\gamma(0)) = \gamma(0)$. By uniqueness of the fixed point of I_x, it follows that $\gamma(0) = x = \tilde{\gamma}(0)$.
Thus, all geodesic segments γ and $\tilde{\gamma}$ coincide after the affine reparameterization By the classical result of Chern-Shen, the metric F is Berwald.

Conformal invariants of finsler metrics

Def. Conformal invariants of (M, F) are functions on M canonically constructed by F and invariant w.r.t. conformal change $F \to \lambda(x) \cdot F$.

In the Riemannian case, it is hard to construct them. In the Finsler case, the metric g_F helps:
We define conformal invariants via the Steiner Formula:

$$Vol(B_F + t \cdot B) = \sum_{j=0}^{n} \binom{n}{j} W_j(B_F) t^j,$$

where B_F is the 1-ball in F, B is the 1-Ball in g_F, Vol is in g_F, and everything is done in one tangent space.
These numbers $W_j(x)$ depend only on $F_{|T_xM}$ and are the same for F and $\lambda(x) \cdot F$!!!!

One can construct two more invariants:

$$M(x) = \max_{\xi \in T_xM} \frac{F(x,\xi)}{\sqrt{g(\xi,\xi)}} \quad \text{and} \quad m(x) = \min_{\xi \in T_xM} \frac{F(x,\xi)}{\sqrt{g(\xi,\xi)}}.$$

Thus, in the generic case we obtain $n+2$ "easy to calculate" scalar invariants.

Thank you for your attention!!!