Split signature conformal metrics and half-dimensional projective structures

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Plan

— remind and compare Nurowski–Sparling and Dunajski–Tod construction,

— generalities on Fefferman-type constructions and intro to the parabolic-geometric view,

— some natural questions, especially, on higher dim analogies,

— some answers, especially, on the model situation and the target space in general,

— various remarks, especially, on the feedback to the initial material.

Nurowski–Sparling co.

Rough content of [NS'03][1]:

— 2-order ODE $y'' = Q(x, y, y')$ mod point transf $\rightsquigarrow$ conformal metric of signature $(2, 2)$,

— treated via Cartan's equivalence method as a different real form of the Fefferman metric co.

[1] Nurowski–Sparling, *3-dim CR structures and 2-order ODEs*, 2003

Nurowski–Sparling co. (detail)

Some detail:

— write the eqn as $p = y'$, $p' = Q(x, y, p)$,

— 1-dim subdistribution in the contact distribution on J^1

$$dp - Q\,dx = 0,\ dy - p\,dx = 0,$$

— assoc (normal) Cartan connection on a principal bundle $\mathcal{G}$

$$\omega = \begin{pmatrix} \frac{1}{3}(2\Omega_2 + \bar{\Omega}_2) & i\bar{\Omega}_3 & -\frac{1}{2}\Omega_4 \\ \theta^1 & \frac{1}{3}(\bar{\Omega}_2 - \Omega_2) & -\frac{1}{2}\Omega_3 \\ 2\theta^3 & 2i\theta^2 & -\frac{1}{3}(2\bar{\Omega}_2 + \Omega_2) \end{pmatrix},$$

where i is a non-zero *real* constant, ...

— in this frame, the metric on a 4-dim quotient $\mathcal{G}/\sim$ given by

$$g_F = 2\theta^1\theta^2 + \frac{2}{3i}\theta^3(\Omega_2 - \bar{\Omega}_2),$$

Nurowski–Sparling co. (detail cont.)

— by construction, g_F is expressible in terms of Q, Q_p, ...

— g_F has signature $(2, 2)$,

— essential curvature invariants on both sides, are nicely proportional one another, in particular,

Corollary

(Half-)trivial eqns $\leftrightsquigarrow$ *(half-)flat Fefferman metrics.*

Dunajski–Tod co.

Rough content of [DT'10][2]:

— general necessary conditions,

— equivalent condition in the ASD case,

— "Riemannian extension" from projective str and link to the metrizability problem.

[2] Dunajski–Tod, *4-dim metrics conformal to Kähler*, 2010

Dunajski–Tod co. (ASD case)

Prolongation and Thm 2.3...

Theorem

4-dim conformal ASD str contains a Kähler metric iff there is a non-zero section of the tractor bundle $\Lambda^3_+ \tilde{T}$ *whose injective part is non-degenerate and which is parallel with respect to an non-normal tractor connection.*

Dunajski–Tod co. (Riemannian extension)

Given 2-dim projective str $[\Gamma]$ on U and local coords (x^i, z^j) on TU.

Riemannian extension of $[\Gamma]$ is the conformal str on TU given by

$$g_R = dz_i dx^i - \Pi_{ij}^k \, z_k \, dx^i dx^j,$$

where $\Pi_{ij}^k = \Gamma_{ij}^k - \frac{1}{3}\Gamma_{li}^l \delta_j^k - \frac{1}{3}\Gamma_{lj}^l \delta_i^k$ are Thomas projective parameters. (sl. 26)

Fact

Riemannian extension has signature $(2, 2)$*, is ASD, and admits a null conformal Killing vector…*

The characterization by Prop 4.2… (sl. 24)

Dunajski–Tod co. (metrizability)

Thm 4.1. states:

Theorem

Projective str on U is metrizable iff its Riemannian extension contains a (para-)Kähler metric.

(sl. 24)

Comparing

Projective structure $[\Gamma]$ on U^2 $\leftrightsquigarrow$ geodesic eqn

$$y'' = A_0 + A_1 y' + A_1 y'^2 + A_3 y'^3,$$

where $A_0 = -\Gamma^2_{11}$, $A_1 = \Gamma^1_{11} - 2\Gamma^2_{11}$, $A_2 = 2\Gamma^1_{12} - \Gamma^2_{22}$, and $A_3 = \Gamma^1_{22}$.
Corresponding Thomas parameters are:
$\Pi^1_{11} = \frac{1}{3}A_1$, $\Pi^1_{12} = \frac{1}{3}A_2$, $\Pi^1_{22} = A_3$, $\Pi^2_{11} = -A_0$, $\Pi^2_{21} = -\frac{1}{3}A_1$,
$\Pi^2_{22} = -\frac{1}{3}A_2$.
Subs into g_F and g_R from [NS'03] and [DT'10], respectively:

Claim

g_F and g_R are conformal.

Fefferman extension (revised)

Original Fefferman co., interpreted as an extension of Cartan geometries, is fully determined by the embedding $SU(2,1) \to SO(4,2)\dots$

Further generalized to any dim and sign, powered by the embedding $G = SU(p+1,q+1) \to SO(2p+2,2q+2) = \tilde{G}$:

— start with $(\mathcal{G} \to M, \omega)$, the normal Cartan geometry of type G/P assoc. to the CR str on M,

— let $\tilde{P}$ the Poincaré subgroup in $\tilde{G}$ and $Q := G \cap \tilde{P}$,

— observe $Q \subset P$ and $G/Q = \tilde{G}/\tilde{P}$,

— denote $\tilde{M} := \mathcal{G}/Q$, the Fefferman space,

— def $\tilde{\mathcal{G}} := \mathcal{G} \times_Q \tilde{P}$ and extend equivariantly ω to $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \tilde{\mathfrak{g}})$,

— altogether, $(\tilde{\mathcal{G}} \to \tilde{M}, \tilde{\omega})$ is a Cartan geometry of type $\tilde{G}/\tilde{P}$.

Fefferman extension (normality)

Necessary control of the normality condition [ČG'07][3]:

Theorem

Let ω be the normal. Then $\tilde{\omega}$ is normal iff ω is torsion-free.

Note that

— torsion-freeness of $\omega \leftrightsquigarrow$ integrability of the CR str,

— automatically satisfied if $\dim M = 3$,

— curvature of $\tilde{\omega}$ is fully determined by the curvature of ω, in particular (and in general), $\tilde{\omega}$ is flat iff ω is flat. (sl. 22)

[3] Čap–Gover, *CR tractors and the Fefferman space*, 2007

Fefferman extension (characterization)

Fefferman metrics from CR str's are nicely characterized [ČG'10][4]:

Theorem

If $\tilde{M}$ admits a parallel and OG complex structure $\mathbb{J}$ on the standard tractor bundle, then $\tilde{M}$ is locally conf. equivalent to the Fefferman space of a CR mfld...

Note that

— orthogonality $\rightsquigarrow$ skew-symmetry of $\mathbb{J}$ $\rightsquigarrow$ parallel section of the adjoint tractor bundle $\rightsquigarrow$ null conformal Killing vector on $\tilde{M}$ which inserts trivially into the curvature tensors,

— these (and consequences of $\mathbb{J} \circ \mathbb{J} = -\mathrm{id}$) yield the Sparling's characterization,

— all the study starts with a good understanding of the model situation! (sl. 22)

[4] Čap–Gover, *A holonomy characterization of Fefferman spaces*, 2010

Natural ideas and questions

[NS'03] provides a split real form of the classical Fefferman co. in 3-dim case; a natural analogy in general dim starts with *Lagrangean contact* structures.

— May also this version be treated in similarly nice manner as the classical one?

— If yes, what is the proper interpretation of the Fefferman space?

— In particular, how to deal in model situation?

[DT'10] provides a characterization of Riemannian/Fefferman extensions from *projective* structures in 2-dim case.

— What can one add to this point?

— In particular, what about possible generalizations and different views?

— What about the metrizability problem?

Lagrangean contact str

$=$ contact structure $H \subset TM$ with a fixed decomposition $H = E \oplus V$ into Lagrangean subspaces (equiv. an almost para-complex str $J \circ J = \mathrm{id}_H$)

$=$ parabolic geometry of type $PGL(n+1, \mathbb{R})/P$, where …

Model $= \mathrm{Flag}_{1,n}(\mathbb{R}^{n+1}) = \mathcal{P}T^*\mathbb{RP}^n$ where

$H =$ canonical contact distribution,
$V =$ vertical subbundle of $\mathcal{P}T^*\mathbb{RP}^n \to \mathbb{RP}^n$, and
$E =$ determ. by the flat projective str on $\mathbb{RP}^n$.

Harmonic curvatures, torsion-freeness vs. integrability, …

Choice $G = SL(n+1, \mathbb{R}) \rightsquigarrow$ an additional geom. data…

From projective to Lagrangean contact

More generally [T'94][5]:

projective structure on $X \rightsquigarrow$ Lagrangean contact str on $\mathcal{P}T^*X$

Correspondence space co. [Č'05][6]:

let $(\mathcal{G} \to X, \omega)$ be normal Cartan geometry of type G/P_1 assoc. to the projective str. on X and let $P \subset P_1$ be the parabolic subgroup as above $\rightsquigarrow$ Cartan geometry $(\mathcal{G} \to \mathcal{G}/P, \omega)$ of type G/P.

Theorem

$\mathcal{G}/P \cong \mathcal{P}T^*X$ and $(\mathcal{G} \to \mathcal{G}/P, \omega)$ is the normal Cartan geometry to the induced Lagrangean contact str; harmonic curvatures $K = T^V = 0$ and $T^E \propto W$, the projective Weyl tensor. Moreover, this provides a local characterization.

Case $n = 2$ is, of course, special...

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[5] Takeuchi, *Lagrangean contact str. on projective cotangent bundles*, 1994

[6] Čap, *Correspondence spaces and twistor spaces for parabolic geom.*, 2005

Para-complex vector space

Let $\mathbb{V} = \mathbb{R}^{2n+2}$ with a real inner product h and a skew-symmetric para-complex structure J, i.e.

$$J \circ J = \mathrm{id} \text{ and } h(J-,-) + h(-,J-) = 0.$$

The compatibility of h and J yields, in particular,

— the eigenspaces $\mathbb{V}_\pm$ of J are isotropic,

— h has split signature,

— $h(X,X) = 0$ iff $h(JX,JX) = 0$ iff $\langle X, JX \rangle$ is isotropic.

Embedding

Given $\mathbb{V} = \mathbb{R}^{2n+2}$, $h \in S^2\mathbb{V}^*$, and compatible $J \in \mathrm{End}(\mathbb{V})$ as above.

$\tilde{G} := SO(h) \cong SO(n+1, n+1)$, def $\bar{G} := \{A \in \tilde{G} : A \circ J = J \circ A\}$.

Hence $\bar{G} \cong GL(n+1, \mathbb{R})$.

Appropriate matrix realization...

Reduce to $G := SL(n+1, \mathbb{R})$...

Note that

— $G \subset \tilde{G}$ is the standard embedding,

— for $n = 2$, it is conjugate to [NS'03], ...

Embedding (cont.)

Denote $\mathcal{N} \subset \mathbb{V}$ the null-cone of h,
remind $\mathbb{V}_\pm \subset \mathcal{N}$, denote $\mathcal{N}_0 := \mathcal{N} \setminus \mathbb{V}_\pm$.
$\mathcal{P}\mathcal{N} \cong \tilde{G}/\tilde{P} =$ conformal (n, n) sphere; consists of three G-orbit:

$$\mathcal{P}\mathcal{N} = \mathcal{P}\mathbb{V}_+ \cup \mathcal{P}\mathcal{N}_0 \cup \mathcal{P}\mathbb{V}_-.$$

Para-complex (null) lines

$=$ real (isotropic) planes of the form $\langle X, JX\rangle$; abbrev. $\bar{\mathbb{C}}$ (null) lines.

Facts:

— $X \in \mathcal{N}_0 \Longrightarrow \langle X, JX\rangle$ is a $\bar{\mathbb{C}}$ null line in $\mathcal{N}$,

— any $\bar{\mathbb{C}}$ null line $\langle X, JX\rangle$ determined by a pair $Y_\pm := X \pm JX \in \mathbb{V}_\pm$,

— that pair is orthogonal, $h(Y_+, Y_-) = 0$.

Denote $\tilde{M} := \mathcal{P}\mathcal{N}_0 = \{\mathbb{R}\text{-lines in } \mathcal{N}_0\}$, define $M := \{\bar{\mathbb{C}}\text{-lines in } \mathcal{N}\}$.

Claim

$\tilde{M} \cong G/Q, \quad M \cong \mathrm{Flag}_{1,n}(\mathbb{R}^{n+1}) \cong G/P,$
$\mathcal{P}\mathbb{V}_+ \cong \mathbb{RP}^n \cong G/P_1$ and $\mathcal{P}\mathbb{V}_- \cong \mathbb{RP}^{n*} \cong G/P_2$,
where $P_1 \cap P_2 = P \subset Q \ldots$

Fefferman space

Fefferman space in general $\tilde{M} := \mathcal{G}/Q$.

Typical fibre of $\tilde{M} \to M$ is $P/Q \cong \mathbb{R} \setminus \{0\}$.

According to standard conventions:

Claim

$\tilde{M} \cong$ (double cover of) the scale bundle $\mathcal{E}(1, -1)$ over M.

If $M = \mathcal{P}T^*X$ then $\tilde{M} \cong T^*X[2]$ (without the zero section).

From Lagrangean contact to conformal

Now launch the extension procedure for $(\mathcal{G} \to M, \omega)$ over the embedding $G = SL(n, \mathbb{R}) \subset SO(n, n) = \tilde{G}$ and mimic selected classical results:

……

Cf., in particular, the normality and the characterization aspects. (sl. 3,4)

From projective to conformal

Compose the previous two steps:

If $n > 2$ then normal projective $X \rightsquigarrow$ normal Lagrangean contact $M = \mathcal{P}T^*X$ with half-torsion $\rightsquigarrow$ “half-normal” conformal Cartan connection on $\tilde{M}$, cf. [HS][7].

If $n = 2$ then go to the next slide.

[7] Hammerl–Sagerschnig, *A non-normal Fefferman-type construction of split-signature conformal structures admitting twistor spinor*, preprint

Back to $n = 2$

— Normal projective $X \rightsquigarrow$ normal conformal $\tilde{M}$ which is ASD and admits a parallel anti-OG para-complex structure on $\tilde{T}$. (sl. 8)

— Both the metrizability and Kählerity is char'd as a solution of an ODS , cf. [BDE'10][8], [DT'10] $\leftrightsquigarrow$ parallel sections of a tractor bundle w.r. to a *non*-normal connection, cf. [HSSŠ'10][9]. (sl. 9)

Namely, the appropriate G-, resp. $\tilde{G}$-bundles are $S^2 T$, resp. $\Lambda^3_+ \tilde{T}$.

Now $G \subset \tilde{G} \rightsquigarrow S^2 T \subset \Lambda^3_+ \tilde{T}, \ldots\ldots!$

[8] Bryant–Dunajski–Eastwood, *Metrizability of 2-dim projective structures*, 2010

[9] Hammerl–Somberg–Souček–Šilhan, *On a new normalization for tractor covariant derivatives*, 2010

Back to the model

Remind the model definitions within $\mathbb{V} = \mathbb{R}^{n+1,n+1}$: (sl. 20)

$\tilde{M} = \{\mathbb{R}\text{-lines in } \mathcal{N}_0\}$,
$M = \{\bar{\mathbb{C}}\text{-lines in } \mathcal{N}\} \cong \mathcal{P}T^*\mathbb{RP}^n$, the model Lagrangean contact str.

— In particular,

$\tilde{M} \subsetneq \{\mathbb{R}\text{-lines in } \mathcal{N}\} = L^{n,n}$, the Lie quadric,
$M \subsetneq \{\text{isotropic 2-planes in } \mathcal{N}\} \cong \mathcal{P}T^*S^{n,n-1}$, the model Lie contact str.

— The correspondence $\mathbb{RP}^n \leftarrow \mathrm{Flag}_{1,n}(\mathbb{R}^{n+1}) \to \mathbb{RP}^{n*}$ is visible within $\mathcal{PN} \cong \tilde{G}/\tilde{P}$ via $(x,\eta) \in \mathbb{RP}^n \times \mathbb{RP}^{n*} \leftrightsquigarrow (X,Y) \in \mathbb{V}_+ \times \mathbb{V}_-$:

$$x \in \ker\eta \text{ iff } h(X,Y) = 0.$$

Thomas projective parameters

Remind the definition of Π_{ij}^{k}, which is somehow related to the Thomas ambient connection… (sl. 8)

What about an ambient reinterpretation of all the story?