TRACTOR BUNDLES IN CR GEOMETRY

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This is an outline of my lecture at the Hayama Symposium on several complex variables on December 21, 2002.

The subject of the talk are new applications of the canonical Cartan connection for CR manifolds. Most of the results I will report on are based on joint research (partly in progress) with A.R. Gover and with J. Slovák and V. Souček. Partly, these results are specific for the class of CR structures but many of them have analogs in the much more general setting of Cartan geometries corresponding to arbitrary parabolic subgroups in semisimple Lie groups, the so–called *parabolic geometries.*

A central ingredient in these new developments is a special class of natural vector bundles called *tractor bundles.* These are equivalent to the Cartan bundle and Cartan connection and directly lead to an invariant calculus.

1. Partially integrable almost CR structures

In this article, we will only be concerned with almost CR structures of hypersurface type. It should be remarked at this point that there is also a relation between parabolic geometries and certain almost CR structures of codimension two, see [13] and [6], but this is an entirely different story.

1.1. **Basic definitions.** An *almost CR manifold of hypersurface type* is a smooth manifold M of Dimension $2n+1$ endowed with a rank n complex subbundle H in the tangent bundle TM. The complex structure on H is denoted by J. Given such a structure, we define $Q := TM/H$ and denote by $q : TM \to Q$ the natural quotient map.

The *Levi–bracket* $\mathcal{L} : H \times H \to Q$ is the tensorial operation induced by the Lie bracket of vector fields, i.e. it is characterized by $\mathcal{L}(\xi,\eta) = q([\xi,\eta])$ for $\xi,\eta \in \Gamma(H)$. The almost CR structure (M,H,J) is called *non–degenerate* if $\mathcal{L}(\xi,\eta) = 0$ for all $\eta \in H$ implies $\xi = 0$ and it is called *partially integrable* if $\mathcal{L}(J\xi,J\eta) = \mathcal{L}(\xi,\eta)$ for all $\xi,\eta \in H$.

Passing to the complexification $TM \otimes \mathbb{C}$ of the tangent bundle, the subbundle $H \otimes \mathbb{C}$ splits into a holomorphic and an anti–holomorphic part, i.e. $H \otimes \mathbb{C} = H^{1,0} \oplus H^{0,1}$. Partial integrability is easily seen to be equivalent to $[\xi,\eta] \in \Gamma(H^{1,0} \oplus H^{0,1})$ for all $\xi,\eta \in \Gamma(H^{0,1})$, so this is indeed a weakening of the usual integrability condition (see below). Assuming partial integrability, the Levi bracket $\mathcal{L}$ is (up to a nonzero factor) the imaginary part of the usual Levi form, so one obtains the usual notion of non–degeneracy. From now on, we will restrict our attention to non–degenerate partially integrable almost CR (abbreviated as p.i.a.CR) manifolds.

The *signature* (p,q) of a p.i.a.CR manifold is defined as the signature of the Levi form. We require $p \geq q$ for this to be unambiguous.

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A p.i.a.CR manifold (M, H, J) is called *integrable* or a *CR manifold* if the subbundle $H^{0,1}$ of $H \otimes \mathbb{C}$ is involutive, i.e. $[\xi, \eta] \in \Gamma(H^{0,1})$ for all $\xi, \eta \in \Gamma(H^{0,1})$. Equivalently, one may require vanishing of the *Nijenhuis–tensor* N which is the section of $\Lambda^2 H^* \otimes H$ characterized by

$$N(\xi, \eta) = [\xi, \eta] - [J\xi, J\eta] + J([J\xi, \eta] + [\xi, J\eta]).$$

1.2. **The homogeneous model.** The basic idea behind the canonical Cartan connection is to view general p.i.a.CR manifolds as "curved analogs" of a homogeneous model. Also the other constructions discussed in this article carry over some aspects of this homogeneous model to general p.i.a.CR manifolds, so a good understanding of this homogeneous model is important for the further developments.

Consider $\mathbb{V} = \mathbb{C}^{p+q+2}$ endowed with a Hermitian form $\langle\ ,\ \rangle$ of signature $(p+1, q+1)$ and the *null cone* $C := \{0 \neq v \in \mathbb{V} : \langle v, v\rangle = 0\}$. Let M be the space of complex lines in C. For $v \in C$, the tangent space T_vC is a real hyperplane in $\mathbb{V}$, and the orthogonal complement $v^\perp$ of v is a complex hyperplane contained in T_vC. It is easy to see that the images of these complex hyperplanes in the tangent spaces of M endow M with an almost CR structure which turns out to be integrable and non–degenerate of signature (p, q). Alternatively, one may deduce this CR structure from the fact that M is a smooth real hypersurface in the projectivization $\mathcal{P}(\mathbb{V}) \cong \mathbb{C}P^{p+q+1}$.

Let $G = SU(\mathbb{V})$ be the special unitary group of $\langle\ ,\ \rangle$ and let $\underline{G} = PSU(\mathbb{V})$ be the associated projective group, i.e. the quotient of G by its center. The standard action of G on $\mathbb{V}$ induces a transitive action of G on M, which factors to a transitive action of $\underline{G}$. From the above description of the CR structure on M it is obvious that these actions are by CR diffeomorphisms, and it turns out that the second one leads to an identification of $\underline{G}$ with the group of all CR diffeomorphisms of M.

Denoting by P and $\underline{P}$ the stabilizers of some base point $x_0 \in M$ and G and $\underline{G}$, we see that $G/P \cong \underline{G}/\underline{P} \cong M$. The map $p : G \to M$, $p(g) := g \cdot x_0$ is a principal fiber bundle with structure group P, and likewise for $\underline{p} : \underline{G} \to M$.

1.3. **The canonical Cartan connection.** Let (M, H, J) be an arbitrary p.i.a.CR manifold of signature (p, q), and let $\mathfrak{g} = \mathfrak{su}(\mathbb{V})$ be the Lie algebra of G and $\underline{G}$. Then there exists a canonical principal $\underline{P}$–bundle $\underline{p} : \underline{\mathcal{G}} \to M$ endowed with a Cartan connection $\underline{\omega} \in \Omega^1(\underline{\mathcal{G}}, \mathfrak{g})$, whose curvature satisfies a normalization condition. Conversely, a principal $\underline{P}$–bundle over any smooth manifold M endowed with a regular Cartan connection (whose existence implies that M has dimension $2p+2q+1$) gives rise to a p.i.a.CR–structure of signature (p, q) on M. Moreover, local CR diffeomorphisms induce principal bundle morphisms which are compatible with the Cartan connections and vice versa, so this leads to an equivalence of categories. The (integrable) CR structures correspond exactly to those normal Cartan connections which are torsion–free.

This result was proved in [8] by E. Cartan for $\dim(M) = 3$, by N. Tanaka in [14, 15] in complete generality, and by S.S. Chern and J. Moser in [9] for CR structures of arbitrary dimension.

Of course, the principal bundle $\underline{p} : \underline{\mathcal{G}} \to M$ is an analog of the bundle $\underline{G} \to \underline{G}/\underline{P}$ for the homogeneous model. The Cartan connection should be thought of as an analog of the left Maurer–Cartan form on the group $\underline{G}$. The point here is that the actions of elements of $\underline{G}$ are characterized among the principal bundle automorphisms of the bundle $\underline{G} \to \underline{G}/\underline{P}$ by the fact that they pull back the Maurer–Cartan form to itself.

In order to pass to the equivalent picture of tractor bundles, it is useful to extend the bundle $\underline{\mathcal{G}}$ to a principal P–bundle $p : \mathcal{G} \to M$ (which is then automatically endowed with a canonical normal Cartan connection). It turns out that such an extension is equivalent to the choice of a complex line bundle $\mathcal{E}(1,0)$ such that $\mathcal{E}(1,0)^{\otimes n+2} \cong \Lambda^{p+q}_{\mathbb{C}} H \otimes Q$, so this boils down to choosing an $n+2$'nd root of the canonical bundle. While existence of such a bundle may be obstructed in general, there are no problems locally and for boundaries of domains. We will usually assume that such a choice has been made.

Having chosen the line bundle $\mathcal{E}(1,0)$, one defines complex line bundles $\mathcal{E}(k,\ell)$ for all $k,\ell \in \mathbb{Z}$ by using tensor powers, dual and conjugate bundles. Further one defines real line bundles $\mathcal{E}[2] := Q$ and then $\mathcal{E}[2k]$ for $k \in \mathbb{Z}$ via tensor powers and duals. The convention is chose in such a way that $\mathcal{E}[2k]$ naturally includes into $\mathcal{E}(k,k)$. Without the choice of $\mathcal{E}(1,0)$, all the real line bundles $\mathcal{E}[2k]$ do exist, but $\mathcal{E}(k,\ell)$ is only available if $k-\ell$ is an integer multiple of $n+2$. All these line bundles, which are usually referred to as *density bundles* are examples of irreducible bundles (see below).

1.4. **Natural vector bundles.** Having the canonical Cartan bundle at hand, there is an obvious notion of natural vector bundles over p.i.a.CR manifolds, namely associated vector bundles to the principal Cartan bundle $\mathcal{G}$. It is well known that such associated vector bundles are in bijective correspondence with representations of the structure group P of the principal bundle. In the case of the homogeneous model G/P, one obtains exactly the homogeneous vector bundles in this way.

Now it turns out that the group P is the semidirect product of a reductive subgroup $G_0 \subset P$ and a nilpotent normal vector subgroup $P_+ \subset P$. The reductive part G_0 is isomorphic to the conformal unitary group $CU(p,q)$ and there is a complex Heisenberg algebra $\mathfrak{p}_+ \cong \mathbb{C}^{p+q} \oplus \mathbb{R} \subset \mathfrak{p}$ such that the exponential map restricts to a diffeomorphism $\mathfrak{p}_+ \to P_+$. Hence the representation theory of P is complicated and general representations tend to be unmanageable, but there are two simple classes of representations.

Completely reducible representations are given by arbitrary representations of G_0 with trivial P_+–action. Any such representation splits into direct sum of irreducible representations, which are classified by their highest weights. These highest weights are best viewed as weights for $\mathfrak{g}$ which satisfy certain dominancy and integrality conditions, see e.g. [2, chapter 3]. Bundles associated to irreducible representations of P are referred to as *irreducible bundles.* Any such bundle can be realized as a subbundle in a tensor product of copies of the bundle H, its conjugate, and density bundles.

Restrictions of representations of G: Of course, any representation of G may be viewed as a representation of P. Such a representation splits into a direct sum of G–irreducibles, which are classified by dominant integral weights for $\mathfrak{g}$. Irreducible G–representations are typically indecomposable as P–representations but have non–trivial P–invariant subspaces. Such representations lead to *tractor bundles*. In the case of the homogeneous model, any tractor bundle is canonically trivial, but this is no longer true for general p.i.a.CR structures.

Any tractor bundle can be realized as a subbundle in some tensor product of copies of the *standard tractor bundle* $\mathcal{T}$, which corresponds to the standard representation of G on $\mathbb{V}$, and its conjugate. Another important example of a tractor bundle is the *adjoint tractor bundle* $\mathcal{A}$ which corresponds to the adjoint representation of G on $\mathfrak{g}$.

Tractor bundles (for conformal and projective structures) have been first introduced by Tracy Thomas in the 1930's as an alternative to Cartan's approach to these structures, see e.g. [16]. They were rediscovered and the theory was developed both for special structures and in a general setting during the last 20 years, see e.g. [1, 5].

1.5. **The standard tractor bundle.** By construction, the standard tractor bundle $\mathcal{T}$ over a p.i.a.CR manifold M is a rank $n+2$ complex vector bundle, which carries a canonical Hermitian bundle metric h of signature $(p+1, q+1)$. Since P stabilizes a null line in $\mathbb{V}$, there is a natural subbundle $\mathcal{T}^1 \subset \mathcal{T}$ consisting of complex null lines. Putting $\mathcal{T}^0 = (\mathcal{T}^1)^\perp$, one obtains a filtration $\mathcal{T} \supset \mathcal{T}^0 \supset \mathcal{T}^1$. One easily verifies that $\mathcal{T}^1 \cong \mathcal{E}(-1,0)$, $\mathcal{T}/\mathcal{T}^0 \cong \mathcal{E}(0,1)$ and $\mathcal{T}^0/\mathcal{T}^1 \cong H \otimes \mathcal{E}(-1,0) =: H(-1,0)$. Finally, there is a canonical global section τ of the complex line bundle $\Lambda^{n+2}\mathcal{T}$.

It turns out that the canonical Cartan connection ω induces on any tractor bundle a canonical linear connection. In particular, we get the *standard tractor connection* $\nabla^{\mathcal{T}}$ on $\mathcal{T}$. By construction this connection is Hermitian with respect to h and τ is parallel for the induced connection. It turns out that the curvature of $\nabla^{\mathcal{T}}$ equals the curvature of ω and that $(\mathcal{G}, \omega)$ can be recovered from $(\mathcal{T}, \mathcal{T}^1, h, \tau, \nabla^{\mathcal{T}})$.

Any choice of a pseudo–Hermitian structure, i.e. a (local) trivialization of the bundle Q, on (M, H, J) induces an isomorphism

$$\mathcal{T} \cong \mathcal{E}(-1,0) \oplus H(-1,0) \oplus \mathcal{E}(0,1).$$

In terms of the resulting triples the filtration components are given by vanishing of the last respectively the last two components, and there is a simple formula for the tractor metric h. The change of this isomorphism under a change of pseudo–Hermitian structure can be easily described explicitly. This point of view is used to define the standard tractor bundle in [11]. There is an explicit formula for $\nabla^{\mathcal{T}}$ in triples in terms of the Webster–Tanaka connection associated to the pseudo–Hermitian structure and its torsion and curvature. Hence choosing a pseudo–Hermitian structure, one obtains a completely explicit description of all data on $\mathcal{T}$. The adjoint tractor bundle $\mathcal{A}$ can be identified with the bundle $\mathfrak{su}(\mathcal{T})$ of tracefree skew Hermitian endomorphisms of $\mathcal{T}$. Under this identification, the natural algebraic Lie bracket on $\mathcal{A}$ is given by the commutator of endomorphisms and the tractor connection $\nabla^{\mathcal{A}}$ is induced from $\nabla^{\mathcal{T}}$. Hence also all relevant data on $\mathcal{A}$ can be described explicitly after the choice of a pseudo–Hermitian structure.

2. Invariant differential operators

The standard tractor connection offers a first step towards a CR–invariant calculus. However it is not satisfactory, since the cotangent bundle is not a tractor bundle and thus there is no way to iterate tractor connections to obtain higher derivatives. It follows from a surprising connection to representation theory that finding CR invariant operators between irreducible bundles is a subtle problem:

2.1. **Invariant operators on the homogeneous model.** As we have noted in 1.4 above, natural vector bundles give rise to homogeneous vector bundles on the homogeneous model G/P. There is a canonical G–action on the space of sections of such a bundle, and one may look for G–equivariant differential operators between sections of two such bundles. Any such operator extends canonically to a natural operator on the subcategory of spherical CR manifolds (i.e. those which are locally CR–diffeomorphic to the homogeneous model). On the other hand it turns out that such invariant operators are in bijective correspondence with homomorphisms between certain induced modules, see [2, chapter 11]. If one restricts to the case of irreducible bundles, then the relevant modules are generalized Verma modules, and representation theory leads to detailed information about homomorphisms between those.

The main restriction on existence of invariant operators comes from the fact that nonzero homomorphisms can exist only between generalized Verma modules which have the same infinitesimal character. By a classical theorem of Harish–Chandra the infinitesimal characters agree if and only if the highest weights lie in the same orbit of the affine action of the Weyl group. The dominancy conditions further restrict the possibilities to a certain subset of the Weyl group, which is best visualized as a directed graph. This graph is called the *Hasse graph* of the parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$. For example, in the case relevant for 7 dimensional p.i.a.CR manifolds, this graph has the form

$$\begin{tikzcd}
 & & & E_{3,0} \arrow[r] \arrow[ddr] & F_{3,0} \arrow[dr] & & & \\
 & & E_{2,0} \arrow[ur] \arrow[dr] & & & F_{2,0} \arrow[dr] & & \\
 & E_{1,0} \arrow[ur] \arrow[dr] & & E_{2,1} \arrow[r] \arrow[uur] \arrow[ddr] & F_{2,1} \arrow[ur] \arrow[dr] & & F_{1,0} \arrow[dr] & \\
E_{0,0} \arrow[ur] \arrow[dr] & & E_{1,1} \arrow[ur] \arrow[dr] & & & F_{1,1} \arrow[ur] \arrow[dr] & & F_{0,0} \\
 & E_{0,1} \arrow[ur] \arrow[dr] & & E_{1,2} \arrow[r] \arrow[uur] \arrow[ddr] & F_{1,2} \arrow[ur] \arrow[dr] & & F_{0,1} \arrow[ur] & \\
 & & E_{0,2} \arrow[ur] \arrow[dr] & & & F_{0,2} \arrow[ur] & & \\
 & & & E_{0,3} \arrow[r] \arrow[uur] & F_{0,3} \arrow[ur] & & &
\end{tikzcd}$$

Here the numbering of the vertices is just for later reference. This pattern has the form of the Rumin complex (decomposed into irreducibles) which computes the complex cohomology.

Given a highest weight, there is a simple algorithm (see [2, section 4]) to compute the affine Weyl orbit. In particular, any Weyl orbit contains a unique dominant weight, so these weights can be used for indexing the different patterns. If that weight lies in the interior of the dominant Weyl chamber ("regular infinitesimal character"), then the affine Weyl orbit looks exactly like the picture above. If it lies in a wall ("singular infinitesimal character"), the pattern collapses in the sense that some vertices have to be removed and/or several vertices correspond to the same weight.

By the work of Bernstein–Gelfand–Gelfand and Lepowsky (see [3, 12]) any arrow in the above pattern really gives rise to a nonzero homomorphism ("standard homomorphisms"). Besides these and their compositions, there may be further homomorphisms, called nonstandard–homomorphisms. In our case, these homomorphisms map $E_{i,j}$ to $F_{i,j}$ for $(i,j) = (0,0)$, $(1,0)$, and $(0,1)$. Essentially all these homomorphisms are unique up to scale.

If the weight indexing the pattern is dominant and integral and has regular infinitesimal character, then there is a unique irreducible representation $\mathbb{W}$ of $\mathfrak{g}$ having the weight corresponding to the initial point in the pattern and in [3, 12] it is shown that one may distribute signs among the homomorphisms in such a way that the whole pattern forms a resolution of the $\mathfrak{g}$–module $\mathbb{W}$ ("BGG–resolutions").

Translated to the language of bundles, this means that given any irreducible bundle there are invariant operators from at most three other and to at most three other irreducible bundles, and these bundles can be determined algorithmically. The orders of these operators are determined by the weights and may be arbitrarily high. Hence if one wants to construct invariant differential operators between such bundles from simpler operators, one has to leave the realm of irreducible bundles.

2.2. **Two basic invariant operators.** Tractor bundles directly lead to two basic sets of CR invariant operators which have the advantage that they both can be iterated in order to obtain higher order operators.

- For any natural vector bundle $E \to M$, there is the *fundamental derivative* or *fundamental D–operator*

$$\Gamma(E) \to \Gamma(\mathcal{A}^* \otimes E),$$

see [5]. This is a family of first order operators which is natural with respect to any bundle map induced by a P–homomorphism and satisfies a Leibniz rule for tensor products, so it is like a covariant derivative but with the adjoint tractor bundle $\mathcal{A}$ replacing the tangent bundle TM (which turns out to be a quotient of $\mathcal{A}$). For subquotients of tractor bundles, the fundamental derivative can be computed from the tractor connection. Since there is no restriction on the bundle E at all, these operators can obviously be iterated.

- *Tractor–D–operators*: Let $\mathcal{V}$ be any tractor bundle, $\mathcal{T}$ the standard tractor bundle, $k, \ell \in \mathbb{Z}$ and consider the weighted tractor bundle $\mathcal{V}(k,\ell) := \mathcal{V} \otimes \mathcal{E}(k,\ell)$. For any such choice there is a second order CR–invariant operator

$$\Gamma(\mathcal{V}(k,\ell)) \to \Gamma(\mathcal{T}^* \otimes \mathcal{V}(k,\ell-1)).$$

There also is a complex conjugate version of this operator. Since the range of these operators are again weighted tractor bundles, they can also be iterated.

The compatibility of the tractor–D–operators with the various projections onto subquotients can be described well. Thus one obtains a calculus which can be used for the construction of natural differential operators, e.g. following the ideas of the curved translation principle. This has been applied in [11]to give a construction of all CR–invariant powers of the sublaplacian that are known to exist.

2.3. **Curved analogs of BGG–resolutions.** For patterns (as described in 2.1 above) with regular infinitesimal character which start with a dominant integral weight, there is a conceptual construction of CR invariant operators generalizing the BGG–resolution on the homogeneous model. This was introduced in [7] and improved in [4].

Let $\mathbb{W}$ be the irreducible representation of $\mathfrak{g}$ with the given highest weight. Then Kostant's version of the Bott–Borel–Weil–theorem shows that the weights in the affine Weyl orbit are exactly the highest weights of the irreducible components of the Lie algebra homology groups of $\mathfrak{p}_+$ with coefficients in $\mathbb{W}$. Let us write $\mathbb{H}_*(\mathfrak{p}_+, \mathbb{W})$ for these homology representations. They can be naturally computed from a standard complex formed of P–homomorphisms $\partial^* : \Lambda^k \mathfrak{p}_+ \otimes \mathbb{W} \to \Lambda^{k-1}\mathfrak{p}_+ \otimes \mathbb{W}$.

Geometrically, $\mathbb{W}$ corresponds to a tractor bundle $\mathcal{W}$, and $\Lambda^k \mathfrak{p}_+ \otimes \mathbb{W}$ corresponds to the bundle $\Lambda^k T^*M \otimes \mathcal{W}$. The P–homomorphisms ∂^* induce bundle maps between the corresponding bundles, so we get natural subbundles

$$im(\partial^*) \subset ker(\partial^*) \subset \Lambda^k T^*M \otimes \mathcal{W}$$

such that $ker(\partial^*)/im(\partial^*)$ is the associated bundle corresponding to the representation $\mathbb{H}_k(\mathfrak{p}_+, \mathbb{W})$. Thus we can realize all the irreducible bundles in our pattern as subquotients of the bundles of $\mathcal{W}$–valued differential forms.

The canonical linear connection $\nabla^{\mathcal{W}}$ on the tractor bundle $\mathcal{W}$ extends to the *covariant exterior derivative* d^∇ on $\mathcal{W}$–valued differential forms and one obtains a twisted de–Rham sequence. On the homogeneous model this is a resolution of the constant sheaf $\mathbb{W}$, while in general it is not a complex.

One constructs computable differential projections from $\Omega^k(M, \mathcal{W})$ onto the spaces of sections of the subbundles $ker(\partial^*)$ and $im(\partial^*)$, which in turn induce a differential splitting of the projection from sections of $ker(\partial^*)$ onto sections of the homology bundles. Using these operations one may compress the d^∇'s to a sequence of higher order invariant differential operators between the homology bundles, called the *BGG–sequence* corresponding to $\mathcal{W}$.

One obtains efficient tools for passing between the two sequences, and in particular, one shows that if the twisted de–Rham sequence is a complex, then so is the BGG–sequence, and both complexes compute the same cohomology, so this is an independent construction for the BGG–resolutions. This correspondence sometimes directly leads to geometric interpretations, e.g. for spherical CR manifolds, the adjoint BGG–sequence is a deformation complex.

One drawback of the curved BGG–sequences is that they are not complexes. Indeed, the composition of two consecutive covariant exterior derivatives is given by the action of the Cartan curvature. This can be used to compute the composition of consecutive operators in a BGG–sequence. In joint work in progress with V. Souček, we use elementary representation theory to conclude the existence of subcomplexes under torsion–freeness and/or semi–flatness assumptions. This applies to the case of CR manifolds (as opposed to p.i.a.CR structures). In the case of dimension 7 the result is that in the pattern displayed before, all subsequences which always go up or down are subcomplexes. (The horizontal arrows are not involved in any subcomplex.) This means that using the notation of the diagram, in each pattern we have subcomplexes corresponding for example to $E_{0,0} \to E_{1,0} \to E_{2,0} \to E_{3,0}$, and to $E_{2,0} \to E_{2,1} \to F_{1,2} \to F_{0,2}$, and so on.

Remarkably, in the case of the adjoint BGG–sequence, the subcomplex in the beginning of the sequence still may be interpreted as a deformation complex, but this time in terms of deformations in the category of CR structures (rather than p.i.a.CR structures):

- The kernel of the first operator is a subspace of $\Gamma(Q)$ isomorphic to the space of infinitesimal automorphisms of the CR structure.
- The first homology of the subcomplex is the formal tangent space to the space of all CR deformations, i.e. the quotient of all infinitesimal CR deformations by the trivial infinitesimal CR deformations.

3. Relations to the Fefferman space and to the ambient metric construction

The contents of this last section are based on joint work (partly in progress) with A.R. Gover.

3.1. **Tractors and the Fefferman space.** For a CR structure (M, H, J) of signature (p, q), the Fefferman space $\tilde{M}$ is the total space of a circle bundle over M, on which the CR structure canonically induces a conformal structure of signature $(2p+1, 2q+1)$. For the homogeneous model G/P as described in 1.2, the Fefferman space is simply space of all *real* null lines in $\mathbb{V}$. It is easy to see that $G = SU(\mathbb{V})$ also acts transitively on that space, so we may view it as $G/\tilde{P}$, where $\tilde{P} \subset P$ is the stabilizer in G of a real null line. The group of all conformal diffeomorphisms of this Fefferman space is then simply the group $SO(\mathbb{V})$.

In general, one may define $\tilde{M}$ as a quotient of the frame bundle of $\mathcal{E}(-1,0)$. The CR Cartan bundle $\mathcal{G}$ can then be viewed as a principal $\tilde{P}$–bundle over $\tilde{M}$. As in the CR case, conformal structures may equivalently described by a standard tractor bundle. It is easy to show that the bundle $\tilde{\mathcal{T}} = \mathcal{G} \times_{\tilde{P}} \mathbb{V} \to \tilde{M}$ can be used as a standard tractor bundle for a conformal structure on $\tilde{M}$, which thus is canonically associated to the CR structure on M. Further, one shows that the normal CR Cartan connection induces the conformal normal standard tractor connection on $\tilde{\mathcal{T}}$.

There is a canonical $U(1)$–action on $\tilde{M}$ with orbit space M. This lifts to an action by vector bundle maps on $\tilde{\mathcal{T}}$ whose orbit space is exactly the CR standard tractor bundle $\mathcal{T} \to M$. This construction leads to a parallel orthogonal complex structure on the conformal standard tractor bundle of $\tilde{M}$, which can be used to decompose the conformal adjoint tractor bundle into pieces that are related (via a $U(1)$–action) to certain tractor bundles on M. These constructions are compatible with the fundamental derivatives and the tractor D–operators. A similar correspondence is available for more general bundles.

From the construction, one immediately concludes that the Cartan curvatures of the conformal structure on $\tilde{M}$ and of the CR structure on M are essentially the same object. Further, one may descend conformally invariant differential operators on $\tilde{M}$ to families of CR invariant differential operators on M (including formulae). One obtains a precise relation between infinitesimal automorphisms of M and of $\tilde{M}$, and one can prove a number of results on the conformal geometry of $\tilde{M}$ (existence of twistor spinors and odd degree conformal Killing forms, etc.). Finally, it can be shown that the orthogonal parallel complex structure on the standard tractor bundle locally characterizes Fefferman spaces among all conformal structures.

3.2. **Tractors and the ambient metric.** Consider an embedded non–degenerate CR manifold $M \subset \mathbb{C}^{n+1}$ of signature (p,q). Fefferman's ambient metric construction (see [10]) mimics the homogeneous model by first extending M to the cone $M_\# := \mathbb{C}^* \times M \subset \mathbb{C}^* \times \mathbb{C}^{n+1}$ and then defining a pseudo–Kähler metric of signature $(p+1,q+1)$ locally around that cone. Starting with a smooth *defining function* r for M, i.e. a smooth real valued function defined locally around M such that $M = r^{-1}(0)$ and dr is nonzero on M, one considers the defining function $r_\#(z_0,z) := |z_0|^2 r(z)$ for $M_\#$. It turns out that this can be used as the potential for a pseudo–Kähler metric g of the right signature. Note that by construction the metric g as well as its Levi–Civita connection and curvature are explicitly computable from r.

Of course, this depends on the choice of the defining function, which has to be normalized in order to make things CR invariant. Fefferman showed that this can be achieved by requiring r to be an approximate solution of the *complex Monge–Ampère equation* $J(r) = 1$ and gave a simple algorithm to construct approximate solutions in the sense that $J(r) = 1 + O(r^{n+2})$.

For any choice of defining function r, one may define a complex line bundle $\mathcal{E}(1,0)$ (with the required properties) as an associated bundle to $M_\# \to M$. After this choice, sections of $\mathcal{E}(k,\ell)$ can be identified with homogeneous functions on $M_\#$. Next, one defines a lift of the natural $\mathbb{C}^*$–action on $M_\#$ to the restriction $T(\mathbb{C}^* \times \mathbb{C}^{n+1})|_{M_\#}$ of the ambient tangent bundle. The orbit space $\mathcal{T}$ of this action is a smooth rank $n+2$ complex vector bundle over $M_\#/\mathbb{C}^* = M$. It is easy to see that the ambient pseudo–Kähler metric g descends to a Hermitian bundle metric h of signature $(p+1,q+1)$ on $\mathcal{T}$. The vertical tangent bundle descends to a complex line bundle $\mathcal{T}^1 \subset \mathcal{T}$ which consists of null–lines and thus induces an appropriate filtration on $\mathcal{T}$. Finally, the Levi–Civita connection of g descends to a Hermitian connection $\nabla^{\mathcal{T}}$ on $\mathcal{T}$.

In this generality $\mathcal{T}$ is not a CR standard tractor bundle, since it is not clear whether $\Lambda^{n+2}\mathcal{T}$ admits a parallel section. However, one proves that $\mathcal{A} = \mathfrak{su}(\mathcal{T})$ is an adjoint tractor bundle and the connection $\nabla^{\mathcal{A}}$ induced by $\nabla^{\mathcal{T}}$ is a tractor connection. More subtle arguments show that this tractor connection is normal iff the Ricci curvature of the ambient metric g vanishes along $M_\#$.

Finally, suppose that we deal with a *normalized* defining function r satisfying $J(r) = 1 + O(r^3)$. Then one directly constructs a global section of $\Lambda^{n+2}\mathcal{T}$ which is parallel for the connection induced by $\nabla^{\mathcal{T}}$. Thus, $\mathcal{T}$ becomes a CR standard tractor bundle and $\nabla^{\mathcal{T}}$ is a tractor connection on $\mathcal{T}$. Moreover, it is well know that $J(r) = 1 + O(r^3)$ implies that the associated ambient metric is Ricci flat along $M_\#$, which implies that $\mathcal{T}$ is the normal standard tractor bundle and $\nabla^{\mathcal{T}}$ is the normal standard tractor connection.

In this way, one obtains explicit formulae for all the ingredients for tractor calculus in terms of the normalized defining function r. By construction, the ambient curvature descends to the tractor curvature, which equals the Cartan curvature. For defining functions r such that $J(r) = 1 + O(r^k)$ for some $k > 3$, one may obtain additional information on the ambient curvature and its covariant derivatives up to a certain order from the tractor curvature. Using $-i\partial r$ as a pseudo–Hermitian structure, the tractor machinery leads to explicit formulae for the underlying structures (Weyl– and Webster–Tanaka connections, Rho–tensors) in terms of r.

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