On the Fefferman Construction

Andreas Čap

(joint with A. Rod Gover)

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Associate to a non–degenerate CR structure of hypersurface type on a manifold M a circle bundle $\tilde{M} \to M$ and a conformal structure on the total space $\tilde{M}$ which is CR invariant.

History:

- Fefferman (1976): For boundaries of strictly pseudoconvex domains via the ambient metric.
- Burns–Diederich–Shnider (1977), Kuranishi: For abstract CR structures using the canonical Cartan connection.
- Lee, Farris (1986): For abstract CR structures in terms of the Webster–Tanaka connection associated to a choice of contact form.

I will discuss a new interpretation in terms of tractor bundles, which combines many advantages of these three approaches and leads to new perspectives and applications. This is based on tools for parabolic geometries which have been introduced during the last years.

Let (M, H, J) be a non–degenerate almost CR structure of hypersurface type and consider the decomposition $H \otimes \mathbb{C} = H^{1,0} \oplus H^{0,1}$. Then the structure is called *partially integrable* if the bracket of any two sections of $H^{0,1}$ is a section of $H \otimes \mathbb{C}$. Equivalently, the tensorial map $\mathcal{L} : H \times H \to TM/H$ induced by the Lie bracket of vector fields has to satisfy $\mathcal{L}(J\xi, J\eta) = \mathcal{L}(\xi, \eta)$ for all $\xi, \eta \in H$. The signature (p, q) of the structure is then the signature of the Hermitian form whose imaginary part is $\mathcal{L}$.

Consider $\mathbb{V} := \mathbb{C}^{p+q+2}$ and let $\langle\ ,\ \rangle$ be a Hermitian form of signature $(p+1, q+1)$ on $\mathbb{V}$. Put $G := SU(\mathbb{V})$ and let $P \subset G$ be the stabilizer of a fixed complex null line in $\mathbb{V}$. Then G acts transitively on the space of $\mathcal{Q}$ of complex null lines in $\mathbb{V}$, so $\mathcal{Q} \cong G/P$. The form $\langle\ ,\ \rangle$ induces a CR structure of signature (p, q) on $\mathcal{Q}$, and the action of G induces an isomorphism between $G/Z(G)$ and the group of CR automorphisms of $\mathcal{Q}$. This is the homogeneous model for CR structures.

For G/P, the Fefferman construction is very easy: Let $\hat{P} \subset G$ be the stabilizer of a real null line in $\mathbb{V}$. Then $\hat{P} \subset P$ and $P/\hat{P} \cong \mathbb{R}P^1$. Hence $G/\hat{P} \to G/P$ is a circle bundle. Since G acts transitively on the space of real null lines, we see that $G/\hat{P}$ is the space of real null lines and this is well known to carry a conformal structure which is even invariant under the action of $SO(\mathbb{V})$.

Using Cartan connections, it is easy to carry this over to the curved case. First, generalizing results of Chern–Moser and Tanaka one obtains

Theorem. Let (M, H, J) be a partially integrable non degenerate almost CR structure endowed with a complex line bundle $\mathcal{E}(1,0)$ such that $\mathcal{E}(1,0)^{\otimes(n+2)}$ is isomorphic to the canonical bundle. Then there exists a canonical principal P–bundle $p : \mathcal{G} \to M$ endowed with a canonical normal Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of G.

The choice of $\mathcal{E}(1,0)$ causes no problems locally and for boundaries of domains.

Define $\tilde{M} := \mathcal{G}/\hat{P}$, which can be identified with the space of real lines in the complex line bundle $\mathcal{E}(1,0)^*$. This is the total space of a circle bundle over M and the obvious projection $\mathcal{G} \to \tilde{M}$ is a $\hat{P}$–principal bundle. The CR Cartan connection ω can be viewed as a Cartan connection on that bundle, so its class in $\Omega^1(\mathcal{G}, \mathfrak{g}/\hat{\mathfrak{p}})$ induces an isomorphism $T\tilde{M} \cong \mathcal{G} \times_{\hat{P}} \mathfrak{g}/\hat{\mathfrak{p}}$. Now the invariant conformal structure on $G/\hat{P}$ corresponds to a $\hat{P}$–invariant conformal class of inner products on $\mathfrak{g}/\hat{\mathfrak{p}}$, and hence gives rise to a canonical conformal structure on $\tilde{M}$.

One shows that the evident inclusion $G \hookrightarrow SO(\mathbb{V})$ lifts to an inclusion into $\tilde{G} := Spin(\mathbb{V})$. Denoting by $\tilde{P} \subset \tilde{G}$ the stabilizer of the chosen real null line, we clearly have $\hat{P} = G \cap \tilde{P}$. Moreover, considering the extended principal bundle $\tilde{\mathcal{G}} := \mathcal{G} \times_{\hat{P}} \tilde{P} \to M$, there is a unique Cartan connection $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \tilde{\mathfrak{g}})$ which restricts to ω on $T\mathcal{G} \subset T\tilde{\mathcal{G}}$. Regardless of any normalization condition, this Cartan connection induces the above conformal structure and a spin structure on $\tilde{M}$.

Tractor interpretation. The (CR) standard tractor bundle is defined as $\mathcal{T} := \mathcal{G} \times_P \mathbb{V} \to M$. By construction, this is a complex vector bundle of rank $p+q+2$ endowed with a Hermitian metric h, and a smooth complex line subbundle $\mathcal{T}^1$ with isotropic fibers. The Cartan connection ω induces a linear connection $\nabla^{\mathcal{T}}$ on $\mathcal{T}$, called the *normal standard tractor connection*, and $(\mathcal{T}, \mathcal{T}^1, h, \nabla^{\mathcal{T}})$ is an equivalent way to encode $(\mathcal{G}, \omega)$.

Likewise, we can consider the bundle $\tilde{\mathcal{T}} := \mathcal{G} \times_{\hat{P}} \mathbb{V} \to \tilde{M}$, which comes with the same data (including a connection $\nabla^{\tilde{\mathcal{T}}}$) as $\mathcal{T}$ above, plus a distinguished real line subbundle $\tilde{\mathcal{T}}^1 \subset \tilde{\mathcal{T}}$. This is a conformal standard tractor bundle for the canonical conformal structure on $\tilde{M}$.

• basic tractor calculus leads to a generalization of J. Lee's explicit formula for a metric in the conformal class to the partially integrable case.
• For boundaries of domains, the CR standard tractor bundle can be constructed from the ambient metric, which shows that we recover Fefferman's original construction.

Normality. The developments so far only needed the fact that $\nabla^{\tilde{\mathcal{T}}}$ is a tractor connection on $\tilde{\mathcal{T}}$ or equivalently that there is a Cartan connection $\tilde{\omega}$ on $\tilde{\mathcal{G}}$ extending ω. For further applications, it is essential to know whether $\nabla^{\tilde{\mathcal{T}}}$ coincides with the canonical normal tractor connection (or equivalently $\tilde{\omega}$ coincides with the canonical Cartan connection) associated to the conformal structure on $\tilde{M}$. Surprisingly, this is not always the case:

Theorem. The connection $\nabla^{\tilde{\mathcal{T}}}$ is the normal standard tractor connection associated to the conformal structure on $\tilde{M}$ iff the Cartan connection ω is torsion free and hence iff (M, H, J) is integrable (CR).

Hence we see that the relation between the original structure on M and the conformal structure on $\tilde{M}$ is much stronger in the integrable case. This leads to additional applications:

• *Chern–Moser chains on M are the projections of null geodesics on $\tilde{M}$.* This was one of the main applications in the original works of Fefferman and Burns–Diederich–Shnider.

• *Conformally invariant operators on $\tilde{M}$ descend to families of CR invariant operators on M.* Sections of $\mathcal{T} \to M$ can be identified with sections of $\tilde{\mathcal{T}} \to \tilde{M}$, which are covariantly constant in the vertical direction. This extends to various other bundles as well as to a relation between tractor calculi. Using this, one shows that certain conformally invariant operators on $\tilde{M}$ map subspaces of sections of bundles on M to each other, thus descending to CR invariant differential operators.

• *Fefferman spaces as special conformal structures.* Any Fefferman space admits an orthogonal complex structure on the standard tractor bundle. This can be interpreted as the conformal holonomy being contained in $G \subset \tilde{G}$. This holonomy reduction implies existence of nontrivial solutions to certain conformally invariant equations, for example twistor spinors and conformal Killing forms of all odd degrees. Conversely, one shows that any conformal structure with conformal holonomy contained in G is locally isomorphic to a Fefferman space.

• *Conformal Killing fields on Fefferman spaces.* By naturality, any CR automorphism of M lifts to a conformal isometry of $\tilde{M}$. Infinitesimally, this means that any infinitesimal CR automorphism of M lifts to a conformal Killing field on $\tilde{M}$, but in general there are other conformal Killing fields.

Using tractor calculus one shows that any conformal Killing field on $\tilde{M}$ can be uniquely written as the sum $\xi_1+\xi_2+\xi_3$ of three conformal Killing fields, such that ξ_1 descends to an infinitesimal CR automorphism of M, and ξ_2 is a constant multiple of a canonical conformal Killing tangent to the fibers. The field ξ_3 induces a section of a certain vector bundle on M, which solves a CR invariant differential equation. Conversely, solutions of that equation give rise to conformal Killing fields on $\tilde{M}$.