Automorphism groups of parabolic geometries

Andreas Čap

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Cartan geometries

Let G be a Lie group, $H \subset G$ a closed subgroup such that G/H is connected, and let $\mathfrak{h} \subset \mathfrak{g}$ be the corresponding Lie algebras. Try to interpret G as the automorphism group of a differential geometric structure on G/H.

Definition. A *Cartan geometry* of type (G, H) on a smooth manifold M is a principal $H-$bundle $p : \mathcal{G} \to M$ together with a one form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ such that

- $(r^h)^*\omega = \mathrm{Ad}(h)^{-1} \circ \omega$ for all $h \in H$.
- $\omega(\zeta_A) = A$ for all $A \in \mathfrak{h}$.
- $\omega(u) : T_u\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism for all $u \in \mathcal{G}$.

A *morphism* between two Cartan geometries $(\mathcal{G} \to M, \omega)$ and $(\tilde{\mathcal{G}} \to \tilde{M}, \tilde{\omega})$ is a principal bundle homomorphism $\Phi : \mathcal{G} \to \tilde{\mathcal{G}}$ such that $\Phi^*\tilde{\omega} = \omega$.

The *homogeneous model* of the geometry is the principal bundle $G \to G/H$ together with the left Maurer–Cartan form ω^{MC}.

Example Let G be the group of rigid motions of $\mathbb{R}^n$ and $H = O(n) \subset G$, so G/H is Euclidean space $\mathbb{R}^n$. For an n–dimensional Riemannian manifold M let $\mathcal{G}$ be the orthonormal frame bundle. The Levi–Civita connection and the soldering form define a Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$. This leads to an equivalence of categories between n–dimensional Riemannian manifolds and a subcategory of Cartan geometries of type (G, H).

Automorphisms

For a Cartan geometry $(p : \mathcal{G} \to M, \omega)$ of some fixed type (G, H) let $\mathrm{Aut}(\mathcal{G}, \omega)$ be the group of automorphisms. The infinitesimal version of an automorphism $\Phi : \mathcal{G} \to \mathcal{G}$ is a vector field ξ on $\mathcal{G}$ such that $(r^h)^*\xi = \xi$ for all $h \in H$ and such that $\mathcal{L}_\xi \omega = 0$. The space $\mathrm{inf}(\mathcal{G}, \omega)$ of all such vector fields evidently is a Lie subalgebra of $\mathfrak{X}(\mathcal{G})$.

For $A \in \mathfrak{g}$ let $\tilde{A}$ be the "constant vector field" characterized by $\omega(\tilde{A}) = A$. In particular, $\tilde{A} = \zeta_A$ for $A \in \mathfrak{h} \subset \mathfrak{g}$. For $\xi \in \mathrm{inf}(\mathcal{G}, \omega)$ the equation $0 = (\mathcal{L}_\xi \omega)(\tilde{A})$ immediately implies $[\xi, \tilde{A}] = 0$, and we obtain:

Proposition. If M is connected, then for any point $u_0 \in \mathcal{G}$ the map $\xi \mapsto \omega(\xi(u_0))$ defines a linear isomorphism from $\inf(\mathcal{G}, \omega)$ onto a linear subspace $\mathfrak{a} \subset \mathfrak{g}$.

If ξ is a complete vector field on $\mathcal{G}$ then the corresponding one–parameter group of diffeomorphisms is contained in $\mathrm{Aut}(\mathcal{G}, \omega)$ if and only if ξ lies in $\inf(\mathcal{G}, \omega)$. Since the latter space is a finite dimensional Lie subalgebra of $\mathfrak{X}(\mathcal{G})$ a theorem of R. Palais implies

Theorem. The group $\mathrm{Aut}(\mathcal{G}, \omega)$ is a Lie group with Lie algebra given by all complete vector fields contained in $\inf(\mathcal{G}, \omega)$. For connected M, one has $\dim(\mathrm{Aut}(\mathcal{G}, \omega)) \leq \dim(G)$.

For example, we obtain that the isometry group of a connected n–dimensional Riemannian manifold is a Lie group of dimension at most $\frac{n(n+1)}{2}$. This bound is attained for the homogeneous model $\mathbb{R}^n$ but also for S^n, which has isometry group $SO(n+1)$, and thus for a non–flat manifold.

Curvature

Two equivalent descriptions: curvature form $K \in \Omega^2(\mathcal{G}, \mathfrak{g})$ and curvature function $\kappa : \mathcal{G} \to L(\Lambda^2\mathfrak{g}, \mathfrak{g})$ defined by

$$K(\xi, \eta) = d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)]$$
$$\kappa(u)(X, Y) = K(u)(\tilde{X}, \tilde{Y})$$

One verifies that K is H–equivariant and horizontal. Correspondingly, κ is H–equivariant and has values in $L(\Lambda^2(\mathfrak{g}/\mathfrak{h}), \mathfrak{g})$. The curvature turns out to be a complete obstruction to local isomorphism with the homogeneous model.

Let $\xi \in \mathfrak{X}(\mathcal{G})$ be a vector field such that $\mathcal{L}_\xi \omega = 0$. From the definitions one easily concludes that then $\mathcal{L}_\xi K = 0$ and $\xi \cdot \kappa = 0$. If in addition $\xi(u)$ is vertical, and $A = \omega(\xi(u))$, then $\xi(u) = \zeta_A(u)$ and equivariancy of κ implies that $(\zeta_A \cdot \kappa)(u)$ coincides with the algebraic action of $A \in \mathfrak{h}$ on $\kappa(u) \in L(\Lambda^2(\mathfrak{g}/\mathfrak{h}), \mathfrak{g})$. Hence for $\mathfrak{a} = \{\omega(\xi(u_0)) : \xi \in \mathrm{inf}(\mathcal{G}, \omega)\} \subset \mathfrak{g}$ we see that all elements of $\mathfrak{a} \cap \mathfrak{h}$ annihilate $\kappa(u_0) \in L(\Lambda^2(\mathfrak{g}/\mathfrak{h}), \mathfrak{g})$.

The Lie bracket on $\mathrm{inf}(\mathcal{G}, \omega)$

The bracket on the Lie algebra of $\mathrm{Aut}(\mathcal{G}, \omega)$ is induced by the negative of the Lie bracket of vector fields on $\mathcal{G}$, which also makes sense on $\mathrm{inf}(\mathcal{G}, \omega)$. For $\xi \in \mathrm{inf}(\mathcal{G}, \omega)$ and $\eta \in \mathfrak{X}(\mathcal{G})$ we compute

$$\begin{aligned} 0 =& (\mathcal{L}_\xi \omega)(\eta) = \xi \cdot \omega(\eta) - \omega([\xi, \eta]) \\ =& d\omega(\xi, \eta) + \eta \cdot \omega(\xi) \\ =& \kappa(\omega(\xi), \omega(\eta)) - [\omega(\xi), \omega(\eta)] + \eta \cdot \omega(\xi). \end{aligned}$$

Hence for fixed $u_0 \in \mathcal{G}$, the above bracket on $\mathrm{inf}(\mathcal{G}, \omega)$ corresponds to the operation

$$(A, B) \mapsto [A, B] - \kappa(u_0)(A, B) \qquad (*)$$

on $\mathfrak{a} = \{\omega(\xi(u_0)) : \xi \in \mathrm{inf}(\mathcal{G}, \omega)\} \subset \mathfrak{g}$.

Hence we may identify $\mathrm{inf}(\mathcal{G}, \omega)$ with the subspace $\mathfrak{a} \subset \mathfrak{g}$ endowed with Lie bracket given by $(*)$. Recall further that any element of $\mathfrak{a} \cap \mathfrak{h}$ annihilates $\kappa(u_0)$.

Parabolic geometries

Cartan geometries corresponding to parabolic subalgebras in semisimple Lie algebras. Let $\mathfrak{g}$ be a semisimple Lie algebra endowed with a grading of the form $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$, put $\mathfrak{h} := \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$. Choose a Lie group G with Lie algebra $\mathfrak{g}$ and let H be the normalizer of $\mathfrak{h}$ in G. This is equivalent to H being a parabolic subgroup of G in the sense of representation theory.

Putting $\mathfrak{g}^i = \mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_k$ defines an H–invariant filtration $\mathfrak{g} = \mathfrak{g}^{-k} \supset \cdots \supset \mathfrak{g}^k$, which makes $\mathfrak{g}$ into a filtered Lie algebra such that $\mathfrak{h} = \mathfrak{g}^0$. A parabolic geometry of type (G, H) is called *regular*, if its curvature function κ satisfies $\kappa(u)(\mathfrak{g}^i, \mathfrak{g}^j) \subset \mathfrak{g}^{i+j+1}$ for all $u \in \mathcal{G}$ and all $i, j = -k, \ldots, -1$.

Geometric structures like conformal, almost quaternionic, hypersurface type CR, quaternionic CR and many others can be identified with subclasses of regular normal parabolic geometries of some type.

Proposition. Let $(\mathcal{G} \to M, \omega)$ be a regular normal parabolic geometry with curvature function κ. If $\kappa \neq 0$, then the lowest homogeneous component of κ has values in a nontrivial, completely reducible representation of H.

This representation can be computed explicitly for any given type. Since this representation is nontrivial, Aut$(\mathcal{G}, \omega)$ may have the maximal possible dimension dim(G) only if $\kappa = 0$ and thus the parabolic geometry is locally isomorphic to the homogeneous model.

Return to the identification of inf$(\mathcal{G}, \omega)$ with a subspace $\mathfrak{a} \subset \mathfrak{g}$ induced by $\xi \mapsto \omega(\xi(u_0))$ for some fixed point $u_0 \in \mathcal{G}$. Define a filtration on $\mathfrak{a}$ by $\mathfrak{a}^i := \mathfrak{a} \cap \mathfrak{g}^i$ for $i = -k, \dots, k$. By regularity this makes $\mathfrak{a}$ into a filtered Lie algebra, and the inclusion induces a Lie algebra homomorphism gr$(\mathfrak{a}) \to$ gr$(\mathfrak{g}) = \mathfrak{g}$ on the level of the associated graded Lie algebras. Hence gr$(\mathfrak{a})$ (which has the same dimension as $\mathfrak{a}$) is (isomorphic to) a graded Lie subalgebra of $\mathfrak{g}$.

Example: 3–dimensional CR structures

These are 3–dimensional contact manifolds together with a complex structure on the contact subbundle. The typical examples of such structures are given by non–degenerate hypersurfaces in $\mathbb{C}^2$. By a theorem of E. Cartan, these structures admit a canonical normal Cartan connection of type (G,H), where $G = PSU(2,1)$ and $H \subset G$ is a Borel subgroup. This construction identifies the category of 3–dimensional CR manifolds with the category of regular normal parabolic geometries of type (G,H).

The homogeneous model in this case is $S^3 \subset \mathbb{C}^2$. Therefore CR–manifolds which are locally isomorphic to the homogeneous model are called *spherical*.

The general results on Cartan geometries imply that the group Aut(M) of CR automorphisms of a 3–dimensional CR manifold M is a Lie group of dimension $\leq \dim G = 8$. We now claim:

Theorem. (1) If $\dim(\mathrm{Aut}(M)) < 8$, then $\dim(\mathrm{Aut}(M)) \leq 5$.
(2) $\dim(\mathrm{Aut}(M)) \leq 3$ if M is not spherical.

The grading of $\mathfrak{g} = \mathfrak{su}(2,1)$ has the form $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \cdots \oplus \mathfrak{g}_2$ with $\mathfrak{g}_{\pm 2} \cong \mathbb{R}$, $\mathfrak{g}_{\pm 1} \cong \mathbb{C}$ and $\mathfrak{g}_0 \cong \mathbb{C}$. The Lie algebra of $\mathrm{Aut}(M)$ must be contained in $\mathrm{inf}(\mathcal{G}, \omega)$, which gives rise to a graded Lie subalgebra $\mathrm{gr}(\mathfrak{a})$ of $\mathfrak{g}$. Hence we can prove (1) by showing that any proper graded Lie subalgebra of $\mathfrak{g}$ has dimension at most 5.

For (2) one verifies that the representation of $\mathfrak{h}$, in which the lowest nonzero homogeneous component of the curvature has its values comes from a faithful representation of $\mathfrak{g}_0 \cong \mathbb{C}$. Thus we can prove (2) by showing that any graded Lie subalgebra of $\mathfrak{g}$ which has a trivial component in degree 0 has dimension at most 3.

For an appropriate choice of Hermitian metric on $\mathbb{C}^2$ we have

$$\mathfrak{g} = \left\{ \begin{pmatrix} \alpha + i\beta & z & i\psi \\ x & -2i\beta & -\bar{z} \\ i\varphi & -\bar{x} & -\alpha + i\beta \end{pmatrix} \right\}$$

with $\alpha, \beta, \varphi, \psi \in \mathbb{R}$ and $x, z \in \mathbb{C}$. From this, one immediately reads off that the brackets $\mathfrak{g}_{\pm 1} \times \mathfrak{g}_{\pm 1} \to \mathfrak{g}_{\pm 2}$ are given by the standard symplectic form on $\mathbb{C}$, while the brackets between the other grading components are essentially induced by complex multiplications.

Suppose that $\mathfrak{b} = \mathfrak{b}_{-2} \oplus \cdots \oplus \mathfrak{b}_2$ is a graded Lie subalgebra of $\mathfrak{g}$, put $n_i = \dim(\mathfrak{b}_i)$ and $n = \dim(\mathfrak{b})$, where all dimensions are over $\mathbb{R}$.

Case 1: $n_{-1} = 2$. This means that $\mathfrak{b}_{-1} = \mathfrak{g}_{-1}$ and then $[\mathfrak{b}_{-1}, \mathfrak{b}_{-1}] = \mathfrak{g}_{-2} \subset \mathfrak{b}$. Suppose there is a nonzero element $z \in \mathfrak{b}_1$. Then $[z, \mathfrak{b}_{-1}] = \mathfrak{g}_0$ and hence $[z, \mathfrak{g}_0] = \mathfrak{g}_1$ are contained in $\mathfrak{b}$, which immediately implies $\mathfrak{b} = \mathfrak{g}$. Hence we conclude that $\mathfrak{b} \neq \mathfrak{g}$ is only possible if $n_1 = 0$. This implies $n_2 = 0$, since for a nonzero element $i\psi \in \mathfrak{g}_2$ the map $\mathrm{ad}_{i\psi} : \mathfrak{g}_{-1} \to \mathfrak{g}_1$ is surjective. Hence $\mathfrak{b} \subset \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$, and we get (1) and (2).

Case 2: $n_{-1} = 1$. For $0 \neq x \in \mathfrak{b}_{-1}$ the map ad_x is a linear isomorphism $\mathfrak{g}_0 \to \mathfrak{g}_{-1}$ and $\mathfrak{g}_1 \to \mathfrak{g}_0$, so we conclude that $n_0 \leq 1$ and then $n_1 \leq 1$, and for $n_0 = 0$ we also must have $n_1 = 0$. This already gives (1) and (2).

Case 3: $n_{-1} = 0$. Since the bracket induces a linear isomorphism $\mathfrak{g}_{-2} \otimes \mathfrak{g}_1 \to \mathfrak{g}_{-1}$ we conclude that either $n_{-2} = 0$ or $n_1 = 0$. This implies (1) and (2) in this last case, and the proof of the theorem is complete.

This theorem reduces the classification of homogeneous 3–dimensional CR manifolds to pure algebra: In the spherical case, the Lie algebra of the automorphism group is a subalgebra of $\mathfrak{g} = \mathfrak{su}(2,1)$, and one can work in the homogeneous model. If M is not spherical, then $\dim(\mathrm{Aut}(M)) = 3$ and fixing a point $x_0 \in M$ the map $f \mapsto f(x_0)$ is a covering $\mathrm{Aut}(M) \to M$. The CR structure on M lifts to a left invariant structure on $\mathrm{Aut}(M)$.

Hence any non–spherical homogeneous 3–dimensional CR structure is covered by a left invariant structure on a Lie group. Determining such left invariant structures is a purely algebraic problem.

For higher dimensional CR structures, similar methods were used by K. Yamaguchi to determine the second largest possible dimension for the automorphism group. He completely classified the CR structures with automorphism group of this second largest dimension.