Infinitesimal deformations of parabolic geometries

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Basic ideas

• Consider isomorphism class of geometric structures of some type on a manifold M, or equivalently the space of all such structures modulo the action of Diff(M). "*Moduli space*" of such structures.

• Localizing leads to deformations and one may pass further to a formal and infinitesimal level. *Formal tangent space* at the given structure to the moduli space: infinitesimal deformations modulo trivial infinitesimal deformations.

Example Riemannian metrics:

• Infinitesimal deformations of g are sections $h = h_{ab}$ of $\mathcal{E}_{(ab)} = S^2(T^*M)$.

• h is trivial, iff $h = \mathcal{L}_\xi g$ for a vector field $\xi = \xi^a \in \mathfrak{X}(M) = \Gamma(\mathcal{E}^a)$. Then $h_{ab} = \nabla_{(a}\xi_{b)}$ and we obtain the Killing operator $D : \mathcal{E}^a \to \mathcal{E}_{(ab)}$

• ker(D) is the space of Killing fields, i.e. infinitesimal isometries of g, and coker(D) is the formal tangent space at g to $\mathcal{M}$.

Cartan geometries

"Curved analogs" of a homogeneous space G/H. For M with $\dim(M) = \dim(G/H)$, a *Cartan geometry* of type (G, H) on M is a principal H–bundle $p : \mathcal{G} \to M$ plus a *Cartan connection* $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, i.e.

(i) $(r^h)^*\omega = \mathrm{Ad}(h)^{-1} \circ \omega$ for all $h \in H$.
(ii) $\omega(\zeta_A) = A$ for all $A \in \mathfrak{h} \subset \mathfrak{g}$.
(iii) $\omega(u) : T_u\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism for all $u \in \mathcal{G}$.

Morphisms are principal bundle maps compatible with the Cartan connections. The *homogeneous model* is $p : G \to G/H$ with the Maurer–Cartan form as a Cartan connection.

Infinitesimal deformations: principal bundles are rigid, so we can only deform ω. Condition (iii) is open, so infinitesimal deformations are given by $\mathfrak{g}$–valued one forms on $\mathcal{G}$, which satisfy (i) and are horizontal. These are equivalent to elements of $\Omega^1(M, \mathcal{A}M)$, where $\mathcal{A}M := \mathcal{G} \times_H \mathfrak{g}$ is the *adjoint tractor bundle*.

Trivial deformations come from pulling back ω by principal bundle automorphisms. Infinitesimally, one obtains right invariant vector fields $\xi \in \mathfrak{X}(\mathcal{G})^H$. The trivial infinitesimal deformation caused by ξ is $\mathcal{L}_\xi \omega$. Applying ω induces an isomorphism

$$\mathfrak{X}(\mathcal{G})^H \cong C^\infty(\mathcal{G}, \mathfrak{g})^H \cong \Gamma(\mathcal{A}M).$$

Via the Cartan connection ω, one has $TM \cong \mathcal{G} \times_H (\mathfrak{g}/\mathfrak{h})$. In particular, there is a natural bundle map $\Pi : \mathcal{A}M \to TM$, which corresponds to projecting a right invariant field to the base.

The Cartan connection ω induces a canonical linear connection ∇ on $\mathcal{A}M$. Further we have the curvature $K \in \Omega^2(\mathcal{G}, \mathfrak{g})$ defined by

$$K(\xi, \eta) = d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)].$$

It is easy to see that K is horizontal and $H-$equivariant, thus defining $\kappa \in \Omega^2(M, \mathcal{A}M)$. For the homogeneous model, $\kappa = 0$ by the Maurer–Cartan equation, and κ is a complete obstruction to local isomorphism to G/H.

A simple computation shows that if $\xi \in \mathfrak{X}(G)^H$ corresponds to $s \in \Gamma(\mathcal{A}M)$, then $\mathcal{L}_\xi \omega$ corresponds to

$$\tilde{\nabla} s := \nabla s + i_{\Pi(s)}\kappa \in \Omega^1(M, \mathcal{A}M)$$

In particular, infinitesimal automorphisms of $(p : \mathcal{G} \to M, \omega)$ are in bijective correspondence with sections of $\mathcal{A}M$ that are parallel for the linear connection $\tilde{\nabla}$.

Cartan geometries are most interesting when they are equivalent to some underlying structure. This needs a normalization condition on the curvature. Analyzing the effect of an infinitesimal deformation on curvature is subtle, since the identification of TM with an associated bundle depends on the Cartan connection. The infinitesimal change of curvature caused by $\varphi \in \Omega^1(M, \mathcal{A}M)$ turns out to be given by

$$d^\nabla \varphi - i_{\Pi \circ \varphi}\kappa = d^{\tilde{\nabla}} \varphi \in \Omega^2(M, \mathcal{A}M),$$

where we use the covariant exterior derivative. Thus infinitesimal deformations of a Cartan geometry are governed by the twisted de-Rham sequence $(\Omega^*(M, \mathcal{A}M), d^{\tilde{\nabla}})$.

The case of parabolic geometries

These are Cartan geometries of type (G, P), where G is semisimple and $P \subset G$ is a parabolic subgroup. Under the conditions of regularity and normality on the curvature, these are equivalent to underlying structures. Among them, there are important examples like conformal, almost quaternionic, CR, and quaternionic contact structures. From now one, we only deal with regular normal geometries.

Here there is a nilpotent ideal $\mathfrak{p}_+ \subset \mathfrak{p}$, which via the Killing form is dual to $\mathfrak{g}/\mathfrak{p}$. Hence for a geometry $(p : \mathcal{G} \to M, \omega)$, we get $\mathcal{G} \times_P \mathfrak{p}_+ \cong T^*M$. The bracket in $\mathfrak{g}$ makes $\mathcal{A}M$ into a bundle of Lie algebras, which contains T^*M as a bundle of subalgebras. Using the (tensorial) Lie bracket on both bundles, one defines natural bundle maps

$$\partial^* : \Lambda^k T^*M \otimes \mathcal{A}M \to \Lambda^{k-1} T^*M \otimes \mathcal{A}M$$

such that $\partial^* \circ \partial^* = 0$. We also denote by ∂^* the induced operators on $\mathcal{A}M$–valued forms. Normality of a parabolic geometry is defined by $\partial^*(\kappa) = 0$.

The BGG–machinery

Now $\operatorname{im}(\partial^*) \subset \ker(\partial^*) \subset \Lambda^k T^*M \otimes \mathcal{A}M$ are natural subbundles, and $\mathcal{H}_k := \ker(\partial^*)/\operatorname{im}(\partial^*)$ is algorithmically computable via Kostant's version of the BBW-theorem. The BGG–machinery relates operators on $\mathcal{A}M$–valued forms to operators defined on these subquotient bundles. First note that there is a natural tensorial map

$$\pi_H : \Omega^k(M, \mathcal{A}M) \supset \Gamma(\ker(\partial^*)) \to \Gamma(\mathcal{H}_k).$$

The core of the machinery is to construct (higher order) invariant differential operators $L_k : \Gamma(\mathcal{H}_k) \to \Omega^k(M, \mathcal{A}M)$, which split these algebraic projections. These splitting operators are characterized by the additional fact that $\partial^* \circ d^\nabla \circ L_k = 0$ for all k. Having these splitting operators at hand, one defines the *BGG–operators* by

$$D_k := \pi_H \circ d^\nabla \circ L_k : \Gamma(\mathcal{H}_k) \to \Gamma(\mathcal{H}_{k+1}).$$

The machinery can also be applied using $d^{\tilde\nabla}$ rather than d^∇ and we will denote the resulting operators by $\tilde{L}_k$ and $\tilde{D}_k$.

Results

1. $\mathcal{H}_0$ is a quotient of TM, and π_h and $\tilde{L}_0$ induced inverse isomorphisms between the set of infinitesimal automorphisms and the kernel of $\tilde{D}_0$.
Under mild conditions, one can prove that $\tilde{D}_0 = D_0$ and $\tilde{L}_0 = L_0$, but this is not true in general.

2. An infinitesimal deformation $\varphi \in \Omega^1(M, \mathcal{A}M)$ is called *normal* if $\partial^*(d^{\tilde{\nabla}}\varphi) = 0$. Then trivial infinitesimal deformations are normal and the operator $\tilde{L}_1$ induces a bijection between the cokernel of $\tilde{D}_0$ and the formal tangent space of the moduli space of normal parabolic geometries.

3. By normality, one has $\partial^*(\kappa) = 0$, so one may define the *harmonic curvature* $\kappa_H := \pi_H(\kappa) \in \Gamma(\mathcal{H}_2)$. This is much simpler than the full curvature κ but it is still a complete obstruction to local flatness. Under the isomorphism from 2., the operator $\tilde{D}_1$ computes the change of harmonic curvature caused by an infinitesimal deformation. In some cases, one can prove that $\tilde{D}_1 = D_1$.

The locally flat case

If $\kappa = 0$, then $\nabla = \tilde{\nabla}$, and $d^{\nabla} \circ d^{\nabla} = 0$, so $(\Omega^*(M, \mathcal{A}M), d^{\nabla})$ is a complex, which is a fine resolution of the sheaf of parallel sections for ∇. The homology groups in degree 0 and 1 are the space of infinitesimal automorphisms respectively the formal tangent space to the moduli space of locally flat structures.

The BGG machinery easily implies that also $(\mathcal{H}_*, D)$ is a complex which computes the same homology. Hence one obtains a deformation complex for locally flat geometries in the picture of the underlying structure.

Semi–flat cases

For some geometries, the bundle $\mathcal{H}_2$ decomposes into a direct sum of natural bundles. Hence there are various components in the harmonic curvature, and semi–flatness corresponds to vanishing of some of these parts.

The most important examples here are self–duality for four–dimensional conformal structures, torsion freeness for almost quaternionic structures, integrability for CR structures, and torsion freeness for quaternionic contact structures in dimension 7. For higher dimensional quaternionic contact structures, regularity can be equivalently characterized as semi–flatness.

In these cases, all the bundles $\mathcal{H}_k$ decompose into direct sums, and in joint work with V. Souček, we have shown that the resulting BGG–patterns contain various subcomplexes. For all the structures listed above, this leads to a deformation complex in the semi–flat category. For quaternionic structures (and in particular self–dual conformal structures in dimension four) this deformation complex is elliptic.