RANDOM WALKS ON GROUPS, 2023 SS
EXERCISES C

(1) Show that a harmonic function on a finite graph is constant.

(2) Show that a superharmonic function on a graph (not necessarily finite) that realizes a minimum value is constant.

(3) (finite Dirichlet problem) Suppose A is a finite subset of a bounded valence graph X. Given a function $f_0 \colon A^c \to \mathbb{R}$, show there is a unique extension $f \colon X \to \mathbb{R}$ that agrees with f_0 on A^c and is harmonic at every point of A, given by $f(x) = \mathbb{E}_x[f_0(Z_{s^{A^c}})]$, where $s^{A^c} = \min\{n \geq 0 \mid Z_n \in A^c\}$.

Let T be a bounded valence tree, and fix a vertex $\mathcal{O}$ as basepoint. Define ∂T to be the set of geodesic rays based at $\mathcal{O}$; that is, infinite edge paths $e_1e_2e_3\dots$ such that $\mathcal{O} = e_1^-$, $e_i^+ = e_{i+1}^-$, and $e_{i+1} \neq \bar{e}_i$ for all i.

(4) Show that the Gromov product $(x \mid y)_z$ defined last time can be extended ∂T as follows. Let $\xi = e_1e_2e_3\dots$ and $\eta = e_1'e_2'e_3'\dots$ be geodesic rays and $x, y \in T$. Show the sequence $(e_i^- \mid x)_{\mathcal{O}}$ stabilizes, and define this to be $(\xi \mid x)_{\mathcal{O}}$. Similarly, show that for all sufficiently large i and j the function $i, j \mapsto (e_i^- \mid (e_j')^-)_{\mathcal{O}}$ is constant, and define this to be $(\xi \mid \eta)_{\mathcal{O}}$. Show $(\xi \mid \eta)_{\mathcal{O}} = \max\{0\} \cup \{n \in \mathbb{N} \mid \forall i \leq n,\, e_i = e_i'\}$.

(5) Show $d(\xi, \eta) = e^{-(\xi|\eta)_{\mathcal{O}}}$ defines a metric on ∂T.

(6) Show that if e is an edge in T then the set of boundary points that contain e in their defining geodesic ray is a subset of ∂T that is both open and closed.

(7) Show ∂T is compact.

(8) Show that if the vertices of T have valence bounded below by 3 then ∂T contains no isolated points. Brouwer's Theorem then says that a tree with vertices of valence bounded above and bounded below by 3 has ∂T homeomorphic to the Cantor space.

(9) Show that if T and T' are bounded valence trees with no leaves and ∂T is isometric to $\partial T'$ then T is isomorphic to T'.