RANDOM WALKS ON GROUPS, 2023 SS
EXERCISES D

(1) Consider a graph with vertex set $\mathbb{N}_0$ with 2^n edges between n and $n+1$. Show the simple random walk on this graph is transient. Conclude that bounded geometry hypotheses are important.

(2) Show there is a quasi-isometry $T_3 \to T_4$ that is bijective on vertices.

If X is a bounded geometry graph, define a *ray* $\gamma\colon \mathbb{N}_0 \to X$ to be a sequence of vertices $v_0, v_1, \ldots$ such that $\forall i$, $v_i \sim v_{i+1}$. A ray is *proper* if for every bounded set $B \subset X$, $\gamma^{-1}(B)$ is bounded. For any proper ray γ and bounded subset $B \subset X$, B has only finitely many complementary connected components, and one of those contains all but finitely many vertices of γ. Say that γ *ends* in C.

(3) Show there is an equivalence relation on proper rays in X defined by $\gamma \sim \gamma'$ if for every bounded set B, γ and γ' end in the same complementary component of B.

Define $Ends(X)$ to be the set of equivalence classes of proper rays.

(4) Show that for any choice of basepoint $\mathcal{O} \in X$ and every $\mathcal{E} \in Ends(X)$ there is a geodesic ray ξ based at $\mathcal{O}$ that belongs to $\mathcal{E}$.

(5) Show that a quasi-isometry between bounded geometry graphs induces a bijection between their ends.

(6) Compute the cardinality of $Ends(\mathbb{Z})$, $Ends(\mathbb{Z}^2)$, and $Ends(T_3)$. Conclude that none of $\mathbb{Z}$, $\mathbb{Z}^2$, and T_3 are quasi-isometric to one another.