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QUANTIZATIONS
OF BRAIDED DIFFERENTIAL OPERATORS

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Abstract. We construct differential operators (d.o.) in braided tensor categories and define a quantization as a natural isomorphism of the tensor product bifunctor with some additional natural conditions. The quantizations preserve the notion of a braided d.o. and produces a natural procedure which allows one to quantize particular differential operators.

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0. Introduction

In this paper a general quantization scheme is proposed.

Having in mind applications to differential equations we begin with a definition of differential operators in braided tensor categories (*braided differential operators* [L]). The concrete choice of the category depends on two reasons. First, any "reach" theory of differential equations requires some (generalized) commutativity conditions. And second, a quantization is, to our opinion, a "controlled deviation" of commutativity law. To be more precise we define a *quantization* in a monoidal category as a natural isomorphism of the tensor product bifunctor with some additional conditions. These conditions generalize the known quantization schemes: see Moyal-Vey [V], Lichnerowicz [BFFLS], Drinfeld-Jimbo [D1], [D2], [J], Reshetikhin [R], etc.

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We relate quantization procedures to the concept of *calculus* in non-commutative algebras. Note, there exists a number of different approaches to the construction of this calculus: the universal construction for associative algebras[C], [K], [DV], fermionic and colour calculus [JK], [BMO], the calculus for quadratic algebras [WZ], [MY2], etc. All of them are based on various notions of derivative and differential forms and actually present scalar calculus of the 1-st order.

In this paper we suggest a new approach based on the notion of a differential operator, or in our case – *a braided differential operator*. Differential forms, jets, and de Rham and Spencer complexes arise in this approach as representative objects (cf. [KLV]).

Here we study a general scheme only. But in a number of concrete situations (=categories) one can actually compute the quantizations and yield an explicit description of quantum differential operators. We are going to consider some of them in forthcoming publications.

1. Quantizations and braidings

Here we define quantizations of monoidal categories and build up quantizations of algebras and modules in the category. We produce a quantization of braidings and commutative objects in monoidal category too.

1.1. Let $\mathcal{C} = (\mathcal{O}b(\mathcal{C}), \mathcal{M}or(\mathcal{C}))$ be a category with objects $\mathcal{O}b(\mathcal{C})$ and morphisms $\mathcal{M}or(\mathcal{C})$. Recall that a monoidal category is a category $\mathcal{C}$ equipped with:

(1) a bifunctor
$$\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C},$$
of (tensor) multiplication,

(2) a unit object k, with natural isomorphisms (*unit constraints*)
$$\rho : X \otimes k \backsimeq X, \qquad \lambda : k \otimes X \backsimeq X, \in \mathcal{O}b(\mathcal{C}),$$
and

(3) a natural isomorphism
$$\alpha = \alpha_{X,Y,Z} : X \otimes (Y \otimes Z) \backsimeq (X \otimes Y) \otimes Z, \quad X, Y, Z \in \mathcal{O}b(\mathcal{C}),$$
(an *associativity constraint*), such that the pentagon diagram is commutative [ML].

A monoidal category is called *strict*, if all constraints λ, ρ, α are identity arrows. It is known that any monoidal category $\mathcal{C}$ is equivalent to a strict one, say $\hat{\mathcal{C}}$. To see this one can use a linguistic proof. Define objects of category $\hat{\mathcal{C}}$ as words in objects of category $\mathcal{C}$ and arrows as words in arrows. The tensor product in the category is given by the word product. This allows one to consider strict monoidal categories only.

1.2. A *braiding* for a monoidal category $\mathcal{C}$ is a natural isomorphism
$$\sigma_{X,Y} : X \otimes Y \longrightarrow Y \otimes X,$$

such that the two hexagon conditions

$$\sigma_{X\otimes Y,Z} = (\sigma_{X,Z} \otimes id_Z) \circ (id_X \otimes \sigma_{Y,Z}),$$

$$\sigma_{X,Y\otimes Z} = (id_Y \otimes \sigma_{X,Z}) \circ (\sigma_{X,Y} \otimes id_Z),$$

hold [JS].

A braiding σ is called *a symmetry* , if

$$\sigma_{X,Y} \circ \sigma_{Y,X} = id_{Y\otimes X}$$

for all $X, Y \in \mathcal{O}b(\mathcal{C})$.

A monoidal category equipped with a braiding is called a *braided tensor* or *quasi-tensor category* [JS]. A braided tensor category is called a *tensor category* if the braiding is a symmetry.

1.3. A *quantization* of a monoidal category $\mathcal{C}$ is a natural isomorphism $\mathcal{Q}$ of the tensor product bifunctor

$$\mathcal{Q}_{X,Y} : X \otimes Y \longrightarrow X \otimes Y,$$

preserving a unit and such that

$$\mathcal{Q}_{X,Y\otimes Z} \circ (id_X \otimes \mathcal{Q}_{Y,Z}) = \mathcal{Q}_{X\otimes Y,Z} \circ (\mathcal{Q}_{X,Y} \otimes id_Z),$$

for all $X, Y, Z \in \mathcal{O}b(\mathcal{C})$, [L].

Theorem [L]. *Let $\mathcal{Q}$ be a quantization of a braided tensor category $\mathcal{C}$ with a braiding σ. Then $\sigma^q_{X,Y}$ given by the diagram*

$$\begin{tikzcd}
X \otimes Y \arrow[r, "\sigma^q_{X,Y}"] \arrow[d, "\mathcal{Q}_{X,Y}"'] & Y \otimes X \arrow[d, "\mathcal{Q}_{Y,X}"] \\
X \otimes Y \arrow[r, "\sigma_{X,Y}"'] & Y \otimes X
\end{tikzcd}$$

is a braiding in $\mathcal{C}$. $\square$

1.4. For any algebra $(A, \mu), \mu : A \otimes A \longrightarrow A$ in a monoidal category $\mathcal{C}$ with a quantization $\mathcal{Q}$ we define a quantization (A_q, μ^q) of the algebra A as the object $A_q = A$ with new multiplication $\mu^q = \mu \circ \mathcal{Q}_{A,A}$. Similarly, for any left (right, bi-) A-module $(X, \mu_X), \mu_X : A \otimes X \longrightarrow X$, we define the quantization (X_q, μ^q_X), where $X_q = X, \mu^q_X = \mu_X \circ \mathcal{Q}_{A,X}$.

Theorem. *Let (A, μ) be an algebra in a monoidal category with a quantization $\mathcal{Q}$ and let (X, μ_X) be a left (right, bi-) A-module in this category. Then (A_q, μ^q) is an algebra and (X_q, μ^q_X) is an A_q-module in $\mathcal{C}$.*

Proof. Show, for example, that (A_q, μ_q) is an associative algebra. Since $\mathcal{Q}$ is a natural transformation, we have

$$\mathcal{Q}_{A,A} \circ (id_A \otimes \mu) = (id_A \otimes \mu) \circ \mathcal{Q}_{A,A\otimes A},$$

$$\mathcal{Q}_{A,A} \circ (\mu \otimes id_A) = (\mu \otimes id_A) \circ \mathcal{Q}_{A\otimes A,A},$$

and therefore

$$\begin{aligned}\mu_q \circ (id_A \otimes \mu_q) &= \mu \circ \mathcal{Q}_{A,A} \circ (id_A \otimes \mu) \circ (id_A \otimes \mathcal{Q}_{A,A}) \\ &= \mu \circ (id_A \otimes \mu) \circ \mathcal{Q}_{A,A\otimes A} \circ (id_A \otimes \mathcal{Q}_{A,A}),\end{aligned}$$

$$\begin{aligned}\mu_q \circ (\mu_q \otimes id_A) &= \mu \circ \mathcal{Q}_{A,A} \circ (\mu \otimes id_A) \circ (\mathcal{Q}_{A,A} \otimes id_A) \\ &= \mu \circ (\mu \otimes id_A) \circ \mathcal{Q}_{A\otimes A,A} \circ (\mathcal{Q}_{A,A} \otimes id_A).\end{aligned}$$

□

2. Braided differential operators

The purpose of this chapter is to build up the main functors of the calculus (cf. [KLV]) in braided categories. To define differential operators in some category we need an analog of commutativity with usual additivity properties. For this reason, we deal with the abelian, braided tensor categories only.

2.1. Let $\mathcal{C}$ be an abelian category. Say that $\mathcal{C}$ is *monoidal*, if the tensor product functor is a biadditive and right exact (i.e. $\mathcal{C}$ is a *tensored* category [ML2]).

Let (A, μ) be an algebra in a monoidal abelian category $\mathcal{C}$ with a braiding σ. Say that A is a *σ-commutative algebra* if $\mu = \mu \circ \sigma_{A,A}$.

We fix a σ-commutative algebra A and consider $(A - A)$-bimodules over A. Let X be such a bimodule with left $\mu^l_X : A \otimes X \longrightarrow X$ and right $\mu^r_X : X \otimes A \longrightarrow X$ multiplications respectively.

An $(A - A)$-bimodule X is called a *σ-symmetric*, if $\mu^l_X = \mu^r_X \circ \sigma_{X,A}$ and $\mu^r_X = \mu^l_X \circ \sigma_{A,X}$.

Proposition. *Let A be a σ-commutative algebra in a braided tensor category $\mathcal{C}$ and let $\mathcal{Q}$ be a quantization of this category. Then*

1. *(A_q, μ^q)-is a σ^q-commutative algebra.*
2. *For any σ-symmetric $(A - A)$-bimodule (X, μ^l_X, μ^r_X) in $\mathcal{C}$ its quantization $(X_q, (\mu^l_X)^q, (\mu^r_X)^q)$ is a σ^q-symmetric $(A_q - A_q)$-bimodule.*

□

2.2. For any $A-A$-bimodule X define *σ-symmetric part* X_σ of X in the following way:

$$X_\sigma = \{x \in X | (\mu^r_X - \mu^l_X \circ \sigma_{X,A})(x \otimes A) = 0, (\mu^l_X - \mu^r_X \circ \sigma_{A,X})(A \otimes x) = 0\}.$$

Proposition. *X_σ is a σ-symmetric bimodule.* □

2.3. Consider a quotient bimodule X/X_σ and define a bimodule $X^{(1)}_\sigma \subset X$ as the inverse image of the bimodule $(X/X_\sigma)_\sigma \subset X/X_\sigma$with respect to the natural projection $X \longrightarrow X/X_\sigma$.

Thus we have an embedding $X^{(1)}_\sigma \supset X_\sigma$. Applying the procedure to the bimodule $X/X^{(1)}_\sigma$ we get a new bimodule $X^{(2)}_\sigma$, and so on.

Proceeding in this way we obtain a filtration of the bimodule X by bimodules $X_\sigma^{(i)}$:

$$0 = X_\sigma^{(-1)} \subset X_\sigma^{(0)} = X_\sigma \subset X_\sigma^{(1)} \subset \ldots \subset X_\sigma^{(i)} \subset X_\sigma^{(i+1)} \subset \ldots \subset X_\sigma^{(*)} \subset X, \qquad (1)$$

where, by definition, $X_\sigma^{(i+1)} \subset X$ is the inverse image of $(X/X_\sigma^{(i)})_\sigma$ with respect to the projection $X \longrightarrow X/X_\sigma^{(i)}$.

We call a bimodule $X_\sigma^{(*)} = \bigcup_{i\geq 0} X_\sigma^{(i)}$ *a differential approximation* of the $(A-A)$-bimodule X.

Theorem [L]. *A graded object*

$$Gr_*(X) = \sum_{i\geq 0} Gr_i(X),$$

where

$$Gr_i(X) = X_\sigma^{(i)}/X_\sigma^{(i-1)},$$

related to any $(A-A)$-bimodule X over a σ-commutative algebra A is a σ-symmetric $(A-A)$-bimodule. $\square$

2.4. Now we apply the above procedure to bimodules of internal homomorphisms and obtain modules of (σ-) differential operators in braided tensor categories.

First we recall the definition of *internal homomorphisms.*

Suppose that the functor $M_{X,Y} : \mathcal{C}^{op} \longrightarrow \mathcal{S}et$ is defined for a pair of objects $X, Y \in \mathcal{O}b(\mathcal{C})$ of a monoidal category $\mathcal{C}$ by

$$M_{X,Y} : Z \longrightarrow \mathcal{M}or(Z \otimes X, Y)$$

is representable.

Then *a representative object* $Hom(X,Y)$ is called the *internal Hom.* A morphism

$$ev_{X,Y} : Hom(X,Y) \otimes X \longrightarrow Y$$

is called an *evaluation map*, if any morphism $f : Z \otimes X \longrightarrow Y$ can be represented by the composition $f = ev_{X,Y} \circ (\hat{f} \otimes id_X)$ for a unique morphism $\hat{f} : Z \longrightarrow Hom(X,Y)$.

In other words we have functorial isomorphisms

$$\mathcal{M}or(Z, Hom(X,Y)) \xrightarrow{j} \mathcal{M}or(Z \otimes X, Y), \qquad (1)$$
$$j : \hat{f} \longmapsto f = ev_{X,Y} \circ (\hat{f} \otimes id_X).$$

Assume that the internal $Hom(X,Y)$ exists for all objects X, Y, or that $\mathcal{C}$ is a *closed category* [ML].

2.5. Let A be a σ-commutative algebra and X, Y be left A-modules. Applying 2.4.(1) to the composition

$$A \otimes Hom(X,Y) \otimes X \overset{id_A \otimes ev_{X,Y}}{\longrightarrow} A \otimes Y \xrightarrow{\mu_Y} Y$$

we get a left A-module structure on the internal Hom:

$$l_A : A \otimes Hom(X,Y) \longrightarrow Hom(X,Y).$$

Similarly we get a right A-module structure

$$r_A : Hom(X,Y) \otimes A \longrightarrow Hom(X,Y)$$

by using 2.4.(1) and the composition

$$Hom(X,Y) \otimes A \otimes X \overset{id \otimes \mu_X}{\longrightarrow} Hom(X,Y) \otimes X \overset{ev_{X,Y}}{\longrightarrow} Y.$$

Morphisms l_A and r_A define an $(A-A)$-bimodule structure on $Hom(X,Y)$ for any left A-modules X and Y.

Definition. *The differential approximations* $(Hom(X,Y))_\sigma^{(i)}$ *of the* $(A-A)$*-bimodule of internal Hom are called modules of* σ*-differential operators of the order* $\leq i$ *and are denoted by* $\mathcal{D}iff_i^\sigma(X,Y)$.

Keeping in mind the definition of differential approximations we construct the modules of differential operators in a direct way.

Denote by

$$\delta_A^l : A \otimes Hom(X,Y) \longrightarrow Hom(X,Y)$$

and

$$\delta_A^r : Hom(X,Y) \otimes A \longrightarrow Hom(X,Y)$$

the following morphisms:

$$\delta_A^l = l_A - r_A \circ \sigma_{A,H}, \quad \delta_A^r = r_A - l_A \circ \sigma_{H,A},$$

where $H = Hom(X,Y)$.

We define modules $\mathcal{D}iff_i^\sigma(X,Y)$ by induction on i.

Set

$$\mathcal{D}iff_0^\sigma(X,Y) = Hom_A^\sigma(X,Y),$$

where $Hom_A^\sigma(X,Y) = Hom^\sigma(X,Y)$ is a bimodule of σ-morphisms, i.e. a maximal submodule with the following conditions:

$$\delta_A^l(A \otimes Hom_A^\sigma(X,Y)) = 0, \quad \delta_A^r(Hom_A^\sigma(X,Y) \otimes A) = 0.$$

Using the induction hypothesis we define $\mathcal{D}iff_{i+1}^\sigma(X,Y)$ as a maximal submodule of $Hom(X,Y)$ such that

$$\delta_A^l(A \otimes \mathcal{D}iff_{i+1}^\sigma(X,Y)) \subset \mathcal{D}iff_i^\sigma(X,Y), \tag{2}$$

$$\delta_A^r(\mathcal{D}iff_{i+1}^\sigma(X,Y) \otimes A) \subset \mathcal{D}iff_i^\sigma(X,Y). \tag{3}$$

Remark. *If the braiding* σ *is a symmetry, we need condition (1) only.*

We shall use the abbreviation **ABC**-*category* for an abelian, braided tensor and closed category.

Theorem. *In* **ABC**-*categories the internal composition*

$$Hom(Y,Z) \otimes Hom(X,Y) \longrightarrow Hom(X,Z)$$

generates a composition of σ*-differential operators*

$$\mathcal{D}iff_i^\sigma(Y,Z) \otimes \mathcal{D}iff_j^\sigma(X,Y) \longrightarrow \mathcal{D}iff_{i+j}^\sigma(X,Z).$$

A module of σ*-differential operators* $\mathcal{D}iff_*^\sigma(X,X)$ *is an algebra in the category under consideration and* $\mathcal{D}iff_*^\sigma(X,Y)$ *is a left* $\mathcal{D}iff_*^\sigma(Y,Y)$*-module and a right* $\mathcal{D}iff_*^\sigma(X,X)$*-module.* $\square$

2.6. Filtration 2.3.(1) defines a filtration of modules σ-differential operators

$$0 \subset \mathcal{D}iff_0^\sigma(X,Y) \subset \mathcal{D}iff_1^\sigma(X,Y) \subset \ldots \subset \mathcal{D}iff_i^\sigma(X,Y) \subset \mathcal{D}iff_{i+1}^\sigma(X,Y) \subset \ldots$$

Corresponding graded object is denoted by

$$\mathcal{S}mbl_*^\sigma(X,Y) = \sum_{i\geq 0} \mathcal{S}mbl_i^\sigma(X,Y),$$

where

$$\mathcal{S}mbl_i^\sigma(X,Y) = \mathcal{D}iff_i^\sigma(X,Y)/\mathcal{D}iff_{i-1}^\sigma(X,Y).$$

We call $\mathcal{S}mbl_*^\sigma(X,Y)$ a *symbol module.*

Theorem.

(1) *Symbol modules* $\mathcal{S}mbl_j^\sigma(X,Y)$ *are* σ*-symmetric* $(A-A)$*-bimodules.*
(2) *A symbol algebra* $\mathcal{S}mbl_*^\sigma(A,A)$ *is a* σ*-commutative algebra.*

□

2.7. Following the classical scheme and using a σ-commutative structure in the symbol algebra $Smbl_*(A,A)$ we can define a braided analog of a Poisson structure. For this purpose take a pair of σ-differential operators

$$f \in \mathcal{D}iff_i^\sigma(A,A), \quad g \in \mathcal{D}iff_j^\sigma(A,A),$$

and consider their symbols

$$smbl(f) \in Smbl_i^\sigma(A,A), \quad smbl(g) \in Smbl_j^\sigma(A,A).$$

The σ-*commutator* of the differential operators

$$[f,g]^\sigma = \mu^D(f \otimes g - \sigma(H,H)(g \otimes f))$$

is of the order $\leq (i+j-1)$. Here H denotes $Hom(A,A)$ and

$$\mu^D : \mathcal{D}iff_*^\sigma(a,A) \otimes \mathcal{D}iff_*^\sigma(A,A) \longrightarrow \mathcal{D}iff_*^\sigma(A,A)$$

is the multiplication (composition) morphism.

Consider the symbol

$$smbl[f,g]^\sigma \in Smbl_{i+j-1}^\sigma(A,A)$$

which actually depends on $smbl(f)$ and $smbl(g)$ only. We get a σ-*Poisson bracket*

$$\{\cdot,\cdot\}^\sigma : Smbl_i^\sigma(A,A) \otimes Smbl_j^\sigma \longrightarrow Smbl_{i+j-1}^\sigma(A,A)$$

by setting

$$\{smbl(f), smbl(g)\}^\sigma = smbl[f,g]^\sigma.$$

3. Quantizations of differential operators

In this Section we link two main facts:

(1) differential operators are defined if some braiding is fixed, and
(2) a quantization preserves the class of braidings in monoidal categories

with the problem of quantization.

To make the relation clearer we describe here some algebra structure on the set of internal homomorphisms which actually is an algebraic realization of the quantum paradigm in our approach.

3.1. First note that any quantization $\mathcal{Q}$ defines isomorphisms

$$\mathcal{Q}^* : \mathcal{M}or(Z \otimes X, Y) \longrightarrow \mathcal{M}or(Z \otimes X, Y),$$

where $\mathcal{Q}^*(f) = f \circ \mathcal{Q}_{Z,X}^{-1}$ for any morphism $f : Z \otimes X \longrightarrow Y$.

Consequently $\mathcal{Q}$ defines the natural isomorphism $\mathcal{Q}_h$

$$(\mathcal{Q}_h)_{X,Y} : Hom(X, Y) \longrightarrow Hom(X, Y),$$

such that the following diagram

$$\begin{array}{ccc} Mor(Z, Hom(X,Y)) & \xrightarrow{j} & Mor(Z \otimes X, Y) \\ {\scriptstyle \mathcal{Q}_h}\downarrow & & \downarrow{\scriptstyle \mathcal{Q}^*} \\ Mor(Z, Hom(X,Y)) & \xrightarrow[j]{} & Mor(Z \otimes X, Y) \end{array}$$

is commutative for any $Z \in \mathcal{O}b(\mathcal{C})$.

To define $\mathcal{Q}_h$ we represent the composition

$$g = ev_{X,Y} \circ \mathcal{Q}_{H,X}^{-1}, \tag{1}$$

where $H = Hom(X, Y)$, by using the isomorphism j, in the following way:

$$g = ev_{X,Y} \circ ((\mathcal{Q}_h)_{X,Y} \otimes id_X). \tag{2}$$

Then the diagram

$$\begin{array}{ccc} Hom(X,Y) \otimes X & \xrightarrow{ev_{X,Y}} & Y \\ \uparrow{\scriptstyle \mathcal{Q}_{H,X}^{-1}} & & {\scriptstyle ev_{X,Y}}\uparrow \\ Hom(X,Y) \otimes X & \xrightarrow[\mathcal{Q}_h \otimes id_X]{} & Hom(X,Y) \end{array}$$

is commutative. Moreover for any morphism $f : Z \longrightarrow Hom(X, Y)$ using the following commutative diagram

$$\begin{array}{ccc} Z \otimes X & \xrightarrow{f \otimes id_X} & Hom(X,Y) \otimes X \\ {\scriptstyle \mathcal{Q}_{Z,X}}\downarrow & & \downarrow{\scriptstyle \mathcal{Q}_{H,X}} \\ Z \otimes X & \xrightarrow{f \otimes id_X} & Hom(X,Y) \otimes X \end{array}$$

we get

$$\begin{aligned} \mathcal{Q}^*(j(f)) &= j(f) \circ \mathcal{Q}_{Z,X}^{-1} = ev_{X,Y} \circ (f \otimes id_X) \circ \mathcal{Q}_{Z,X}^{-1} \\ &= ev_{X,Y} \circ \mathcal{Q}_{H,X}^{-1} \circ (f \otimes id_X) = ev_{X,Y} \circ ((\mathcal{Q}_h)_{X,Y} \otimes id_X) \circ (f \otimes id_X). \end{aligned}$$

3.2. Denote by

$$\mu^H : Hom(Y, Z) \otimes Hom(X, Y) \longrightarrow Hom(X, Y),$$

$$\mu^H : f \otimes g \mapsto f * g,$$

composition of internal homomorphisms defined in(2.5). The product is an associative partially defined multiplication.

By using the right unit constraint $\rho : X \otimes k \longrightarrow X$ and the isomorphism

$$j : \mathcal{M}or(X, Hom(k, X)) \longrightarrow \mathcal{M}or(X \otimes k, X)),$$

we get the isomorphism

$$j^{-1}(\rho) = l_X : X \simeq Hom(k, X).$$

In terms of isomorphisms l the evaluation maps can be considered as a result of the multiplication $*$.

Indeed

$$\mu^H \circ (id \otimes l_X) = l_Y \circ ev_{X,Y}. \tag{1}$$

If one uses the usual notation for the evaluation map, $ev_{X,Y}(f \otimes x) := f(x)$, then from (1) we get

$$f(x) = f * x, \tag{1´}$$

where we identified the elements $x \in X$ and $f(x) \in Y$ with the internal homomorphisms $l_X(x) \in Hom(k, X)$ and $l_Y(f(x)) \in Hom(k, Y)$ respectively.

Now definition 2.5 can be reformulated by means of $(1')$ in the following way:

$$(f * g) * x := f * (g * x), \tag{2}$$

for all $f \in Hom(Y, Z), \quad g \in Hom(X, Y)$.

Similar to 1.4, we quantize the multiplication μ^H and get the new multiplication

$$f *_q g := \mu^H(\mathcal{Q}_{H_1,H_2}(f \otimes g)), \tag{3}$$

where $H_1 = Hom(Y, Z), \quad H_2 = Hom(X, Y)$.

Now the definition of isomorphisms $\mathcal{Q}_h$ is the following

$$\mathcal{Q}_h^{-1}(f) * x = f *_q x. \tag{4}$$

Lemma. *For any quantization $\mathcal{Q}$ one has*

(1) $\mathcal{Q}_h^{-1}(f) * \mathcal{Q}_h^{-1}(g) = \mathcal{Q}_h^{-1}(f *_q g)$
(2) $\mathcal{Q}_h(x) = x$

for all $x \in X \simeq Hom(k, X), \quad f \in Hom(Y, Z) \quad g \in Hom(X, Y)$.

Proof. Using (2) and (4) we get

$$(\mathcal{Q}_h^{-1}(f) * \mathcal{Q}_h^{-1}(g)) * x = \mathcal{Q}_h^{-1}(f) * (\mathcal{Q}_h^{-1}(g) * x) =$$
$$= f *_q (g *_q x) = (f *_q g) *_q x = \mathcal{Q}_h^{-1}(f *_q g) * x.$$

□

3.3. Let A be an algebra, X and Y be left A-modules.

Any element $a \in A$ defines a pair of internal isomorphisms

$$a_X \in Hom(X,X), \qquad a_Y \in Hom(Y,Y)$$

via isomorphisms

$$\mathcal{M}or(A \otimes X, X) \xrightarrow{j_X^{-1}} \mathcal{M}or(A, Hom(X,X)),$$

where

$$a_X = j^{-1}(\mu_X)(a), \quad a_Y = j^{-1}(\mu_Y)(a).$$

To define the δ-operations we will use the Sweedler notations [Sw]:

$$\sigma_{H,X}(f \otimes x) = \sum x' \otimes f', \quad \sigma_{X,H}(x \otimes f) = \sum f'' \otimes x'',$$

where $x, x', x'' \in X; f, f'f'' \in H = Hom(X,Y)$. Then

$$\delta_a^l(f) = a_Y * f - \sum f' * a'_X, \quad \delta_a^r(f) = f * a_X - \sum a''_Y * f'',$$

and from Lemma 3.2. we obtain the following.

Theorem. *Let $\mathcal{Q}$ be a quantization. Then*

(1) *The isomorphisms $\mathcal{Q}_h^{-1}$ preserves the composition and defines isomorphisms of braided differential operators modules*

$$\mathcal{Q}_h^{-1} : \mathcal{D}iff_i^{\sigma^q}(X,Y) \longrightarrow \mathcal{D}iff_i^{\sigma}(X,Y).$$

(2) *The symbols of $\mathcal{Q}_h^{-1}$ define isomorphisms of symbol modules (***quasiclassical approximation***)*

$$Smbl_*(\mathcal{Q}_h) : Smbl_*^{\sigma^q}(X,Y) \longrightarrow Smbl_*^{\sigma}(X,Y).$$

(3) *In the case $X = Y = A$ the morphism $Smbl_*(\mathcal{Q}_h^{-1})$ is a morphism of braided Poisson algebras.*

□

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