Virtual Eigenvalues of the High Order Schrödinger Operator I

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Vienna, Preprint ESI 1705 (2005)

October 3, 2005

Supported by the Austrian Federal Ministry of Education, Science and Culture
Available via <http://www.esi.ac.at>

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Abstract

We consider the Schrödinger operator $H_\gamma = (-\Delta)^l + \gamma V(\mathbf{x})$ acting in the space $L_2(\mathbb{R}^d)$, where $2l \geq d$, $V(\mathbf{x}) \geq 0$, $V(\mathbf{x})$ is continuous and is not identically zero, and $\lim_{|\mathbf{x}| \rightarrow \infty} V(\mathbf{x}) = 0$. We obtain an asymptotic expansion as $\gamma \uparrow 0$ of the bottom negative eigenvalue of H_γ , which is born at the moment $\gamma = 0$ from the lower bound $\lambda = 0$ of the spectrum $\sigma(H_0)$ of the unperturbed operator $H_0 = (-\Delta)^l$ (a virtual eigenvalue). To this end we develop a supplement to the Birman-Schwinger theory on the process of the birth of eigenvalues in the gap of the spectrum of the unperturbed operator H_0 . Furthermore, we extract a finite-rank portion $\Phi(\lambda)$ from the Birman-Schwinger operator $X_V(\lambda) = V^{\frac{1}{2}} R_\lambda(H_0) V^{\frac{1}{2}}$, which yields the leading terms for the desired asymptotic expansion.

Mathematics Subject Classification 2000, Primary: 47F05, Secondary: 47E05, 35Pxx

Keywords. Schrödinger operator, virtual eigenvalues, coupling constant, asymptotic behavior of virtual eigenvalues, Birman-Schwinger principle

1 Introduction

In the present work, divided into two parts (part I is the present paper, and part II is [Ar-Z2]), we consider the elliptic differential operator of order $2l$ ($l \in \mathbb{N}$)

$$H_\gamma = (-\Delta)^l + \gamma V(\mathbf{x}). \quad (1.1)$$

acting in the space $L_2(\mathbb{R}^d)$. Here $V(\mathbf{x}) \cdot$ is the multiplication operator in $L_2(\mathbb{R}^d)$ by the continuous, real-valued function $V(\mathbf{x})$ on $\mathbb{R}^d$, which is assumed to be non-negative, not identically zero, and tends to zero sufficiently fast as $|\mathbf{x}| \rightarrow \infty$. We shall denote this operator briefly by V . We assume that the “coupling constant” γ is real. As it is conventional in the literature, we call the operator H_γ the *Schrödinger operator of order $2l$* and we call the function $V(\mathbf{x})$ the *potential*.

We consider the so-called *virtual eigenvalues* of the operator H_γ . These are the negative eigenvalues which are born at the moment $\gamma = 0$ from the endpoint $\lambda = 0$

*Both authors were partially supported from the Israel Science Foundation (ISF), grant number 585/00, and from the German-Israeli Foundation (GIF), grant number I-619-17.6/2001. The second author was partially supported also by the KAMEA Project for Scientific Absorption in Israel.

of the gap $(-\infty, 0)$ of the spectrum $\sigma(H_0)$ of the unperturbed operator $H_0 = (-\Delta)^l$, while γ varies from 0 to a small negative value γ_0 (see Definitions 3.14 and 3.13). Notice that in the literature the asymptotic behavior of the discrete negative spectrum of the operator H_γ is well studied for $|\gamma| \rightarrow \infty$ ([Bi2], [Bi-So], [Sob]). In the paper of M. Sh. Birman [Bi1] (1961) a variational approach has been worked out for the study of birth of eigenvalues in the gap of the continuous spectrum for a small potential $V(\mathbf{x})$. In the 1970's the asymptotic behavior of virtual eigenvalues of the Schrödinger operator as $\gamma \uparrow 0$ was studied in the cases $l = 1$ and $d = 1, 2, 3$ with the help of analytical methods ([Re-Si], [S], [S1], [Kl], [B-G-S]). These investigations were based on the so-called Birman-Schwinger principle, which describes a behavior of the discrete spectrum of the perturbed operator in the gaps of the spectrum of the unperturbed one ([Bi3], [Sc], [Re-Si], [S]).

The interest in this subject was renewed in the last decade. In [W] T. Weidl has developed the Birman approach for the study of the existence of virtual eigenvalues for a wide class of elliptic differential operators of high order and even for indefinite perturbations. In particular (see [W], Corollary 6.1): if $2l \geq d$, the potential $V(\mathbf{x})$ is continuous, non-negative, not identically zero and $V(\mathbf{x}) \rightarrow 0$ as $|\mathbf{x}| \rightarrow \infty$ sufficiently fast, then the operator H_γ of the form (1.1) has exactly $r = \binom{m+d}{d}$ virtual eigenvalues at the point $\lambda = 0$, where $m = l - \lfloor \frac{d+1}{2} \rfloor$.

The paper [N-W] is devoted to a generalization of Lieb-Thirring inequalities (obtained in [L-Thr] for the Schrödinger operator of second order) to the operator H_γ of a high order $2l$. Furthermore, an asymptotic representation as $\gamma \uparrow 0$ for the bottom virtual eigenvalue of the operator H_γ has been obtained in [N-W], in which the leading coefficient has been evaluated explicitly ([N-W], Lemma 5.1).

In the present paper we obtain asymptotic expansion for the bottom virtual eigenvalue of the operator H_γ as $\gamma \uparrow 0$, which is more precise than the corresponding asymptotic formula in [N-W] (see Remark 5.8). In [Ar-Z2] we obtain asymptotic estimates for non-bottom virtual eigenvalues.

Unlike the variational approach of [W], we use an analytical method in combination with some variational arguments. The idea behind this approach follows B. Simon's paper [S1] (see also [Re-Si] and [S]). Studying the Birman-Schwinger operator

$$X_V(\lambda) = V^{\frac{1}{2}} ((-\Delta)^l - \lambda I)^{-1} V^{\frac{1}{2}} \quad (\lambda < 0),$$

we extract a finite-rank portion from the kernel

$$\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{\exp(i\mathbf{p} \cdot (\mathbf{x} - \mathbf{y})) d\mathbf{p}}{|\mathbf{p}|^{2l} + |\lambda|}.$$

of the resolvent $((-\Delta)^l - \lambda I)^{-1}$. We obtain this finite-rank portion as corresponding to first terms of the Taylor expansion of the exponent at $\mathbf{p} = \mathbf{0}$. We choose the number of these terms such that for the corresponding finite-rank portion $\Phi(\lambda)$ of the Birman-Schwinger operator $X_V(\lambda)$ the norm of the remainder $X_V(\lambda) - \Phi(\lambda)$ is uniformly bounded with respect to λ in $(-\delta, 0)$ for some $\delta > 0$. Afterwards we obtain asymptotic expansions, with respect to a small $\lambda < 0$, of eigenvalues of the operator $\Phi(\lambda)$ which grow to $+\infty$ as $\lambda \uparrow 0$. The inversion of these asymptotic expansions yields the leading terms of the desired asymptotic representation of the virtual eigenvalues of the operator H_γ for $\gamma \uparrow 0$. Unlike our approach, in [N-W] for the Schrödinger operator of high order only rank-one portion has been extracted from the operator $X_V(\lambda)$.

Hence, in general the corresponding remainder does not have there the property mentioned above. This advantage of our approach enables us to obtain an asymptotic estimate for the bottom virtual eigenvalue of the operator H_γ , which is more precise than the estimate obtained in [N-W]. This approach also enables us to get asymptotic estimates for the non-bottom virtual eigenvalues in Part II of the work.

The paper is divided into five sections. After this introduction (Section 1), we give in Section 2 the list of notation used in the paper.

Section 3 is devoted to a supplement to the Birman-Schwinger theory, in which we study the process of the birth of eigenvalues in a gap of the spectrum of the unperturbed operator for a small coupling constant. This is a generalization (to the case of relatively compact perturbations) of the theory developed in our earlier paper [Ar-Z] for the case of finite-rank perturbations. The concept of main characteristic branches of an operator H_0 with respect to a perturbing operator V plays a pivotal role in these considerations (see Definition 3.9 and Definition 3.2). It enables us to get asymptotic estimates of virtual eigenvalues of the operator H_γ for $\gamma \uparrow 0$.

Section 4 is devoted to the above mentioned extraction of a finite-rank portion $\Phi(\lambda)$ from the Birman-Schwinger operator $X_V(\lambda)$ (see Propositions 4.4 and 4.8).

In Section 5 we obtain asymptotic expansion for the bottom virtual eigenvalue of the operator H_γ with respect to a small coupling constant $\gamma < 0$ (Theorems 5.3 and 5.6). To this end we carry out an asymptotic expansion for the maximal eigenvalue $\tilde{\mu}_0(\lambda)$ of the finite-rank portion $\Phi(\lambda)$ of the Birman-Schwinger operator $X_V(\lambda)$. In the case of $\mathbb{R}^d$ with d odd $\tilde{\mu}_0(-t^{2l})$ admits expansion in rational Laurent series near the point $t = 0$, because the operator function $\Phi(-t^{2l})$ is meromorphic (Lemma 5.1). Hence the asymptotic expansion of the bottom virtual eigenvalue has a power form in the case of an odd d (Theorem 5.3). The expansion of $\tilde{\mu}_0(-t^{2l})$ can be easily derived with the help of a simple version of the Schrödinger method ([Bau], Ch. 3, n° 3.1.2) thanks the fact that the quantity $t^{2l-d}\tilde{\mu}_0(-t^{2l})$ is born at the moment $t = 0$ from a *simple* eigenvalue of the operator

$$\Phi_0 = \lim_{t \downarrow 0} t^{2l-d} \Phi(-t^{2l})$$

(see Lemma 5.1). In the case of d even the operator function $\Phi(-t^{2l})$ is not meromorphic, because it contains summands with $\ln(\frac{1}{t})$ in its expansion near the point $t = 0$. Hence, the asymptotic expansion of the bottom virtual eigenvalue is more complicated for an even d . We use in this case a modification of the method mentioned above (see Lemma 5.4 and Theorem 5.6).

2 Notation

In this section we give a list of notation used in the present paper.

$\mathbb{Z}$ is the ring of all integers;

$\mathbb{N}$ is the set of all natural numbers $1, 2, \dots$;

$\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$;

$\mathbb{R}$ is the field of all real numbers;

$\mathbb{R}_+ = [0, \infty)$;

$\mathbb{C}$ is the field of all complex numbers;

$\Re(z)$, $\Im(z)$ are the real and the imaginary parts of a number $z \in \mathbb{C}$.

$\#S$ is the number of elements of a finite set S .

$\mathcal{O}(x)$ is generic notation for a neighborhood of a point x ;

$\text{cl}(\mathcal{S})$ is the closure of a set $\mathcal{S}$.

If M is a metric space, then $\text{dist}(x, y)$ and $\text{dist}(x, Y)$ are the distance between points $x, y \in M$ and the distance between a point $x \in M$ and a set $Y \subseteq M$.

$$\mathcal{C}^d = \times_{j=1}^d \mathcal{C}; \quad \mathbb{R}^d = \times_{j=1}^d \mathbb{R}; \quad \mathbb{Z}_+^d = \times_{j=1}^d \mathbb{Z}_+.$$

$\mathbf{x} \cdot \mathbf{y} = \sum_{j=1}^d x_j y_j$ is the canonical inner product of vectors $\mathbf{x} = (x_1, x_2, \dots, x_d)$

and $\mathbf{y} = (y_1, y_2, \dots, y_d)$ belonging to $\mathbb{R}^d$;

$|\mathbf{x}| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$ is the Euclidean norm in $\mathbb{R}^d$;

$|\mathbf{k}| = \sum_{j=1}^d |k_j|$ is the l_1 -norm of a multi-index $\mathbf{k} = (k_1, k_2, \dots, k_d) \in \mathbb{Z}_+^d$.

σ_d is the $d - 1$ -dimensional volume measure of the unit sphere

$$\{\mathbf{x} \in \mathbb{R}^d \mid |\mathbf{x}| = 1\}.$$

$\mathbf{x}^{\mathbf{k}} = \prod_{j=1}^d x_j^{k_j}$, where $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$ and $\mathbf{k} = (k_1, \dots, k_d) \in \mathbb{Z}_+^d$;

$\mathbf{k}! = \prod_{j=1}^d k_j!$, where $\mathbf{k} = (k_1, \dots, k_d) \in \mathbb{Z}_+^d$.

Let $f : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ be a mapping. We denote:

$\text{Dom}(f) = \mathcal{A}_1$ is the domain of f ;

$\text{Im}(f) = f(\mathcal{A}_1)$ is the image (range) of f ;

$f|_{\mathcal{G}}$ is the restriction of f on a subset $\mathcal{G} \subset \mathcal{A}_1$.

If A is a closed linear operator acting in a Hilbert space $\mathcal{H}$, then:

$\ker(A)$ is the kernel of A ;

$\mathcal{R}(A)$ is the resolvent set of A , that is the set of all $\lambda \in \mathcal{C}$ such that $A - \lambda I$ is continuously invertible;

$R_\lambda(A)$ ($\lambda \in \mathcal{R}(A)$) is the resolvent of A , that is $R_\lambda(A) = (A - \lambda I)^{-1}$;

$\sigma(A) = \mathcal{C} \setminus \mathcal{R}(A)$ is the spectrum of A .

P_G is the orthogonal projection on a closed subspace G of a Hilbert space $\mathcal{H}$.

$\mathcal{S}_2$ is the Hilbert-Schmidt class of operators acting in a Hilbert space $\mathcal{H}$;

$\|T\|_2$ is the Hilbert-Schmidt norm of the operator $T \in \mathcal{S}_2$.

$\hat{f}(\mathbf{p}) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(\mathbf{x}) e^{-i\mathbf{x} \cdot \mathbf{p}} d\mathbf{x}$ is the Fourier transform of a function f ;

$(f \star g)(\mathbf{x}) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(\mathbf{x} - \mathbf{y}) g(\mathbf{y}) d\mathbf{y}$ is the convolution of f and g .

For a power series $p(\epsilon) = \sum_{k=0}^{\infty} p_k \epsilon^k$ and $j \in \mathbb{Z}_+$ we denote

$$(p(\epsilon))_j = \sum_{k=0}^j p_k \epsilon^k.$$

3 General results on the birth of eigenvalues in a gap of the unperturbed spectrum

Let H_0, V be self-adjoint operators acting in a Hilbert space $\mathcal{H}$, such that $D = \text{Dom}(H_0) \subseteq \text{Dom}(V)$ and D is dense in $\mathcal{H}$. For $\gamma \in \mathbb{R}$ (a “coupling constant”)

consider the perturbed operator

$$H_\gamma = H_0 + \gamma V. \quad (3.1)$$

In this section we shall assume that the following conditions are satisfied:

(A) (a, b) $(-\infty \leq a < b \leq +\infty)$ is a gap of the spectrum $\sigma(H_0)$ of the unperturbed operator H_0 .

(B) The operator V is bounded¹, $V \geq 0$ and for some $\lambda_0 \in (a, b)$ the operator $V^{\frac{1}{2}}R_{\lambda_0}(H_0)$ is compact.

The unperturbed operator H_0 may be unbounded.

The theory developed in this section answers the following questions: how many eigenvalues of the operator H_γ are born in the gap (a, b) at the moment $\gamma = 0$ and what is the rate of this birth with respect to a small γ ?

3.1°. We shall use the following well known statement concerning the structure of the set

$$\sigma_\gamma(a, b) = \sigma(H_\gamma) \cap (a, b) \quad (3.2)$$

(see, for instance, [Bi2], Proposition 1.5):

Proposition 3.1. *The set $\sigma_\gamma(a, b)$ consists of at most countable number of eigenvalues of finite multiplicities of the operator H_γ and these eigenvalues can cluster only to endpoints of the gap (a, b) . Furthermore, this set coincides with the spectrum $\sigma(\Phi(\gamma))$ of the pencil of operators*

$$\Phi(\gamma) = \{I + \gamma X_V(\lambda)\}_{\lambda \in (a, b)}, \quad (3.3)$$

where

$$X_V(\lambda) = V^{\frac{1}{2}}R_\lambda(H_0)V^{\frac{1}{2}}. \quad (3.4)$$

Moreover, the operator function $X_V(\lambda)$ is holomorphic in $\mathcal{R}(H_0)$ in the operator norm, each of the operators $X_V(\lambda)$ is compact and any point $\lambda \in \sigma_\gamma(a, b)$ is an eigenvalue of the pencil $\Phi(\gamma)$ such that

$$\dim(\ker(H_\gamma - \lambda I)) = \dim(\ker(I + \gamma X_V(\lambda))). \quad (3.5)$$

As it is conventional in the literature, we shall call the operator $X_V(\lambda)$, defined by (3.4), the *Birman-Schwinger operator*.

We shall investigate some properties of the operator pencil $\Phi(\gamma)$, defined by (3.3), (3.4), and define some new concepts. Since, by Proposition 3.1, each operator $X_V(\lambda)$ ($\lambda \in (a, b)$) is self-adjoint and compact, its spectrum consists of at most a countable number of real eigenvalues which can cluster only to the point 0. Furthermore, each of the non-zero eigenvalues has finite multiplicity. Let us number all the positive eigenvalues $\mu_k^+(\lambda)_{k \in \mathbb{N}}$ in the non-increasing ordering

$$\mu_1^+(\lambda) \geq \mu_2^+(\lambda) \geq \dots \geq \mu_k^+(\lambda) \geq \dots \quad (3.6)$$

and all the negative ones $\mu_k^-(\lambda)_{k \in \mathbb{N}}$ in the non-decreasing ordering

$$\mu_1^-(\lambda) \leq \mu_2^-(\lambda) \leq \dots \leq \mu_k^-(\lambda) \leq \dots \quad (3.7)$$

¹Apparently, it is possible to replace this condition by a less restrictive one, but this is sufficient for our applications.

(each eigenvalue is repeated according to its multiplicity). So, by such ordering we have chosen one-valued branches of eigenvalues of the operator function $X_V(\lambda)$. Since the positive and negative branches can “go to zero” at some points of the gap (a, b) , each branch has its domain which will be denoted by $\text{Dom}(\mu_k^+)$ and $\text{Dom}(\mu_k^-)$. Let us introduce the following

Definition 3.2. We call the positive and negative branches (3.6), (3.7) of eigenvalues of the operator function $X_V(\lambda)$, defined by (3.4), the *characteristic branches* (positive and negative) of the operator H_0 with respect to the operator V on a gap (a, b) of $\sigma(H_0)$.

3.2°. Before studying some properties of the characteristic branches, we shall prove two lemmas which are versions of the comparison theorem for eigenvalues of self-adjoint compact operators, based on the minimax characterization of the eigenvalues ([Ri-Nag], Ch. VI, Sect. 1).

Lemma 3.3. *Let A_1, A_2 be self-adjoint compact operators acting in a Hilbert space $\mathcal{H}$, such that*

$$A_1 < A_2,^2$$

and

$$\mu_{k,j}^+ \quad (j = 1, 2, k = 1, 2, \dots, N_j^+), \quad \mu_{k,j}^- \quad (j = 1, 2, k = 1, 2, \dots, N_j^-) \quad (3.8)$$

be the positive and negative eigenvalues of the operator A_j ($j = 1, 2$) arranged by the non-increasing ordering of their absolute values (multiplicity counted). Then the following inequalities are valid:

$$N_2^+ \geq N_1^+, \quad (3.9)$$

$$N_2^- \leq N_1^-, \quad (3.10)$$

$$\mu_{k,1}^+ < \mu_{k,2}^+ \quad (k = 1, 2, \dots, N_1^+), \quad (3.11)$$

$$\mu_{k,2}^- < \mu_{k,1}^- \quad (k = 1, 2, \dots, N_2^-).$$

Proof. We shall consider only the positive eigenvalues, because for the negative ones the proof is analogous. By the condition of the lemma,

$$(A_1 x, x) < (A_2 x, x) \quad \forall x \in \mathcal{H}, x \neq 0. \quad (3.12)$$

By the minimax characterization of eigenvalues ([Ri-Nag], Ch. VI, Sect. 1, n° 95),

$$\mu_{k,j}^+ = \min_{L_{k-1} \in \mathcal{L}_{k-1}} \max_{x \in L_{k-1}^\perp, \|x\|=1} (A_j x, x) \quad (j = 1, 2), \quad (3.13)$$

where $\mathcal{L}_k$ is the set of all k -dimensional subspaces of $\mathcal{H}$. Let us take $1 \leq k \leq N_2^+$. Then for some $L_{k-1} \in \mathcal{L}_{k-1}$

$$\mu_{k,2}^+ = \max_{x \in L_{k-1}^\perp, \|x\|=1} (A_2 x, x). \quad (3.14)$$

²In the well known version of the comparison theorem the weak inequality $\leq$ is used, and accordingly one obtains the weak inequality for eigenvalues.

Hence, if for $k \in \{1, 2, \dots, N_2^+\}$ there exists $\mu_{k,1}^+$, we obtain from (3.13), taking into account inequality (3.12), that

$$\begin{aligned} \mu_{k,1}^+ &\leq \max_{x \in L_{k-1}^\perp, \|x\|=1} (A_1 x, x) < \\ &< \max_{x \in L_{k-1}^\perp, \|x\|=1} (A_2 x, x) = \mu_{k,2}^+, \end{aligned}$$

i.e, we have obtained the inequalities (3.9), (3.11). $\square$

Lemma 3.4. *Let A_1, A_2 be self-adjoint compact operators acting in a Hilbert space $\mathcal{H}$, such that for some $\epsilon > 0$*

$$A_2 - \epsilon I \leq A_1 \leq A_2 + \epsilon I. \quad (3.15)$$

*Then for the positive and negative eigenvalues (3.8) of the operators A_1 and A_2 the following properties are valid*³:

(a) *if for $k \in \{1, 2, \dots, N_2^+\}$ there exists $\mu_{k,1}^+$, then*

$$\mu_{k,1}^+ \leq \mu_{k,2}^+ + \epsilon;$$

(b) *if for $k \in \{1, 2, \dots, N_1^+\}$ there exists $\mu_{k,2}^+$, then*

$$\mu_{k,2}^+ \leq \mu_{k,1}^+ + \epsilon;$$

(c) *if for $k \in \{1, 2, \dots, N_2^-\}$ there exists $\mu_{k,1}^-$, then*

$$\mu_{k,2}^- \leq \mu_{k,1}^- + \epsilon;$$

(d) *if for $k \in \{1, 2, \dots, N_1^-\}$ there exists $\mu_{k,2}^-$, then*

$$\mu_{k,1}^- \leq \mu_{k,2}^- + \epsilon.$$

Proof. Assume that $k \in \{1, 2, \dots, N_1^+\}$. Making use of the minimax characterization of eigenvalues, we choose a subspace $L_{k-1} \in \mathcal{L}_{k-1}$ such that (3.14) is valid and, furthermore, in view of (3.15),

$$\mu_{k,1}^+ \leq \max_{x \in L_{k-1}^\perp, \|x\|=1} (A_1 x, x) < \max_{x \in L_{k-1}^\perp, \|x\|=1} (A_2 x, x) + \epsilon = \mu_{k,2}^+ + \epsilon.$$

So, property (a) is proven. Property (b) follows from (a) by the interchange of A_1 and A_2 and properties (c) and (d) follow from (a) and (b) applied to the operators $-A_1$ and $-A_2$. The lemma is proven. $\square$

3.3°. We now turn to properties of the characteristic branches. They are based on the following

³Here we cannot use immediately the comparison theorem, because $A_2 - \epsilon I, A_2 + \epsilon I$ are not compact operators

Lemma 3.5. *For any $\lambda \in (a, b)$ the subspace*

$$\mathcal{H}_V = \mathcal{H} \ominus \ker(V). \quad (3.16)$$

is invariant under the operator $X_V(\lambda)$, defined by (3.4), and

$$\sigma(X_V(\lambda)) = \begin{cases} \sigma(\tilde{X}_V(\lambda)), & \text{if } \ker(V) = \{\mathbf{0}\}, \\ \sigma(\tilde{X}_V(\lambda)) \cup \{0\}, & \text{if } \ker(V) \neq \{\mathbf{0}\}, \end{cases} \quad (3.17)$$

where

$$\tilde{X}_V(\lambda) = X_V(\lambda)|_{\mathcal{H}_V} \quad (3.18)$$

Furthermore, if the operator $\tilde{X}_V(\lambda)$ is considered as acting in the space $\mathcal{H}_V$, then the corresponding operator function increases on (a, b) , that is

$$\lambda_1 < \lambda_2 \text{ implies } \tilde{X}_V(\lambda_1) < \tilde{X}_V(\lambda_2) \quad \forall \lambda_1, \lambda_2 \in (a, b). \quad (3.19)$$

Proof. As it is known,

$$\ker(V^{\frac{1}{2}}) = \ker(V), \quad (3.20)$$

that is

$$\text{cl}(\text{Im}(V^{\frac{1}{2}})) = (\ker(V^{\frac{1}{2}}))^{\perp} = \text{cl}(\text{Im}(V)) = \mathcal{H}_V.$$

From the latter fact and (3.4) we see that $\text{Im}(X_V(\lambda)) \subseteq \mathcal{H}_V$, hence the subspace $\mathcal{H}_V$ is invariant for the operator $X_V(\lambda)$. Furthermore, we see from (3.4), (3.20) that $\ker(V) \subseteq \ker(X_V(\lambda))$. Since the operator $X_V(\lambda)$ is self-adjoint, we obtain from the above circumstances and from (3.16) that (3.17) is valid. Let us prove property (3.19). Observe that, by Proposition 3.1,

$$X_V(\cdot) \in C^1((a, b), \mathcal{B}(\mathcal{H}_V)). \quad (3.21)$$

In view of (3.4), we obtain:

$$\frac{d}{d\lambda} X_V(\lambda) = V^{\frac{1}{2}}(R_{\lambda}(H_0))^2 V^{\frac{1}{2}}.$$

Then for any $x \in \mathcal{H}_V$, $x \neq 0$, $\lambda \in (a, b)$

$$\frac{d}{d\lambda} (\tilde{X}_V(\lambda)x, x) = (R_{\lambda}(H_0)V^{\frac{1}{2}}x, R_{\lambda}(H_0)V^{\frac{1}{2}}x) > 0,$$

because $\ker(R_{\lambda}(H_0)) = \{0\}$ and, in view of (3.20) and (3.16), $V^{\frac{1}{2}}x \neq 0$. The latter inequality implies (3.19). $\square$

Remark 3.6. In view of (3.17), the union of the sequences (3.6) and (3.7) coincides with the set of the non-zero eigenvalues of the operator $\tilde{X}_V(\lambda)$ defined by (3.18).

The following properties of the characteristic branches (Definition 3.2) are valid:

Proposition 3.7. *All the positive and negative characteristic branches $\mu_k^+(\lambda)$, $\mu_k^-(\lambda)$ of H_0 , with respect to V on the gap (a, b) , are continuous and increasing on their domains, these domains have the form*

$$\text{Dom}(\mu_k^+) = (\eta_k^+, b), \quad \eta_k^+ \in [a, b], \quad (3.22)$$

$$\text{Dom}(\mu_k^-) = (a, \eta_k^-), \eta_k^- \in [a, b]. \quad (3.23)$$

and the following property is valid:

$$\eta_k^+ \in (a, b) \Rightarrow \lim_{\lambda \downarrow \eta_k^+} \mu_k^+(\lambda) = 0, \quad (3.24)$$

$$\eta_k^- \in (a, b) \Rightarrow \lim_{\lambda \uparrow \eta_k^-} \mu_k^-(\lambda) = 0. \quad (3.25)$$

Furthermore, the sequence $\{\eta_k^+\}$ is non-decreasing and the sequence $\{\eta_k^-\}$ is non-increasing.

Proof. By (3.19), Lemma 3.3 and Remark 3.6, if $\lambda_1, \lambda_2 \in (a, b)$, $\lambda_1 < \lambda_2$ and there exists $\mu_k^+(\lambda_1)$, then there exists $\mu_k^+(\lambda_2)$ and

$$\mu_k^+(\lambda_1) < \mu_k^+(\lambda_2).$$

This means that either (3.22) or

$$\text{Dom}(\mu_k^+) = [\eta_k^+, b), \eta_k^+ \in (a, b]$$

is valid, and, furthermore, the function $\mu_k^+(\lambda)$ increases on $\text{Dom}(\mu_k^+)$. Let us show that only the case (3.22) is realized. Take $\lambda_0 \in \text{Dom}(\mu_k^+)$. In view of (3.21), for any $\epsilon \in (0, \mu_k^+(\lambda_0))$ there exists $\delta > 0$, such that $(\lambda_0 - \delta, \lambda_0 + \delta) \subset (a, b)$ and

$$X_V(\lambda_0) - \epsilon I < X_V(\lambda) < X_V(\lambda_0) + \epsilon I. \quad (3.26)$$

Then, by Lemma 3.4, for any $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$ there exists $\mu_k^+(\lambda)$ and

$$\mu_k^+(\lambda_0) - \epsilon < \mu_k^+(\lambda) < \mu_k^+(\lambda_0) + \epsilon.$$

This means that $(\lambda_0 - \delta, \lambda_0 + \delta) \subset \text{Dom}(\mu_k^+)$. So, we have proved that the set $\text{Dom}(\mu_k^+)$ is open, that is (3.22) is valid, and, furthermore, the function $\mu_k^+(\lambda)$ is continuous on $\text{Dom}(\mu_k^+)$.

Let us prove relation (3.24). Assume, on the contrary, that

$$\mu_k^+(\eta_k^+ + 0) = \lim_{\lambda \downarrow \eta_k^+} \mu_k^+(\lambda) > 0$$

for $\eta_k^+ \in (a, b)$. Then from the left inequality (3.26), with $\lambda_0 = \eta_k^+$, and Lemma 3.3 we obtain that for any $\epsilon \in (0, \mu_k^+(\eta_k^+ + 0))$ there exists $\delta > 0$, such that $(\eta_k^+ - \delta, \eta_k^+] \subset \text{Dom}(\mu_k^+)$. This contradicts equality (3.22). So, (3.24) is valid.

Since for any fixed $\lambda \in (\eta_k^+, b)$ the sequence $\mu_k^+(\lambda)$ is non-increasing, then $(\eta_k^+, b) \subseteq (\eta_j^+, b)$ for $j < k$, that is the sequence η_k^+ is non-decreasing.

The corresponding properties (3.23) and (3.25) of the negative branches $\mu_k^-(\lambda)$ are proved analogously. $\square$

Corollary 3.8. *For the characteristic branches of H_0 with respect to V on the gap (a, b) there exist the limits (finite or infinite):*

$$\mu_{k,b}^+ = \lim_{\lambda \uparrow b} \mu_k^+(\lambda), \quad (3.27)$$

$$\mu_{k,a}^- = \lim_{\lambda \downarrow a} \mu_k^-(\lambda). \quad (3.28)$$

We can introduce the following

Definition 3.9. Consider $l(a), l(b) \in \mathbb{Z}_+ \cup \{+\infty\}$ defined by the conditions:

$$\begin{aligned} -\infty &= \mu_{1,a}^- = \mu_{2,a}^- = \cdots = \mu_{l(a),a}^- < \mu_{l(a)+1,a}^- \leq \mu_{l(a)+2,a}^- \leq \cdots, \\ +\infty &= \mu_{1,b}^+ = \mu_{2,b}^+ = \cdots = \mu_{l(b),b}^+ > \mu_{l(b)+1,b}^+ \geq \mu_{l(b)+2,b}^+ \geq \cdots, \end{aligned}$$

were the quantities $\mu_{k,a}^-$, $\mu_{k,b}^+$ are defined by (3.27) and (3.28). We shall call $l(a)$ and $l(b)$ the *asymptotic multiplicities of the endpoints a and b* of the gap (a, b) of $\sigma(H_0)$ with respect to the operator V and denote them

$$l(a) = M(a, H_0, V), \quad l(b) = M(b, H_0, V).$$

The characteristic branches

$$\{\mu_k^-(\lambda)\}_{k=1}^{l(a)}, \quad \{\mu_k^+(\lambda)\}_{k=1}^{l(b)}$$

are called the *main characteristic branches of the operator H_0 with respect to the operator V near the endpoints a and b* respectively.

The following property of the main characteristic branches holds:

Proposition 3.10. *If*

$$M(b, H_0, V) = \infty, \tag{3.29}$$

then all the positive characteristic branches $\mu_k^+(\lambda)$ of H_0 with respect to V on (a, b) are the main characteristic branches near the endpoint b . If $M(a, H_0, V) = \infty$, then all the negative characteristic branches $\mu_k^-(\lambda)$ of H_0 with respect to V on (a, b) are the main characteristic branches near the endpoint a .

Proof. Assume, on the contrary, that a positive branch $\mu_{k_0}^+(\lambda)$ is not the main one, i.e, the quantity $\mu_{k_0,b}^+$, defined by (3.27), is finite (see Definition 3.9). Since for any fixed $\lambda \in (a, b)$ the sequence $\mu_k^+(\lambda)$ is non-increasing, then for any $k > k_0$ $\mu_{k,b}^+ < \infty$. This contradicts the assumption (3.29). The case of the negative characteristic branches is treated analogously. $\square$

3.4°. We now turn to the study of the eigenvalues of the operator H_γ , defined by (3.1), which appear in the gap (a, b) of $\sigma(H_0)$ for a small γ . First of all, we shall describe a general picture of the birth of the eigenvalues in the gap (a, b) .

Proposition 3.11. *(i) The set $\sigma_\gamma(a, b)$, defined by (3.2), has the representation:*

$$\sigma_\gamma(a, b) = \begin{cases} \bigcup_{k=1}^{\infty} \{\lambda_k^+(\gamma)\} & \text{for } \gamma < 0, \\ \bigcup_{k=1}^{\infty} \{\lambda_k^-(\gamma)\} & \text{for } \gamma > 0, \end{cases} \tag{3.30}$$

where

$$\lambda_k^+(\gamma) = (\mu_k^+)^{-1} \left(-\frac{1}{\gamma} \right), \tag{3.31}$$

$$\lambda_k^-(\gamma) = (\mu_k^-)^{-1} \left(-\frac{1}{\gamma} \right) \tag{3.32}$$

and $(\mu_k^+)^{-1}$, $(\mu_k^-)^{-1}$ are the inverses of the functions $\mu_k^+(\lambda)$, $\mu_k^-(\lambda)$, the positive and negative characteristic branches of H_0 with respect to V on the gap (a, b) .

(ii) Each of the functions $\lambda_k^+(\gamma)$ has a domain of the form

$$\text{Dom}(\lambda_k^+) = (\gamma_k^+, \tilde{\gamma}_k^+) \quad (\gamma_k^+ < \tilde{\gamma}_k^+ \leq 0), \quad (3.33)$$

it is continuous and increasing there, and satisfies the conditions:

$$\lim_{\gamma \uparrow \tilde{\gamma}_k^+} \lambda_k^+(\gamma) = b \quad (3.34)$$

$$\lim_{\gamma \downarrow \gamma_k^+} \lambda_k^+(\gamma) = \eta_k^+ \quad (3.35)$$

(recall that $\text{Dom}(\mu_k^+) = (\eta_k^+, b)$). Furthermore,

$$\eta_k^+ \in (a, b) \Rightarrow \gamma_k^+ = -\infty \quad (3.36)$$

and

$$\eta_k^+ = a = -\infty \Rightarrow \gamma_k^+ = -\infty. \quad (3.37)$$

The sequences $\{\gamma_k^+\}$, $\{\tilde{\gamma}_k^+\}$ are non-increasing.

(iii) In the analogous manner, each of the functions $\lambda_k^-(\gamma)$ has domain of the form

$$\text{Dom}(\lambda_k^-) = (\tilde{\gamma}_k^-, \gamma_k^-) \quad (0 \leq \tilde{\gamma}_k^- < \gamma_k^-),$$

it is continuous and increasing there, and satisfies the conditions:

$$\lim_{\gamma \downarrow \tilde{\gamma}_k^-} \lambda_k^-(\gamma) = a \quad (3.38)$$

$$\lim_{\gamma \uparrow \gamma_k^-} \lambda_k^-(\gamma) = \eta_k^-$$

(recall that $\text{Dom}(\mu_k^-) = (a, \eta_k^-)$). Furthermore,

$$\eta_k^- \in (a, b) \Rightarrow \gamma_k^- = +\infty.$$

and

$$\eta_k^- = b = +\infty \Rightarrow \gamma_k^- = +\infty$$

The sequences $\{\gamma_k^-\}$, $\{\tilde{\gamma}_k^-\}$ are non-decreasing.

(iv) For any fixed $\gamma < 0$ the sequence $\{\lambda_k^+(\gamma)\}$ is non-decreasing and for any fixed $\gamma > 0$ the sequence $\{\lambda_k^-(\gamma)\}$ is non-increasing. Furthermore, for $\gamma < 0$ the cluster point of the set $\sigma_\gamma(a, b)$ can be only the endpoint b of the gap (a, b) and for $\gamma > 0$ this cluster point can be only the endpoint a .

Proof. Consider the pencil of operators $\Phi(\gamma)$, defined by (3.3). By Proposition (3.1), the set $\sigma_\gamma(a, b)$ coincides with the spectrum $\sigma(\Phi(\gamma))$ of the pencil $\Phi(\gamma)$ and, furthermore,

$$\dim(\ker(H_\gamma - \lambda I)) = \dim(\ker(I + \gamma X_V(\lambda))) \quad \forall \lambda \in \sigma_\gamma(a, b) \quad (3.39)$$

Let $\mu_k^+(\lambda)$, $\mu_k^-(\lambda)$ be positive and negative characteristic branches of the operator H_0 with respect to the operator V on the gap (a, b) (see (3.6), (3.7) and Definition 3.2).

The above arguments imply that the set $\sigma_\gamma(a, b)$ coincides with the union of the sets of solutions of the equations:

$$\mu_k^+(\lambda) = -\frac{1}{\gamma}, \quad (k = 1, 2, \dots). \quad (3.40)$$

$$\mu_k^-(\lambda) = -\frac{1}{\gamma} \quad (k = 1, 2, \dots). \quad (3.41)$$

Notice that, in view of (3.39), the following property is valid: if $\lambda(\gamma)$ is a solution of the equation (3.40), then the multiplicity of the eigenvalue $\lambda(\gamma)$ of the operator H_γ coincides with the number of repetitions of the number $\mu_k^+(\lambda(\gamma))$ in the sequence (3.6) with $\lambda = \lambda(\gamma)$. The analogous property is valid for the equation (3.41).⁴ Observe that in the case $\gamma < 0$ we need to consider the equation (3.40) only, because the equation (3.41) has no solution for any k . By a similar argument, in case $\gamma > 0$ we need to consider only the equation (3.41).

Consider the case of a negative γ . By Proposition 3.7, all the characteristic branches $\mu_k^+(\lambda)$ increase on their domains (η_k^+, b) and are continuous there. This fact implies that the equation (3.40) has a unique solution $\lambda_k^+(\gamma)$ if and only if $\gamma \in (\gamma_k^+, \tilde{\gamma}_k^+)$, where

$$\tilde{\gamma}_k^+ = -\frac{1}{\lim_{\lambda \uparrow b} \mu_k^+(\lambda)}, \quad (3.42)$$

$$\gamma_k^+ = -\frac{1}{\lim_{\lambda \downarrow \eta_k^+} \mu_k^+(\lambda)}. \quad (3.43)$$

This means that for $\gamma < 0$ representation (3.30) is valid, in which the domain $\text{Dom}(\lambda_k^+)$ has the form (3.33). Furthermore, it is clear that $\lambda_k^+(\gamma)$ is continuous and increasing on this domain. Relations (3.34) and (3.35) are obvious. Property (3.36) follows from property (3.24) of characteristic branches (Proposition 3.7). Property (3.37) follows from the fact: if $\eta_k^+ = -\infty$, then, in view of estimate

$$\|X_V(\lambda)\| \leq \|V^{\frac{1}{2}}\|^2 \|R_\lambda(H_0)\| \leq \frac{\|V\|}{\text{dist}(\lambda, \sigma(H_0))} = \frac{\|V\|}{b - \lambda} \quad (\lambda < b),$$

the relation holds

$$\lim_{\lambda \downarrow -\infty} \mu_k^+(\lambda) = 0. \quad (3.44)$$

The non-increase of the sequence $\{\mu_k^+(\lambda)\}$ for any fixed $\lambda \in (a, b)$ implies the non-decrease of the sequence $\{\lambda_k^+(\gamma)\}$ for any fixed $\gamma < 0$ and the non-increase of the sequences $\{\gamma_k^+\}$, $\{\tilde{\gamma}_k^+\}$ defined by (3.42), (3.43). In view of Proposition 3.1, the sequence $\{\lambda_k^+(\gamma)\}$ can cluster only to the endpoint b .

The case $\gamma > 0$ is treated analogously. $\square$

Remark 3.12. Proposition 3.11 says that, while γ varies from 0 to $-\infty$, each of the eigenvalues $\lambda_k^+(\gamma)$ of the operator H_γ enters the gap (a, b) across the endpoint b at the moment $\gamma = \tilde{\gamma}_k^+$ and afterwards it either stays in (a, b) for any $\gamma < 0$ and tends to a point $\eta_k^+ \in (a, b)$ as $\gamma \downarrow -\infty$, or (if $a = -\infty$) it tends to $-\infty$ as $\gamma \downarrow -\infty$, or (if $a > -\infty$) it leaves the gap (a, b) across the endpoint a at the moment $\gamma = \gamma_k^+ > -\infty$.

⁴It is clear that $\lambda(\gamma)$ cannot be a solution of the equations (3.40) and (3.41) simultaneously.

In the analogous manner, while γ varies from 0 to $+\infty$, each of the eigenvalues $\lambda_k^-(\gamma)$ of the operator H_γ enters the gap (a, b) across the endpoint a at the moment $\gamma = \tilde{\gamma}_k^-$ and afterwards it either stays in (a, b) for any $\gamma > 0$ and tends to a point $\eta_k^- \in (a, b)$ as $\gamma \uparrow +\infty$, or (if $b = +\infty$) it tends to $+\infty$ as $\gamma \downarrow +\infty$, or (if $b < +\infty$) it leaves the gap (a, b) across the endpoint b at the moment $\gamma = \gamma_k^- < +\infty$.

Let us introduce the following

Definition 3.13. We call the function $\lambda_k^+(\gamma)$, defined by (3.31), the k -th branch of eigenvalues of the operator H_γ which enter the gap (a, b) of $\sigma(H_0)$ across the endpoint b at the moment $\gamma = \tilde{\gamma}_k^+$ in the sense that (3.34) is valid. In the analogous manner, we call the function $\lambda_k^-(\gamma)$, defined by (3.32), the k -th branch of eigenvalues of the operator H_γ which enter the gap (a, b) of $\sigma(H_0)$ across the endpoint a at the moment $\gamma = \tilde{\gamma}_k^-$ in the sense that (3.38) is valid.

Definition 3.14. Let $\lambda_k^+(\gamma)$ be the k -th branch of eigenvalues of the operator H_γ which enters the gap (a, b) of $\sigma(H_0)$ across the endpoint b at the moment $\gamma = 0$. We call it a *branch of virtual eigenvalues of the operator H_γ at the endpoint b of (a, b)* . This means that

$$\lim_{\gamma \uparrow 0} \lambda_k^+(\gamma) = b.$$

In the analogous manner we define a *branch of virtual eigenvalues $\lambda_k^-(\gamma)$ of the operator H_γ at the endpoint a of (a, b)* . This branch satisfies the condition:

$$\lim_{\gamma \downarrow 0} \lambda_k^-(\gamma) = a.$$

Remark 3.15. If $a = -\infty$ and $\gamma < 0$, the branches of eigenvalues $\lambda_k^+(\gamma)$ of the operator H_γ can be defined by the following ordering of the set $\sigma_\gamma(a, b)$ defined by (3.2):

$$\lambda_1^+(\gamma) \leq \lambda_2^+(\gamma) \leq \dots \leq \lambda_k^+(\gamma) \leq \dots$$

It is clear that all the branches of virtual eigenvalues at the endpoint b (if they exist) occur at the beginning of this sequence. If $b = +\infty$ and $\gamma > 0$, the branches of eigenvalues $\lambda_k^-(\gamma)$ of the operator H_γ can be defined analogously and the branches of virtual eigenvalues at the endpoint a have the analogous property.

The following statement about the branches of virtual eigenvalues is valid:

Proposition 3.16. Assume that the endpoint b of the gap (a, b) of $\sigma(H_0)$ is finite. If

$$l(b) = M(b, H_0, V) < \infty, \quad (3.45)$$

then the operator H_γ has exactly $l(b)$ branches

$$a < \lambda_1^+(\gamma) \leq \lambda_2^+(\gamma) \leq \dots \leq \lambda_{l(b)}^+(\gamma)$$

of virtual eigenvalues at the endpoint b of (a, b) . Furthermore, for any $1 \leq k \leq l(b)$ and $\gamma \in \text{Dom}(\lambda_k^+)$,

$$\lambda_k^+(\gamma) = (\mu_k^+)^{-1} \left(-\frac{1}{\gamma} \right), \quad (3.46)$$

where $(\mu_k^+)^{-1}$ is the inverse of the function $\mu_k^+(\lambda)$ (the main characteristic branch of H_0 with respect to V near the endpoint b , see Definition 3.9). If (3.45) holds and $a = -\infty$, then

$$\text{Dom}(\lambda_k^+) = (-\infty, 0) \quad \forall k \in \{1, 2, \dots, l(b)\}. \quad (3.47)$$

Hence the operator H_γ has at least $l(b)$ eigenvalues in the gap $(-\infty, b)$ for any $\gamma < 0$.

If

$$M(b, H_0, V) = \infty, \quad (3.48)$$

then the number of the branches of eigenvalues $\{\lambda_k^+(\gamma)$ of the operator H_γ , which go into the gap (a, b) across the endpoint b , is infinite, each of them is a virtual eigenvalue and the property is valid for them

$$(-\theta_k, 0) \subseteq \text{Dom}(\lambda_k^+), \text{ where } \theta_k \uparrow +\infty \text{ for } k \rightarrow \infty. \quad (3.49)$$

The latter fact means that the operator H_γ has the infinite number of eigenvalues in the gap (a, b) for any $\gamma < 0$. These eigenvalues cluster to the endpoint b only and formula (3.46) with $l(b) = \infty$ is valid for them.

For the branches of virtual eigenvalues of the operator H_γ at the endpoint a of (a, b) the analogous assertions are valid as in the case of the endpoint b .

Proof. Assume that (3.45) is valid. By Definition 3.9 of the asymptotic multiplicity $l(b)$ and the main characteristic branches $\{\mu_k^+(\lambda)\}_{k=1}^{l(b)}$,

$$\mu_{k,b}^+ = \lim_{\lambda \uparrow b} \mu_k^+(\lambda) \begin{cases} = +\infty, & \text{if } 1 \leq k \leq l(b), \\ < +\infty, & \text{if } k > l(b). \end{cases} \quad (3.50)$$

We see from this relation and (3.31) that

$$\text{Dom}(\lambda_k^+) = (\gamma_k^+, 0) \quad \forall k \in \{1, 2, \dots, l(b)\}, \quad (3.51)$$

hence the functions $\{\lambda_k^+(\gamma)\}_{k=1}^{l(b)}$ are branches of virtual eigenvalues of the operator H_γ at the endpoint b of (a, b) . Taking into account the increase of the characteristic branches, we see from (3.50) that for $\gamma \in \left(-\frac{1}{\mu_{l(b)+1,b}^+}, 0\right)$ and $k > l(b)$ the equation (3.40) has no solution in the gap (a, b) . This means that for $k > l(b)$ the equation (3.40) gives no virtual branch of eigenvalues of H_γ at the endpoint b of (a, b) . So, we have proved that all the branches of virtual eigenvalues of H_γ at the endpoint b of (a, b) are exhausted by the sequence $\{\lambda_k^+(\gamma)\}_{k=1}^{l(b)}$, and, in view of (3.31), they are expressed by formula (3.46).

Assume that $a = -\infty$ and that, as above, (3.45) is valid. In view of (3.36), (3.37) and (3.51), for the branches of virtual eigenvalues of H_γ at the endpoint b the equalities (3.47) are valid.

We now turn to the case (3.48). Assume that $a > -\infty$. By Proposition 3.10, all the positive characteristic branches are the main characteristic branches near the endpoint b , i.e., they have the property:

$$\lim_{\lambda \uparrow b} \mu_k^+(\lambda) = +\infty \quad \forall k \in \mathbb{N}. \quad (3.52)$$

As in the case of a finite $l(b)$, this property implies that all the functions $\lambda_k^+(\gamma)$ ($k \in \mathbb{N}$), defined by (3.31), are branches of virtual eigenvalues of H_γ at the endpoint b

of (a, b) . Let us prove property (3.49). By Proposition 3.7, the sequence $\{\eta_k^+\}_{k \in \mathbb{N}}$, defined by (3.22), is non-decreasing. Consider two cases:

$$\lim_{k \rightarrow \infty} \eta_k^+ = b \quad (3.53)$$

and

$$\eta_\infty^+ = \lim_{k \rightarrow \infty} \eta_k^+ < b \quad (3.54)$$

In the case (3.53) there exists $k_0 \in \mathbb{N}$, such that for any $k \geq k_0$ $\eta_k^+ \in (a, b)$, hence, in view of (3.44) and (3.52), $\text{Im}(\mu_k^+) = (0, \infty)$ for these k . The latter fact implies that

$$\text{Dom}(\lambda_k^+) = (-\infty, 0) \text{ for } k \geq k_0,$$

hence property (3.49) is valid. In case (3.54) holds a point $\lambda_0 \in (\eta_\infty^+, b)$ belongs to $\text{Dom}(\mu_k^+)$ for any $k \in \mathbb{N}$ and the numbers $\mu_k^+(\lambda_0)$ tend to zero for $k \rightarrow \infty$, because they are the eigenvalues of the compact operator $X_V(\lambda_0)$. Since, in view of (3.52),

$$(\mu_k^+(\lambda_0), +\infty) \subseteq \text{Im}(\mu_k^+),$$

then property (3.49) is valid. This property and Proposition 3.11 imply that for any $\gamma < 0$ the set $\sigma_\gamma(a, b)$, defined by (3.2), consists of infinite sequence of the form $\{\lambda_k^+(\gamma)\}_{k \geq k_0(\gamma)}$ whose terms are expressed by formula (3.46), and this sequence can cluster to the endpoint b only. The case $a = -\infty$ is treated in the same manner as in the case of a finite $l(b)$. $\square$

The following proposition deals with estimates from below of the asymptotic multiplicity and the main characteristic branches.

Proposition 3.17. *Assume that there exists a subspace $L_n \subset \mathcal{H}$ of a finite dimension n and an increasing function $\mu(\lambda)$ defined on (λ_0, b) ($\lambda_0 \in (a, b)$) such that*

$$\mu(\lambda) > 0 \quad \forall \lambda \in (\lambda_0, b), \quad (3.55)$$

$$\mu(\lambda) \uparrow \infty \text{ as } \lambda \uparrow b, \quad (3.56)$$

and

$$\inf_{x \in L_n, \|x\|=1} (X_V(\lambda)x, x) \geq \mu(\lambda), \quad (3.57)$$

where $X_V(\lambda)$ is defined by (3.4). Then

$$M(b, H_0, V) \geq n, \quad (3.58)$$

and for the first n main characteristic branches $\{\mu_k^+(\lambda)\}_{k=1}^n$ of H_0 (with respect to V near b) the property

$$(\lambda_0, b) \subseteq \text{Dom}(\mu_k^+) \quad \forall k \in \{1, 2, \dots, n\} \quad (3.59)$$

and the estimates

$$\mu_k^+(\lambda) \geq \mu(\lambda) \quad \forall \lambda \in (\lambda_0, 0), \quad k \in \{1, 2, \dots, n\} \quad (3.60)$$

are valid.

Proof. In order to prove property (3.59), we need to show that for any $\lambda \in (\lambda_0, b)$ there exist at least n positive eigenvalues of the operator $X_V(\lambda)$. Indeed, otherwise the invariant subspace $G_\lambda^- \subseteq \mathcal{H}$ of the operator $X_V(\lambda)$, corresponding to its non-positive eigenvalues, has codimension less than n . Then there exist a vector $g \neq 0$ belonging to $L_n \cap G^-(\lambda)$. Hence, in view of (3.55) and (3.57), $(X_V(\lambda)g, g) > 0$. On the other hand, by the choice of the subspace $G^-(\lambda)$, $(X_V(\lambda)g, g) \leq 0$. This contradiction proves property (3.59).

Let us prove property (3.60). Take $\lambda \in (\lambda_0, b)$ and consider the invariant subspace $E_{n-1}(\lambda)$ of the operator $X_V(\lambda)$ corresponding to the first $n-1$ positive eigenvalues of it $\{\mu_k^+(\lambda)\}_{k=1}^{n-1}$. Denote $G(\lambda) = (E_{n-1}(\lambda))^\perp$. Since $\text{codim}(G(\lambda)) = n-1$, there exists a vector $e \in L_n \cap G(\lambda)$, $\|e\| = 1$. Then, if $\mu_n^+(\lambda)$ is the n -th positive eigenvalue of $X_V(\lambda)$, we have, taking into account condition (3.57),

$$\mu_n^+(\lambda) = \max_{x \in G, \|x\|=1} (X_V(\lambda)x, x) \geq (X_V(\lambda)e, e) \geq \inf_{x \in L_n, \|x\|=1} (X_V(\lambda)x, x) \geq \mu(\lambda).$$

This estimate and condition (3.56) imply that $\{\mu_k^+(\lambda)\}_{k=1}^n$ are the main characteristic branches of H_0 near the endpoint b of the gap (a, b) , hence estimate (3.58) is valid. Moreover, we have proved estimate (3.60). $\square$

Propositions 3.17, 3.11 and 3.16 imply

Corollary 3.18. *Assume that all the conditions of Proposition 3.17 are satisfied. Then the first n branches of eigenvalues of the operator H_γ , gone into the gap (a, b) of $\sigma(H_0)$ across the endpoint b , are virtual and the following estimate is valid for them for a small enough $\gamma < 0$:*

$$\lambda_k^+(\gamma) \leq \mu^{-1} \left(-\frac{1}{\gamma} \right) \quad \forall k \in \{1, 2, \dots, n\},$$

where μ^{-1} is the inverse of the function $\mu(\lambda)$ used in condition (3.57).

4 Representations for the Birman-Schwinger operator of the Schrödinger operator

4.1°. We return to the Schrödinger operator H_γ , defined by (1.1). Recall that the domain $\text{Dom}(H_\gamma)$ of the operator H_γ is described in the following manner:

$$\text{Dom}(H_\gamma) = \{u \in L_2(\mathbb{R}^d) \mid u \in W_{2,loc}^{2l}(\mathbb{R}^d), H_\gamma u \in L_2(\mathbb{R}^d)\}.$$

In the sequel we shall assume that the potential $V(x)$ satisfies the conditions:

$$V(\cdot) \in C(\mathbb{R}^d), \tag{4.1}$$

$$V(\mathbf{x}) \geq 0 \quad \forall \mathbf{x} \in \mathbb{R}^d, \tag{4.2}$$

$$\lim_{|\mathbf{x}| \rightarrow \infty} V(\mathbf{x}) = 0. \tag{4.3}$$

Notice that the “unperturbed” operator $H_0 = (-\Delta)^l$ is self-adjoint, because the Fourier transform on $\mathbb{R}^d$ establishes a unitary equivalence between it and the multiplication operator by the function $|\mathbf{p}|^{2l}$ (where “ $\mathbf{p}$ ” denotes the variable in $\mathbb{R}^d$). In

view of conditions (4.1) and (4.3), the operator $V(\mathbf{x})\cdot$ is bounded in $L_2(\mathbb{R}^d)$. Furthermore, it is self-adjoint, because the potential $V(\mathbf{x})$ is real-valued. These circumstances imply that the Schrödinger operator H_γ , defined by (1.1), is self-adjoint. Observe that the spectrum $\sigma(H_0)$ of the unperturbed operator H_0 coincides with the image of the function $|\mathbf{p}|^{2l}$, that is $\sigma(H_0) = [0, \infty)$. Hence the spectrum $\sigma(H_0)$ has the unique gap $(-\infty, 0)$. We shall study the eigenvalues of the operator H_γ in this gap. In order to use the general theory (developed in Section 3), we need to show that, under conditions (4.1)-(4.3) imposed on the potential $V(\mathbf{x})$, condition (B) (formulated in the beginning of Section 3) holds. Recall that we denote briefly the operator $V(\mathbf{x})\cdot$ by V . Observe that, in view of condition (4.2), $V \geq 0$.

In what follows we shall need the following well known statement (see [Gl]):

Proposition 4.1. *For $\lambda_0 < 0$ the operator $V^{\frac{1}{2}}R_{\lambda_0}(H_0)$ is compact.*

From the above proposition and the fact that the operator V is bounded we obtain the following

Corollary 4.2. *Under conditions (4.1), (4.2), (4.3) the operators H_0 , V satisfy condition (B) of Section 3.*

In the next two subsections of this section we deal with an extraction of a finite-rank portion from the Birman-Schwinger operator $X_V(\lambda)$ of the operator H_γ .

4.2°. Consider the case of $\mathbb{R}^d$ with d odd. We shall need the following result.

Lemma 4.3. *If d is odd, $2l > d$ and $\lambda < 0$, then the resolvent $R_\lambda(H_0)$ of the operator*

$$H_0 = (-\Delta)^l \quad (4.4)$$

is the convolution operator

$$R_\lambda(H_0)f = K \star f, \quad (4.5)$$

where $K = K(\mathbf{x}, \lambda)$ is a continuous function belonging to $L_2(\mathbb{R}^d)$ and having the following representation:

$$K(\mathbf{x} - \mathbf{y}, \lambda) = F(\mathbf{x}, \mathbf{y}, \lambda) + \Theta(\mathbf{x} - \mathbf{y}, \lambda), \quad (4.6)$$

where

$$F(\mathbf{x}, \mathbf{y}, \lambda) = (2\pi)^{\frac{d}{2}} \sum_{|\mathbf{k}+\mathbf{n}| \leq 2l-d-1} |\lambda|^{\frac{|\mathbf{k}+\mathbf{n}|+d}{2l}-1} \frac{(i\mathbf{x})^{\mathbf{k}}}{\mathbf{k}!} \frac{(-i\mathbf{y})^{\mathbf{n}}}{\mathbf{n}!} \xi_{\mathbf{k}+\mathbf{n}}, \quad (4.7)$$

$$\xi_{\mathbf{k}} = (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{\mathbf{s}^{\mathbf{k}} d\mathbf{s}}{|\mathbf{s}|^{2l+1}}$$

and for the function $\Theta(\mathbf{x}, \lambda)$ the estimate

$$|\Theta(\mathbf{x}, \lambda)| \leq M|\mathbf{x}|^{2l-d} \quad (\lambda < 0), \quad (4.8)$$

is valid, in which $M > 0$ does not depend on $\mathbf{x}$, λ .

Proof. Since the Fourier transform on $\mathbb{R}^d$ establishes a unitary equivalence between the operator H_0 , defined by (4.4), and the multiplication operator $\hat{H}_0 = |\mathbf{p}|^{2l}$, the resolvent $R_\lambda(H_0)$ is the convolution operator (4.5), in which

$$K = K(\mathbf{x}, \lambda) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \frac{\exp(i\mathbf{p} \cdot \mathbf{x}) d\mathbf{p}}{|\mathbf{p}|^{2l} + |\lambda|}. \quad (4.9)$$

Since $\frac{1}{|\mathbf{p}|^{2l} + |\lambda|} \in L_2(\mathbb{R}^d) \cap L_1(\mathbb{R}^d)$, the function $K(\mathbf{x}, \lambda)$ is continuous and belongs to $L_2(\mathbb{R}^d)$. Using the Taylor expansion of the exponent, we obtain:

$$K(\mathbf{x}, \lambda) = \tilde{F}(\mathbf{x}, \lambda) + \Theta(\mathbf{x}, \lambda), \quad (4.10)$$

where

$$\tilde{F}(\mathbf{x}, \lambda) = (2\pi)^{-\frac{d}{2}} \sum_{k=0}^{2l-d-1} \frac{1}{k!} \int_{\mathbb{R}^d} \frac{(i\mathbf{p} \cdot \mathbf{x})^k d\mathbf{p}}{|\mathbf{p}|^{2l} + |\lambda|}, \quad (4.11)$$

$$\Theta(\mathbf{x}, \lambda) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \frac{\left(\exp(i\mathbf{p} \cdot \mathbf{x}) - \sum_{k=0}^{2l-d-1} \frac{(i\mathbf{p} \cdot \mathbf{x})^k}{k!} \right) d\mathbf{p}}{|\mathbf{p}|^{2l} + |\lambda|}. \quad (4.12)$$

It is not difficult to see that the integrals in (4.11), (4.12) converge absolutely. Furthermore,

$$\Theta(\mathbf{x}, \lambda) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \frac{\left(\cos(\mathbf{p} \cdot \mathbf{x}) - \sum_{\nu=0}^{l-\frac{d+1}{2}} (-1)^\nu \frac{(\mathbf{p} \cdot \mathbf{x})^{2\nu}}{(2\nu)!} \right) d\mathbf{p}}{|\mathbf{p}|^{2l} + |\lambda|},$$

because in (4.12) the integral of the odd part of the integrand vanishes. Changing the variables in the latter integral $\mathbf{s} = |\mathbf{x}|\mathbf{p}$ and rotating the space $\mathbb{R}^d$ such that the direction of the vector $\mathbf{x}$ transforms to the direction of the axis s_1 , we obtain the estimate $|\Theta(\mathbf{x}, \lambda)| \leq M|\mathbf{x}|^{2l-d}$, where

$$M = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \frac{\left| \cos s_1 - \sum_{\nu=0}^{l-\frac{d+1}{2}} (-1)^\nu \frac{s_1^{2\nu}}{(2\nu)!} \right| ds}{|\mathbf{s}|^{2l}}. \quad (4.13)$$

Observe that the latter integral converges, because

$$\left| \cos s - \sum_{\nu=0}^{l-\frac{d+1}{2}} (-1)^\nu \frac{s^{2\nu}}{(2\nu)!} \right| \leq \begin{cases} C_1 |s|^{2l-d+1} & \text{for } |s| \leq 1, \\ C_2 |s|^{2l-d-1} & \text{for } |s| \geq 1, \end{cases}$$

where $C_1, C_2 > 0$ do not depend on s . Thus the function $\Theta(\mathbf{x}, \lambda)$ satisfies estimate (4.8) in which M is expressed by (4.13).

In view of (4.10), in order to prove representation (4.6), it remains only to show that

$$\tilde{F}(\mathbf{x} - \mathbf{y}, \lambda) = F(\mathbf{x}, \mathbf{y}, \lambda). \quad (4.14)$$

One has:

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{(i\mathbf{p} \cdot (\mathbf{x} - \mathbf{y}))^k d\mathbf{p}}{|\mathbf{p}|^{2l} + |\lambda|} &= |\lambda|^{\frac{k+d}{2l}-1} \int_{\mathbb{R}^d} \frac{(i\mathbf{s} \cdot (\mathbf{x} - \mathbf{y}))^k ds}{|\mathbf{s}|^{2l} + 1} = \\ &= |\lambda|^{\frac{k+d}{2l}-1} \int_{\mathbb{R}^d} \frac{(\sum_{\nu=1}^d s_\nu (ix_\nu - iy_\nu))^k ds}{|\mathbf{s}|^{2l} + 1} = \\ &= k! |\lambda|^{\frac{k+d}{2l}-1} \sum_{|\mathbf{k}+\mathbf{n}|=k} \frac{(i\mathbf{x})^{\mathbf{k}}}{\mathbf{k}!} \frac{(-i\mathbf{y})^{\mathbf{n}}}{\mathbf{n}!} \int_{\mathbb{R}^d} \frac{\mathbf{s}^{\mathbf{k}+\mathbf{n}} ds}{|\mathbf{s}|^{2l} + 1}. \end{aligned} \quad (4.15)$$

The latter equality and (4.11), (4.7) imply (4.14). The lemma is proven. $\square$

We now turn to the main statement of this subsection.

Proposition 4.4. *Assume that d is odd, $2l > d$ and that the potential $V(\mathbf{x})$ satisfies the additional condition:*

$$\int_{\mathbb{R}^d} |\mathbf{x}|^{2(2l-d)} V(\mathbf{x}) d\mathbf{x} < \infty. \quad (4.16)$$

Then the Birman-Schwinger operator

$$X_V(\lambda) = V^{\frac{1}{2}} R_\lambda(H_0) V^{\frac{1}{2}}$$

of the operator H_γ has the representation for $\lambda < 0$:

$$X_V(\lambda) = \Phi(\lambda) + T(\lambda),$$

where $\Phi(\lambda)$ is an integral operator with the kernel

$$\Phi(\mathbf{x}, \mathbf{y}, \lambda) = (2\pi)^{-\frac{d}{2}} (V(\mathbf{x}))^{\frac{1}{2}} F(\mathbf{x}, \mathbf{y}, \lambda) (V(\mathbf{y}))^{\frac{1}{2}}, \quad (4.17)$$

$F(\mathbf{x}, \mathbf{y}, \lambda)$ is defined by (4.7), $T(\lambda)$ is an operator belonging to the Hilbert-Schmidt class $\mathcal{S}_2$ and its Hilbert-Schmidt norm is bounded uniformly with respect to $\lambda < 0$, that is

$$\exists \bar{T} > 0 \quad \forall \lambda < 0 : \quad \|T(\lambda)\|_2 \leq \bar{T}. \quad (4.18)$$

Proof. Let $\Theta(\mathbf{x}, \lambda)$ be the function used in representation (4.6) of the kernel $K(\mathbf{x} - \mathbf{y}, \lambda)$ of the integral operator $(2\pi)^{\frac{d}{2}} R_\lambda(H_0)$. Then, in order to prove the proposition, we should show that for any $\lambda < 0$ the integral operator $T(\lambda)$ with the kernel

$$T(\mathbf{x}, \mathbf{y}, \lambda) = (2\pi)^{-\frac{d}{2}} (V(\mathbf{x}))^{\frac{1}{2}} \Theta(\mathbf{x} - \mathbf{y}, \lambda) (V(\mathbf{y}))^{\frac{1}{2}}$$

has the desired property. In view of estimate (4.8), it is enough to show that the kernel $(V(\mathbf{x}))^{\frac{1}{2}} |\mathbf{x} - \mathbf{y}|^{2l-d} (V(\mathbf{y}))^{\frac{1}{2}}$ belongs to the Hilbert-Schmidt class. We have:

$$\int_{\mathbb{R}^d} V(\mathbf{x}) d\mathbf{x} \int_{\mathbb{R}^d} |\mathbf{x} - \mathbf{y}|^{2(2l-d)} V(\mathbf{y}) d\mathbf{y} \leq \int_{\mathbb{R}^d} V(\mathbf{x}) d\mathbf{x} \int_{\mathbb{R}^d} (|\mathbf{x}| + |\mathbf{y}|)^{2(2l-d)} V(\mathbf{y}) d\mathbf{y}$$

By virtue of condition (4.16), the latter integral converges. $\square$

It turns out that the numbers $\xi_{\mathbf{k}}$, used in formula (4.7), can be calculated by a standard method. Let us formulate the corresponding statement.

Proposition 4.5. *Let $\mathbf{k} \in \mathbb{Z}_+^d$, $l > 0$ be such that $|\mathbf{k}| < 2l - d$. Consider the integral*

$$\xi_{\mathbf{k}} = (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{\mathbf{s}^{\mathbf{k}} d\mathbf{s}}{|\mathbf{s}|^{2l+1}} \quad (|\mathbf{k}| < 2l - d).$$

Then:

- (i) *For any permutation $\Pi \mathbf{k}$ of components of the multi-index $\mathbf{k}$: $\xi_{\Pi \mathbf{k}} = \xi_{\mathbf{k}}$;*
- (ii) *If at least one component of the multi-index $\mathbf{k}$ is odd then $\xi_{\mathbf{k}} = 0$;*

(iii) If all the components of the multi-index $\mathbf{k}$ are even, we write $\mathbf{k} = 2\mathbf{m}$ with $\mathbf{m} = (m_1, m_2, \dots, m_d) \in \mathbb{Z}_+^d$, and then

$$\xi_{\mathbf{k}} = (2\pi)^{-d} \frac{\pi}{l \sin\left(\frac{\pi}{l} \left(\frac{d}{2} + |\mathbf{m}|\right)\right)} \frac{\prod_{j=1}^d \Gamma\left(m_j + \frac{1}{2}\right)}{\Gamma\left(\frac{d}{2} + |\mathbf{m}|\right)}. \quad (4.19)$$

4.3°. We now turn to the case of $\mathbb{R}^d$ with d even. The following lemma plays the same role as Lemma 4.3 in the case of an odd dimension:

Lemma 4.6. *If d is even, $2l \geq d$ and $\lambda < 0$, then the resolvent $R_\lambda(H_0)$ of the operator H_0 , defined by (4.4), is the convolution operator (4.5), in which the function $K = K(\mathbf{x}, \lambda)$ belongs to $L_2(\mathbb{R}^d)$, and for $2l > d$ it has the following representation:*

$$K(\mathbf{x} - \mathbf{y}, \lambda) = F(\mathbf{x}, \mathbf{y}, \lambda) + \Theta(\mathbf{x} - \mathbf{y}, \lambda), \quad (4.20)$$

where

$$\begin{aligned} F(\mathbf{x}, \mathbf{y}, \lambda) = & 2l(2\pi)^{\frac{d}{2}} \sum_{|\mathbf{k}+\mathbf{n}| \leq 2l-d-1} \frac{|\lambda|^{\frac{|\mathbf{k}+\mathbf{n}|+d}{2l}-1}}{2l-d-|\mathbf{k}+\mathbf{n}|} \frac{(i\mathbf{x})^{\mathbf{k}}}{\mathbf{k}!} \frac{(-i\mathbf{y})^{\mathbf{n}}}{\mathbf{n}!} \eta_{\mathbf{k}+\mathbf{n}} + \\ & + (2\pi)^{\frac{d}{2}} \ln\left(\frac{1}{|\lambda|}\right) \sum_{|\mathbf{k}+\mathbf{n}|=2l-d} \frac{(i\mathbf{x})^{\mathbf{k}}}{\mathbf{k}!} \frac{(-i\mathbf{y})^{\mathbf{n}}}{\mathbf{n}!} \eta_{\mathbf{k}+\mathbf{n}}, \end{aligned} \quad (4.21)$$

or in another form:

$$\begin{aligned} F(\mathbf{x}, \mathbf{y}, \lambda) = & (2\pi)^{\frac{d}{2}} \sum_{|\mathbf{k}+\mathbf{n}| \leq 2l-d-1} |\lambda|^{\frac{|\mathbf{k}+\mathbf{n}|+d}{2l}-1} \frac{(i\mathbf{x})^{\mathbf{k}}}{\mathbf{k}!} \frac{(-i\mathbf{y})^{\mathbf{n}}}{\mathbf{n}!} \xi_{\mathbf{k}+\mathbf{n}} + \\ & + (2\pi)^{\frac{d}{2}} \ln\left(\frac{1}{|\lambda|}\right) \sum_{|\mathbf{k}+\mathbf{n}|=2l-d} \frac{(i\mathbf{x})^{\mathbf{k}}}{\mathbf{k}!} \frac{(-i\mathbf{y})^{\mathbf{n}}}{\mathbf{n}!} \eta_{\mathbf{k}+\mathbf{n}}, \end{aligned} \quad (4.22)$$

where

$$\eta_{\mathbf{k}} = (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{\mathbf{s}^{\mathbf{k}} d\mathbf{s}}{(|\mathbf{s}|^{2l} + 1)^2}$$

and the numbers $\xi_{\mathbf{k}}$ are defined by (4.19). Furthermore, for the function $\Theta(\mathbf{x}, \lambda)$ the estimate

$$|\Theta(\mathbf{x}, \lambda)| \leq M(1 + |\mathbf{x}|^{2l-d})(1 + \ln(|\mathbf{x}| + 1)) \quad (4.23)$$

holds for a small enough $\lambda < 0$, in which $M > 0$ does not depend on $\mathbf{x}$, λ .

If $2l = d$, then the representation

$$K(\mathbf{x} - \mathbf{y}, \lambda) = F_1(\lambda) + \Theta_1(\mathbf{x} - \mathbf{y}) + \Theta_2(\mathbf{x} - \mathbf{y}, \lambda) \quad (4.24)$$

is valid, where

$$F_1(\lambda) = (2\pi)^{\frac{d}{2}} \ln\left(\frac{1}{|\lambda|}\right) \eta_0, \quad (4.25)$$

the function $\Theta_1(x)$ belongs to $L_2(\mathbb{R}^d)$ and for the function $\Theta_2(\mathbf{x}, \lambda)$ the estimate

$$|\Theta_2(\mathbf{x}, \lambda)| \leq \tilde{M}(1 + |\mathbf{x}|^{2l-d})(1 + \ln(|\mathbf{x}| + 1)) \quad (4.26)$$

holds for a small enough $\lambda < 0$, in which $\tilde{M} > 0$ does not depend on $\mathbf{x}$, λ .

Proof. Like in the proof of Lemma 4.3, we show that the resolvent $R_\lambda(H_0)$ is the convolution operator (4.5) in which $K \in L_2(\mathbb{R}^d)$. But the further arguments of the proof of Lemma 4.3 fail, because for $|\mathbf{k} + \mathbf{n}| = 2l - d$ the integral $\int_{\mathbb{R}^d} \frac{\mathbf{s}^{\mathbf{k}+\mathbf{n}} d\mathbf{s}}{|\mathbf{s}|^{2l+1}}$ diverges. Therefore we are forced to use some trick in order to overcome this difficulty.

First assume that $2l > d$. Denote $t = |\lambda|^{\frac{1}{2l}}$. Then we have from (4.9):

$$\begin{aligned} K'(\mathbf{x}, t) &= \frac{\partial}{\partial t} K(\mathbf{x}, -t^{2l}) = \frac{\partial}{\partial t} (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \frac{\exp(i\mathbf{p} \cdot \mathbf{x}) d\mathbf{p}}{|\mathbf{p}|^{2l} + t^{2l}} = \\ &= -2lt^{2l-1} (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \frac{\exp(i\mathbf{p} \cdot \mathbf{x}) d\mathbf{p}}{(|\mathbf{p}|^{2l} + t^{2l})^2}. \end{aligned} \quad (4.27)$$

Observe that this differentiation is justified because the above integrals converge uniformly with respect to $t \in [\delta, \infty)$ for any $\delta > 0$. We have from (4.27) using the Taylor expansion for the exponent and the change of variables $\mathbf{p} = t\mathbf{s}$ in the integral:

$$\begin{aligned} K'(\mathbf{x} - \mathbf{y}, t) &= \frac{-2l}{(2\pi)^{\frac{d}{2}}} \sum_{k=0}^{2l-d} \frac{1}{k!} \frac{1}{t^{2l+1-d-k}} \int_{\mathbb{R}^d} \frac{(i\mathbf{s} \cdot (\mathbf{x} - \mathbf{y}))^k d\mathbf{s}}{(|\mathbf{s}|^{2l} + 1)^2} + \Theta'(\mathbf{x} - \mathbf{y}, t) = \\ &= \frac{-2l}{(2\pi)^{\frac{d}{2}}} \sum_{k=0}^{2l-d} \frac{1}{t^{2l+1-d-k}} \sum_{|\mathbf{k}+\mathbf{n}|=k} \frac{(i\mathbf{x})^{\mathbf{k}}}{\mathbf{k}!} \frac{(-i\mathbf{y})^{\mathbf{n}}}{\mathbf{n}!} \int_{\mathbb{R}^d} \frac{\mathbf{s}^{\mathbf{k}+\mathbf{n}} d\mathbf{s}}{(|\mathbf{s}|^{2l} + 1)^2} \\ &+ \Theta'(\mathbf{x} - \mathbf{y}, t), \end{aligned} \quad (4.28)$$

where

$$\Theta'(\mathbf{x}) = -\frac{2l}{t^{2l-d+1}} (I(\mathbf{x}, t) + I_\infty(\mathbf{x}, t)), \quad (4.29)$$

$$I(\mathbf{x}, t) = \int_{|\mathbf{s}| \leq (|\mathbf{x}|t)^{-1}} \frac{C(t, \mathbf{s}, \mathbf{x}) d\mathbf{s}}{(|\mathbf{s}|^{2l} + 1)^2}, \quad (4.30)$$

$$I_\infty(\mathbf{x}, t) = \int_{|\mathbf{s}| \geq (|\mathbf{x}|t)^{-1}} \frac{C(t, \mathbf{s}, \mathbf{x}) d\mathbf{s}}{(|\mathbf{s}|^{2l} + 1)^2}, \quad (4.31)$$

and

$$C(t, \mathbf{s}, \mathbf{x}) = \cos(t\mathbf{s} \cdot \mathbf{x}) - \sum_{\nu=0}^{l-\frac{d}{2}} (-1)^\nu \frac{(t\mathbf{s} \cdot \mathbf{x})^{2\nu}}{(2\nu)!}. \quad (4.32)$$

Taking into account (4.27) and (4.15), we can write $K'(\mathbf{x} - \mathbf{y}, t)$ in another form:

$$\begin{aligned} K'(\mathbf{x} - \mathbf{y}, t) &= (2\pi)^{-\frac{d}{2}} \left\{ \frac{\partial}{\partial t} \sum_{|\mathbf{k}+\mathbf{n}| \leq 2l-d-1} t^{|\mathbf{k}+\mathbf{n}|-2l+d} \frac{(i\mathbf{x})^{\mathbf{k}}}{\mathbf{k}!} \frac{(-i\mathbf{y})^{\mathbf{n}}}{\mathbf{n}!} \right. \\ &\times \left. \int_{\mathbb{R}^d} \frac{\mathbf{s}^{\mathbf{k}+\mathbf{n}} d\mathbf{s}}{|\mathbf{s}|^{2l} + 1} - \frac{2l}{t} \sum_{|\mathbf{k}+\mathbf{n}|=2l-d} \frac{(i\mathbf{x})^{\mathbf{k}}}{\mathbf{k}!} \frac{(-i\mathbf{y})^{\mathbf{n}}}{\mathbf{n}!} \int_{\mathbb{R}^d} \frac{\mathbf{s}^{\mathbf{k}+\mathbf{n}} d\mathbf{s}}{(|\mathbf{s}|^{2l} + 1)^2} \right\} \\ &+ \Theta'(\mathbf{x} - \mathbf{y}, t). \end{aligned} \quad (4.33)$$

It is easy to check that

$$C(t, \mathbf{s}, \mathbf{x}) \leq \begin{cases} C_1(t|\mathbf{s}||\mathbf{x}|)^{2l-d+2}, & \text{if } t|\mathbf{s}||\mathbf{x}| \leq 1 \\ C_1(t|\mathbf{s}||\mathbf{x}|)^{2l-d}, & \text{if } t|\mathbf{s}||\mathbf{x}| > 1. \end{cases}$$

Using this estimate, we obtain from (4.30):

$$|I(\mathbf{x}, t)| \leq (t|\mathbf{x}|)^{2l-d+2} \sigma_d \int_0^{(t|\mathbf{x}|)^{-1}} \frac{r^{2l+1} dr}{(r^{2l} + 1)^2}, \quad (4.34)$$

$$|I_\infty(\mathbf{x}, t)| \leq (t|\mathbf{x}|)^{2l-d} \sigma_d \int_{(t|\mathbf{x}|)^{-1}}^\infty \frac{r^{2l-1} dr}{(r^{2l} + 1)^2} = \frac{(t|\mathbf{x}|)^{2l-d} \sigma_d}{2l} \frac{(t|\mathbf{x}|)^{2l}}{(t|\mathbf{x}|)^{2l} + 1}. \quad (4.35)$$

Take $t_0 > 0$. In view of (4.27), (4.28) and (4.29), we have for $t \in (0, t_0)$:

$$\begin{aligned} K(\mathbf{x} - \mathbf{y}, -t^{2l}) &= K(\mathbf{x} - \mathbf{y}, -t_0^{2l}) - \int_t^{t_0} K'(\mathbf{x} - \mathbf{y}, \xi) d\xi = \\ &= F(\mathbf{x}, \mathbf{y}, -t^{2l}) + \Theta(\mathbf{x} - \mathbf{y}, -t^{2l}), \end{aligned} \quad (4.36)$$

where $F(\mathbf{x}, \mathbf{y}, \lambda)$ is defined by (4.21),

$$\Theta(\mathbf{x}, \lambda) = \tilde{I}(\mathbf{x}, t) + \tilde{I}_\infty(\mathbf{x}, t) - \tilde{F}(\mathbf{x}) + K(\mathbf{x}, -t_0^{2l}), \quad (4.37)$$

$$\tilde{I}(\mathbf{x}, t) = 2l \int_t^{t_0} \frac{I(\mathbf{x}, \xi) d\xi}{\xi^{2l-d+1}}, \quad (4.38)$$

$$\tilde{I}_\infty(\mathbf{x}, t) = 2l \int_t^{t_0} \frac{I_\infty(\mathbf{x}, \xi) d\xi}{\xi^{2l-d+1}} \quad (4.39)$$

and

$$\begin{aligned} \tilde{F}(\mathbf{x}) &= \frac{2l}{(2\pi)^{\frac{d}{2}}} \sum_{k=0}^{2l-d-1} \frac{t_0^{k+d-2l}}{k!(k+d-2l)} \int_{\mathbb{R}^d} \frac{(i\mathbf{s} \cdot \mathbf{x})^k d\mathbf{s}}{(|\mathbf{s}|^{2l} + 1)^2} + \\ &\quad + \frac{\ln t_0}{(2\pi)^{\frac{d}{2}} (2l-d)!} \int_{\mathbb{R}^d} \frac{(i\mathbf{s} \cdot \mathbf{x})^{2l-d} d\mathbf{s}}{(|\mathbf{s}|^{2l} + 1)^2}. \end{aligned} \quad (4.40)$$

Observe that, in view of (4.33), $F(\mathbf{x}, \mathbf{y}, \lambda)$ can be written also in the form (4.22). From (4.38) and (4.34) we obtain for $t \in (0, t_0)$ changing the order of the integration:

$$\begin{aligned} |\tilde{I}(\mathbf{x}, t)| &\leq 2l|\mathbf{x}|^{2l-d+2} \int_t^{t_0} \xi d\xi \int_0^{(|\mathbf{x}|\xi)^{-1}} \frac{r^{2l+1} dr}{(r^{2l} + 1)^2} \\ &= l|\mathbf{x}|^{2l-d+2} \left((t_0^2 - t^2) \int_0^{(|\mathbf{x}|t_0)^{-1}} \frac{r^{2l+1} dr}{(r^{2l} + 1)^2} \right. \\ &\quad \left. + \int_{(|\mathbf{x}|t_0)^{-1}}^{(|\mathbf{x}|t)^{-1}} \frac{r^{2l+1}}{(r^{2l} + 1)^2} ((|\mathbf{x}|r)^{-2} - t^2) dr \right) \\ &= l|\mathbf{x}|^{2l-d+2} \left(t_0^2 \int_0^{(|\mathbf{x}|t_0)^{-1}} \frac{r^{2l+1} dr}{(r^{2l} + 1)^2} \right. \\ &\quad \left. - t^2 \int_0^{(|\mathbf{x}|t)^{-1}} \frac{r^{2l+1} dr}{(r^{2l} + 1)^2} + |\mathbf{x}|^{-2} \int_{(|\mathbf{x}|t_0)^{-1}}^{(|\mathbf{x}|t)^{-1}} \frac{r^{2l-1}}{(r^{2l} + 1)^2} dr \right). \end{aligned} \quad (4.41)$$

Carrying out the integration by parts, we obtain:

$$\int_0^{(|\mathbf{x}|t)^{-1}} \frac{r^{2l+1} dr}{(r^{2l} + 1)^2} = -\frac{(|\mathbf{x}|t)^{-2}}{2l((|\mathbf{x}|t)^{-2l} + 1)} + \frac{1}{l} \int_0^{(|\mathbf{x}|t)^{-1}} \frac{r dr}{r^{2l} + 1}. \quad (4.42)$$

Furthermore, we have:

$$\frac{(|\mathbf{x}|t)^{-2}}{(|\mathbf{x}|t)^{-2l} + 1} \leq (|\mathbf{x}|t)^{-2} \quad \text{and} \quad \int_0^{(|\mathbf{x}|t)^{-1}} \frac{r \, dr}{r^{2l} + 1} \leq \frac{1}{2}(|\mathbf{x}|t)^{-2}.$$

The latter estimates, estimate (4.41) and equality (4.42) imply

$$|\tilde{I}(\mathbf{x}, t)| \leq C_1 |\mathbf{x}|^{2l-d}, \quad (4.43)$$

where $C_1 > 0$ does not depend on $(\mathbf{x}, t) \in \mathbb{R}^d \times (0, t_0)$.

Let us estimate the integral $\tilde{I}_\infty(\mathbf{x}, t)$ (defined by (4.39)) using (4.35):

$$|\tilde{I}_\infty(\mathbf{x}, t)| \leq \sigma_d |\mathbf{x}|^{2l-d} \int_t^{t_0} \frac{|\mathbf{x}|^{2l} \xi^{2l-1} d\xi}{(|\mathbf{x}|\xi)^{2l} + 1} = \frac{\sigma_d |\mathbf{x}|^{2l-d}}{2l} \ln(|\mathbf{x}|\xi)^{2l} + 1 \Big|_t^{t_0}.$$

Then for a small enough t_0 and $t \in (0, t_0)$ we obtain:

$$|\tilde{I}_\infty(\mathbf{x}, t)| \leq C_2 |\mathbf{x}|^{2l-d} (1 + \ln(|\mathbf{x}| + 1)), \quad (4.44)$$

where $C_2 > 0$ does not depend on $\mathbf{x}, t$.

Observe that

$$|K(\mathbf{x}, -t_0^{2l})| = \left| (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \frac{\exp(i\mathbf{p} \cdot \mathbf{x}) d\mathbf{p}}{|\mathbf{p}|^{2l} + t_0^{2l}} \right| \leq (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \frac{d\mathbf{p}}{|\mathbf{p}|^{2l} + t_0^{2l}} < \infty, \quad (4.45)$$

because $2l > d$. Furthermore, we have from (4.40):

$$|\tilde{F}(\mathbf{x})| \leq C_3 (1 + |\mathbf{x}|^{2l-d}),$$

where $C_3 > 0$ does not depend on $\mathbf{x}$. This estimate and estimates (4.43), (4.44) and (4.45) imply estimate (4.23) for the function $\Theta(\mathbf{x}, \lambda)$ defined by (4.37). Furthermore, in view of (4.36), we obtain representation (4.20).

Consider the case $2l = d$. In the same manner as in the previous case we obtain representation (4.24), in which $F_1(\lambda)$ is defined by (4.25),

$$\Theta_1(\mathbf{x}) = K(\mathbf{x}, -t_0^{2l}), \quad \Theta_2(\mathbf{x}, \lambda) = \tilde{I}(\mathbf{x}, t) + \tilde{I}_\infty(\mathbf{x}, t) - \tilde{F}(\mathbf{x}),$$

where $\tilde{I}(\mathbf{x}, t), \tilde{I}_\infty(\mathbf{x}, t)$ are defined by (4.38), (4.39), (4.30), (4.31), (4.32) with $k = 2l - d$ and

$$\tilde{F}(\mathbf{x}) = \ln t_0 (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \frac{ds}{(|\mathbf{s}|^{2l} + 1)^2}.$$

In the same manner as above, we obtain estimate (4.26). We have only to justify the possibility of differentiation under integral sign in (4.27) and of reconstruction of $K(\mathbf{x}, -t^{2l})$ by formula (4.36), because for $2l = d$ this integral does not converges absolutely. This possibility follows from the obvious membership

$$\frac{1}{|\mathbf{p}|^{2l} + t^{2l}} \in C^1((0, \infty), L_2(\mathbb{R}^d))$$

and from the fact that the Fourier transform on $\mathbb{R}^d$ is an isometric linear automorphism of the space $L_2(\mathbb{R}^d)$. The lemma is proven. $\square$

Remark 4.7. For the numbers $\eta_{\mathbf{k}}$, defined by

$$\eta_{\mathbf{k}} = (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{\mathbf{s}^{\mathbf{k}} d\mathbf{s}}{(|\mathbf{s}|^{2l} + 1)^2} \quad (|\mathbf{k}| < 4l - d)$$

assertions (i) and (ii) of Proposition 4.5 are valid. By a standard method it is possible to show that if $\mathbf{k} = 2\mathbf{m}$ with $\mathbf{m} = (m_1, m_2, \dots, m_d) \in \mathbb{Z}_+^d$, then

$$\eta_{\mathbf{k}} = (2\pi)^{-d} = (2\pi)^{-d} \times \begin{cases} \frac{\left(1 - \frac{\frac{d}{2} + |\mathbf{m}|}{l}\right) \pi}{l \sin\left(\frac{\pi}{l} \left(\frac{d}{2} + |\mathbf{m}| \right)\right)} \frac{\prod_{j=1}^d \Gamma(m_j + \frac{1}{2})}{\Gamma(\frac{d}{2} + |\mathbf{m}|)}, & \text{if } |\mathbf{k}| < 2l - d, \\ \frac{\prod_{j=1}^d \Gamma(m_j + \frac{1}{2})}{\Gamma(\frac{d}{2} + |\mathbf{m}|)^l}, & \text{if } |\mathbf{k}| = 2l - d. \end{cases} \quad (4.46)$$

We now turn to the main statement of this subsection.

Proposition 4.8. *Assume that d is even, $2l \geq d$ and the potential $V(\mathbf{x})$ satisfies the additional condition:*

$$\int_{\mathbb{R}^d} |\mathbf{x}|^{2(2l-d)} (\ln(1 + |\mathbf{x}|))^2 V(\mathbf{x}) d\mathbf{x} < \infty. \quad (4.47)$$

Then the Birman-Schwinger operator

$$X_V(\lambda) = V^{\frac{1}{2}} R_\lambda(H_0) V^{\frac{1}{2}}$$

of the operator H_γ has the representation for $\lambda < 0$:

$$X_V(\lambda) = \Phi(\lambda) + T(\lambda),$$

where $\Phi(\lambda)$ is an integral operator with the kernel

$$\Phi(\mathbf{x}, \mathbf{y}, \lambda) = (2\pi)^{-\frac{d}{2}} (V(\mathbf{x}))^{\frac{1}{2}} F(\mathbf{x}, \mathbf{y}, \lambda) (V(\mathbf{y}))^{\frac{1}{2}}, \quad (4.48)$$

$F(\mathbf{x}, \mathbf{y}, \lambda)$ is defined by (4.21), $T(\lambda)$ is an operator belonging to the Hilbert-Schmidt class $\mathcal{S}_2$ and its Hilbert-Schmidt norm is bounded uniformly with respect to a small enough $\lambda < 0$, that is

$$\exists \bar{T} > 0, \delta > 0 : \|T(\lambda)\|_2 \leq \bar{T} \quad \forall \lambda \in (-\delta, 0).$$

Proof. Consider the case $2l > d$. Let $\Theta(\mathbf{x}, \lambda)$ be the function used in representation (4.20) of the kernel $K(\mathbf{x} - \mathbf{y}, \lambda)$ of the integral operator $(2\pi)^{\frac{d}{2}} R_\lambda(H_0)$. Then we should show that for a small enough $\lambda < 0$ the integral operator $T(\lambda)$ with the kernel

$$T(\mathbf{x}, \mathbf{y}, \lambda) = (2\pi)^{-\frac{d}{2}} (V(\mathbf{x}))^{\frac{1}{2}} \Theta(\mathbf{x} - \mathbf{y}, \lambda) (V(\mathbf{y}))^{\frac{1}{2}}$$

has the desired property. In view of estimate (4.23), it is enough to show that the kernel

$$(V(\mathbf{x}))^{\frac{1}{2}} (1 + |\mathbf{x} - \mathbf{y}|^{2l-d}) (1 + \ln(1 + |\mathbf{x} - \mathbf{y}|)) (V(\mathbf{y}))^{\frac{1}{2}}$$

belongs to the Hilbert-Schmidt class. We have:

$$\int_{\mathbb{R}^d} V(\mathbf{x}) d\mathbf{x} \int_{\mathbb{R}^d} (1 + |\mathbf{x} - \mathbf{y}|^{2l-d})^2 (1 + \ln(1 + |\mathbf{x} - \mathbf{y}|))^2 V(\mathbf{y}) d\mathbf{y} \leq$$

$$\int_{\mathbb{R}^d} V(\mathbf{x}) d\mathbf{x} \int_{\mathbb{R}^d} (1 + |\mathbf{x} - \mathbf{y}|^{2l-d})^2 (1 + \ln(1 + |\mathbf{x}|) + 1 + \ln(1 + |\mathbf{y}|))^2 V(\mathbf{y}) d\mathbf{y}.$$

Then, in view of condition (4.47), we obtain the desired property of the operator $T(\lambda)$.

We now turn to the case $2l = d$. Let $\Theta_1(\mathbf{x})$, $\Theta_2(\mathbf{x}, \lambda)$ be the functions used in representation (4.24) of the kernel $K(\mathbf{x} - \mathbf{y}, \lambda)$. We should show that the integral operator T_1 with the kernel

$$T_1(\mathbf{x}, \mathbf{y}) = (2\pi)^{-\frac{d}{2}} (V(\mathbf{x}))^{\frac{1}{2}} \Theta_1(\mathbf{x} - \mathbf{y}) (V(\mathbf{y}))^{\frac{1}{2}}$$

belongs to the Hilbert-Schmidt class $\mathcal{S}_2$, for a small enough $\lambda < 0$ the integral operator with the kernel

$$T_2(\mathbf{x}, \mathbf{y}, \lambda) = (2\pi)^{-\frac{d}{2}} (V(\mathbf{x}))^{\frac{1}{2}} \Theta_2(\mathbf{x} - \mathbf{y}, \lambda) (V(\mathbf{y}))^{\frac{1}{2}}$$

belongs to the class $\mathcal{S}_2$ too and its Hilbert-Schmidt norm is bounded uniformly with respect to $\lambda \in (-\delta, 0)$ for some $\delta > 0$. The operator $T_2(\lambda)$ is treated in the same manner as the operator $T(\lambda)$. Consider the operator T_1 . Observe that, in view of conditions (4.1) and (4.3), the potential $V(\mathbf{x})$ is bounded on $\mathbb{R}^d$. Then we have, taking into account condition (4.47) and the fact that $\Theta_1 \in L_2(\mathbb{R}^d)$:

$$\int_{\mathbb{R}^d} V(\mathbf{x}) d\mathbf{x} \int_{\mathbb{R}^d} \Theta_1^2(\mathbf{x} - \mathbf{y}) V(\mathbf{y}) d\mathbf{y} \leq \sup_{\mathbf{x} \in \mathbb{R}^d} V(\mathbf{x}) \int_{\mathbb{R}^d} \Theta_1^2(\mathbf{x}) d\mathbf{x} \int_{\mathbb{R}^d} V(\mathbf{y}) d\mathbf{y} < \infty.$$

Thus, $T_1 \in \mathcal{S}_2$. □

5 Asymptotic expansion for the bottom virtual eigenvalue

In this section we obtain asymptotic expansion for the bottom virtual eigenvalue of the operator H_γ with respect to a small coupling constant $\gamma < 0$. To this end we carry out an asymptotic expansion for the maximal eigenvalue $\tilde{\mu}_0(\lambda)$ of the finite-rank portion $\Phi(\lambda)$, extracted from the Birman-Schwinger operator $X_V(\lambda)$ in Section 4. In the case of $\mathbb{R}^d$ with d odd $\tilde{\mu}_0(-t^{2l})$ admits expansion in a rational Laurent series near the point $t = 0$, because the operator function $\Phi(-t^{2l})$ is meromorphic. Hence the asymptotic expansion of the bottom virtual eigenvalue has a power form for an odd d . In the case of $\mathbb{R}^d$ with d even the operator function $\Phi(-t^{2l})$ is not meromorphic, because it contains summands with $\ln(\frac{1}{t})$ in its expansion near the point $t = 0$. Hence the asymptotic expansion of the bottom virtual eigenvalue is more complicated in this case.

5.1° First consider the case of $\mathbb{R}^d$ with d odd. Assume that $2l > d$. Let $\Phi(\lambda)$ ($\lambda < 0$) be the integral operator acting in $L_2(\mathbb{R}^d)$ and having the kernel of the form (4.17), where $F(\mathbf{x}, \mathbf{y}, \lambda)$ is defined by (4.7). In other words, $\Phi(\lambda)$ has the form:

$$\Phi(\lambda) = \sum_{\mathbf{k}, \mathbf{n}: |\mathbf{k} + \mathbf{n}| \leq 2m} |\lambda|^{\frac{|\mathbf{k} + \mathbf{n}| - 2m - 1}{2l}} \xi_{\mathbf{k} + \mathbf{n}}(\cdot, h_{\mathbf{n}}) h_{\mathbf{k}}, \quad (5.1)$$

where

$$m = l - \frac{d+1}{2}, \quad (5.2)$$

$\xi_{\mathbf{k}}$ is defined by (4.19) and

$$h_{\mathbf{k}}(\mathbf{x}) = \frac{(i\mathbf{x})^{\mathbf{k}}}{\mathbf{k}!} (V(\mathbf{x}))^{\frac{1}{2}} \quad (|\mathbf{k}| \leq 2m). \quad (5.3)$$

Observe that if the potential $V(\mathbf{x})$ satisfies condition (4.16), then the functions $h_{\mathbf{k}}(\mathbf{x})$ ($|\mathbf{k}| \leq 2m$) belong to the space $L_2(\mathbb{R}^d)$. Taking into account that, by Proposition 4.5, $\xi_{\mathbf{k}} = 0$ if at least one component of the multi-index $\mathbf{k}$ is odd, we can write formula (5.1) in another form:

$$\Phi(\lambda) = \frac{1}{|\lambda|^{\frac{m+1}{t}}} \sum_{j=0}^m |\lambda|^{\frac{j}{t}} \Phi_j \quad (5.4)$$

and

$$\Phi_j = \sum_{\mathbf{k}, \mathbf{n}: |\mathbf{k}+\mathbf{n}|=2j} \xi_{\mathbf{k}+\mathbf{n}}(\cdot, h_{\mathbf{n}}) h_{\mathbf{k}} \quad (j \in \{0, 1, \dots, m\}). \quad (5.5)$$

The following statement about an asymptotic expansion of the maximal eigenvalue $\tilde{\mu}_0(\lambda)$ of the operator $\Phi(\lambda)$ is valid:

Lemma 5.1. *Assume that d is odd, the number m , defined by (5.2), is positive and in addition to conditions (4.1)-(4.3) the potential $V(\mathbf{x})$ is not identically zero and satisfies condition (4.16). Then for $\lambda \uparrow 0$ the following asymptotic expansion holds for the maximal eigenvalue $\tilde{\mu}_0(\lambda)$ of the operator $\Phi(\lambda)$:*

$$\tilde{\mu}_0(\lambda) = \frac{\nu_0}{|\lambda|^{\frac{m+1}{t}}} + \frac{\nu_1}{|\lambda|^{\frac{m-\frac{1}{2}}{t}}} + \frac{\nu_2}{|\lambda|^{\frac{m-\frac{3}{2}}{t}}} + \dots + \frac{\nu_m}{|\lambda|^{\frac{1}{2t}}} + O\left(|\lambda|^{\frac{1}{2t}}\right), \quad (5.6)$$

where the numbers ν_k ($k = 0, 1, \dots, m$) have the form:

$$\nu_0 = \xi_0 \|h_0\|^2 = \xi_0 \int_{\mathbb{R}^d} V(\mathbf{s}) \, d\mathbf{s}, \quad (5.7)$$

$$\nu_1 = (\Phi_1 X_0, X_0), \quad (5.8)$$

$$X_0 = \frac{h_0}{\|h_0\|} = \frac{\sqrt{V(\mathbf{x})}}{\sqrt{\int_{\mathbb{R}^d} V(\mathbf{s}) \, d\mathbf{s}}}, \quad (5.9)$$

$$\nu_2 = (\Phi_2 X_0, X_0) + (\Phi_1 X_1, X_0), \quad (5.10)$$

$$X_1 = -\frac{1}{\nu_0}(\nu_1 X_0 - \Phi_1 X_0), \quad (5.11)$$

.....,

$$\nu_j = (\Phi_j X_0, X_0) + (\Phi_{j-1} X_1, X_0) + \dots + (\Phi_1 X_{j-1}, X_0), \quad (5.12)$$

$$\begin{aligned} X_{j-1} &= -\frac{1}{\nu_0}(\nu_1 X_{j-2} + \nu_2 X_{j-3} + \dots + \nu_{j-1} X_0 - \Phi_1 X_{j-2} \\ &\quad - \Phi_2 X_{j-3} - \dots - \Phi_{j-1} X_0), \end{aligned} \quad (5.13)$$

.....,

$$\nu_m = (\Phi_m X_0, X_0) + (\Phi_{m-1} X_1, X_0) + \cdots + (\Phi_1 X_{m-1}, X_0), \quad (5.14)$$

$$\begin{aligned} X_{m-1} &= -\frac{1}{\nu_0}(\nu_1 X_{m-2} + \nu_2 X_{m-3} + \cdots + \nu_{m-1} X_0 - \Phi_1 X_{m-2} \\ &\quad - \Phi_2 X_{m-3} - \cdots - \Phi_{m-1} X_0) \end{aligned} \quad (5.15)$$

and the operators Φ_j are defined by (5.5).

Proof. Denote $t = |\lambda|^{\frac{1}{t}}$ and consider the polynomial operator function:

$$\Psi(t) = t^{m+\frac{1}{2}} \Phi(-t^l) = \Phi_0 + \sum_{j=1}^m t^j \Phi_j. \quad (5.16)$$

In view of (5.5), the operators Φ_j are bounded, self-adjoint and have a finite rank. By Rellich Theorem ([Bau], Ch. 3, n° 3.5.3, Theorem 4), the branches of eigenvalues $\mu_k(t)$ of the operator $\Psi(t)$ can be chosen to be analytic in a neighborhood of $t = 0$ and they are connected with the branches $\tilde{\mu}_k(\lambda)$ of eigenvalues of the operator $\Phi(\lambda)$ via the formula:

$$\tilde{\mu}_k(\lambda) = \frac{1}{|\lambda|^{\frac{m+\frac{1}{2}}{t}}} \mu_k(|\lambda|^{\frac{1}{t}}).$$

Observe that

$$\Psi(0) = \Phi_0 = \xi_0(\cdot, h_0)h_0, \quad (5.17)$$

hence the operator $\Psi(0)$ has a unique non-zero eigenvalue ν_0 , defined by (5.7), this eigenvalue is positive and simple and the normalized eigenvector X_0 , defined by (5.9), corresponds to ν_0 . Hence the analytic branch $\mu_0(t)$ of eigenvalues of $\Psi(t)$, having the property $\mu_0(0) = \nu_0$, is a maximal eigenvalue of $\Psi(t)$ for a small enough $t \in \mathbb{R}$. Therefore

$$\tilde{\mu}_0(\lambda) = \frac{1}{|\lambda|^{\frac{m+\frac{1}{2}}{t}}} \mu_0(|\lambda|^{\frac{1}{t}}). \quad (5.18)$$

is a maximal eigenvalue of the operator $\Phi(\lambda)$ for a small enough $\lambda < 0$. Taking into account that for a small enough t $\mu_0(t)$ is a simple eigenvalue of the operator $\Psi(t)$, we can find $\mu_0(t)$ making use of a simple version of the method of power series of E. Schrödinger ([Bau], Ch. 3, n° 3.1.2). We search for $\mu_0(t)$ and the corresponding eigenvector $X(t)$ in the form:

$$\mu_0(t) = \nu_0 + \nu_1 t + \nu_2 t^2 + \dots,$$

$$X(t) = X_0 + tX_1 + t^2X_2 + \dots,$$

imposing the following normalization condition:

$$(X(t), X_0) = 1.$$

Since $\|X_0\| = 1$, the latter condition is equivalent to the following one:

$$X_k \perp X_0 \quad \forall k \in \mathbb{N}. \quad (5.19)$$

Substituting the above series into the equation

$$\Psi(t)X(t) = \mu_0(t)X(t),$$

taking into account (5.16) and comparing the coefficients near the same powers of t , we obtain the equations:

$$\Phi_0 X_0 = \nu_0 X_0,$$

$$\Phi_0 X_1 - \nu_0 X_1 = \nu_1 X_0 - \Phi_1 X_0, \quad (5.20)$$

$$\Phi_0 X_2 - \nu_0 X_2 = \nu_1 X_1 + \nu_2 X_0 - \Phi_1 X_1 - \Phi_2 X_0, \quad (5.21)$$

.....

Already we have found ν_0 and X_0 . Let us find ν_1 and X_1 . Observe that, since $\text{Im}(\Phi_0 - \nu_0 I) = X_0^\perp$, equation (5.20) has a solution if and only if

$$\nu_1 X_0 - \Phi_1 X_0 \perp X_0,$$

that is ν_1 is expressed by the formula (5.8). In view of (5.19), we search for the vector X_1 in the subspace $X_0^\perp$. Taking into account that, in view of (5.17),

$$\forall X \in X_0^\perp : \Phi_0 X - \nu_0 X = -\nu_0 X,$$

we obtain from (5.20) the expression (5.11) for X_1 . Analogously, equation (5.21) is solvable if and only if

$$\nu_1 X_1 + \nu_2 X_0 - \Phi_1 X_1 - \Phi_2 X_0 \perp X_0,$$

that is we get the expression (5.10) for ν_2 . Furthermore, as above, we obtain from (5.21) the corresponding expression for X_2 . Proceeding with this process and taking into account (5.18), we obtain all the desired formulas. The lemma is proven. $\square$

Since in the sequel we shall invert the asymptotic expansion of the form (5.6), we need the following

Lemma 5.2. *Consider the equation*

$$t = \epsilon \theta(t), \quad (5.22)$$

where

$$\theta(t) = \left(\nu_0 + \nu_1 t + \cdots + \nu_m t^m + \psi t^{m+\frac{1}{2}} \right)^{\frac{2}{2m+1}}, \quad (5.23)$$

$m \in \mathbb{N}$ and $\nu_k, \psi \in \mathbb{R}$, $\nu_0 > 0$. Let $\{t_j(\epsilon)\}_{j=0}^{m+1}$ be a sequence of functions defined by the following recurrence relations:

$$t_0(\epsilon) = 0, \quad t_j(\epsilon) = \epsilon(\theta_{j-1}(t_{j-1}(\epsilon)))_{j-1}, \quad (5.24)$$

where

$$\theta_j(t) = \left(\nu_0 + \nu_1 t + \cdots + \nu_j t^j \right)^{\frac{2}{2m+1}}. \quad (5.25)$$

Then:

(i) Equation (5.22) has a unique solution $t = t(\epsilon)$ for ϵ belonging to a neighborhood $\mathcal{O}(0)$ of $\epsilon = 0$ and this solution is continuous in $\mathcal{O}(0)$;

(ii) For any $j \in \{0, 1, \dots, m\}$

$$t(\epsilon) - t_j(\epsilon) = O(\epsilon^{j+1}) \quad (5.26)$$

and

$$t(\epsilon) - t_{m+1}(\epsilon) = O(\epsilon^{m+\frac{3}{2}}). \quad (5.27)$$

Proof. As is easily seen, the function $\theta(t)$ defined by (5.23) satisfies Lipschitz condition in any neighborhood $(-\delta, \delta)$:

$$\forall \delta > 0 \quad \exists L_\delta > 0 \quad \forall t_1, t_2 \in (-\delta, \delta) : |\theta(t_1) - \theta(t_2)| \leq L_\delta |t_1 - t_2|. \quad (5.28)$$

This circumstance and the form of the equation (5.22) imply assertion (i) of the lemma and the fact that the successive approximations

$$\tilde{t}_0(t) = 0, \tilde{t}_1(t) = \epsilon \theta(\tilde{t}_0(t)), \dots, \tilde{t}_j(t) = \epsilon \theta(\tilde{t}_{j-1}(t)), \dots \quad (5.29)$$

converge to the solution $t(\epsilon)$ of this equation uniformly in some neighborhood $\mathcal{O}(0)$ of the point $\epsilon = 0$. Making use of (5.28), we have:

$$|\tilde{t}_{k+1}(\epsilon) - \tilde{t}_k(\epsilon)| \leq L^k \epsilon^{k+1},$$

hence

$$|t(\epsilon) - \tilde{t}_j(\epsilon)| \leq \sum_{k=j+1}^{\infty} |\tilde{t}_{k+1}(\epsilon) - \tilde{t}_k(\epsilon)| \leq L^j \epsilon^{j+1} (1 + L\epsilon + L^2 \epsilon^2 + \dots).$$

This means that

$$t(\epsilon) - \tilde{t}_j(\epsilon) = O(\epsilon^{j+1}). \quad (5.30)$$

Let us return to the sequence $t_j(\epsilon)$, defined by (5.24). Our immediate goal is to prove that

$$\forall j \in \{0, 1, \dots, m\} : t_j(\epsilon) - \tilde{t}_j(\epsilon) = O(\epsilon^{j+1}). \quad (5.31)$$

Let us make use of the induction method. Recall that $t_0(\epsilon) = \tilde{t}_0(\epsilon) = 0$. Assume that (5.31) is valid for some $j \in \{0, 1, \dots, m-1\}$ and prove it for j replaced by $j+1$. Taking into account (5.24 and (5.29), we have:

$$\begin{aligned} |\tilde{t}_{j+1}(\epsilon) - t_{j+1}(\epsilon)| &\leq \epsilon \{ |\theta(\tilde{t}_j(\epsilon)) - \theta(t_j(\epsilon))| + \\ &|\theta(t_j(\epsilon)) - \theta_j(t_j(\epsilon))| + |\theta_j(t_j(\epsilon)) - (\theta_j(t_j(\epsilon)))_j| \}. \end{aligned} \quad (5.32)$$

The property (5.28) and the assumption of the induction imply:

$$\theta(\tilde{t}_j(\epsilon)) - \theta(t_j(\epsilon)) = O(\tilde{t}_j(\epsilon) - t_j(\epsilon)) = O(\epsilon^{j+1}). \quad (5.33)$$

Observe that, in view of (5.24) and the fact that $\nu_0 > 0$, the functions $t_j(\epsilon)$ have the form:

$$t_j(\epsilon) = \sigma_0 \epsilon + O(\epsilon^2) \quad (\sigma_0 > 0, j = 0, 1, \dots, m+1). \quad (5.34)$$

Taking into account this circumstance and (5.23), (5.25), let us estimate for $j \in \{0, 1, \dots, m-1\}$:

$$\begin{aligned} \theta(t_j(\epsilon)) - \theta_j(t_j(\epsilon)) &= \\ &\left(\nu_0 + \nu_1 t_j(\epsilon) + \dots + \nu_j (t_j(\epsilon))^j + \nu_{j+1} (t_j(\epsilon))^{j+1} + \dots + \psi(t_j(\epsilon))^{m+\frac{1}{2}} \right)^{\frac{2}{2m+1}} - \\ &(\nu_0 + \nu_1 t_j(\epsilon) + \dots + \nu_j (t_j(\epsilon))^j)^{\frac{2}{2m+1}} = O((t_j(\epsilon))^{j+1}) = O(\epsilon^{j+1}). \end{aligned} \quad (5.35)$$

Furthermore, by the sense of the notation $(\dots)_j$,

$$\theta_j(t_j(\epsilon)) - (\theta_j(t_j(\epsilon)))_j = O(\epsilon^{j+1}). \quad (5.36)$$

The latter estimate and estimates (3.9), (5.35) and (5.32) imply that

$$t_{j+1}(\epsilon) - \tilde{t}_{j+1}(\epsilon) = O(\epsilon^{j+2}).$$

So, we have proved estimate (5.31), which together with estimate (5.30) yields the desired estimate (5.26).

Let us prove estimate (5.27). Observe that estimate (5.32) is valid also for $j = m$, that is

$$\begin{aligned} |\tilde{t}_{m+1}(\epsilon) - t_{m+1}(\epsilon)| &\leq \epsilon \{ |\theta(\tilde{t}_m(\epsilon)) - \theta(t_m(\epsilon))| + \\ |\theta(t_m(\epsilon)) - \theta_m(t_m(\epsilon))| &+ |\theta_m(t_m(\epsilon)) - (\theta_m(t_m(\epsilon)))_m| \}. \end{aligned} \quad (5.37)$$

The property (5.28) and estimate (5.31) imply that

$$\theta(\tilde{t}_m(\epsilon)) - \theta(t_m(\epsilon)) = O(\epsilon^{j+1}). \quad (5.38)$$

By (5.36) with $j = m$ we get:

$$\theta_m(t_m(\epsilon)) - (\theta_m(t_m(\epsilon)))_m = O(\epsilon^{m+1}). \quad (5.39)$$

Furthermore, taking into account (5.34) with $j = m$, we have:

$$\begin{aligned} \theta(t_m(\epsilon)) - \theta_m(t_m(\epsilon)) &= \\ &\left(\nu_0 + \nu_1 t_m(\epsilon) + \cdots + \nu_m (t_m(\epsilon))^m + \psi(t_m(\epsilon))^{m+\frac{1}{2}} \right)^{\frac{2}{2m+1}} - \\ &(\nu_0 + \nu_1 t_m(\epsilon) + \cdots + \nu_m (t_m(\epsilon))^m)^{\frac{2}{2m+1}} = O\left((t_m(\epsilon))^{m+\frac{1}{2}} \right) = O(\epsilon^{m+\frac{1}{2}}). \end{aligned}$$

The latter estimate, estimates (5.37), (5.38), (5.39) and estimate (5.30) with $j = m+1$ imply the desired estimate (5.27). The lemma is proven. $\square$

We now turn to the main result of this subsection.

Theorem 5.3. *Assume that d is odd, $2l > d$, and in addition to conditions (4.1)-(4.3) the potential $V(\mathbf{x})$ is not identically zero and satisfies condition (4.16). Denote $m = l - \frac{d+1}{2}$. Then:*

(i) *If $m = 0$, the operator H_γ has an unique virtual eigenvalue $\lambda_0(\gamma)$ at $\lambda = 0$ and the following asymptotic representation is valid for $\gamma \uparrow 0$:*

$$\lambda_0(\gamma) = -c_0 |\gamma|^{2l} (1 + O(\gamma)),$$

where

$$c_0 = \left(\xi_0 \int_{\mathbb{R}^d} V(\mathbf{x}) d\mathbf{x} \right)^{2l}$$

and

$$\xi_0 = (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{ds}{|s|^{2l} + 1}.$$

Recall that ξ_0 is calculated by (4.19) with $\mathbf{k} = \mathbf{0}$;

(ii) *If $m > 0$, for the bottom virtual eigenvalue $\lambda_0(\gamma)$ of the operator H_γ at $\lambda = 0$ the following asymptotic expansion is valid for $\gamma \uparrow 0$:*

$$\lambda_0(\gamma) = -|\gamma|^{\frac{2l}{2m+1}} \left(\delta_0 + \delta_1 |\gamma|^{\frac{2}{2m+1}} + \cdots + \delta_m |\gamma|^{\frac{2m}{2m+1}} + O(\gamma) \right)^l, \quad (5.40)$$

where the numbers $\delta_0, \delta_1, \dots, \delta_m$ are coefficients of a polynomial

$$p(\epsilon) = \epsilon(\delta_0 + \delta_1\epsilon + \dots + \delta_m\epsilon^m),$$

which is calculated by the following procedure:

(A) The functions

$$\theta_j(t) = (\nu_0 + \nu_1 t + \dots + \nu_j t^j)^{\frac{2}{2m+1}} \quad (j = 0, 1, \dots, m)$$

are considered, where the numbers $\nu_0, \nu_1, \dots, \nu_m$ are defined by equalities (5.7)-(5.15), in which the operators Φ_j are defined by (5.5), (5.3) and (4.19);

(B) The polynomials $t_j(\epsilon)$ ($j = 0, 1, 2, \dots, m+1$) are constructed in the following manner:

$$t_0(\epsilon) = 0, \quad t_j(\epsilon) = \epsilon(\theta_{j-1}(t_{j-1}(\epsilon)))_{j-1} \quad (j = 1, 2, \dots, m+1);$$

(C) We put $p(\epsilon) = t_{m+1}(\epsilon)$.

Proof. First we shall prove assertion (ii). Assume that $m > 0$. Let $\mu_0^+(\lambda)$ be the maximal characteristic branch of the operator H_0 with respect to V in the gap $(-\infty, 0)$ of $\sigma(H_0)$, that is, this is the maximal positive eigenvalue of the Birman-Schwinger operator $X_V(\lambda)$ (see Definition 3.2). Let $\tilde{\mu}_0(\lambda)$ be the maximal eigenvalue of the finite-rank portion $\Phi(\lambda)$ of the operator $X_V(\lambda)$. By Proposition 4.4 and Lemma 3.4, we have for some $\bar{T} > 0$:

$$\tilde{\mu}_0(\lambda) - \bar{T} \leq \mu_0^+(\lambda) \leq \tilde{\mu}_0(\lambda) + \bar{T} \quad (\lambda < 0).$$

These estimates and the asymptotic expansion (5.6) for $\lambda \uparrow 0$ (Lemma 5.1) imply that for some $\bar{S} > 0$ and a small enough $\lambda < 0$

$$\sigma(\lambda) - \bar{S} \leq \mu_0^+(\lambda) \leq \sigma(\lambda) + \bar{S},$$

where the function $\sigma(\lambda)$ is defined on $(-\infty, 0)$ and has the form:

$$\sigma(\lambda) = \frac{\nu_0}{|\lambda|^{\frac{m+1}{2}}} + \frac{\nu_1}{|\lambda|^{\frac{m-1}{2}}} + \frac{\nu_2}{|\lambda|^{\frac{m-3}{2}}} + \dots + \frac{\nu_m}{|\lambda|^{\frac{1}{2}}}. \quad (5.41)$$

The latter estimates and Proposition 3.16 imply that $\mu_0^+(\lambda)$ is a main characteristic branch of the operator H_0 , with respect to V , near the endpoint $\lambda = 0$ of $(-\infty, 0)$ and, furthermore, for the bottom virtual eigenvalue $\lambda_0(\gamma)$ of H_γ the estimates are valid for a small enough $\gamma < 0$:

$$\Lambda_+ \left(\frac{1}{|\gamma|} \right) \leq \lambda_0(\gamma) \leq \Lambda_- \left(\frac{1}{|\gamma|} \right), \quad (5.42)$$

where Λ_+ and Λ_- are the inverses of the functions $\sigma(\lambda) + \bar{S}$ and $\sigma(\lambda) - \bar{S}$, respectively. In order to find the bounds $\Lambda_+ \left(\frac{1}{|\gamma|} \right)$ and $\Lambda_- \left(\frac{1}{|\gamma|} \right)$ in (5.42), we should solve the equation of the form:

$$\sigma(\lambda) + \psi = \frac{1}{|\gamma|}, \quad (5.43)$$

where ψ takes one of the values: $\bar{S}$ or $-\bar{S}$. Denote

$$t = |\lambda|^{\frac{1}{l}} \quad (5.44)$$

and

$$\epsilon = |\gamma|^{\frac{2}{2m+1}}. \quad (5.45)$$

In view of (5.41), equation (5.43) acquires the form:

$$t = \epsilon \theta(t), \quad (5.46)$$

where

$$\theta(t) = \left(\nu_0 + \nu_1 t + \cdots + \nu_m t^m + \psi t^{m+\frac{1}{2}} \right)^{\frac{2}{2m+1}}.$$

By Lemma 5.2, the solution of equation (5.46) has the form:

$$t(\epsilon) = p(\epsilon) + O\left(\epsilon^{m+\frac{3}{2}}\right),$$

where the polynomial $p(\epsilon)$ is calculated according to the procedure (A), (B) and (C). This circumstance, notation (5.44), (5.45) and estimates (5.42) imply the desired asymptotic expansion (5.40).

We now turn to assertion (i) of the theorem. Observe that in the case $m = 0$ the finite-rank portion $\Phi(\lambda)$ of the Birman-Schwinger operator $X_V(\lambda)$ is of rank one and has the form:

$$\Phi(\lambda) = \frac{\xi_0}{|\lambda|^{\frac{1}{2l}}}(\cdot, h_0)h_0.$$

The latter operator has only one non-zero branch of eigenvalues:

$$\mu_0(\lambda) = \frac{\xi_0}{|\lambda|^{\frac{1}{2l}}} \|h_0\|^2.$$

Using the same arguments, as in the proof of assertion (ii), we prove assertion (i). The theorem is proven. $\square$

5.2° We now turn to the case of $\mathbb{R}^d$ with d even. Assume that $2l > d$. Let $\Phi(\lambda)$ ($\lambda < 0$) be the integral operator acting in the space $L_2(\mathbb{R}^d)$ and having the kernel $\Phi(\mathbf{x}, \mathbf{y}, \lambda)$, defined by (4.48), where $F(\mathbf{x}, \mathbf{y}, \lambda)$ is defined by (4.21). In other words, $\Phi(\lambda)$ has the form:

$$\Phi(\lambda) = \sum_{\mathbf{k}, \mathbf{n}: |\mathbf{k}+\mathbf{n}| \leq 2m-1} \frac{\xi_{\mathbf{k}+\mathbf{n}}}{|\lambda|^{\frac{2m-|\mathbf{k}+\mathbf{n}|}{2l}}}(\cdot, h_{\mathbf{n}})h_{\mathbf{k}} + \ln\left(\frac{1}{|\lambda|}\right) \sum_{\mathbf{k}, \mathbf{n}: |\mathbf{k}+\mathbf{n}|=2m} \eta_{\mathbf{k}+\mathbf{n}}(\cdot, h_{\mathbf{n}})h_{\mathbf{k}}, \quad (5.47)$$

where

$$m = l - \frac{d}{2}, \quad (5.48)$$

the functions $h_{\mathbf{k}}(\mathbf{x})$ are defined by (5.3) and the numbers $\xi_{\mathbf{k}}$ and $\eta_{\mathbf{k}}$ are defined by (4.19) and (4.46), respectively. In this subsection we assume that the potential $V(\mathbf{x})$ satisfies condition (4.47), which in particular ensures the membership of the functions

$h_{\mathbf{k}}(\mathbf{x})$ ($|\mathbf{k}| \leq 2m$) to the class $L_2(\mathbb{R}^d)$. Taking into account Proposition 4.5, we can write formula (5.47) in another form:

$$\Phi(\lambda) = \frac{1}{|\lambda|^{\frac{m}{t}}} \sum_{j=0}^{m-1} |\lambda|^{\frac{j}{t}} \Phi_j + \ln \left(\frac{1}{|\lambda|} \right) \Psi_m, \quad (5.49)$$

where

$$\Phi_j = \sum_{\mathbf{k}, \mathbf{n}: |\mathbf{k}+\mathbf{n}|=2j} \xi_{\mathbf{k}+\mathbf{n}}(\cdot, h_{\mathbf{n}}) h_{\mathbf{k}} \quad (j \in \{0, 1, \dots, m-1\}) \quad (5.50)$$

and

$$\Psi_m = \sum_{\mathbf{k}, \mathbf{n}: |\mathbf{k}+\mathbf{n}|=2m} \eta_{\mathbf{k}+\mathbf{n}}(\cdot, h_{\mathbf{n}}) h_{\mathbf{k}}. \quad (5.51)$$

The following statement about an asymptotic expansion of the maximal eigenvalue $\tilde{\mu}_0(\lambda)$ of the operator $\Phi(\lambda)$ is valid:

Lemma 5.4. *Assume that d is even, the number m , defined by (5.48), is positive and in addition to conditions (4.1)-(4.3) the potential $V(\mathbf{x})$ is not identically zero and satisfies condition (4.47). Then for $\lambda \uparrow 0$ the following asymptotic expansion holds for the maximal eigenvalue $\tilde{\mu}_0(\lambda)$ of the operator $\Phi(\lambda)$:*

$$\tilde{\mu}_0(\lambda) = \frac{\nu_0}{|\lambda|^{\frac{m}{t}}} + \frac{\nu_1}{|\lambda|^{\frac{m-1}{t}}} + \frac{\nu_2}{|\lambda|^{\frac{m-2}{t}}} + \dots + \frac{\nu_{m-1}}{|\lambda|^{\frac{1}{t}}} + \nu_m \ln \left(\frac{1}{|\lambda|} \right) + O(1), \quad (5.52)$$

where the numbers ν_k ($k = 0, 1, \dots, m$) have the form:

$$\nu_0 = \xi_0 \|h_0\|^2 = \xi_0 \int_{\mathbb{R}^d} V(\mathbf{s}) d\mathbf{s}, \quad (5.53)$$

$$\nu_1 = (\Phi_1 X_0, X_0), \quad (5.54)$$

$$X_0 = \frac{h_0}{\|h_0\|} = \frac{\sqrt{V(\mathbf{x})}}{\sqrt{\int_{\mathbb{R}^d} V(\mathbf{s}) d\mathbf{s}}}, \quad (5.55)$$

$$\nu_2 = (\Phi_2 X_0, X_0) + (\Phi_1 X_1, X_0), \quad (5.56)$$

$$X_1 = -\frac{1}{\nu_0}(\nu_1 X_0 - \Phi_1 X_0), \quad (5.57)$$

.....,

$$\nu_j = (\Phi_j X_0, X_0) + (\Phi_{j-1} X_1, X_0) + \dots + (\Phi_1 X_{j-1}, X_0), \quad (5.58)$$

$$\begin{aligned} X_{j-1} &= -\frac{1}{\nu_0}(\nu_1 X_{j-2} + \nu_2 X_{j-3} + \dots + \nu_{j-1} X_0 - \Phi_1 X_{j-2} \\ &\quad - \Phi_2 X_{j-3} - \dots - \Phi_{j-1} X_0), \end{aligned} \quad (5.59)$$

.....,

$$\nu_{m-2} = (\Phi_{m-2} X_0, X_0) + (\Phi_{m-3} X_1, X_0) + \dots + (\Phi_1 X_{m-3}, X_0), \quad (5.60)$$

$$\begin{aligned}
X_{m-3} &= -\frac{1}{\nu_0}(\nu_1 X_{m-4} + \nu_2 X_{m-5} + \cdots + \nu_{m-3} X_0 - \Phi_1 X_{m-4} \\
&\quad - \Phi_2 X_{m-5} - \cdots - \Phi_{m-3} X_0),
\end{aligned} \tag{5.61}$$

$$\nu_{m-1} = (\Phi_{m-1} X_0, X_0) + (\Phi_{m-2} X_1, X_0) + \cdots + (\Phi_1 X_{m-2}, X_0), \tag{5.62}$$

$$\begin{aligned}
X_{m-2} &= -\frac{1}{\nu_0}(\nu_1 X_{m-3} + \nu_2 X_{m-4} + \cdots + \nu_{m-2} X_0 - \Phi_1 X_{m-3} \\
&\quad - \Phi_2 X_{m-4} - \cdots - \Phi_{m-2} X_0),
\end{aligned} \tag{5.63}$$

$$\nu_m = (\Psi_m X_0, X_0) \tag{5.64}$$

and the operators Φ_j and Ψ_m are defined by (5.50) and (5.51).

Proof. Let us denote

$$t = |\lambda|^{\frac{1}{l}} \tag{5.65}$$

and consider the operator function:

$$\Psi(t) = t^m \Phi(-t^l) = \Phi_0 + \sum_{j=1}^{m-1} t^j \Phi_j + l t^m \ln\left(\frac{1}{t}\right) \Psi_m. \tag{5.66}$$

Unlike the case of an odd d , this function is not a polynomial one. We shall reduce it to a polynomial operator function of two variables t, v putting

$$v = t \ln\left(\frac{1}{t}\right).$$

In other words, we represent:

$$\Psi(t) = F\left(t, t \ln\left(\frac{1}{t}\right)\right), \tag{5.67}$$

where

$$F(t, v) = \Phi_0 + \sum_{j=1}^{m-1} t^j \Phi_j + l t^{m-1} v \Psi_m \tag{5.68}$$

Our immediate goal is to find a power expansion of the maximal eigenvalue $\omega_0(t, v)$ of the operator $F(t, v)$. Observe that

$$F(0, 0) = \Phi_0 = \xi_0(\cdot, h_0)h_0,$$

hence the operator $F(0, 0)$ has a unique non-zero eigenvalue ν_0 , defined by (5.53), this eigenvalue is positive and simple and the normalized eigenvector X_0 , defined by (5.55), corresponds to ν_0 . Hence, as is known (see [Bau], Supplement, S2, Corollary 1), there exists a branch $\omega_0(t, v)$ of eigenvalues of the operator function $F(t, v)$, such that $\omega_0(0, 0) = \nu_0$ and it is analytic with respect to the variables t, v in a neighborhood $\mathcal{O}(0, 0) \subset \mathcal{C}^2$ of the point $(0, 0)$ and the corresponding branch of normalized eigenvectors $X(t, v)$ is analytic in $\mathcal{O}(0, 0)$ too. Furthermore, it is possible to choose $\mathcal{O}(0, 0)$ such that $\omega_0(t, v)$ is a maximal eigenvalue of the operator $F(t, v)$ for any $(t, v) \in \mathcal{O}(0, 0)$. Thus, we can search for $\omega_0(t, v)$ and $X(t, v)$ in the form:

$$\omega_0(t, v) = \nu_0 + \rho_1(v)t + \cdots + \rho_{m-2}(v)t^{m-2} + \rho_{m-1}(v)t^{m-1} + t^m r(t, v) \tag{5.69}$$

$$\omega_0(t, v) = \nu_0 + tX_1(v) + \cdots + t^{m-2}X_{m-2}(v) + t^{m-1}X_{m-1}(v) + t^m R(t, v), \quad (5.70)$$
$$(X(t, v), X_0) = 1 \quad \forall (t, v) \in \mathcal{O}(0, 0).$$
$$F(t, v)X(t, v) = \omega(t, v)X(t, v),$$
$$\Phi_0 X_1(v) - \nu_0 X_1(v) = \rho_1(v) X_0 - \Phi_1 X_0,$$

$$\begin{aligned}\Phi_0 X_{m-2}(v) - \nu_0 X_{m-2}(v) &= \rho_1(v) X_{m-3}(v) + \cdots + \rho_{m-2}(v) X_0 \\ &- \Phi_1 X_{m-3}(v) - \cdots - \Phi_{m-2} X_0,\end{aligned}$$

$$\begin{aligned}\Phi_0 X_{m-1}(v) - \nu_0 X_{m-1}(v) &= \rho_1(v) X_{m-2}(v) + \cdots + \rho_{m-1}(v) X_0 \\ &- \Phi_1 X_{m-2}(v) - \cdots - (\Phi_{m-1} + l v \Psi_m) X_0.\end{aligned}$$

$$\forall j \in \{1, 2, \dots, m-2\} : \rho_j(v) = \nu_j$$
$$\rho_{m-1}(v) = \nu_{m-1} + lv\nu_m,$$
$$\omega(t, v) = \nu_0 + \nu_1 t + \cdots + \nu_{m-2} t^{m-2} + \nu_{m-1} t^{m-1} + l v \nu_m t^{m-1} + t^m r(t, v),$$

As in the case of an odd d , in the sequel we shall invert the asymptotic expansion of the form (5.52). To this end we need the following lemma, which is analogous to Lemma 5.2:

$$t = \epsilon \theta(t), \quad (5.71)$$
$$\theta(t) = \left(\nu_0 + \nu_1 t + \cdots + \nu_{m-1} t^{m-1} + l \nu_m t^m \ln \left(\frac{1}{t} \right) + \psi t^m \right)^{\frac{1}{m}}, \quad (5.72)$$

$m \in \mathbb{N}$ and $\nu_k, \psi, l \in \mathbb{R}$, $\nu_0 > 0$. Let $\{t_j(\epsilon)\}_{j=0}^{m+1}$ be a sequence of functions defined by the following recurrence relations:

$$t_0(\epsilon) = 0, \quad t_j(\epsilon) = \epsilon(\theta_{j-1}(t_{j-1}(\epsilon)))_{j-1} \quad (j = 1, 2, \dots, m) \quad (5.73)$$

and

$$t_{m+1}(\epsilon) = \epsilon Q_m(t_m(\epsilon)), \quad (5.74)$$

where

$$\theta_j(t) = (\nu_0 + \nu_1 t + \dots + \nu_j t^j)^{\frac{1}{m}} \quad (j = 0, 1, \dots, m-1) \quad (5.75)$$

and

$$\begin{aligned} Q_m(t_m(\epsilon)) &= \left\{ (\nu_0 + \nu_1 t_m(\epsilon) + \dots + \nu_{m-1} (t_m(\epsilon))^{m-1})_{m-1} \right. \\ &\quad \left. + l \nu_m ((t_m(\epsilon))_1)^m \ln \left(\frac{1}{\epsilon} \right) \right\}^{\frac{1}{m}} \end{aligned} \quad (5.76)$$

Then:

(i) Equation (5.46) has a unique solution $t = t(\epsilon)$ for ϵ belonging to a neighborhood $\mathcal{O}(0)$ of $\epsilon = 0$ and this solution is continuous in $\mathcal{O}(0)$;

(ii) For any $j \in \{0, 1, \dots, m-1\}$

$$t(\epsilon) - t_j(\epsilon) = O(\epsilon^{j+1}) \quad (5.77)$$

and

$$t(\epsilon) - t_m(\epsilon) = O\left(\epsilon^{m+1} \ln \left(\frac{1}{\epsilon} \right)\right), \quad (5.78)$$

$$t(\epsilon) - t_{m+1}(\epsilon) = O(\epsilon^{m+1}). \quad (5.79)$$

Proof. As is easily seen, the function $\theta(t)$ defined by (5.72) satisfies Lipschitz condition in any neighborhood $(-\delta, \delta)$:

$$\forall \delta > 0 \quad \exists L_\delta > 0 \quad \forall t_1, t_2 \in (-\delta, \delta) : |\theta(t_1) - \theta(t_2)| \leq L_\delta |t_1 - t_2|. \quad (5.80)$$

This circumstance and the form of the equation (5.46) imply assertion (i) of the lemma.

Along with the functions $t_j(\epsilon)$, defined by (5.73) and (5.74), consider the successive approximations

$$\tilde{t}_0(t) = 0, \quad \tilde{t}_1(t) = \epsilon \theta(\tilde{t}_0(t)), \dots, \tilde{t}_j(t) = \epsilon \theta(\tilde{t}_{j-1}(t)), \dots \quad (5.81)$$

for the solution of equation (5.46). In the same manner, as in the proof of Lemma 5.2, we prove the estimates

$$t(\epsilon) - \tilde{t}_j(\epsilon) = O(\epsilon^{j+1}), \quad (5.82)$$

$$t_j(\epsilon) - \tilde{t}_j(\epsilon) = O(\epsilon^{j+1}) \quad \text{for } j \in \{0, 1, \dots, m-1\} \quad (5.83)$$

and estimate (5.77) of assertion (ii) of the proposition. Furthermore, in view of (5.73), (5.74) and the fact that $\nu_0 > 0$, the functions $t_j(\epsilon)$ have the form:

$$t_j(\epsilon) = \sigma_0 \epsilon + O(\epsilon^2) \quad (\sigma_0 > 0, \quad j = 0, 1, \dots, m+1). \quad (5.84)$$

In view of (5.81) and (5.73),

$$\begin{aligned} |\tilde{t}_m(\epsilon) - t_m(\epsilon)| &\leq \epsilon \{ |\theta(\tilde{t}_{m-1}(\epsilon)) - \theta(t_{m-1}(\epsilon))| \\ &\quad + |\theta(t_{m-1}(\epsilon)) - \theta_{m-1}(t_{m-1}(\epsilon))| + |\theta_{m-1}(t_{m-1}(\epsilon)) \\ &\quad - (\theta_{m-1}(t_{m-1}(\epsilon)))_{m-1}| \}. \end{aligned} \quad (5.85)$$

By the sense of the notation $(\cdots)_{m-1}$,

$$|\theta_{m-1}(t_{m-1}(\epsilon)) - (\theta_{m-1}(t_{m-1}(\epsilon)))_{m-1}| = O(\epsilon^m). \quad (5.86)$$

Furthermore, in view of (5.80) and estimate (5.83), we have:

$$\theta(\tilde{t}_{m-1}(\epsilon)) - \theta(t_{m-1}(\epsilon)) = O(\epsilon^m). \quad (5.87)$$

Let us estimate, taking into account (5.84):

$$\begin{aligned} \theta(t_{m-1}(\epsilon)) - \theta_{m-1}(t_m(\epsilon)) &= \\ &\{ \nu_0 + \nu_1 t_{m-1}(\epsilon) + \cdots + \nu_{m-1} (t_{m-1}(\epsilon))^{m-1} + \\ &l\nu_m \ln \left(\frac{1}{t_{m-1}(\epsilon)} \right) (t_{m-1}(\epsilon))^m + \psi(t_{m-1}(\epsilon))^m \}^{\frac{1}{m}} - \\ &\{ \nu_0 + \nu_1 t_{m-1}(\epsilon) + \cdots + \nu_{m-1} (t_{m-1}(\epsilon))^{m-1} \}^{\frac{1}{m}} = O \left(\epsilon^m \ln \left(\frac{1}{\epsilon} \right) \right). \end{aligned} \quad (5.88)$$

Estimates (5.85), (5.86), (5.87) and (5.88) imply estimate

$$\tilde{t}_m(\epsilon) - t_m(\epsilon) = O \left(\epsilon^{m+1} \ln \left(\frac{1}{\epsilon} \right) \right), \quad (5.89)$$

which together with (5.82) implies the desired estimate (5.78).

We now turn to the proof of estimate (5.79). We have, taking into account (5.81) and (5.74):

$$\begin{aligned} |\tilde{t}_{m+1}(\epsilon) - t_{m+1}(\epsilon)| &\leq \epsilon \{ |\theta(\tilde{t}_m(\epsilon)) - \theta(t_m(\epsilon))| + \\ &|\theta(t_m(\epsilon)) - \theta_m(t_m(\epsilon))| + |\theta_m(t_m(\epsilon)) - Q_m(t_m(\epsilon))| \}, \end{aligned} \quad (5.90)$$

where

$$\theta_m(t) = \left\{ \nu_0 + \nu_1 t + \cdots + \nu_{m-1} t^{m-1} + l\nu_m \ln \left(\frac{1}{t} \right) t^m \right\}^{\frac{1}{m}}. \quad (5.91)$$

Making use of property (5.80) and estimate (5.89), we get:

$$\theta(\tilde{t}_m(\epsilon)) - \theta(t_m(\epsilon)) = O \left(\epsilon^{m+1} \ln \left(\frac{1}{\epsilon} \right) \right). \quad (5.92)$$

Furthermore, taking into account (5.84), we have:

$$\begin{aligned} \theta(t_m(\epsilon)) - \theta_m(t_m(\epsilon)) &= \{ \nu_0 + \nu_1 t_m(\epsilon) + \cdots + \nu_{m-1} (t_m(\epsilon))^{m-1} + \\ l\nu_m \ln \left(\frac{1}{t_m(\epsilon)} \right) (t_m(\epsilon))^m &+ \psi(t_m(\epsilon))^m \}^{\frac{1}{m}} - \\ \{ \nu_0 + \nu_1 t_m(\epsilon) &+ \cdots + \nu_{m-1} (t_m(\epsilon))^{m-1} \\ &+ l\nu_m \ln \left(\frac{1}{t_m(\epsilon)} \right) (t_m(\epsilon))^m \}^{\frac{1}{m}} = O(\epsilon^m). \end{aligned} \quad (5.93)$$

Let us estimate the third summand in the right hand side of (5.90). Taking into account definitions (5.91) and (5.76), we get:

$$\begin{aligned}\theta_m(t_m(\epsilon)) - Q_m(t_m(\epsilon)) &= O(\epsilon^m) + O\left(\ln\left(\frac{1}{t_m(\epsilon)}\right)(t_m(\epsilon))^m\right. \\ &\quad \left.- \ln\left(\frac{1}{\epsilon}\right)((t_m(\epsilon))_1)^m\right).\end{aligned}\quad (5.94)$$

On the other hand, in view of (5.84), we have:

$$\begin{aligned}\ln\left(\frac{1}{t_m(\epsilon)}\right)(t_m(\epsilon))^m &- \ln\left(\frac{1}{\epsilon}\right)((t_m(\epsilon))_1)^m = \epsilon^m \sigma_0^m (1 + O(\epsilon)) \left(\ln\left(\frac{1}{\epsilon}\right)\right. \\ &\quad \left.- \ln \sigma_0 + O(\epsilon) - \epsilon^m \sigma_0^m \ln\left(\frac{1}{\epsilon}\right)\right) = O(\epsilon^m).\end{aligned}$$

The latter estimate and estimates (5.94), (5.92), (5.93), (5.90) and (5.82) imply the desired estimate (5.79). The lemma is proven. $\square$

We now turn to the main result of this subsection.

Theorem 5.6. *Assume that d is even, $2l \geq d$, and in addition to conditions (4.1)-(4.3) the potential $V(\mathbf{x})$ is not identically zero and satisfies condition (4.47). Denote $m = l - \frac{d}{2}$. Then:*

(i) *If $m = 0$, the operator H_γ has an unique virtual eigenvalue $\lambda_0(\gamma)$ at $\lambda = 0$ and the following asymptotic representation is valid for $\gamma \uparrow 0$:*

$$\ln\left(\frac{1}{|\lambda_0(\gamma)|}\right) = \frac{f}{|\gamma|}(1 + O(\gamma)), \quad (5.95)$$

where

$$f = \left(\eta_0 \int_{\mathbb{R}^d} V(\mathbf{x}) d\mathbf{x}\right)^{-1} \quad (5.96)$$

and

$$\eta_0 = (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{ds}{(|\mathbf{s}|^{2l} + 1)^2}. \quad (5.97)$$

Recall that η_0 is calculated by (4.46) with $\mathbf{k} = \mathbf{0}$;

(ii) *If $m > 0$, for the bottom virtual eigenvalue $\lambda_0(\gamma)$ of the operator H_γ at $\lambda = 0$ the following asymptotic expansion is valid for $\gamma \uparrow 0$:*

$$\lambda_0(\gamma) = -|\gamma|^{\frac{1}{m}} \left(\eta \left(|\gamma|^{\frac{1}{m}}\right) + O(\gamma)\right)^l, \quad (5.98)$$

where the function $\eta(\epsilon)$ has the form:

$$\eta(\epsilon) = \left(\delta_0 + \delta_1 \epsilon + \dots + \delta_{m-1} \epsilon^{m-1} + \delta_m \epsilon^m \ln\left(\frac{1}{\epsilon}\right)\right)^{\frac{1}{m}}$$

and it is calculated by the following procedure:

(A) The functions

$$\theta_j(t) = (\nu_0 + \nu_1 t + \dots + \nu_j t^j)^{\frac{1}{m}} \quad (j = 0, 1, \dots, m-1)$$

are considered, where the numbers $\nu_0, \nu_1, \dots, \nu_m$ are defined by equalities (5.53)-(5.63), in which the operators Φ_j are defined by (5.50), (5.3) and (4.19);

(B) The polynomials $t_j(\epsilon)$ ($j = 0, 1, 2, \dots, m$) are constructed in the following manner:

$$t_0(\epsilon) = 0, \quad t_j(\epsilon) = \epsilon(\theta_{j-1}(t_{j-1}(\epsilon)))_{j-1} \quad (j = 1, 2, \dots, m);$$

(C) We put

$$\begin{aligned} \eta(\epsilon) &= \left\{ (\nu_0 + \nu_1 t_m(\epsilon) + \dots + \nu_{m-1} (t_m(\epsilon))^{m-1})_{m-1} \right. \\ &\quad \left. + l\nu_m ((t_m(\epsilon))_1)^m \ln \left(\frac{1}{\epsilon} \right) \right\}^{\frac{1}{m}}, \end{aligned}$$

where ν_m is defined by equality (5.64), in which the operator Ψ_m is defined by (5.51), (5.3) and (4.46).

Proof. First we shall prove assertion (ii). Assume that $m > 0$. Let $\mu_0^+(\lambda)$ be the maximal characteristic branch of the operator H_0 with respect to V in the gap $(-\infty, 0)$ of $\sigma(H_0)$ and $\tilde{\mu}_0(\lambda)$ be the maximal eigenvalue of the finite-rank portion $\Phi(\lambda)$ of the Birman-Schwinger operator $X_V(\lambda)$. By Lemma 5.4, $\tilde{\mu}_0(\lambda)$ has the asymptotic expansion (5.52) for $\lambda \uparrow 0$. Making use of Proposition 4.8 and the same arguments, as in the proof of Theorem 5.3, we have estimates for some $\bar{S} > 0$ and a small enough $\lambda < 0$:

$$\sigma(\lambda) - \bar{S} \leq \mu_0^+(\lambda) \leq \sigma(\lambda) + \bar{S},$$

where the function $\sigma(\lambda)$ is defined on $(-\infty, 0)$ and has the form:

$$\sigma(\lambda) = \frac{\nu_0}{|\lambda|^{\frac{m}{t}}} + \frac{\nu_1}{|\lambda|^{\frac{m-1}{t}}} + \frac{\nu_2}{|\lambda|^{\frac{m-2}{t}}} + \dots + \frac{\nu_{m-1}}{|\lambda|^{\frac{1}{t}}} + \nu_m \ln \left(\frac{1}{|\lambda|} \right).$$

Then, making use of Proposition 3.16, we get the following estimates for the bottom virtual eigenvalue $\lambda_0(\gamma)$ of H_γ for a small enough $\gamma < 0$:

$$\Lambda_+ \left(\frac{1}{|\gamma|} \right) \leq \lambda_0(\gamma) \leq \Lambda_- \left(\frac{1}{|\gamma|} \right), \quad (5.99)$$

where Λ_+ and Λ_- are the inverses of the functions $\sigma(\lambda) + \bar{S}$ and $\sigma(\lambda) - \bar{S}$, respectively. Like in the proof of Theorem 5.3, in order to find the bounds $\Lambda_+ \left(\frac{1}{|\gamma|} \right)$ and $\Lambda_- \left(\frac{1}{|\gamma|} \right)$ in (5.99), we should solve the equation:

$$t = \epsilon \theta(t),$$

where

$$\theta(t) = \left(\nu_0 + \nu_1 t + \dots + \nu_{m-1} t^{m-1} + l\nu_m t^m \ln \left(\frac{1}{t} \right) + \psi t^m \right)^{\frac{1}{m}},$$

$t = |\lambda|^{\frac{1}{t}}$, $\epsilon = |\gamma|^{\frac{1}{m}}$ and ψ takes one of the values: $\bar{S}$ or $-\bar{S}$. Then, making use of Lemma 5.5 and estimates (5.99), we obtain assertion (ii) of the theorem.

We now turn to assertion (i). Observe that in the case $m = 0$ the finite-rank portion $\Phi(\lambda)$ of the Birman-Schwinger operator $X_V(\lambda)$ is of rank one and has the form:

$$\Phi(\lambda) = \ln \left(\frac{1}{|\lambda|} \right) \eta_0(\cdot, h_0) h_0.$$

The latter operator has only one non-zero branch of eigenvalues:

$$\mu_0(\lambda) = \ln \left(\frac{1}{|\lambda|} \right) \eta_0 \|h_0\|^2.$$

Using the same arguments, as in the proof of assertion (ii), we prove assertion (i). The theorem is proven. $\square$

Remark 5.7. The asymptotic formula (5.95)-(5.97) is well known in the case $l = 1$ and $d = 2$ (see [S], Ch. 7 and [S1]).

Remark 5.8. In [N-W] under conditions $2l > d$, $0 \leq V \in L_1(\mathbb{R}^d)$ an asymptotic formula for the bottom virtual eigenvalue of the operator H_γ has been obtained, which has the following form in our notation:

$$\lambda_0(\gamma) = -|\gamma|^{\frac{2l}{2l-d}} \left\{ \left(\xi_0 \int_{\mathbb{R}^d} V(\mathbf{x}) d\mathbf{x} \right)^{\frac{2l}{2l-d}} + o(1) \right\} \text{ for } \gamma \uparrow 0. \quad (5.100)$$

We see that the remainder of this asymptotic formula has the order $o\left(|\gamma|^{\frac{2l}{2l-d}}\right)$. On the other hand, we see that the remainder of the asymptotic expansions (5.40) and (5.98) has the order $O\left(|\gamma|^{\frac{4l-d}{2l-d}}\right)$. This means that for $2l > d$ Theorems 5.3 and 5.6 yield more precise asymptotic representations of the bottom virtual eigenvalue of the operator H_γ than the asymptotic representation (5.100) of [N-W].

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