ESI The Erwin Schrödinger International Institute for Mathematical Physics
Boltzmanngasse 9
A-1090 Wien, Austria

On the Path Integral in Non–Commutative (NC) QFT

Christoph Dehne

Vienna, Preprint ESI 2010 (2008)

February 22, 2008

Supported by the Austrian Federal Ministry of Education, Science and Culture
Available via http://www.esi.ac.at

ON THE PATH INTEGRAL IN NON–COMMUTATIVE (NC) QFT

CHRISTOPH DEHNE*

Institut für Theoretische Physik, Universität Leipzig,
Postfach 100 920, D – 04009 Leipzig

As is generally known, different quantization schemes applied to field theory on NC spacetime lead to Feynman rules with different physical properties, if time does not commute with space. In particular, the Feynman rules that are derived from the path integral corresponding to the $T^{\star}$–product (the so–called naïve Feynman rules) violate the causal time ordering property.
Within the Hamiltonian approach to quantum field theory, we show that we can (formally) modify the time ordering encoded in the above path integral. The resulting Feynman rules are identical to those obtained in the canonical approach via the Gell–Mann-Low formula (with T–ordering). They preserve thus unitarity and causal time ordering.

MSC: 70 S 05, 70 S 99, 81 T 08, 81 T 10, 81 T 18, 81 T 99

Keywords: non–commutative spacetime, causal time–ordering (T–ordering), path integral, Hamiltonian approach, naïve Feynman rules, $T^{\star}$–product, time–ordered perturbation theory, Wick rotation, reflexion positivity

1. Introductory remarks on set–up of QFT on NC spacetime

In the last 15 years, much work and effort has been devoted to the construction and study of quantum field theories on NC spacetime. The increase in research activity in this field can be traced back to the appearance of the seminal work by Doplicher, Fredenhagen and Roberts,[1] to an important discovery in string theory[2] and last, but not least, to its relation to non–commutative geometry,[3] in general.
The nowadays most popular idea how to implement the non–commutativity of spacetime in field theory is based on the Weyl–Moyal correspondence. The formerly pointwise product between fields $f_1(x)$ and $f_2(x)$ is then replaced by the so–called star product:

$$(f_1 * f_2)(x) := \big[\exp(\frac{i}{2}\theta^{\mu\nu}\partial^x_\mu\partial^y_\nu) f_1(x) f_2(y)\big]_{y=x}. \tag{1}$$

Here, $\theta^{\mu\nu}$ is defined via $[\hat{x}_\mu, \hat{x}_\nu] =: i\theta_{\mu\nu}\mathbb{1}$; $\hat{x}_\mu$, $\hat{x}_\nu$ are coordinate operators; $\theta_{\mu\nu}$ is a real, antisymmetric, constant matrix ($d = 1 + 3$). The field theoretic change to a

*Christoph.Dehne@itp.uni-leipzig.de

physical system with a, say Φ^3 self–interaction is then given by the following action:

$$S_{class}[\Phi;\theta^{\mu\nu}] = \int d^4x\Big(\frac{1}{2}\partial_\mu\Phi * \partial^\mu\Phi(x) - \frac{m^2}{2}\Phi * \Phi(x) - \frac{g}{3!}\Phi * \Phi * \Phi(x)\Big). \qquad (2)$$

Since the star product is cyclic under the trace ($\int d^4x\ f * g(x) = \int d^4x\ g * f(x)$, $f, g \,\epsilon\, \mathcal{S}(\mathbb{R}^4)$), it follows that the quantum theory of the kinetic part is the free theory of ordinary quantum field theory. However, as for the interacting theory, a perturbative expansion of Green's functions leads to Feynman rules that depend on the starting point for quantization and are no longer equivalent (already at tree level). In the following, we will see how a (slightly) different set–up of the generating functional formula (path integral) leads to Feynman rules with different physical properties!

2. Path integral in NC QFT corresponding to T^*–ordering

The easiest way to set up the path integral formula for the kind of non–local model considered here is to take over the formula of the generating functional $Z(J)$ from the local case and replace in the interaction term $\mathcal{L}_{int}(\Phi)$ the local field products by the star products (The free theory remains unchanged.). The resulting formula is then given by

$$Z[J] = \exp\big[i\int d^4z\mathcal{L}_{int}(\frac{\delta}{i\delta J(z)})_*\big]\exp\big[\frac{-1}{2}\int d^4x\int d^4y J(x)\Delta_c(x-y)J(y)\big], \quad (3)$$

where $\mathcal{L}_{int}(\Phi)_*$ reads for our before mentioned example $\Phi * \Phi * \Phi(x)$ and $\Delta_c(z) := \int \frac{d^4p}{(2\pi)^4}\frac{i\exp(-ip\cdot z)}{p^2-m^2+i\varepsilon}$ is the causal propagator of the free field. A perturbative expansion and a subsequent setting to zero of the external sources $J(x)$ leads to the so–called naïve Feynman rules.[4] For example, the NC analogon of the "fishgraph" in momentum space reads

$$\frac{-1}{4(p^2-m^2+i\varepsilon)^2}\int \frac{d^4k}{(2\pi)^4}\frac{1+\cos(\frac{1}{2}k^\mu\theta_{\mu\nu}p^\nu)}{((p-k)^2-m^2+i\varepsilon)(k^2-m^2+i\varepsilon)}. \qquad (4)$$

It is important to note that the same Feynman rules are derived within the canonical approach by starting from the Gell–Mann - Low formula and applying the T^*–operator. The latter is defined as follows:[6] All time derivatives associated to the star product act after the time ordering has been carried out (multiplication by step function.). Although these Feynman rules preserve the properties of the action related to the spacetime symmetry, one can show that these Feynman rules violate causal time ordering.

3. Path integral in NC QFT corresponding to T–ordering

Since, as stated in the section before, the naïve Feynman rules violate causal time ordering, one may wonder whether it is possible to modify the derivation of the above formula for the generating functional $Z(J)$ such that the resulting Feynman rules preserve causality. It turns out that such a modification is possible by means of the introduction of derivative shift brackets:

$$Z[J] = \exp\Big[i\int d^4z\big[\mathcal{L}_{int}(\frac{\delta}{i\delta J(z)})\big]^{\rightarrow}_{\theta,z}\Big]\exp\Big[\frac{-1}{2}\int d^4x\int d^4y J(x)T\Delta_+(x-y)J(y)\Big]. \tag{5}$$

Here, $T\Delta_+(x-y)$ is defined by $\vartheta(x^0-y^0)\Delta_+(x-y)+\vartheta(y^0-x^0)\Delta_+(y-x)$, $\Delta_+(x-y)$ is the positive frequency solution of the Klein–Gordon equation and $\vartheta(x^0-y^0)$ Heavyside's step function. $\big(\frac{\delta}{i\delta J(z)})\big)^{\rightarrow}_{*,z}$ means the following: For each time–ordered configuration ($\Delta_+(x-y)$ or $\Delta_+(y-x)$), shift all time derivatives associated with θ^{0i} through the step function which is to the right of this shift bracket. Then, realize the time ordering by multiplying with a step function.
Finally, the resulting Feynman rules are the same as those of old–fashioned perturbation theory (OTO).[5] The latter are derived by starting from the Gell–Mann - Low formula and applying the T–operator (T–operator: All time derivatives associated with the star product act before the time ordering is applied.). For example, the fishgraph amplitude now reads ($(a,b,c) := a\wedge b + a\wedge c + b\wedge c$, $a\wedge b := \frac{a_\mu\theta^{\mu\nu}b_\nu}{2}$):

$$\sum_{\lambda_{1,2}\epsilon\{-,+\}}\int\frac{d^3q_1}{\omega_{\vec{q}_1}}\int\frac{d^3q_2}{\omega_{\vec{q}_2}}\frac{-i}{4}(1+\frac{\lambda_1 p^0}{\omega_{\vec{p}}})(1+\frac{\lambda_2 p^0}{\omega_{\vec{p}}})\delta^{(3)}(\vec{q}_1+\vec{q}_2-\vec{p})\times$$

$$\times\Big[\frac{(\sum_{sym}e^{-i(-p_{\lambda_1},q_{1+},q_{2+})}e^{-i(-p_{\lambda_2},q_{1+},q_{2+})})}{p^0-\omega_{\vec{q}_1}-\omega_{\vec{q}_2}+i\varepsilon}+\frac{(\sum_{sym}e^{-i(-p_{\lambda_1},q_{1-},q_{2-})}e^{-i(-p_{\lambda_2},q_{1-},q_{2-})})}{-p^0-\omega_{\vec{q}_1}-\omega_{\vec{q}_2}+i\varepsilon}\Big], \tag{6}$$

where $p_\pm := (\pm\omega_{\vec{p}}, p^1, p^2, p^3)^\tau$. It has been shown that these Feynman rules maintain unitarity. By construction, they preserve also causal time ordering.

4. Summary and outlook

In this article, we tried to clarify that, within the Hamiltonian approach (We start from a Hamilton density $\mathcal{H}$ with $\pi := \dot{\Phi}$.), the time ordering is *not* rigidly implemented in the path integral.
We close this article by commenting on an aspect that has only been mentioned at the end of the talk. As the time ordering in the path integral seems to be better understood, one can then try to take over all formal manipulations from the Wick rotation of local quantum field theory. However, it is not clear whether one should also rotate θ^{0i} ($i\,\epsilon\,\{1,2,3\}$). It turns out that a nonlocal generalization of reflexion positivity can be derived and that θ^{0i} has to be rotated to $\pm i\theta^{0i}$, correspondingly, in order to assure reflexion positivity. These interesting findings and further results will be reported on in future publications.[7]

Acknowledgements

The author is grateful to Prof. Sibold for constructive criticism and to Prof. Belinski for giving the opportunity to present results at the 11th Marcel Grossmann

meeting.

This article (without the MSC numbers and the keywords) has already been submitted to the proceedings of the Eleventh Marcel Grossmann Meeting on General Relativity (Berlin).
The author feels highly indebted to Prof. Helmuth Hüffel (University of Vienna) and Dr. Jean–Bernard Bru (University of Vienna) for the invitation as a supported junoir scientist to the fourth "Vienna Central European Seminar on Particle Physics and Quantum Field Theory". Financial support from the Austrian Federal Ministry of Science and Research, the High Energy Physics Institute of the Austrian Academy of Sciences and the Erwin Schrödinger International Institute of Mathematical Physics is gratefully acknowledged.

References

1. S. Doplicher, K. Fredenhagen, J. E. Roberts, Commun. Math. Phys. 172 (1995) 187.
2. A. Connes, M. R. Douglas, A. Schwarz, JHEP 02 (1998) 003 ⟨arXiv:hep-th/9711162⟩; M. R. Douglas and C. Hull, *ibid.* 02 (1998) 008 ⟨arXiv:hep-th/9711165⟩; N. Seiberg and E. Witten, *ibid.* 09 (1999) 032 ⟨arXiv:hep-th/9908142⟩; V. Schomerus, JHEP 9906 (1999) 030 ⟨arXiv:hep-th/9903205⟩.
3. A. Connes, J. Lott, Nucl. Phys. Proc. Suppl. **18B** (1991) 29; V. Gayral, J. M. Gracia-Bondia, B. Iochum, T. Schucker, J. C. Varilly, Commun. Math. Phys. **246** (2004) 569 ⟨arXiv:hep-th/0307241⟩.
4. J. Gomis, T. Mehen, NP B591 (2000) 265 ⟨arXiv:hep-th/0005129⟩.
5. Y. Liao, K. Sibold, Eur. Phys. J. C **25** (2002) 469 ⟨arXiv:hep-th/0205269⟩; Y. Liao, K. Sibold, Eur. Phys. J. C **25**, 479 (2002) ⟨arXiv:hep-th/0206011⟩.
6. P. Heslop, K. Sibold, Eur. Phys. J. C **41** (2005) 545 ⟨arXiv:hep-th/0411161⟩.
7. C. Dehne, to appear.