Analytic Continuation
of Weighted Bergman Kernels

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ABSTRACT. We show that the Bergman kernel $K_\alpha(x,y)$ on a smoothly bounded strictly pseudoconvex domain with respect to the weight ρ^α, where $-\rho$ is a defining function and $\alpha > -1$, extends meromorphically in α to the entire complex plane. This is somewhat reminiscent of scattering poles or resonances in scattering theory. With a small change, the assertion remains valid also for functions ρ of slightly more general type.

1. INTRODUCTION

Let Ω be a bounded strictly pseudoconvex domain in $\mathbf{C}^n$ with smooth boundary and ρ a positively-signed defining function for Ω, i.e. $\rho \in C^\infty(\overline{\Omega})$, $\rho > 0$ on Ω, and $\rho = 0$, $\|\nabla\rho\| > 0$ on $\partial\Omega$. For $\alpha > -1$, consider the weighted Bergman space $A_\alpha^2 = L_{\text{hol}}^2(\Omega, \rho^\alpha)$ of all holomorphic functions in $L^2(\Omega, \rho^\alpha)$. The assumption $\alpha > -1$ guarantees that A_α^2 is nontrivial (it contains the constants). By the mean value property of holomorphic functions, it follows in the standard way that A_α^2 has bounded point evaluations and thus possesses a reproducing kernel — the weighted Bergman kernel $K_\alpha(x,y)$. Namely, for each $x \in \Omega$, $K_{\alpha,x} \equiv K_\alpha(\cdot, x)$ belongs to A_α^2 and

$$\int_\Omega f(y) \, K_\alpha(x,y) \, \rho(y)^\alpha \, dy = f(x), \qquad \forall f \in A_\alpha^2.$$

(Here dy stands for the Lebesgue volume measure on $\mathbf{C}^n$.)

The main result of the present paper asserts that as a function of α, $K_\alpha(x,y)$ extends from the interval $\alpha > -1$ to a meromorphic function on the entire complex plane.

Theorem 1. *Let Ω and ρ be as above. Then there exists a set $U \subset \mathbf{C}$ without an accumulation point such that for any fixed $x,y \in \Omega$, the function*

$$\alpha \longmapsto K_\alpha(x,y)$$

extends to a holomorphic function on $\mathbf{C} \setminus U$, and has at most poles at the points of U.

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The well-known prototypical situation is, of course, that of the unit disc $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$ with the defining functions $\rho(z) = 1 - |z|^2$: the weighted Bergman spaces A_α^2 then consist of all holomorphic functions $f(z) = \sum_{k=0}^\infty f_k z^k$ on $\mathbf{D}$ whose Taylor coefficients satisfy

$$(1) \qquad \|f\|_{A_\alpha^2}^2 = \pi \Gamma(\alpha + 1) \sum_{k=0}^\infty \frac{k!}{\Gamma(k + \alpha + 2)} |f_k|^2 < \infty,$$

and the weighted kernels are given by

$$K_\alpha(x, y) = \frac{1}{\pi} \sum_{k=0}^\infty \frac{\Gamma(k + \alpha + 2)}{k! \Gamma(\alpha + 1)} (x\overline{y})^k = \frac{\alpha + 1}{\pi} (1 - x\overline{y})^{-\alpha - 2}.$$

Similarly for the unit ball $\mathbf{B}^n$ of $\mathbf{C}^n$, with the defining function $\rho(z) = 1 - \|z\|^2$,

$$(2) \qquad K_\alpha(x, y) = \frac{(\alpha + 1) \ldots (\alpha + n)}{\pi^n} (1 - \langle x, y \rangle)^{-\alpha - n - 1}.$$

In both cases, the pole-set U is thus empty, in fact the extended kernels $K_\alpha(x, y)$ have *zeroes* (rather than poles) at $\alpha = -1, -2, \ldots, -n$.

The only existing result (up to the author's knowledge) in the direction of Theorem 1 is an earlier theorem due to the present author [12], to the effect that there exist equivalent norms on the spaces A_α^2, $\alpha > -1$, such that the corresponding reproducing kernels $K^{(\alpha)}(x, y)$ admit, for all $x, y \in \Omega$, a holomorphic continuation to the entire complex plane. In the above example of the disc, the equivalent norms would be

$$\|f\|_{(\alpha)}^2 := \sum_{k=0}^\infty \frac{|f_k|^2}{(k + 1)^{\alpha + 1}}$$

(which is indeed equivalent to (1) since $\Gamma(k + \alpha + 1)/k! \asymp (k + 1)^{\alpha + 1}$ by Stirling's formula), with kernels

$$K^{(\alpha)}(x, y) = \sum_{k=0}^\infty (k + 1)^{\alpha + 1} (x\overline{y})^k$$

holomorphic in $x, \overline{y}, \alpha$ on all of $\mathbf{D} \times \mathbf{D} \times \mathbf{C}$. However, passing to an equivalent norm changes the kernel completely (see the examples in §8.1 in [12]), so these kernels $K^{(\alpha)}$ have in general very little in common with the genuine weighted Bergman kernels K_α which we are interested in.

The assertion of Theorem 1 has been around as a sort of folklore conjecture for some time, and is vaguely reminiscent of the various results concerning e.g. the analytic continuation of the zeta functions of an elliptic operator (see e.g. the book by Shubin [37], or the numerous literature on regularized traces such as Grubb [18], Paycha [33] or Lesch [30], for instance), or the resonances in scattering theory (see e.g. Guillopé and Zworski [22], Guillarmou [20], Borthwick and Perry [5]). In fact, it will become apparent below that these topics and our Theorem 1 are not totally unrelated.

In analogy with these and other results of this kind, one might also expect the pole-set U to typically contain the negative integers (at least, this was what the

present author was expecting). It is therefore perhaps mildly surprising that U can in fact assume quite diverse and picturesque forms; a few examples are given in Section 7.2.

One of the main applications of the ordinary (i.e. unweighted) Bergman kernel concerns the problem of biholomorphic equivalence of strictly pseudoconvex domains. The idea, originating in the work of Fefferman, is to use the description of the boundary singularity of the Bergman kernel [14] for a construction of boundary invariants, of a geometric nature, that are preserved by biholomorphic maps [15]. For some later developments see e.g. Graham [17], Bailey, Eastwood and Graham [1], or Hirachi, Komatsu and Nakazawa [25]. Hirachi and Komatsu [24] studied an analytic continuation of the boundary singularity (as a microfunction), which coincides with the boundary singularity of $K_\alpha(x, y)$ when α is a nonnegative integer.

Our second result in this paper is that, for $\alpha \notin U$, our analytic continuation of the weighted Bergman kernels $K_\alpha(x, y)$ has indeed the same singularity at the boundary diagonal $x = y \in \partial\Omega$ as the one in [24]. In other words, Komatsu's and Hirachi's "local Sobolev-Bergman kernels" exist not only as microfunctions, but as boundary singularities of genuine holomorphic functions of $x, \overline{y}$ on $\Omega \times \Omega$.

Theorem 2. *Let Ω, ρ, K_α ($\alpha > -1$) and U be as in Theorem 1; abusing notation slightly, let us denote by the same symbol K_α also the analytic continuation of K_α to $\alpha \in \mathbf{C} \setminus U$. Then for each fixed $\alpha \in \mathbf{C} \setminus U$, there exist functions $a_\alpha, b_\alpha \in C^\infty(\overline{\Omega} \times \overline{\Omega})$ such that, on $\Omega \times \Omega$,*

$$K_\alpha(x, y) = \begin{cases} \dfrac{a_\alpha(x, y)}{\rho(x, y)^{n+\alpha+1}} + b_\alpha(x, y) & \text{if } n + \alpha \notin \mathbf{Z}; \\[3mm] \dfrac{a_\alpha(x, y)}{\rho(x, y)^{n+\alpha+1}} + b_\alpha(x, y) \log \rho(x, y) & \text{if } n + \alpha \in \mathbf{Z}_{\geq 0}; \\[3mm] \dfrac{a_\alpha(x, y)}{\rho(x, y)^{n+\alpha+1}} \log \rho(x, y) + b_\alpha(x, y) & \text{if } n + \alpha \in \mathbf{Z}_-. \end{cases}$$

Moreover, for all $x \in \partial\Omega$,

$$a_\alpha(x, x) = \begin{cases} \dfrac{(\alpha+1) \dots (\alpha+n) J[\rho](x)}{\pi^n} & \text{if } n + \alpha \notin \mathbf{Z}_-, \\[3mm] 0 & \text{if } n + \alpha \in \mathbf{Z}_-. \end{cases}$$

Here $\rho(x, y)$ is a fixed almost-sesquianalytic extension of $\rho(x)$ (see Section 5 below for the precise definition), and $J[\rho]$ is the Monge-Ampére determinant

$$J[\rho] = (-1)^n \det \begin{bmatrix} \rho & \partial\rho \\ \overline{\partial}\rho & \partial\overline{\partial}\rho \end{bmatrix},$$

whose positivity on $\partial\Omega$ follows from the strict-pseudoconvexity of Ω.

The pole-set U as well as the kernels $K_\alpha(x, y)$ themselves depend heavily on the choice of the defining function. From the point of view of biholomorphic equivalence, this is a serious drawback, since it is well known that it is impossible to choose a defining function for a strictly-pseudoconvex domain in a biholomorphically-invariant way [24]. On the other hand, there exist various quantities in abundance

which are biholomorphically invariant but miss being a defining function due to not being smooth up to the boundary; instead, they have a logarithmic singularity there at some lower-order terms. An example is the solution u of the Monge-Ampere equation

$$(3) \qquad J[u] = 1 \quad \text{on } \Omega, \qquad u = 0, \ \|\nabla u\| > 0 \quad \text{on } \partial\Omega,$$

which, as shown by Lee and Melrose [28], has boundary singularity of the form

$$(4) \qquad u \approx \rho \sum_{j=0}^{\infty} (\rho^{n+1} \log \rho)^j \, \eta_j, \qquad \eta_j \in C^{\infty}(\overline{\Omega}),$$

where "$\approx$" means that the difference between the left-hand side and a partial sum of the right-hand side is continuous on $\overline{\Omega}$ together with as many derivatives as and vanishes at $\partial\Omega$ to an order as high as the next term of the series, i.e.

$$u - \rho \sum_{j=0}^{N-1} (\rho^{n+1} \log \rho)^j \, \eta_j \in C_0^{(n+1)N}(\overline{\Omega}), \qquad \forall N = 0, 1, 2, \ldots.$$

(One often speaks of this as "equality in the sense of resolution of singularities".) The same kind of boundary behaviour prevails for the power $K_0(x,x)^{-1/(n+1)}$ of the unweighted Bergman kernel on the diagonal.

Our final result here is that a variant of Theorem 1 remains in force even for weights like u^α, with the only difference that the points of the pole-set U can then accumulate at negative integers.

Theorem 3. *Let $v \in C^{\infty}(\Omega)$ be a positive function on Ω such that, at the boundary,*

$$(5) \qquad v \approx \rho \sum_{j=0}^{\infty} \rho^j \sum_{k=0}^{M_j} (\log \rho)^k \eta_{jk}, \qquad \eta_{jk} \in C^{\infty}(\overline{\Omega}),$$

where $M_j < \infty$, $M_0 = 0$ and $\eta_{00} > 0$ on $\partial\Omega$. For $\alpha > -1$, let $K_{v^\alpha}(x,y)$ be the reproducing kernel of the weighted Bergman space $L^2_{\mathrm{hol}}(\Omega, v^\alpha)$. Then there exists a set $U \subset \mathbf{C} \setminus \mathbf{Z}_-$, consisting of isolated points, such that for any fixed $x, y \in \Omega$, the function

$$\alpha \longmapsto K_{v^\alpha}(x,y)$$

extends to a holomorphic function on $\mathbf{C} \setminus \mathbf{Z}_- \setminus U$, and has at most poles at the points of U.

With some modifications, Theorem 2 also remains in force for weights of the above form; see Theorem 14 below for the precise statement.

As in the earlier papers [12] and [13] by the present author, our proof of Theorem 1 uses the theory of Boutet de Monvel and Guillemin of generalized Toeplitz operators on the Hardy space of $\partial\Omega$. Namely, one first of all employs the Poisson extension operator $\mathbf{K}$ (the solution operator for the Dirichlet problem on Ω) to identify the reproducing kernels $K_\alpha(x,y) \equiv K_{\alpha,x}(y)$, $\alpha > -1$, $x, y \in \Omega$, with their boundary values ($y \in \partial\Omega$). The problem then reduces to finding an analytic continuation of the Hardy-space Toeplitz operator with symbol $\mathbf{K}^* \rho^\alpha \mathbf{K}$, and of its

inverse. Now $\mathbf{K}^* \rho^\alpha \mathbf{K}$ is a pseudodifferential operator (or ΨDO for short) on $\partial\Omega$ governed by the Boutet de Monvel calculus, and one can get its analytic continuation using a simple recurrence formula building on an idea akin to Bell's [2]. A standard "renormalization" (familiar also from scattering theory, cf. e.g. [4]) transforms the corresponding holomorphic family of ΨDO's into a holomorphic function whose values are bounded operators, and whose invertibility (except for a set U of isolated points) is therefore guaranteed by a classical theorem of Gohberg from 1950's. This settles Theorem 1; Theorem 2 is then obtained from Fefferman's [14] and Boutet de Monvel's and Sjöstrand's [10] description of the boundary singularity of the unweighted kernel K_0 in the same way as in [13], while Theorem 3 follows in a similar manner upon admitting also ΨDOs which are not classical but have logarithmically polyhomogeneous symbols.

The paper is organized as follows. In Section 2, we review the necessary background material on pseudodifferential operators, Boutet de Monvel's calculus and on the generalized Toeplitz operators of Boutet de Monvel and Guillemin. The proof of Theorem 1 occupies Section 3, the extension to "logarithmic" weights Section 4. The boundary behaviour of the kernels (Theorem 2) is addressed in Section 5, and some applications to construction of biholomorphically invariant domain functionals are described in Section 6. The final Section 7 contains miscellaneous concluding remarks, examples, supplementary results and open problems.

NOTATION. Throughout the paper, $\langle\cdot,\cdot\rangle_\alpha$, $\langle\cdot,\cdot\rangle_\Omega$ and $\langle\cdot,\cdot\rangle_{\partial\Omega}$ denote the inner products in A_α^2, $L^2(\Omega)$ and $L^2(\partial\Omega)$, respectively (the last being understood with respect to the $(2n-1)$-dimensional Hausdorff measure).

2. PRELIMINARIES

2.1 ΨDOs. Let $L^2(\partial\Omega)$ be the Lebesgue space on $\partial\Omega$ with respect to the surface measure. The Hardy space $H^2(\partial\Omega)$ is the subspace in $L^2(\partial\Omega)$ of functions whose Poisson extension is holomorphic in Ω; or, equivalently, the closure in $L^2(\partial\Omega)$ of $C_{\mathrm{hol}}^\infty(\partial\Omega)$, the space of boundary values of all the functions in $C^\infty(\overline{\Omega})$ that are holomorphic on Ω. We will also denote by $W^s(\partial\Omega)$, $s \in \mathbf{R}$, the Sobolev spaces on $\partial\Omega$, and by $W_{\mathrm{hol}}^s(\partial\Omega)$ the corresponding subspaces of nontangential boundary values of functions holomorphic in Ω. (Thus $W^0(\partial\Omega) = L^2(\partial\Omega)$ and $W_{\mathrm{hol}}^0(\partial\Omega) = H^2(\partial\Omega)$.)

As usual, by a *classical* pseudodifferential operator on $\partial\Omega$ we will mean an operator whose total symbol in any local coordinate system has an asymptotic expansion

$$p(x,\xi) \sim \sum_{j=0}^\infty p_{m-j}(x,\xi),$$

where p_{m-j} is C^∞ in x,ξ, and is positive homogeneous of degree $m-j$ in ξ for $|\xi| > 1$. Here j runs through nonnegative integers, but m can be any complex number; and the symbol "$\sim$" means that the difference between p and $\sum_{j=0}^{k-1} p_{m-j}$ should belong to the Hörmander class $S^{\mathrm{Re}\, m-k}$, for each $k = 0, 1, 2, \ldots$. The set of all classical ΨDOs on $\partial\Omega$ as above (i.e. of order m) will be denoted by Ψ_{cl}^m.

The (larger) class of all (not necessarily classical) ΨDOs whose total symbol in any local coordinate chart belongs to the Hörmander class $S^m = S^m_{1,0}$, $m \in \mathbf{R}$, will be denoted by Ψ^m; and we set, as usual, $\Psi_{\mathrm{cl}} := \bigcup_{m \in \mathbf{C}} \Psi^m_{\mathrm{cl}}$, $\Psi := \bigcup_{m \in \mathbf{R}} \Psi^m$, and $\Psi^{-\infty} := \bigcap_{m \in \mathbf{C}} \Psi^m_{\mathrm{cl}} = \bigcap_{m \in \mathbf{R}} \Psi^m$. The operators in $\Psi^{-\infty}$ are precisely the *smoothing* operators, i.e. those given by a C^∞ Schwartz kernel; and for any $P, Q \in \Psi$, we will write $P \sim Q$ if $P - Q$ is smoothing. Note that $\Psi^m_{\mathrm{cl}} \subset \Psi^{\mathrm{Re}\, m}$, and if $P \in \Psi^m$, then P is continuous from $W^s(\partial\Omega)$ into $W^{s-m}(\partial\Omega)$, for any $s \in \mathbf{R}$.

In addition to classical ΨDOs, we will need the more general class $\Psi_{\log}$ of *log-polyhomogeneous* ΨDOs, whose total symbol in any local coordinates satisfies

$$(6) \qquad\qquad p(x, \xi) \sim \sum_{j=0}^\infty p_{m-j}(x, \xi)$$

where p_{m-j} is of the form

$$(7) \qquad\qquad p_{m-j}(x, \xi) = \sum_{k=0}^{k_j} p_{m-j,k}\left(x, \frac{\xi}{|\xi|}\right) |\xi|^{m-j} (\log |\xi|)^k$$

for $|\xi| > 1$, for some (finite) integers k_j. We denote the class of ΨDOs of this form $\Psi^m_{\log}$, and we again also set $\Psi_{\log} = \bigcup_{m \in \mathbf{C}} \Psi^m_{\log}$. Note that $\Psi^m_{\log}$ is not contained in $\Psi^{\mathrm{Re}\, m}$, but only in $\Psi^{\mathrm{Re}\, m + \epsilon}$ for any $\epsilon > 0$; accordingly, the "$\sim$" in (6) now means that $p - \sum_{j=0}^{N-1} p_{m-j} \in \Psi^{\mathrm{Re}\, m - N + \epsilon}$ for any $\epsilon > 0$ and $N = 0, 1, 2, \dots$. We will call P *pure* if $k_0 = 0$, and *pure elliptic* if $k_0 = 0$ and $p_m(x, \xi) \neq 0$ for $\xi \neq 0$. More generally, we will denote by $\Psi^{m,k}_{\log}$ the class of all ΨDOs with symbol of the form (6), (7) where $k_0 = k$; so pure symbols correspond to $\Psi^{m,0}_{\log}$. For $P \in \Psi^m_{\log}$ such that p_m does not vanish identically, we still call $m =: \mathrm{ord}(P)$ the order of P (as before, this can be any complex number), and $p_m =: \sigma(P)$ the (principal) symbol of P; this clearly agrees with the corresponding notions for classical ΨDOs.

The operators in $\Psi_{\log}$ naturally arise as logarithms of complex powers of classical ΨDOs; more precisely, each operator in $\Psi^{m,k}_{\log}$ arises, modulo lower order terms, as $(\frac{\partial}{\partial z})^k A^z B$ for some $A, B \in \Psi_{\mathrm{cl}}$. Recall that if A is a positive selfadjoint elliptic classical ΨDO of order $m > 0$ on $\partial\Omega$, then A^{-1} is compact, hence the spectrum of A consists of isolated eigenvalues $0 < \lambda_1 < \lambda_2 < \dots$ of finite multiplicity. We can therefore define for any $z \in \mathbf{C}$ the operator A^z by the spectral theorem, i.e.

$$A^z = \sum_j \lambda_j^z P_j$$

where P_j is the projection onto the eigenspace corresponding to λ_j. Alternatively, one can define A^z for $\mathrm{Re}\, z < 0$ by the contour integral

$$A^z = \oint_{\lambda_1/2 - i\infty}^{\lambda_1/2 + i\infty} \lambda^z (\lambda - A)^{-1} \, d\lambda$$

(with the branch of λ^z defined in the right half-plane so that $1^z = 1$). For $\mathrm{Re}\, z \geq 0$, one then sets

$$A^z = A^k A^{z-k}, \qquad k > \mathrm{Re}\, z;$$

this is unambiguous since $AA^z = A^{z+1}$ for $\text{Re } z < -1$. For a positive self-adjoint elliptic classical ΨDO of order $m < 0$, one then defines A^z as $(A^{-1})^{-z}$, the right-hand side being defined as above. In both cases ($m < 0$ and $m > 0$), the operator A^z so defined is normal for any $z \in \mathbf{C}$, and self-adjoint and positive if z is real.

It is then a result going back to Seeley [36] (see also Shubin [37], Bucicovschi [11] or Schrohe [35]), that the operator A^z defined as above is again a classical ΨDO, of order mz, and with symbol $\sigma(A)^z$. Furthermore, the total symbol of A^z, in any local coordinate system, depends holomorphically on z (i.e. each $(mz - j)$-th homogeneous component does). Differentiating with respect to z, we see that for any $k = 0, 1, 2, \ldots$,

$$A^z (\log A)^k = \sum_j \lambda_j^z (\log \lambda_j)^k P_j$$

is an operator in $\Psi_{\log}^{mz,k}$, with principal symbol $\sigma(A)^z (\log \sigma(A))^k$.

The standard reference for log-polyhomogeneous ΨDOs is Schrohe [35]; see also Lesch [29] and Paycha and Scott [34].

2.2 Generalized Toeplitz operators. For $P \in \Psi^m$, the *generalized Toeplitz operator* $T_P : W_{\text{hol}}^m(\partial\Omega) \to H^2(\partial\Omega)$ is defined as

$$T_P = \Pi P,$$

where $\Pi : L^2(\partial\Omega) \to H^2(\partial\Omega)$ is the orthogonal projection (the Szegö projection). Alternatively, one may view T_P as the operator

$$T_P = \Pi P \Pi$$

on all of $W^m(\partial\Omega)$. Then T_P maps continuously $W^s(\partial\Omega)$ into $W_{\text{hol}}^{s-m}(\partial\Omega)$, for each $s \in \mathbf{R}$, because Π is bounded on $W^s(\partial\Omega)$ for any $s \in \mathbf{R}$ (see [10]).

The microlocal structure of generalized Toeplitz operators was described by Boutet de Monvel and Guillemin [8] [9]. It was shown there that the generalized Toeplitz operators T_P, $P \in \Psi_{\text{cl}}$, have the following properties, which were extended to $P \in \Psi_{\log}$ in [13]. The notation Σ below refers to the half-line bundle

$$\Sigma := \{(x, \xi) \in \mathcal{T}^*(\partial\Omega) : \xi = t\eta_x, \ t > 0\},$$

where η is the restriction to $\partial\Omega$ of the 1-form $\text{Im}(-\partial\rho) = (\overline{\partial}\rho - \partial\rho)/2i$. The strict pseudoconvexity of Ω guarantees that η is a contact form, i.e. the half-line bundle Σ is a symplectic submanifold of the cotangent bundle $\mathcal{T}^*(\partial\Omega)$.

(P1) For any T_P, $P \in \Psi_{\log}^{m,k}$, there in fact exists $Q \in \Psi_{\log}^{m,k}$ such that $T_P = T_Q$ and Q commutes with Π. (Hence, $T_P = T_Q$ is just the restriction of Q to the Hardy space. It follows, in particular, that generalized Toeplitz operators T_P, $P \in \Psi_{\log}$, form an algebra.)

(P2) It can happen that $T_P = T_Q$ for two different ΨDOs P and Q. If $\text{ord}(P) - \text{ord}(Q) \notin \mathbf{R}$, then $T_P = T_R$ for some $R \sim 0$. If $\text{ord}(P) - \text{ord}(Q) > 0$, then the restriction of the principal symbol $\sigma(P)$ of P to Σ identically vanishes. If $\text{ord}(P) = \text{ord}(Q)$, then the restrictions of $\sigma(P)$ and $\sigma(Q)$ to the cone Σ coincide.

One can thus define unambiguously the order of T_Q as $\text{ord}(Q) + \min\{\text{ord}(P) - \text{ord}(Q) : T_P = T_Q\}$, and the symbol of T_Q as $\sigma(T_Q) := \sigma(Q)|_\Sigma$ if $\text{ord}(Q) = \text{ord}(T_Q)$.

(P3) The order and the symbol obey the usual laws: $\operatorname{ord}(T_Q T_{Q'}) = \operatorname{ord}(T_Q) + \operatorname{ord}(T_{Q'})$ and $\sigma(T_Q T_{Q'}) = \sigma(T_Q)\sigma(T_{Q'})$.

(P4) If $\operatorname{Re}\operatorname{ord}(P) = 0$ and P is pure, then T_P is a bounded operator on $L^2(\partial\Omega)$; if $\operatorname{Re}\operatorname{ord}(P) < 0$, then it is even compact.

(P5) If $P \in \Psi^m_{\log}$ and $\sigma(T_P) = 0$, then there exists $Q \in \Psi^{m-1}_{\log}$ with $T_Q = T_P$. In particular, if $T_P \sim T_Q$, then there exists a ΨDO $R \sim 0$ such that $T_P - T_Q = T_R$.

(P6) We will say that a generalized Toeplitz operator T_P is *pure elliptic* if P is pure and $\sigma(P)|_\Sigma = \sigma(T_P)$ does not vanish. (Note that, by (P2), this implies $\operatorname{ord}(T_P) = \operatorname{ord}(P) \neq -\infty$.) Then T_P has a parametrix, i.e. there exists a pure elliptic Toeplitz operator T_Q of order $-\operatorname{ord}(T_P)$, with $\sigma(T_Q) = \sigma(T_P)^{-1}$, such that $T_P T_Q \sim I^2_H(\partial\Omega) \sim T_Q T_P$.

2.3 Holomorphic families of ΨDOs. Recall that a function $A(z)$, which is defined for z in some domain $\mathcal{G} \subset \mathbf{C}$ and whose values are operators from a fixed Banach space X into another Banach space Y, is called holomorphic if the derivative

$$A'(z) := \lim_{h \to 0} \frac{A(z+h) - A(z)}{h}$$

exists for each $z \in \mathcal{G}$ in the operator norm topology. By the uniform boundedness principle, this is in fact equivalent to the existence of $\lim_{h\to 0}(A(z+h)x - A(z)x)/h$ for each $x \in X$ in the norm topology of Y, and even to the holomorphy of the number-valued functions $\phi(A(z)x)$ for each $x \in X$ and each continuous linear functional ϕ on Y. Following Kato [26], we will call such operator-valued functions $A(z)$ *boundedly-holomorphic*. Clearly, sums and products of boundedly-holomorphic functions are again boundedly holomorphic, and so is the inverse of an invertible boundedly-holomorphic function.

We will need one more notion of holomorphy, suited for families of pseudodifferential (hence, in general, unbounded) operators. There exist several definitions in the literature (Guillemin [21, (3.17) and (3.18)], Ouedraogo and Paycha [32, Appendix], Paycha [33, Definition 43], Paycha and Scott [34, §1.3], Kontsevich and Vishik [27, §3] and Lesch [29]); we follow the one from §11 of Shubin [37] (cf. Definition 11.2 there).

Let X be a domain in $\mathbf{R}^N$ and $a(x,\xi,z) \in C^\infty(X \times \mathbf{R}^N \times \mathcal{G})$ a function which is holomorphic in z. We will write $a(x,\xi,z) \in \mathcal{O}(\mathcal{G}, S^m(X))$ if for any multiindices α,β, any integer $k \geq 0$ and any compact subsets $K_1 \subset X$, $K_2 \subset \mathcal{G}$ there exists a finite constant C such that

$$(8) \qquad |\partial_\xi^\alpha \partial_x^\beta \partial_z^k a(x,\xi,z)| \leq C\,(1+|\xi|^2)^{(m-|\alpha|)/2} \qquad \forall x \in K_1, z \in K_2, \xi \in \mathbf{R}^N.$$

If $A(z)$ is a ΨDO in X depending on $z \in \mathcal{G}$, we will say that $A(z) \in \mathcal{O}(\mathcal{G}, \Psi^m(X))$ if $A(z) = A_1(z) + R(z)$, where $A_1(z)$ is a proper ΨDO on X with total symbol $a(x,\xi,z) \in \mathcal{O}(\mathcal{G}, S^m(X))$ and the operator $R(z)$ has Schwartz kernel $R(x,y,z) \in C^\infty(X \times X \times \mathcal{G})$ holomorphic in z. Finally, we will say that $A(z)$ belongs to $\mathcal{O}(\mathcal{G}, \Psi^m(\partial\Omega)) \equiv \mathcal{O}(\mathcal{G}, \Psi^m)$ if $A(z)$ is a ΨDO on $\partial\Omega$, depending on $z \in \mathcal{G}$, such that for any local chart $\kappa : X_\kappa \to \partial\Omega$ (with X_κ a domain in $\mathbf{R}^{2n-1}$), the induced operators $A^\kappa(z)$ on X_κ belong to $\mathcal{O}(\mathcal{G}, \Psi^m(X_\kappa))$.

Remark 4. The inclusion $A(z) \in \mathcal{O}(\mathcal{G}, S^m(X))$ means precisely that A is boundedly-holomorphic as a function from $\mathcal{G}$ into the space $S^m(X)$ equipped with its usual Frechet topology given by the seminorms implicit in (8). $\square$

The set $\mathcal{O}(\mathcal{G}, \Psi^m(\partial\Omega))$ is a vector space and obeys the expected composition law, i.e. $A(z) \in \mathcal{O}(\mathcal{G}, \Psi^m(\partial\Omega))$, $B(z) \in \mathcal{O}(\mathcal{G}, \Psi^k(\partial\Omega))$ implies $A(z)B(z) \in \mathcal{O}(\mathcal{G}, \Psi^{m+k})$. Also, it is enough to check the inclusions $A^\kappa(z) \in \mathcal{O}(\mathcal{G}, \Psi^m(X_\kappa))$ just for some fixed atlas $\{X_\kappa\}_\kappa$ of coordinate charts covering $\partial\Omega$.

Let now $A(z)$ be a family of classical ΨDOs on $\partial\Omega$, depending on $z \in \mathcal{G}$, such that $A(z) \in \Psi_{cl}^{d(z)}$ for some $d(z) \in \mathbf{C}$. Fix an atlas $\{X_\kappa\}_\kappa$ of local charts as above. Let $\{\phi_\kappa\}$ be a smooth partition of unity with $\operatorname{supp}\phi_\kappa \subset \kappa(X_\kappa)$; choose $\psi_\kappa \in C^\infty(\partial\Omega)$ such that $\operatorname{supp}\psi_\kappa \subset \kappa(X_\kappa)$ and $\psi_\kappa = 1$ in a neighbourhood of $\operatorname{supp}\phi_\kappa$, and denote by Φ_κ, Ψ_κ the operators of multiplication by ϕ_κ and ψ_κ, respectively. For each κ, let

$$\tag{9} a^{(z),\kappa}(x,\xi) \sim \sum_{j=0}^\infty a_{d(z)-j}^{(z),\kappa}(x,\xi)$$

be the homogeneous expansion of the total symbol of $A^\kappa(z)$. Take the operator on X_κ with symbol $a_{d(z)-j}^{(z),\kappa}(x,\xi)$ and let $A_{(\kappa);d(z)-j}^{(z)}$ be the corresponding operator induced on $\kappa(X_\kappa)$ via the diffeomorphism κ. Set

$$\tag{10} A_{d(z)-j}^{(z)} = \sum_\kappa \Phi_\kappa A_{(\kappa);d(z)-j}^{(z)} \Psi_\kappa$$

and

$$A_{(N)}^{(z)} = \sum_{j=0}^{N-1} A_{d(z)-j}^{(z)}.$$

We will say that $A(z)$ is *holomorphic of order* $d(z)$ if $d(z)$ is a holomorphic function of z on $\mathcal{G}$ and for each $N \geq 0$ and $t \in \mathbf{R}$,

$$\tag{11} A(z) - A_{(N)}^{(z)} \in \mathcal{O}(\mathcal{G}_t, \Psi^{t-N}(\partial\Omega)),$$

where $\mathcal{G}_t := \{z \in \mathcal{G} : \operatorname{Re} d(z) < t\}$.

The definition can be modified in an obvious way to accommodate also the case of log-polyhomogeneous families $A(z) \in \Psi_{\log}^{d(z)}$ on $\partial\Omega$: namely, instead of (9) one takes the log-polyhomogeneous expansion (6), (7) of the symbol, and in (11) one has to replace "Ψ^{t-N}" by "$\Psi^{t-N+\epsilon}$ for all $\epsilon > 0$". In particular, if $A(z) \in \Psi_{\log}^{d(z),0}$ is pure for all z (which will be the only case of interest to us here), we will have

$$\tag{12} \begin{aligned} A(z) &\in \mathcal{O}(\mathcal{G}_t, \Psi^t), \quad \text{and} \\ A(z) - A_{(1)}^{(z)} &\in \mathcal{O}(\mathcal{G}_t, \Psi^{t-1+\epsilon}) \quad \forall\epsilon > 0, \end{aligned}$$

for all $t \in \mathbf{R}$.

The most important example of a holomorphic family are the complex powers of elliptic ΨDOs from the preceding subsection: namely, if $A \in \Psi_{cl}^m$, $m \neq 0$, is positive selfadjoint and elliptic, then A^z is a holomorphic family of order mz. For differential operators, this is the content of Theorem 11.4 in [37]; the general case, including an extension to pure log-polyhomogeneous ΨDOs (i.e. $A \in \Psi_{\log}^{m,0}$), can be found in [35].

It is clear that the sum of two holomorphic families of the same order $d(z)$ is again holomorphic of order $d(z)$; also, if $B \in \Psi_{\mathrm{cl}}^m$ then it follows from the familiar composition formula for ΨDOs

$$(13) \qquad \sigma_{AB}(x,\xi) \sim \sum_{\alpha \text{ multiindex}} \frac{i^{|\alpha|}}{\alpha!} \, \partial_\xi^\alpha \sigma_A(x,\xi) \, \partial_x^\alpha \sigma_B(x,\xi)$$

that $A(z)B$ and $BA(z)$ are holomorphic of order $d(z)+m$. For non-constant holomorphic families $B(z)$, however, holomorphy of the composition $A(z)B(z)$ cannot be expected unless the level sets of their orders match nicely. The following proposition takes care of the case that we will need here.

Proposition 5. *If $A(z)$ is holomorphic on $\mathbf{C}$ of order $a - z$, $B(z)$ is holomorphic on $\mathbf{C}$ of order $b + z$, where a, b are fixed complex numbers, and both $A(z)$ and $B(z)$ are pure, then $A(z)B(z)$ is pure and holomorphic on $\mathbf{C}$ of order $a + b$.*

Proof. Replacing $A(z)$, $B(z)$ by $A_0 A(z)$ and $B(z)B_0$, respectively, with some fixed elliptic $A_0 \in \Psi_{\mathrm{cl}}^{-a}$ and $B_0 \in \Psi_{\mathrm{cl}}^{-b}$, we can assume that $a = b = 0$. By (12) we then have, for any $s, t \in \mathbf{R}$, $N \geq 0$ and $\epsilon > 0$,

$$A(z) - \sum_{j=0}^{N-1} A_{z-j}^{(z)} \in \mathcal{O}(\operatorname{Re} z < t, \Psi^{t-N+\frac{\epsilon}{2}}),$$

$$B(z) - \sum_{j=0}^{N-1} B_{-z-j}^{(z)} \in \mathcal{O}(\operatorname{Re} z > s, \Psi^{-s-N+\frac{\epsilon}{2}}),$$

with

$$A_z^{(z)} \in \Psi_{\mathrm{cl}}^z, \qquad B_{-z}^{(z)} \in \Psi_{\mathrm{cl}}^{-z},$$

$$A_{z-j}^{(z)} \in \Psi_{\log}^{z-j}, \quad B_{-z-j}^{(z)} \in \Psi_{\log}^{-z-j}.$$

Setting $C_{-j;z} := \sum_{l=0}^{j} A_{z-l}^{(z)} B_{-z-j+l}^{(z)}$, we thus see from (13) that

$$A(z)B(z) - \sum_{j=0}^{N-1} C_{-j;z} \in \mathcal{O}(s < \operatorname{Re} z < t, \Psi^{t-s-N+\epsilon}),$$

with $C_{0;z} \in \Psi_{\mathrm{cl}}^0$, $C_{-j;z} \in \Psi_{\log}^{-j}$ being built (as in (10)) from total symbols which depend holomorphically on z. Now $(AB)_{-j}^{(z)}$ is a linear function of $C_{-j;z}$ and (the corresponding lower-order terms of) $C_{-l;z}$, $0 \leq l \leq j$ (this follows from (10) and (9)); taking N so large that $t - s - N$ is less than a given integer $-M$, it follows that

$$(14) \qquad (AB)_0^{(z)} \in \mathcal{O}(s < \operatorname{Re} z < t, \Psi^0),$$

$$(15) \qquad (AB)_{-j}^{(z)} \in \mathcal{O}(s < \operatorname{Re} z < t, \Psi^{-j+\epsilon}),$$

$$(16) \qquad A(z)B(z) - \sum_{j=0}^{N-1} (AB)_{-j}^{(z)} \in \mathcal{O}(s < \operatorname{Re} z < t, \Psi^{-M+\epsilon}).$$

Owing to (15), we can replace the upper summation limit $N-1$ in (16) by $M-1$. Taking in particular $M = 1$, it then follows from (14) that $A(z)B(z) \in \mathcal{O}(s < \operatorname{Re} z < t, \Psi^0)$ (i.e. not only $\Psi^{-\epsilon}$). Since s, t and M can be taken arbitrary, this completes the proof. $\square$

Since zeroth-order ΨDOs on a compact manifold are bounded, we obtain the following corollary.

Corollary 6. *In the situation from the preceding proposition, if $a + b = 0$ then $A(z)B(z)$ is boundedly-holomorphic on $L^2(\partial\Omega)$.*

Proof. Immediate from Theorem 6.2 in [37] in combination with (8) and the elementary fact that any function $R(x, y, z)$ on $\partial\Omega \times \partial\Omega \times \mathbf{C}$, C^∞ in x, y and holomorphic in z, must necessarily be locally bounded. □

Remark 7. The last proposition and corollary apply, for instance, to operators like

$$(I + \Delta_{LB})^{-z/2} B(z)$$

with $B(z) \in \Psi^z_{\mathrm{cl}}$ holomorphic on $\mathbf{C}$ of order z, taking values in ΨDOs on a compact Riemannian manifold, and Δ_{LB} the Laplace-Beltrami operator on the manifold. Note that a brute-force application of the definition (11) only yields that $A(z)B(z)$ is holomorphic of order $-\delta$, for any $\delta > 0$ (as a consequence of $A(z) \in \mathcal{O}(\mathrm{Re}\, z < t, \Psi^t)$ and $B(z) \in \mathcal{O}(\mathrm{Re}\, z > s, \Psi^{-s})$ for any $s, t \in \mathbf{R}$, upon taking $t = s + \delta$ and letting s vary), thus missing, in particular, Corollary 6. Up to the author's knowledge, this seems not to have been very explicitly noticed in the literature, an exception being Lemma 3.2 in [19] (which, however, requires the additional assumption of continuity of $A(z)B(z)$ in operator norm). □

2.4 Boutet de Monvel calculus. Let $\mathbf{K}$ denote the Poisson extension operator on Ω, i.e. $\mathbf{K}$ solves the Dirichlet problem

$$(17) \qquad\qquad \Delta\mathbf{K}u = 0 \quad \text{on } \Omega, \qquad \mathbf{K}u|_{\partial\Omega} = u.$$

(Thus $\mathbf{K}$ acts from functions on $\partial\Omega$ into functions on Ω. Here Δ is the ordinary Laplace operator.) By the standard elliptic regularity theory (see e.g. [31]), $\mathbf{K}$ acts continuously from $W^s(\partial\Omega)$ onto the subspace $W^{s+1/2}_{\mathrm{harm}}(\Omega)$ of all harmonic functions in $W^{s+1/2}(\Omega)$. In particular, it is continuous from $L^2(\partial\Omega)$ into $L^2(\Omega)$, and thus has a continuous Hilbert space adjoint $\mathbf{K}^* : L^2(\Omega) \to L^2(\partial\Omega)$. The composition

$$\mathbf{K}^*\mathbf{K} =: \Lambda$$

is known to be an elliptic positive ΨDO on $\partial\Omega$ of order -1. We have

$$(18) \qquad\qquad \Lambda^{-1}\mathbf{K}^*\mathbf{K} = I_{L^2(\partial\Omega)},$$

while

$$\mathbf{K}\Lambda^{-1}\mathbf{K}^* = \mathbf{\Pi}_{\mathrm{harm}},$$

the orthogonal projection in $L^2(\Omega)$ onto the subspace $L^2_{\mathrm{harm}}(\Omega)$ of all harmonic functions. (Indeed, from (18) it is immediate that the left-hand side acts as the identity on the range of $\mathbf{K}$, while it trivially vanishes on $\mathrm{Ker}\,\mathbf{K}^* = (\mathrm{Ran}\,\mathbf{K})^\perp$.) Comparing (18) with (17), we also see that the restriction

$$\gamma := \Lambda^{-1}\mathbf{K}^*|_{L^2_{\mathrm{harm}}(\Omega)}$$

is the operator of "taking the boundary values" of a harmonic function. Again, by elliptic regularity, γ extends to a continuous operator from $W^s_{\mathrm{harm}}(\Omega)$ onto $W^{s-1/2}(\partial\Omega)$, for any $s \in \mathbf{R}$, which is the inverse of $\mathbf{K}$.

The operators

$$\Lambda_w := \mathbf{K}^* w \mathbf{K},$$

with w a smooth function on $\overline{\Omega}$, are governed by a calculus developed by Boutet de Monvel in [7]. It was shown there that for w of the form

$$w = \rho^\alpha g, \qquad \operatorname{Re}\alpha > -1, \ g \in C^\infty(\overline{\Omega}),$$

Λ_w is a ΨDO on $\partial\Omega$ of order $-\alpha - 1$, with symbol

$$\sigma(\Lambda_w)(x,\xi) = \frac{\Gamma(\alpha+1)}{2|\xi|^{\alpha+1}} \, g(x) \, \|\eta_x\|^\alpha.$$

(In particular, $\sigma(\Lambda)(x,\xi) = 1/2|\xi|$.) We will need the following additional fact about Λ_w, which is not readily available in the literature.

Proposition 8. *For any $g \in C^\infty(\overline{\Omega})$, $A(z) = \mathbf{K}^* \rho^z g \mathbf{K}$ is a holomorphic family of ΨDOs of order $-z - 1$ on $\{z : \operatorname{Re} z > -1\}$.*

Proof. Invoking the atlas $\{X_\kappa\}_\kappa$ of local coordinate systems, it is again enough to prove the assertion for each coordinate chart; this reduces the problem to the case of the operators $B(z) = \psi \mathbf{P}^* \rho^z g \mathbf{P} \phi$ on the upper half-space $\Omega = \{x \in \mathbf{R}^m : x_m > 0\}$, $m = 2n$, where $\phi, \psi \in C^\infty(\partial\Omega)$ have compact support, $\rho(x) = x_m h(x)$ with $h \in C^\infty(\overline{\Omega})$ not vanishing on $\partial\Omega \cong \mathbf{R}^{m-1}$, $g \in C^\infty(\overline{\Omega})$, and $\mathbf{P} : L^2(\partial\Omega) \to L^2(\Omega)$ the solution operator to the Dirichlet problem for some elliptic second-order differential operator on Ω; we need to show that $B(z) \in \mathcal{O}(\operatorname{Re} z > -1, \Psi^0(\mathbf{R}^{m-1}))$, and more generally that the truncated operator $B(z) - B_{(N)}^{(z)}$, whose total symbol is obtained from the total symbol of $B(z)$ upon removing the top N leading terms in its homogeneous expansion, belongs to $\mathcal{O}(\operatorname{Re} z > -1, \Psi^{-N}(\mathbf{R}^{m-1}))$. Now by §1 in [6] (see also [7]), the operator $\mathbf{P}$ is given by the oscillatory integral

$$\mathbf{P}u(x) = \iint e^{i\langle x' - y', \xi'\rangle} k(x,\xi') u(y') \, dy' \, d\xi',$$

where the kernel $k(x,\xi') \in C^\infty(\overline{\Omega} \times \mathbf{R}^{m-1})$, $x = (x', x_m) \in \Omega$, $\xi' \in \mathbf{R}^{m-1}$, is rapidly decreasing in ξ' and admits the asymptotic expansion

$$k(x,\xi') \sim \sum_{j=0}^\infty k_j(x,\xi'),$$

with components $k_j(x,\xi')$ (also rapidly decreasing) satisfying the homogeneity conditions

$$k_j(x', \lambda^{-1}x_n, \lambda\xi') = \lambda^{-j} k_j(x,\xi'), \qquad \lambda > 0, \ |\xi'| > 1,$$

in the sense that

$$x_m^p \partial_x^\alpha \partial_{\xi'}^\beta \Big[k(x,\xi') - \sum_{j=0}^{N-1} k_j(x,\xi') \Big] = O(|\xi'|^{-N - |\beta| - p + \alpha_m})$$

as $|\xi'| \to +\infty$, uniformly for x in compact subsets of $\overline{\Omega}$. Taking adjoints yields a similar formula for $\mathbf{P}^*$, and using the Taylor expansion for h in the x_m-variable,

$$h(x', x_m) = \sum_{j=0}^{N-1} \frac{x_m^j}{j!} \frac{\partial^j h}{\partial x_m^j}(x', 0) + O(x_m^N),$$

which holds uniformly on compact subsets of $\overline{\Omega}$, and similarly for g, we obtain an integral representation for $B(z)$, from which, upon carrying out the x_m integration, one can read off the polyhomogeneous expansion of the total symbol of $B(z)$. The estimate of the remainder term $B(z) - B_{(N)}^{(z)}$ is then accomplished in the same manner as in the proof of Theorem 1.16 in [6]. $\square$

2.5 A theorem from operator theory. Recall that a bounded operator from one Hilbert space into another is called *Fredholm* if its kernel has finite dimension and its range is closed and of finite codimension. Equivalently, an operator is Fredholm if and only if it is invertible modulo compact operators; that is, if and only if there exists a bounded operator B such that $AB - I$ and $BA - I$ are both compact. In particular, operators of the form $A = E + L$, with L compact and E invertible, are Fredholm.

Elliptic ΨDOs of order m on a compact manifold are Fredholm as operators from W_{hol}^s into W_{hol}^{s-m}, for any $s \in \mathbf{R}$. This is a consequence of the existence of a parametrix for such operators, as is the fact that the kernel of such operator is contained in C^∞ (and hence does not depend on s).

We will need the following theorem on boundedly-holomorphic families of Fredholm operators, which is due to Gohberg.

Theorem. ([16], Chapter I, Theorem 5.1) *Let $F(z)$ be a boundedly-holomorphic operator function on some domain $\mathcal{G} \subset \mathbf{C}$ such that $I - F(z)$ is a compact operator, for all z; and assume that there exists $z_0 \in \mathcal{G}$ for which $F(z)$ has a bounded inverse.*

Then for all $z \in \mathcal{G}$, except possibly for some isolated points, $F(z)$ is boundedly invertible (and $F(z)^{-1}$ is boundedly-holomorphic).

Furthermore, at the points of the possible exceptional set where the inverse fails to exist, $F(z)^{-1}$ has poles whose principal parts are finite-rank operators; this can be seen as follows. Let $a \in \mathcal{G}$ be such a point; since $F(a)$ is a compact perturbation of the identity, the spaces $\operatorname{Ker} F(a)$ and $(\operatorname{Ran} F(a))^\perp$ are of finite and equal dimension. Let

$$F(z) := \begin{bmatrix} A(z) & B(z) \\ C(z) & D(z) \end{bmatrix}$$

be the block decomposition of $F(z)$ with respect to the splittings $\operatorname{Ker} F(a) \oplus (\operatorname{Ker} F(a))^\perp$ and $(\operatorname{Ran} F(a))^\perp \oplus \operatorname{Ran} F(a)$. Then $A(z)$ is a finite square matrix, $B(z), C(z)$ are finite rank operators, $A(a) = B(a) = C(a) = 0$, $D(a)$ is boundedly invertible, and so is $F(z)$ in some punctured neighbourhood of a. By continuity, $D(z)$ is also boundedly invertible in a neighbourhood of a. In view of the decomposition

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & BD^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} I & 0 \\ D^{-1}C & I \end{bmatrix},$$

we thus see that in some punctured neighbourhood of a, $A - BD^{-1}C$ is boundedly invertible and

$$F^{-1} = \begin{bmatrix} I & \\ -D^{-1}C & I \end{bmatrix} \begin{bmatrix} (A - BD^{-1}C)^{-1} & 0 \\ 0 & D^{-1} \end{bmatrix} \begin{bmatrix} I & -BD^{-1} \\ 0 & I \end{bmatrix}.$$

However, by linear algebra, the inverse of a finite-size matrix is of the form (the adjoint of that matrix)/(its determinant). As A, B, C and D^{-1} are boundedly holomorphic in a neighbourhood of a, it follows that near a, F^{-1} is of the form

$$F^{-1} = \frac{\text{(a boundedly holomorphic function)}}{\det(A - BD^{-1}C)}.$$

Thus F^{-1} has a pole at a of order at most $\dim \operatorname{Ker} F(a)$, whose principal part is an operator of rank not exceeding $\dim \operatorname{Ker} F(a)$.

Remark 9. The assertion concerning poles must be well known, but including the above short proof here proved easier than finding a reference. Operator functions of this kind, i.e. boundedly-holomorphic except for isolated poles with finite-rank principal parts, are commonly called *finite-meromorphic* (e.g. in the literature on scattering theory, cf. [19]).   □

3. Proof of Theorem 1

The Hardy space $H^2(\partial\Omega)$ also has a reproducing kernel, namely the *Szegö kernel* $S(x, y) \equiv S_y(x) = \overline{S_x(y)}$, $x, y \in \Omega$, which satisfies $S_y \in H^2(\partial\Omega)$ $\forall y \in \Omega$ and

$$\mathbf{K}u(x) = \langle u, S_x \rangle_{\partial\Omega} = \int_{\partial\Omega} u(y) S(x, y), \qquad \forall x \in \Omega, \ u \in H^2(\partial\Omega).$$

For $\alpha > -1$, $x \in \Omega$ and $u \in C^\infty_{\mathrm{hol}}(\partial\Omega)$ (the subspace in $C^\infty(\partial\Omega)$ of functions whose Poisson extension into Ω is holomorphic), we thus have

$$\begin{aligned}
\langle u, S_x \rangle_{\partial\Omega} &= \mathbf{K}u(x) \\
&= \langle \mathbf{K}u, K_{\alpha,x} \rangle_\alpha \\
&= \langle \rho^\alpha \mathbf{K}u, K_{\alpha,x} \rangle_\Omega \\
&= \langle u, \mathbf{K}^* \rho^\alpha K_{\alpha,x} \rangle_{\partial\Omega} \\
&= \langle u, \Pi \mathbf{K}^* \rho^\alpha K_{\alpha,x} \rangle_{\partial\Omega}.
\end{aligned}$$

Consequently,

$$(19) \qquad S_x = \Pi \mathbf{K}^* \rho^\alpha K_{\alpha,x} = \Pi \mathbf{K}^* \rho^\alpha \mathbf{K} \gamma K_{\alpha,x} = T_{\mathbf{K}^* \rho^\alpha \mathbf{K}} \gamma K_{\alpha,x}.$$

By §2.4, we know $\mathbf{K}^* \rho^\alpha \mathbf{K}$ to be a classical elliptic ΨDO on $\partial\Omega$ of order $-\alpha-1$; by the property (P6) of generalized Toeplitz operators, it follows that the corresponding generalized Toeplitz operator $T_{\mathbf{K}^* \rho^\alpha \mathbf{K}}$ is Fredholm as an operator from $W^s_{\mathrm{hol}}(\partial\Omega)$ into $W^{s+\alpha+1}_{\mathrm{hol}}(\partial\Omega)$, for any $s \in \mathbf{R}$. On the other hand, for any $u \in H^2(\partial\Omega) \setminus \{0\}$,

$$(20) \qquad \langle T_{\mathbf{K}^* \rho^\alpha \mathbf{K}} u, u \rangle_{\partial\Omega} = \int_\Omega \rho^\alpha |\mathbf{K}u|^2 > 0,$$

so $T_{\mathbf{K}^* \rho^\alpha \mathbf{K}}$ is injective and positive selfadjoint as an operator on $H^2(\partial\Omega)$. It follows that it is in fact an isomorphism of $W^s_{\text{hol}}(\partial\Omega)$ onto $W^{s+\alpha+1}_{\text{hol}}(\partial\Omega)$, for all $s \in \mathbf{R}$; hence, also of $C^\infty_{\text{hol}}(\partial\Omega) = \bigcap_{s\in\mathbf{R}} W^s_{\text{hol}}(\partial\Omega)$ onto itself. As $S_x \in C^\infty_{\text{hol}}(\partial\Omega)$, we thus see that $(T_{\mathbf{K}^* \rho^\alpha \mathbf{K}})^{-1} S_x$ is defined and

$$(21) \qquad \gamma K_{\alpha,x} = (T_{\mathbf{K}^* \rho^\alpha \mathbf{K}})^{-1} S_x,$$

and, hence,

$$(22) \qquad K_\alpha(x,y) = \langle \gamma K_{\alpha,y}, S_x \rangle_{\partial\Omega} = \langle (T_{\mathbf{K}^* \rho^\alpha \mathbf{K}})^{-1} S_y, S_x \rangle_{\partial\Omega},$$

for all $x, y \in \Omega$ and $\alpha > -1$.

Our plan now is to establish, first of all, an analytic continuation for $T_{\mathbf{K}^* \rho^\alpha \mathbf{K}}$; and then, to show that it is invertible if α does not belong to the exceptional set U, the inverse is holomorphic and its domain contains $C^\infty_{\text{hol}}(\partial\Omega)$. An application of (22) then gives the desired analytic continuation for $K_\alpha(x,y)$.

We begin by establishing an analytic continuation for the operators $\mathbf{K}^* \rho^\alpha \mathbf{K}$.

Proposition 10. *There exists a family of classical ΨDOs $R(z)$ on $\partial\Omega$, holomorphic of order $-z - 1$ on all of $\mathbf{C}$, such that*

$$R(\alpha) = \frac{1}{\Gamma(\alpha+1)} \mathbf{K}^* \rho^\alpha \mathbf{K} \qquad for \ \operatorname{Re}\alpha > -1.$$

In particular, the principal symbol of $R(z)$ is $\|\eta_x\|^z/2|\xi|^{z+1}$.

Proof. The harmonic Bergman space

$$L^2_{\text{harm}}(\Omega) := \{f \in L^2(\Omega) : \ \Delta f = 0\}$$

has a reproducing kernel — the *harmonic Bergman kernel* $H(x,y) \equiv H_y(x)$. It is well-known that H is real-valued and symmetric, i.e. $H(x,y) = H(y,x) = \overline{H(x,y)}$. For each $x \in \Omega$, the mean value property of harmonic functions implies that

$$f(x) = \int_\Omega f \, \phi_x \qquad \forall f \in L^2_{\text{harm}}(\Omega),$$

for any function $\phi_x \in C^\infty(\Omega)$ whose support is contained in Ω, which has total mass 1 and is such that $\phi_x(y)$ depends only on $|x - y|$. Thus

$$H_x = \mathbf{\Pi}_{\text{harm}}\phi_x = \mathbf{K}\Lambda^{-1}\mathbf{K}^*\phi_x.$$

From the mapping properties of $\mathbf{K}$, $\mathbf{K}^*$ and Λ on Sobolev spaces it therefore follows that

$$H_x \in C^\infty(\overline{\Omega}), \qquad \forall x \in \Omega.$$

Let now v be an arbitrary function in $C^\infty(\overline{\Omega})$. By Green's formula, for any $\epsilon > 0$ we have

$$\int_{\rho>\epsilon} H_x \cdot \Delta(\rho^{\alpha+2}v) = \int_{\rho>\epsilon} H_x \cdot \Delta(\rho^{\alpha+2}v) - \rho^{\alpha+2}v \cdot \Delta H_x$$

$$= \oint_{\rho=\epsilon} H_x \frac{\partial(\rho^{\alpha+2}v)}{\partial n} - \rho^{\alpha+2}v \frac{\partial H_x}{\partial n}.$$

(Here $\partial/\partial n$ denotes the normal derivative.) The last integrand is of the form $\rho^{\alpha+1} \cdot$ (a function in $C^\infty(\overline{\Omega})$), hence approaches zero as $\epsilon \searrow 0$ if $\mathrm{Re}\,\alpha > -1$. Thus

$$\int_\Omega H_x\, \Delta(\rho^{\alpha+2} v) = 0 \qquad \text{for } \mathrm{Re}\,\alpha > -1.$$

Consequently, for any $f \in C^\infty(\overline{\Omega})$ and $\mathrm{Re}\,\alpha > -1$, we have

$$(\mathbf{\Pi}_{\mathrm{harm}}\rho^\alpha f)(x) = \int_\Omega \rho^\alpha f H_x$$
$$= \int_\Omega H_x\, [\rho^\alpha f - \Delta(\rho^{\alpha+2} v)],$$

or

$$\mathbf{\Pi}_{\mathrm{harm}}\rho^\alpha f = \mathbf{\Pi}_{\mathrm{harm}}[\rho^\alpha f - \Delta(\rho^{\alpha+2} v)]$$
$$= \mathbf{\Pi}_{\mathrm{harm}}[\rho^\alpha f - \rho^{\alpha+2}\Delta v - (\alpha+2)\rho^{\alpha+1}(\partial\rho \cdot \overline{\partial} v + \partial v \cdot \overline{\partial}\rho)$$
$$- (\alpha+2)\rho^{\alpha+1} v\Delta\rho - (\alpha+2)(\alpha+1)\rho^\alpha v\partial\rho \cdot \overline{\partial}\rho],$$

since $\Delta\rho^{\alpha+2} = (\alpha+2)(\rho^{\alpha+1}\Delta\rho + (\alpha+1)\rho^\alpha\partial\rho \cdot \overline{\partial}\rho)$.

Let us now fix a function $\phi \in C^\infty(\overline{\Omega})$ which is identically 1 in a neighbourhood of $\partial\Omega$, and vanishes in a neighbourhood of the set where $\partial\rho = 0$. Then

$$\Psi := \frac{\phi}{\partial\rho \cdot \overline{\partial}\rho} = \frac{\phi}{\|\eta\|^2}$$

is a function in $C^\infty(\overline{\Omega})$. Set

$$v = \frac{f\Psi}{(\alpha+1)(\alpha+2)}.$$

Then the last formula becomes

$$\mathbf{\Pi}_{\mathrm{harm}}\rho^\alpha f = \mathbf{\Pi}_{\mathrm{harm}}\rho^{\alpha+1}\Big[wf - \frac{\rho\,\Delta(f\Psi)}{(\alpha+1)(\alpha+2)} - \frac{f\Psi\Delta\rho}{\alpha+1}$$
$$- \frac{\overline{\partial}\rho \cdot \partial(f\Psi) + \partial\rho \cdot \overline{\partial}(f\Psi)}{\alpha+1}\Big],$$

where

$$w := \frac{1-\phi}{\rho} \in C^\infty(\overline{\Omega})$$

vanishes identically near $\partial\Omega$.

Let us now take $f = g\mathbf{K}u$, where $g \in C^\infty(\overline{\Omega})$ and $u \in C^\infty(\partial\Omega)$ (so $\mathbf{K}u \in C^\infty(\overline{\Omega})$). After a small manipulation, we get

$$\mathbf{\Pi}_{\mathrm{harm}}\rho^\alpha g\mathbf{K}u = \mathbf{\Pi}_{\mathrm{harm}}\rho^{\alpha+1} wg\mathbf{K}u$$
$$- \frac{1}{(\alpha+1)(\alpha+2)}\mathbf{\Pi}_{\mathrm{harm}}\rho^{\alpha+2}\Big[(\Delta(\Psi g))\mathbf{K}u + \partial(\Psi g) \cdot \overline{\partial}\mathbf{K}u + \overline{\partial}(\Psi g) \cdot \partial\mathbf{K}u\Big]$$
$$- \frac{1}{\alpha+1}\mathbf{\Pi}_{\mathrm{harm}}\rho^{\alpha+1}\Psi g(\Delta\rho)\mathbf{K}u$$
$$- \frac{1}{\alpha+1}\mathbf{\Pi}_{\mathrm{harm}}\rho^{\alpha+1}\Big[(\overline{\partial}\rho \cdot \partial(g\Psi) + \partial\rho \cdot \overline{\partial}(g\Psi))\,\mathbf{K}u + g\Psi\overline{\partial}\rho \cdot \partial\mathbf{K}u + g\Psi\partial\rho \cdot \overline{\partial}\mathbf{K}u\Big].$$

Introduce the operators $R_j, \overline{R}_j$ on $\partial\Omega$ by

$$R_j := \gamma\partial_j\mathbf{K}, \quad \overline{R}_j := \gamma\overline{\partial}_j\mathbf{K}, \qquad j = 1, 2, \ldots, n,$$

so that $\partial_j\mathbf{K} = \mathbf{K}R_j$ and $\overline{\partial}_j\mathbf{K} = \mathbf{K}\overline{R}_j$. Then $R_j, \overline{R}_j$ are commuting (since $\partial_j, \overline{\partial}_j$ commute on Ω) classical ΨDOs on $\partial\Omega$ of order 1. Denoting

$$\mathcal{R}_{\alpha,g} := \frac{1}{\Gamma(\alpha+1)}\gamma\mathbf{\Pi}_{\mathrm{harm}}\rho^\alpha g\mathbf{K} = \frac{1}{\Gamma(\alpha+1)}\Lambda^{-1}\mathbf{K}^*\rho^\alpha g\mathbf{K},$$

we thus finally obtain

$$
\begin{aligned}
(23) \quad \mathcal{R}_{\alpha,g} = {}&(\alpha+1)\mathcal{R}_{\alpha+1,wg} - \mathcal{R}_{\alpha+2,\Delta(\Psi g)} \\
&- \sum_{j=1}^n \left(\mathcal{R}_{\alpha+2,\partial_j(\Psi g)}\overline{R}_j + \mathcal{R}_{\alpha+2,\overline{\partial}_j(\Psi g)}R_j\right) \\
&- \mathcal{R}_{\alpha+1,\Psi g\Delta\rho} - \mathcal{R}_{\alpha+1,\overline{\partial}\rho\cdot\partial(\Psi g)+\partial\rho\cdot\overline{\partial}(\Psi g)} \\
&- \sum_{j=1}^n \left(\mathcal{R}_{\alpha+1,\Psi g\overline{\partial}_j\rho}R_j + \mathcal{R}_{\alpha+1,\Psi g\partial_j\rho}\overline{R}_j\right),
\end{aligned}
$$

on $C^\infty(\partial\Omega)$ for $\mathrm{Re}\,\alpha > -1$.

Now from Proposition 10 we know that for any $g \in C^\infty(\overline{\Omega})$, the operators $\mathcal{R}_{\alpha,g}$ form a holomorphic family of ΨDOs on $\mathrm{Re}\,\alpha > -1$ of order $-\alpha$, with principal symbol

$$(24) \qquad \frac{\|\eta_x\|^\alpha}{|\xi|^\alpha}\, g(x).$$

However, this implies that all the terms on the right-hand side of (23) are, in fact, holomorphic of appropriate orders on the half-plane $\mathrm{Re}\,\alpha > -2$; hence also the left-hand side $\mathcal{R}_{\alpha,g}$ extends, in fact, holomorphically to $\mathrm{Re}\,\alpha > -2$. Secondly, taking principal symbols on both sides of (23) we get, for $\mathrm{Re}\,\alpha > -1$,

$$
\begin{aligned}
(25) \quad \frac{\|\eta_x\|^\alpha}{|\xi|^\alpha}\, g(x) = {}&0 - 0 - 0 - 0 - 0 \\
&- \sum_{j=1}^n \frac{\|\eta_x\|^{\alpha+1}}{|\xi|^{\alpha+1}}\frac{g(x)}{\|\eta_x\|^2}\left(\overline{\partial}_j\rho\cdot\sigma(R_j) + \partial_j\rho\cdot\sigma(\overline{R}_j)\right).
\end{aligned}
$$

Consequently,

$$(26) \qquad \sum_{j=1}^n \left(\overline{\partial}_j\rho\cdot\sigma(R_j) + \partial_j\rho\cdot\sigma(\overline{R}_j)\right) = -|\xi|\,\|\eta_x\|$$

and (25) holds, in fact, for all complex α. Inserting this back into (23), it follows that the principal symbol of $\mathcal{R}_{\alpha,g}$ will still be equal to (24) even in the extended domain $\mathrm{Re}\,\alpha > -2$. This means that the right-hand side of (23) is in fact holomorphic for $\mathrm{Re}\,\alpha > -3$, and its principal symbol is still given by (24) for such α; hence the same is true for the left-hand side, etc. Continuing this bootstrapping argument,

we conclude that $\mathcal{R}_{\alpha,g}$ actually extends to a holomorphic family of ΨDOs on all of $\mathbf{C}$, of order $-\alpha$ and with principal symbol (24). Since $R(\alpha) = \Lambda\mathcal{R}_{\alpha,\mathbf{1}}$, this completes the proof. $\square$

Remark 11. The proof in fact shows that even

$$(27) \qquad R_g(\alpha) := \frac{1}{\Gamma(\alpha+1)}\mathbf{K}^*\rho^\alpha g\mathbf{K},$$

with any $g \in C^\infty(\overline{\Omega})$, extends from $\operatorname{Re}\alpha > -1$ to a holomorphic family of classical ΨDOs of order $-\alpha - 1$ on all of $\mathbf{C}$. $\square$

Remark 12. Unlike $\mathbf{K}$, the adjoint $\mathbf{K}^*$ and the operator $\Lambda = \mathbf{K}^*\mathbf{K}$ depend on the choice of measure on $\partial\Omega$: if, instead of the surface measure $d\sigma$ we are using, we switch to $w\,d\sigma$ with some smooth density w, then $\mathbf{K}^*$ and Λ get multiplied by $\frac{1}{w}$. However, the operator $\Lambda^{-1}\mathbf{K}^*$, and, hence, also the operators $\mathcal{R}_{\alpha,g}$ are independent of the density w. $\square$

Remark 13. The operator

$$\vartheta := -2\sum_{j=1}^n \left(\frac{\partial_j\rho}{\|\eta\|}\,\overline{R}_j + \frac{\overline{\partial}_j\rho}{\|\eta\|}\,R_j\right) = \gamma\frac{\partial}{\partial n}\mathbf{K}$$

is nothing but the familiar Dirichlet-to-Neumann operator. As an immediate consequence of (26), we get the formula

$$\sigma(\vartheta) = 2|\xi| = \sigma(\Lambda^{-1})$$

for its principal symbol. $\square$

Continuing our program, as the next step we need to handle the invertibility of the generalized Toeplitz operator $T_{\mathbf{K}^*\rho^\alpha\mathbf{K}} = \Gamma(\alpha+1)T_{R(\alpha)}$.

Recall that $\Lambda = \mathbf{K}^*\mathbf{K}$ is an elliptic classical ΨDO on $\partial\Omega$ of order -1, and, by (20), T_Λ is injective and positive. By the property (P1) of generalized Toeplitz operators, there exists an elliptic $\Upsilon \in \Psi_{\mathrm{cl}}^{-1}$ such that $\Upsilon\Pi = \Pi\Upsilon = \Pi\Lambda\Pi$. Since changing the total symbol away from a neighbourhood of Σ results in a change of $\Pi\Lambda\Pi$ by a smoothing operator (by the property (P5)), we can actually take Υ to be positive and injective as well (see the proof of Proposition 16 in [12] for the detailed argument).

Similarly, there exists an injective positive elliptic $Q \in \Psi_{\mathrm{cl}}^{-1}$ which commutes with Π and

$$Q\Pi = \Pi Q = \Pi\Lambda^{-1/2}\mathbf{K}^*\rho\mathbf{K}\Lambda^{-1/2}\Pi = \Pi\Lambda^{-1/2}R(1)\Lambda^{-1/2}\Pi.$$

As was reviewed in §2.2, the complex powers Q^α, $\alpha \in \mathbf{C}$, constructed via the Spectral Theorem, then form a family of classical elliptic ΨDOs on $\partial\Omega$ which is holomorphic of order $-\alpha$. Set

$$G(\alpha) := Q^{-\alpha}\Upsilon^{-1}R(\alpha).$$

By Proposition 10 and Corollary 6, $G(\alpha)$ form a family of elliptic classical ΨDOs which is holomorphic of order 0, and, hence, boundedly-holomorphic on $L^2(\partial\Omega)$. The corresponding family of generalized Toeplitz operators

$$F(\alpha) := T_{G(\alpha)} = Q^{-\alpha}\Upsilon^{-1}T_{R(\alpha)}$$

(the second equality stems from the fact that Q and Υ commute with Π) is boundedly holomorphic on $H^2(\partial\Omega)$. Furthermore, since $\sigma(\Upsilon)|_\Sigma = \sigma(\Lambda)|_\Sigma$ and similarly for Q, we have

$$\sigma(G(\alpha))|_\Sigma = \sigma(\Lambda^{-1/2}R(1)\Lambda^{-1/2})^{-\alpha}\sigma(\Upsilon)^{-1}\sigma(R(\alpha))|_\Sigma$$
$$= \frac{\|\eta\|^{-\alpha}}{|\xi|^{-\alpha}} \cdot 2|\xi| \cdot \frac{\|\eta\|^\alpha}{2|\xi|^{\alpha+1}} = 1.$$

By the properties (P5) and (P4), $F(\alpha) - I$ is a generalized Toeplitz operator of order -1, and, hence, compact. Consequently, $I - F(\alpha)$, $\alpha \in \mathbf{C}$, is a boundedly-holomorphic family of compact operators. Finally,

$$F(0) = \Upsilon^{-1}T_\Lambda = I$$

is the identity operator, hence, in particular, boundedly invertible. Applying Gohberg's theorem from §2.5 (and the remarks after it), we thus conclude that except possibly for α in some set $U \subset \mathbf{C}$ consisting of isolated points, $F(\alpha)$ is boundedly invertible, and $F(\alpha)^{-1}$ is boundedly-holomorphic in $\mathbf{C} \setminus U$, while at the points of U it has poles with finite-rank principal parts (i.e. $F(\alpha)^{-1}$ is finite-meromorphic).

Proof of Theorem 1. Set

$$E_x(\alpha) := F(\alpha)^{-1}Q^{-\alpha}\Upsilon^{-1}S_x, \qquad x \in \Omega,\ \alpha \in \mathbf{C} \setminus U.$$

From the fact that $Q^{-\alpha}$ is a holomorphic family of ΨDOs of order α, it follows that for any fixed $f \in W^s(\partial\Omega)$, $Q^{-\alpha}f$ is a holomorphic function from $\{\alpha : \operatorname{Re}\alpha < t\}$ into $W^{s+t}(\partial\Omega)$, for any $s, t \in \mathbf{R}$. Since $S_x \in C^\infty_{\mathrm{hol}}(\partial\Omega)$ and Q and Υ both commute with Π, we conclude that $Q^{-\alpha}\Upsilon^{-1}S_x$ is holomorphic as a function from $\mathbf{C}$ into any $W^s_{\mathrm{hol}}(\partial\Omega)$; hence, in particular, into $H^2(\partial\Omega)$. Thus for each $x \in \Omega$, $E_x(\alpha)$ is holomorphic from $\mathbf{C} \setminus U$ into $H^2(\partial\Omega)$, with poles at the points of U. However, for $\alpha > -1$ we have by (19)

$$E_x(\alpha) = F(\alpha)^{-1}Q^{-\alpha}\Upsilon^{-1}T_{\mathbf{K}^*\rho^\alpha\mathbf{K}}\gamma K_{\alpha,x}$$
$$= \Gamma(\alpha+1)F(\alpha)^{-1}Q^{-\alpha}\Upsilon^{-1}T_{R(\alpha)}\gamma K_{\alpha,x}$$
$$= \Gamma(\alpha+1)F(\alpha)^{-1}F(\alpha)\gamma K_{\alpha,x}$$
$$= \Gamma(\alpha+1)\gamma K_{\alpha,x}.$$

Thus $\frac{1}{\Gamma(\alpha+1)}E_x(\alpha)$ is a holomorphic $H^2(\partial\Omega)$-valued function on $\mathbf{C} \setminus U$, with poles at points of U, which coincides with $\gamma K_{\alpha,x}$ for $\alpha > -1$, and

$$(28) \qquad\qquad \frac{1}{\Gamma(\alpha+1)} \langle E_y(\alpha), S_x\rangle_{\partial\Omega}$$

is, for each $x, y \in \Omega$, a holomorphic function on $\mathbf{C} \setminus U$, with poles at the points of U, which coincides with $K_\alpha(x, y)$ for $\alpha > -1$. This completes the proof of Theorem 1. $\square$

4. Logarithmic weights

The proof of Theorem 3 parallels that of Theorem 1, except that one needs to use the operators in $\Psi_{\log}$ instead of the classical ΨDOs; this is fine as long as the operators are pure. Most of the necessary tools have been developed in [13]; we will therefore be brief.

Proof of Theorem 3. As in (19), we have

$$(29) \qquad S_x = \Pi \mathbf{K}^* v^\alpha \mathbf{K} \gamma K_{v^\alpha, x} = T_{\mathbf{K}^* v^\alpha \mathbf{K}} \gamma K_{v^\alpha, x}.$$

In (5) we may assume without loss of generality (inserting some zero terms if needed) that $M_1 \leq M_2 \leq M_3 \leq \dots$. Rewriting it as

$$v \approx \rho \eta_{00} \left[1 + \sum_{j=1}^{\infty} \rho^j \sum_{k=0}^{M_j} (\log \rho)^k \frac{\eta_{jk}}{\eta_{00}} \right],$$

it transpires that, for any $\alpha \in \mathbf{C}$,

$$(30) \qquad v^\alpha \approx \rho^\alpha \sum_{j=0}^{\infty} \rho^j \sum_{k=0}^{N_j} (\log \rho)^k \, \eta_{jk}(\alpha),$$

with $N_j = jM_j$, $\eta_{00}(\alpha) = \eta_{00}^\alpha$, and $\eta_{jk}(\alpha)/\eta_{00}^\alpha$ given by a polynomial (depending only on j, k) in α and $\frac{\eta_{lm}}{\eta_{00}}$, $0 \leq l \leq j$, $0 \leq m \leq k$. (In effect, due to the definition of the symbol "$\approx$", one just uses the identity $[1+X]^\alpha = \sum_{l=0}^{\infty} \frac{(-\alpha)_l}{l!} A^l$, viewing A as a formal power series in ρ with coefficients in the ring $C^\infty(\overline{\Omega})[\log \rho]$ of polynomials in $\log \rho$ with $C^\infty(\overline{\Omega})$ coefficients.) In particular, each $\eta_{jk}(\alpha)$ depends holomorphically on α in a rather simple way.

By differentiating (27) (or making the appropriate modifications in the proof of Proposition 10, which however is somewhat more laborious), one sees that for each $g \in C^\infty(\overline{\Omega})$ and $m = 0, 1, 2, \dots$,

$$\frac{1}{\Gamma(\alpha+1)^{m+1}} \mathbf{K}^* \rho^\alpha (\log \rho)^m g \mathbf{K} \in \Psi_{\log}^{-\alpha-1, m}, \qquad \operatorname{Re} \alpha > -1,$$

extends to a holomorphic family of order $-\alpha - 1$ on all of $\mathbf{C}$. (The extra power at $\frac{1}{\Gamma(\alpha+1)}$ was introduced since $\Gamma^{(j)}$ has the same poles as Γ^{j+1}.) Similarly, one checks that if g_α, $\alpha \in \mathbf{C}$, is a holomorphic family of functions in $C^\infty(\Omega)$ such that for some $m \in \mathbf{R}$ and $N \geq 0$,

$$|X_1 \dots X_k g_\alpha| \leq C_{X_1 \dots X_k}(\alpha) \, \rho^{\operatorname{Re} \alpha + m} |\log \rho|^N \qquad \text{on } \Omega$$

for all smooth tangential (i.e. annihilating ρ) vector fields $X_1, \dots, X_k$, $k \geq 0$, on Ω with some locally bounded real-valued functions $C_{X_1 \dots X_k}(\alpha)$, then

$$\mathbf{K}^* g_\alpha \mathbf{K} \in \mathcal{O}(-\operatorname{Re} \alpha - 1 < t, \Psi^{t-m+\epsilon})$$

for any $\epsilon > 0$. (In the model case of the upper half-space $\mathbf{R}_+^{n+1} := \{(x, y) : x \in \mathbf{R}^n, y > 0\}$ and g_α depending only on y, this amounts to checking that

$$\sigma_{\text{total}}(\mathbf{K}^* g_\alpha \mathbf{K})(\xi) \equiv \sigma(\xi) = \int_0^\infty e^{-2y|\xi|} g_\alpha(y) \, dy$$

satisfies

$$|\partial_\xi^\beta \sigma(\xi)|^2 \le C_{\alpha,\beta,\epsilon}(1+|\xi|^2)^{-\operatorname{Re}\alpha-|\beta|-m-1+\epsilon}$$

locally uniformly in α. Since, by a simple inductive argument,

$$\partial_\xi^\beta \sigma(\xi) = \int_0^\infty |\xi|^{-|\beta|}\, p_\beta(\tfrac{\xi}{|\xi|}, y|\xi|)\, e^{-2y|\xi|}\, g_\alpha(y)\, dy$$

for some polynomials p_β of degree at most $|\beta|$ in each variable, the desired claim follows by the elementary estimates

$$\int_1^\infty y^k |\xi|^k e^{-2y|\xi|} g_\alpha(y)\, dy = O(e^{-|\xi|}) = O(|\xi|^{-\infty}) \qquad \text{as } |\xi| \to +\infty$$

and

$$\begin{aligned}
\left| \int_0^1 y^k |\xi|^k e^{-2y|\xi|} g_\alpha(y)\, dy \right| &= |\xi|^{-1} \left| \int_0^{|\xi|} y^k e^{-2y} g_\alpha(\tfrac{y}{|\xi|})\, dy \right| \\
&\le C(\alpha) C_\epsilon |\xi|^{-1-\operatorname{Re}\alpha-m+\epsilon} \int_0^{|\xi|} y^k e^{-2y} y^{\operatorname{Re}\alpha+m-\epsilon}\, dy \\
&\le \frac{C(\alpha) C_\epsilon \Gamma(k+\operatorname{Re}\alpha+m-\epsilon+1)}{2^{k+1+\operatorname{Re}\alpha+m-\epsilon}} |\xi|^{-1-\operatorname{Re}\alpha-m+\epsilon} \\
&\le C_{\alpha,\beta,\epsilon} |\xi|^{-1-\operatorname{Re}\alpha-m+\epsilon},
\end{aligned}$$

where $C_{\alpha,\beta,\epsilon}$ is the maximum of the constants on the preceding line over all $k = 0, 1, \ldots, |\beta|$.)

Fixing an entire function $\Xi(\alpha)$ on $\mathbf{C}$ (for instance, a suitable Hadamard product) whose only zeroes are at $\alpha = -j-1$ with multiplicity N_j, $j = 0, 1, 2, \ldots$, we thus infer from (30) that

$$\frac{\Xi(\alpha)}{\Gamma(\alpha+1)} \mathbf{K}^* v^\alpha \mathbf{K} \in \Psi_{\log}^{-\alpha-1,0}, \qquad \operatorname{Re}\alpha > -1,$$

extends to a family of ΨDOs on $\partial\Omega$ of order $-\alpha-1$ holomorphic on all of $\mathbf{C}$, with the principal symbol equal to $\Xi(\alpha)\|\eta_x\|^\alpha \eta_{00}(x)^\alpha/2|\xi|^{\alpha+1}$. Introducing Υ and Q analogously as before (i.e. $\Pi\Upsilon = \Upsilon\Pi = \Pi\Lambda\Pi$, $\Pi Q = Q\Pi = \Pi\Lambda^{-1/2}\mathbf{K}^* v\mathbf{K}\Lambda^{-1/2}\Pi$) and setting

$$G(\alpha) := \frac{\Xi(\alpha)\Xi(1)^\alpha}{\Gamma(\alpha+1)} Q^{-\alpha}\Upsilon^{-1}\mathbf{K}^* v^\alpha \mathbf{K},$$

we get that $G(\alpha)$, $\alpha \in \mathbf{C}$, form a holomorphic family of pure elliptic operators in $\Psi_{\log}^{0,0}$ of order 0, with

$$\sigma(G(\alpha))|_\Sigma = \Xi(\alpha).$$

It follows that

$$F(\alpha) := \frac{1}{\Xi(\alpha)}\, T_{G(\alpha)} = \frac{\Xi(1)^\alpha}{\Gamma(\alpha+1)} Q^{-\alpha}\Upsilon^{-1}\mathbf{K}^* v^\alpha \mathbf{K}$$

is a boundedly-holomorphic family on $\mathbf{C} \setminus \mathbf{Z}_-$ such that $I - F(\alpha)$ is compact. Applying Gohberg's theorem, we see that except possibly for α in some set U of

isolated points in $\mathbf{C} \setminus \mathbf{Z}_-$, $F(\alpha)^{-1}$ exists and is boundedly-holomorphic, with poles of finite rank at the points of U; and

$$E_x(\alpha) := F(\alpha)^{-1}\Xi(1)^\alpha Q^{-\alpha}\Upsilon^{-1}T_{\mathbf{K}^*v^\alpha\mathbf{K}}\gamma K_{v^\alpha,x}$$

is holomorphic on the same set (with poles at points of U) and coincides with $\Gamma(\alpha+1)\gamma K_{v^\alpha,x}$ for $\alpha > -1$. Finally, $\frac{1}{\Gamma(\alpha+1)}\langle E_y(\alpha), S_x\rangle$ is holomorphic on $\mathbf{C}\setminus\mathbf{Z}_-\setminus U$, with poles at the points of U, and coincides with $K_{v^\alpha}(x,y)$ for $\alpha > -1$. The proof is complete. $\square$

We present an example below (see §7.4) of a function v of the form (5) on the disc for which the points of the pole-set U of K_{v^α} really do accumulate at some negative integers. Thus the different conclusions of Theorems 1 and 3 are not an artefact of our method of proof, but reflect a real difference between the smooth and nonsmooth weight cases.

5. BOUNDARY BEHAVIOUR

It turns out that the Szegö kernel S again extends to be smooth up to the boundary of $\overline{\Omega \times \Omega}$ except for the boundary diagonal diag $\partial\Omega = \{(x,x) : x \in \partial\Omega\}$. More precisely, let $\rho(x,y) \in C^\infty(\overline{\Omega} \times \overline{\Omega})$ be an *almost-sesquianalytic extension* of $\rho(x)$ (i.e. $\rho(x,x) = \rho(x)$ $\forall x \in \overline{\Omega}$ and $\overline{\partial}_x\rho(x,y)$ and $\partial_y\rho(x,y)$ vanish to infinite order on the diagonal $x = y \in \overline{\Omega}$), satisfying the symmetry and positivity conditions

$$\rho(x,y) = \overline{\rho(y,x)},$$
$$2\operatorname{Re}\rho(x,y) \geq \rho(x) + \rho(y) + c|x-y|^2,$$

for all $x,y \in \overline{\Omega}$, with some $c > 0$ (independent of x and y); it follows from the second, in particular, that one can define single-valued branches of $\log\rho(x,y)$ and $\rho(x,y)^\nu$, $\nu \in \mathbf{C}$, on $\Omega \times \Omega$. (The existence of such sesquianalytic extension follows from strict pseudoconvexity.) Then there exist $a,b \in C^\infty(\overline{\Omega} \times \overline{\Omega})$ such that

$$(31) \qquad S(x,y) = \frac{a(x,y)}{\rho(x,y)^n} + b(x,y)\log\rho(x,y).$$

It is convenient to view the boundary values $S|_{\partial\Omega\times\partial\Omega}$ of $S(x,y)$ on $\partial\Omega \times \partial\Omega$ also in the distributional sense, i.e. as the limit for $\epsilon \searrow 0$ of $S(x,y)|_{\rho(x)=\rho(y)=\epsilon}$ in $C^\infty(\partial\Omega \times \partial\Omega)'$. In this sense, $S|_{\partial\Omega\times\partial\Omega}$ is a (classical) Fourier integral distribution which is the distributional kernel of the Szegö projector $\Pi : L^2(\partial\Omega) \to H^2(\partial\Omega)$.

Using the well-known formulas

$$\int_1^\infty e^{-tp}t^s\,dt = \begin{cases} \dfrac{\Gamma(s+1)}{p^{s+1}} + \mathcal{O}(p), & s \in \mathbf{C} \setminus \{-1,-2,\dots\}, \\[2ex] \dfrac{(-1)^{k+1}}{k!}p^k(\log p + \mathcal{O}(p)), & s = -1-k,\ k \in \mathbf{Z}_{\geq 0}, \end{cases}$$

valid for $\operatorname{Re} p > 0$, where $\mathcal{O}(p)$ denotes a function of p which is smooth (in fact — holomorphic) in a neighbourhood of the origin, the boundary singularity (31)

of S can also be represented as the oscillatory integral with complex-valued phase function [10]

$$(32) \qquad S(x,y) \sim \int_0^\infty e^{-t\rho(x,y)}\, b(x,y,t)\, dt, \qquad x,y \in \partial\Omega,$$

where b is a classical symbol in $S^{n-1}(\partial\Omega \times \partial\Omega \times \mathbf{R}_+)$ with asymptotic expansion

$$b(x,y,t) \sim \sum_{j=0}^\infty t^{n-1-j}\, b_j(x,y) \qquad \text{for } t > 1,$$

with some functions $b_j \in C^\infty(\partial\Omega \times \partial\Omega)$. In particular,

$$b_0(x,x) = \frac{J[\rho](x)}{2\|\eta\|\pi^n}, \qquad \forall x \in \partial\Omega.$$

Here and below, "$f \sim g$" for two elements of $C^\infty(\partial\Omega \times \partial\Omega)'$ means that $f - g$ belongs to $C^\infty(\partial\Omega \times \partial\Omega)$.

We recall now the following fact, which was proved in Theorems 4 and 5 (and their proofs) in [13]. Let T_Q, $Q \in \Psi_{\log}^{-2s,0}$, be an elliptic generalized Toeplitz operator on $H^2(\partial\Omega)$ of order $-2s$, $s \in \mathbf{R}$; we may assume that Q commutes with Π. Let further $K_Q(x,y)$ be a holomorphic function of $x, \overline{y}$ on $\Omega \times \Omega$ whose distributional boundary values $K_Q|_{\partial\Omega \times \partial\Omega}$ satisfy

$$K_Q|_{\partial\Omega \times \partial\Omega} \sim (Q \otimes I)S|_{\partial\Omega \times \partial\Omega}$$

where $Q \otimes I$ means that Q applies to the first variable. Then it follows from (32) and the standard symbol calculus rules for ΨDOs that

$$(33) \qquad K_Q \approx \sum_{j=0}^\infty \rho^{[j+2s-n]} \sum_{k=0}^{k_j} (\log \rho)^k\, v_{jk} + v_\infty \qquad \text{on } \Omega \times \Omega,$$

where $k_j < \infty$, $k_0 = 0$, $v_{jk}, v_\infty \in C^\infty(\overline{\Omega} \times \overline{\Omega})$, and

$$\rho^{[m]} := \begin{cases} \rho^m, & \text{if } m \in \mathbf{C} \setminus \{0,1,2,\dots\}, \\ \rho^m \log \rho, & \text{if } m = 0,1,2,\dots. \end{cases}$$

Furthermore,

$$(34) \qquad v_{00}(x,x) = \begin{cases} \dfrac{\Gamma(n-2s)}{2\|\eta\|\pi^n}\, J[\rho](x)\, \sigma(T_Q)(x,\eta_x) & \text{if } n - 2s \notin \mathbf{Z}_{\leq 0}, \\[2ex] \dfrac{(-1)^{k+1}}{k!2\|\eta\|\pi^n}\, J[\rho](x)\, \sigma(T_Q)(x,\eta_x) & \text{if } n - 2s = -k \in \mathbf{Z}_{\leq 0}, \end{cases}$$

for $x \in \partial\Omega$. Here, $k_1 \leq k_2 \leq \dots$ are the numbers from the expansion (7) for Q, and the "$\approx$" in (33) is again understood in the sense of "resolution of singularities" as in (4), except that the continuity of (many) derivatives is meant on $\overline{\Omega} \times \overline{\Omega}$, while the vanishing (to high order) is meant only at diag $\partial\Omega$. (Also, abusing the notation

slightly, the ρ in (33) stands for $\rho(x, y)$, while in (4) it stood for $\rho(x)$.) If $n - 2s$ is a positive integer, the term v_∞ can be omitted.

In particular, if $Q \in \Psi_{\mathrm{cl}}^{2s}$ is classical, so that $k_j = 0$ for all j, we may rewrite (33) as

$$(35) \qquad K_Q(x, y) = \begin{cases} \dfrac{a_Q(x, y)}{\rho(x, y)^{n-2s}} + b_Q(x, y), & 2s \notin \mathbf{Z}, \\[2ex] \dfrac{a_Q(x, y)}{\rho(x, y)^{n-2s}} + b_Q(x, y) \log \rho(x, y), & n > 2s \in \mathbf{Z}, \\[2ex] \dfrac{a_Q(x, y)}{\rho(x, y)^{n-2s}} \log \rho(x, y) + b_Q(x, y), & n \leq 2s \in \mathbf{Z}, \end{cases}$$

with $a_Q(x, x)$, $x \in \partial\Omega$, given by (34).

After these preparations, the proofs of Theorem 2 and of its "logarithmic" variant become almost a triviality.

Proof of Theorem 2. Looking back at (28), we observe that our holomorphic continuation of the reproducing kernels K_α was actually obtained in the form

$$K_\alpha|_{\partial\Omega \times \partial\Omega} = \frac{1}{\Gamma(\alpha + 1)}(F(\alpha)^{-1} Q^{-\alpha} \Upsilon^{-1} \otimes I) S|_{\partial\Omega \times \partial\Omega}.$$

Since $F(\alpha)$ is a classical elliptic generalized Toeplitz operator of order 0, with principal symbol 1, its inverse $F(\alpha)^{-1}$ enjoys the same properties (since it differs by a smoothing operator from the parametrix of $F(\alpha)$, for which those properties are guaranteed by (P6) of §2.3). Hence $F(\alpha)^{-1} Q^{-\alpha} \Upsilon^{-1}$ is a classical elliptic generalized Toeplitz operator of order $\alpha + 1$ and with principal symbol $2|\xi|^{\alpha+1}/\|\eta\|^\alpha$. An application of (35), with $s = \frac{-\alpha-1}{2}$, completes the proof. $\square$

Likewise, the corresponding assertion for the more general "logarithmic" weights follows upon using (33) instead of (35).

Theorem 14. *Let v and $U \subset \mathbf{C} \setminus \mathbf{Z}_-$ be as in Theorem 3, and let us denote by the same symbol K_{v^α} also the analytic continuation of K_{v^α} to $\alpha \in \mathbf{C} \setminus \mathbf{Z}_- \setminus U$. Then for each fixed $\alpha \in \mathbf{C} \setminus \mathbf{Z}_- \setminus U$, there exist $v_{jk\alpha}, v_{\infty\alpha} \in C^\infty(\overline{\Omega} \times \overline{\Omega})$, $j = 0, 1, 2, \ldots$, $0 \leq k \leq jk_j$, such that*

$$K_{v^\alpha}(x, y) \approx \sum_{j=0}^\infty \rho^{[j-n-\alpha-1]} \sum_{k=0}^{jk_j} (\log \rho)^k v_{jk\alpha} + v_{\infty\alpha} \qquad \textit{on } \Omega \times \Omega.$$

Furthermore,

$$v_{00}(x, x) = \frac{(\alpha + 1) \ldots (\alpha + n) J[\rho](x)}{\pi^n \eta_{00}(x)^\alpha}$$

for $x \in \partial\Omega$.

6. SOME APPLICATIONS

Recall that a *domain functional* is a map $\Omega \mapsto f_\Omega$ assigning to each bounded strictly pseudoconvex domain $\Omega \subset \mathbf{C}^n$ with smooth boundary a function f_Ω on Ω. Examples of domain functionals are

$$\Omega \mapsto K_\Omega(x) := K(x, x),$$

the restriction to the diagonal of the Bergman kernel of Ω (with respect to the Lebesgue measure); or

$$\Omega \mapsto S_\Omega(x) := S(x,x),$$

the restriction to the diagonal of the Szegö kernel of Ω (with respect to the $(2n-1)$-dimensional Hausdorff measure on $\partial\Omega$).

The domain functional is said to be *invariant of weight* α, $\alpha \in \mathbf{C}$, if

$$f_\Omega = |J_\phi|^{2\alpha/(n+1)} f_{\phi\Omega} \circ \phi$$

for any biholomorphic map $\phi : \Omega \to \phi\Omega$; here J_ϕ denotes the complex Jacobian of ϕ. For instance, the Bergman kernel K_Ω above is invariant of weight $n+1$. This follows from the well-known transformation rule for the Bergman kernel

$$(36) \qquad K_\Omega(x,y) = J_\phi(x) \, \overline{J_\phi(y)} \, K_{\phi\Omega}(\phi(x), \phi(y)).$$

The Szegö kernel S_Ω as defined above is not invariant, but can be made so upon using instead of the Hausdorff measure an appropriately chosen "invariant" surface element on $\partial\Omega$; then S_Ω is of weight n. The solution $u = u_\Omega$ of the Monge-Ampere equation (3) is an invariant domain functional of weight -1. For further examples and discussion of invariant domain functionals, we refer to Hirachi and Komatsu [24] and Hirachi [23].

A product of two invariant domain functionals is again an invariant domain functional (with the weights adding up), and similarly for powers. One can also get new invariants from old ones by means of weighted Bergman kernels.

Proposition 15. ([13], Proposition 10) *If f_Ω is a positive domain functional which is invariant of weight α, $\alpha \in \mathbf{R}$, then the weighted Bergman kernel $K_{f_\Omega}(x,x)$ of $L^2(\Omega, f_\Omega)$ restricted to the diagonal is an invariant domain functional of weight $n+1-\alpha$.*

Indeed, if $\phi : \Omega \to \phi\Omega$ is a biholomorphism, then

$$
\begin{aligned}
K_{f_\Omega}^\Omega &= K_{|J_\phi|^{2\alpha/(n+1)} f_{\phi\Omega} \circ \phi}^\Omega \\
&= |J_\phi|^{-\frac{2\alpha}{n+1}} K_{f_{\phi\Omega} \circ \phi}^\Omega \\
&= |J_\phi|^{2 - \frac{2\alpha}{n+1}} K_{f_{\phi\Omega}}^{\phi\Omega} \circ \phi.
\end{aligned}
$$

Here the second equality used the fact that

$$K_{|g|^2 w}(x,y) = g(x)^{-1} \, \overline{g(y)^{-1}} \, K_w(x,y)$$

for any zero-free holomorphic function g, and the third follows from the simple generalization

$$(37) \qquad K_{w \circ \phi}^\Omega(x,y) = J_\phi(x) \, \overline{J_\phi(y)} \, K_w^{\phi\Omega}(\phi(x), \phi(y))$$

of the transformation rule (36).

Thus, for instance,

$$\Omega \mapsto K_{u_\Omega^\alpha}(x,x), \qquad \alpha > -1,$$

is an invariant domain functional of weight $n+1+\alpha$, and similarly for $K_{K_\Omega^{-\alpha/(n+1)}}(x,x)$.

By Theorem 3, there exists a set U^Ω consisting of isolated points in $\mathbf{C} \setminus \mathbf{Z}_-$ such that $\alpha \mapsto K_{u_\Omega^\alpha}(x,x)$ extends from $\alpha > -1$ to a holomorphic function (still denoted $K_{u_\Omega^\alpha}$) on $\mathbf{C} \setminus \mathbf{Z}_- \setminus U^\Omega$ and has at most poles at the points of U^Ω. We thus arrive at the following corollary.

Corollary 16. *The set U^Ω is a biholomorphic invariant of Ω. For any $\alpha \in \mathbf{C} \setminus \mathbf{Z}_- \setminus U^\Omega$,*

$$\Omega \mapsto K_{u^\alpha_\Omega}(x,x)$$

is an invariant domain functional of weight $n+1+\alpha$. Likewise, for each $\alpha \in U^\Omega$, the order of the pole at α is a biholomorphic invariant, and the strength of the pole is an invariant domain functional of weight $n+\alpha+1$.

Proof. By Proposition 15, for any biholomorphism ϕ

$$(38) \qquad K^\Omega_{u^\alpha_\Omega} = |J_\phi|^{2+\frac{2\alpha}{n+1}} \, K^{\phi\Omega}_{u^\alpha_{\phi\Omega}} \circ \phi, \qquad \alpha > -1.$$

Since $\alpha \mapsto |J_\phi|^{2+\frac{2\alpha}{n+1}}$ is holomorphic on the entire complex plane, we see that an analytic continuation of $\alpha \mapsto K^{\phi\Omega}_{u^\alpha_{\phi\Omega}}$ immediately yields also an analytic continuation for $\alpha \mapsto K^\Omega_{u^\alpha_\Omega}$, and vice versa. Thus indeed $U^\Omega = U^{\phi\Omega}$, proving the first claim. Similarly, the validity of (38) for $\alpha > -1$ implies, by analytic continuation, that it must remain in force also for all $\alpha \in \mathbf{C} \setminus \mathbf{Z}_- \setminus U^\Omega$, proving the remaining claims. $\square$

An analogous assertion, of course, holds also for the weighted kernels $K_{K^{-\alpha/(n+1)}_\Omega}$ and $K_{S^{-\alpha/n}_\Omega}$.

Similarly to the famous expansion of Fefferman [14],

$$K_\Omega(x,y) = \frac{a(x,y)}{\rho(x,y)^{n+1}} + b(x,y) \log \rho(x,y), \qquad a,b \in C^\infty(\overline{\Omega} \times \overline{\Omega}),$$

or, in other words,

$$(39) \qquad K_\Omega \approx \sum_{j=0}^\infty \rho^{[j-n-1]}\eta_j, \qquad \eta_j \in C^\infty(\overline{\Omega} \times \overline{\Omega}),$$

our Theorem 14 describes the boundary singularities of the kernels $K_{u^\alpha_\Omega}$ (and of their analytic continuation in α) from the previous proposition. One could thus examine the corresponding CR-invariants occurring in the boundary singularities of (the analytic continuation of) these kernels, as was done for the unweighted Bergman kernel e.g. in Hirachi, Komatsu and Nakazawa [25], and for some weighted kernels in Hirachi and Komatsu [24]. (This again applies, of course, also to $K_{K^{-\alpha/(n+1)}_\Omega}$ and $K_{S^{-\alpha/n}_\Omega}$.) This yields a plethora of invariants of all complex weights $\alpha \in \mathbf{C} \setminus \mathbf{Z}_-$. The problem is, however, that these invariants seldom seem to be local, i.e. to depend, at a point $x \in \partial\Omega$, only on the jet of the boundary at x (i.e. on the coefficients at x of the Chern-Moser normal form for $\partial\Omega$). The reason is that, for instance, even though the boundary values of the η_j in (39) are determined locally, this is no longer the case for the powers K^α_Ω, $\alpha \neq 1$. Further study of all these questions is desirable.

7. Concluding remarks

7.1 Holomorphic families. Another definition for holomorphic families of unbounded operators appears in Kato's book [26]: $A(z)$ is holomorphic if there exist two boundedly-holomorphic functions $B(z), C(z)$ such that $B(z)$ is an isomorphism

onto the domain of $A(z)$ and $A(z)B(z) = C(z)$. Unlike ours, Kato's definition behaves well under unitary equivalence (i.e. if $A(z)$ is holomorphic, then so is $U^*A(z)U$ for any unitary U). On the other hand, Kato's definition does not behave so well with respect to taking products: there exists families $A(z), B(z)$ holomorphic in Kato's sense, and a unitary U such that $A(z) = B(z)^{-1}$ for all z, yet $A(z)U^*B(z)U$ is not even bounded. Thus it is not so clear if there is any analogue of our Proposition 5 for families holomorphic in Kato's sense.

The above-mentioned example goes as follows: consider the unit disc $\mathbf{D}$, and let $A(z) = (2\Lambda)^z$ and $B(z) = A(z)^{-1}$, where Λ is our operator $\mathbf{K}^*\mathbf{K}$ on $\partial\mathbf{D}$. In terms of the standard orthonormal basis $e_n(e^{i\theta}) := \frac{1}{\sqrt{2\pi}}e^{ni\theta}$, $n \in \mathbf{Z}$, of $L^2(\partial\mathbf{D})$, $A(z)$ is a diagonal operator $A(z)e_n = (|n| + 1)^{-z}e_n$. It follows that $A(z)$ is a holomorphic family in our sense (of order $-z$) as well as in Kato's sense (with the above $B(z)$ and $C(z)$ the constant function I) in the left half-plane $\operatorname{Re} z < 0$; and $B(z)$ is even boundedly-holomorphic there. Let now U be the unitary operator which interchanges $e_{a_{2j}}$ and $e_{a_{2j+1}}$, where $a_j = 2^{2^j} - 1$, $j = 0, 1, 2, \ldots$, and leaves the other e_k, $k \in \mathbf{Z}$, unchanged. Then $A(z)U^*B(z)U$ maps each e_k into $t_k e_k$, where $t_{a_{2j}} = 2^{-4^j z}$, $t_{a_{2j+1}} = 2^{4^j z}$, and $t_k = 1$ for $k \notin \{a_0, a_1, a_2, \ldots\}$. Thus neither $A(z)U^*B(z)U$ nor its inverse are bounded.

7.2 The pole-sets. In the prototype example of the unit ball with the standard defining function $\rho(z) = 1 - |z|^2$, the pole-set U of K_α is empty; in fact, K_α even has zeroes at $\alpha = -1, -2, \ldots, -n$ (cf. (2)). It might be tempting to expect that also in the general case, the pole-sets will be something simple, like e.g. the negative integers. It turns out that U can have quite diverse forms.

Consider the unit disc $\Omega = \mathbf{D}$ with a radial defining function, i.e. $\rho(z) = \phi(|z|^2)$ for some $\phi \in C^\infty[0, 1]$, positive on $[0, 1)$ and vanishing at 1, with $\phi'(1) \neq 0$. A simple computation in polar coordinates shows that the reproducing kernels are given by

$$(40) \qquad K_\alpha(x, y) = \frac{1}{\pi}\sum_{k=0}^\infty \frac{(x\overline{y})^k}{c_k(\alpha)}$$

where

$$c_k(\alpha) = \int_0^1 \phi(t)^\alpha\, t^k\, dt, \qquad \operatorname{Re}\alpha > -1.$$

For any $\delta < 1$, ϕ is bounded below by a positive constant on $[0, \delta]$, hence $\int_0^\delta \phi(t)^\alpha t^k\, dt$ is a holomorphic function of α on the entire complex plane (by an elementary application of Morera's and Fubini's theorems). On the other hand, choosing δ so close to 1 that $\phi' \neq 0$ on $[\delta, 1]$, we have by partial integration

$$(41) \qquad \begin{aligned} \int_\delta^1 \phi(t)^\alpha\, t^k\, dt &= \int_\delta^1 \left(\frac{\phi(t)^{\alpha+1}}{\alpha+1}\right)' \cdot \frac{t^k}{\phi'(t)}\, dt \\ &= -\frac{\phi(\delta)^{\alpha+1}\delta^k}{(\alpha+1)\phi'(\delta)} - \frac{1}{\alpha+1}\int_\delta^1 \phi(t)^{\alpha+1}\left(\frac{t^k}{\phi'(t)}\right)'\, dt. \end{aligned}$$

The last integral is holomorphic not only for $\operatorname{Re}\alpha > -1$, but for $\operatorname{Re}\alpha > -2$. Repeating this argument, we see that each $c_k(\alpha)$ extends meromorphically to all
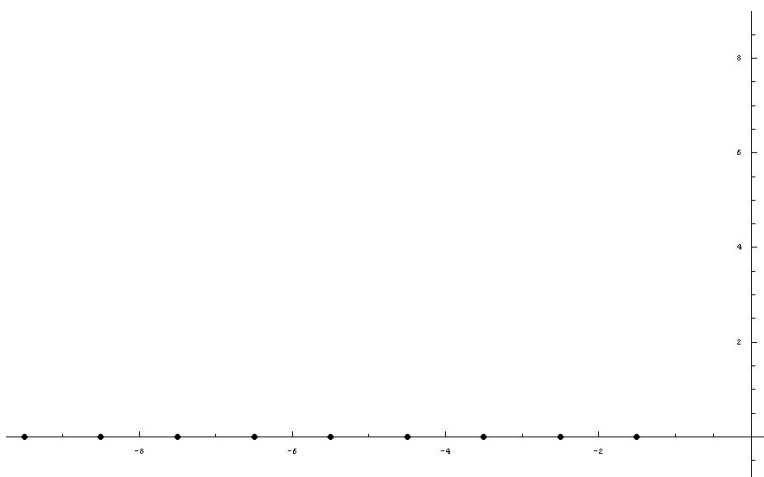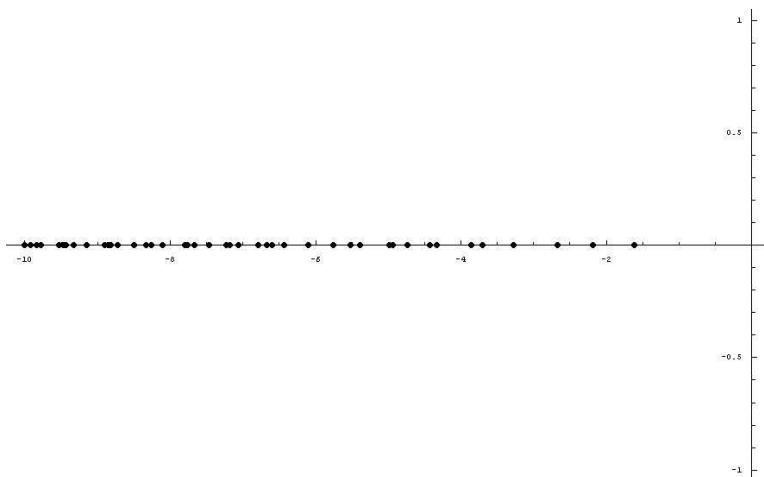
Fig. 1

Fig. 2

of **C**, with poles at negative integers. Thus $K_\alpha(x, y)$ extends (for each $x, y \in \mathbf{D}$) meromorphically to **C**, with poles at

$$(42) \qquad U = \bigcup_{k=0}^{\infty} \{\alpha \in \mathbf{C} : \ c_k(\alpha) = 0\}.$$

These pole-sets are not difficult to depict for various choices of ϕ. For instance, for $\phi(t) = 1 - t^2$ (corresponding to $\rho(z) = 1 - |z|^4$), one gets poles at $\alpha = -k - \frac{1}{2}$, $k = 1, 2, 3, \ldots$ (see Fig. 1). Taking $\phi(t) = 2 - t - t^2$ yields poles on the negative real axis, with increasing density (there are roughly k poles between $-k$ and $-k - 1$, $k = 1, 2, \ldots$); see Fig. 2. For $\phi(t) = \sqrt{2} - \sqrt{1 + t}$, one finds poles arrayed on half-lines (almost) in the left half-plane (Fig. 3). Finally, for $\phi(t) = \min(1, 2 - 2t)$ there are poles arrayed on (roughly) parabolic arcs, lying in all four quadrants (Fig. 4).

We remark that in the last example, the function ϕ was not C^∞; however, it can be approximated by smooth functions, and each point of U then arises as
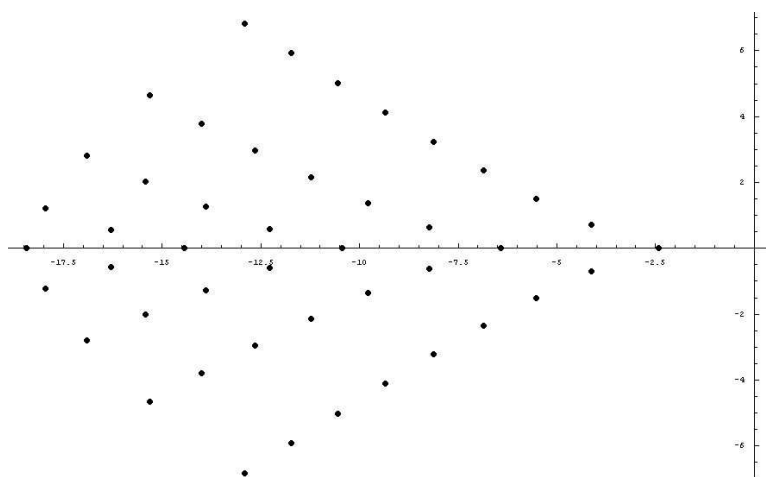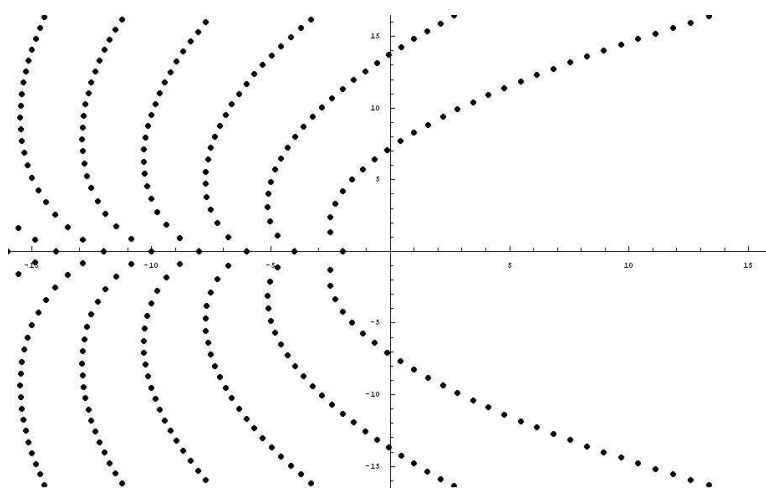
Fig. 3

Fig. 4

an accumulation point of points in the pole-sets of the approximating functions, by Rouché's theorem; thus one can get a pole-set resembling U even for some $\phi \in C^\infty[0,1]$. (By a similar argument, one can achieve that in the first example $\rho(z) = 1 - |z|^4$ is replaced by a defining function for which $-\log\rho$ is strictly plurisubharmonic on the whole disc, including the origin.)

The above examples also show that for some pairs of points $x, y \in \Omega$, $K_\alpha(x, y)$ may become regular for some $\alpha \in U$ (i.e. some of the poles may disappear for special choices of x, y). Indeed, if $x = 0$ (or $y = 0$), then the pole-set of $K_\alpha(0, y)$ is just $\{\alpha \in \mathbf{C} : c_0(\alpha) = 0\}$, which is — with the exception of the first example — strictly smaller than (42).

Finally, taking for Ω the bidisc $\mathbf{D}^2$ with weights of the form $\rho(z_1, z_2) = \rho_1(z_1)\rho_2(z_2)$, the resulting pole-set will be the union of the pole-sets for ρ_1^α and ρ_2^α on $\mathbf{D}$. In this way, even more bizarre pole-sets can be constructed. (And again, using Rouché's theorem it is possible to get such examples even with smooth boundary by approximating $\mathbf{D}^2$ by smoothly bounded strictly pseudoconvex domains.)

Remark 17. Using Hadamard's formula, it is easily checked that the analytic continuation of c_k, obtained via the partial integration (41), still gives the same radius of convergence (namely, 1) for the series (40), thus providing a direct proof of Theorem 1 for radial weights on the unit disc. $\square$

7.3 Generic pole-sets are nonempty. In the context of the preceding examples, one can prove the following result, in some way perhaps reminiscent of the one of Boas for the Lu Qi Keng conjecture [3].

Proposition 18. *Let $\Omega = \mathbf{D}$ and $\rho(z) = 1 - |z|^2 + \epsilon\psi(|z|^2)$, where $\psi \in C^\infty(0,1)$ is supported on a compact subset of $(0,1)$. Then there is $\epsilon_0 > 0$ such that for $0 < |\epsilon| < \epsilon_0$, the pole-set of K_{ρ^α} is nonempty.*

Proof. It is enough to exhibit a zero of $c_0(\alpha)$. Let $\phi(t) = 1 - t + \epsilon\psi(t)$, so that $\rho(z) = \phi(|z|^2)$. Set $\epsilon_0 = 1/\|\psi'\|_\infty$. Then for $|\epsilon| < \epsilon_0$, $\phi' < 0$ on $[0,1]$, so ϕ has an inverse — σ, say. Making the change of variable from t to $r = \phi(t)$, we get

$$c_0(\alpha) = \int_0^1 \phi(t)^\alpha \, dt = -\int_0^1 \sigma'(r) \, r^\alpha \, dr, \qquad \operatorname{Re}\alpha > -1.$$

(This is essentially the Mellin transform of $-\sigma'$.) Now $\sigma' + 1$ is supported on a compact subset of $(0,1)$ — say, in $[\delta, 1-\delta]$, $\delta > 0$. Thus

$$c_0(\alpha) = \frac{1}{\alpha+1} - \int_\delta^{1-\delta} (\sigma'(r) + 1) \, r^\alpha \, dr =: \frac{1}{\alpha+1} + G(\alpha).$$

This gives a meromorphic continuation of c_0 to all of $\mathbf{C}$, with a single pole at -1. Suppose that it has no zeroes. Since

$$|G(\alpha)| \leq \|\sigma' + 1\|_\infty \, \delta^{-|\operatorname{Re}\alpha|}, \qquad \forall\alpha \in \mathbf{C},$$

it follows that

$$f(\alpha) := (\alpha+1)c_0(\alpha)$$

is zero-free on $\mathbf{C}$ and satisfies

$$|f(\alpha)| \leq e^{C_1|\alpha|+C_2}$$

for some constants C_1 and C_2. Consequently, $f = e^g$ for some entire g, $|g(z)| \leq C_1|z| + C_2$. Applying Liouville's theorem to $\frac{g(z)-g(0)}{z}$, we conclude that g is linear:

$$g(z) = az + b, \qquad \forall z \in \mathbf{C}.$$

Since $f(-1) = 1$, we get $f(\alpha) = e^{(\alpha+1)a}$ and

$$(43) \qquad\qquad G(\alpha) = \frac{e^{(\alpha+1)a} - 1}{\alpha+1}.$$

However, making the change of variable $r = e^x$ in the integral defining $G(\alpha)$, we get

$$G(\alpha) = -\int_{\log\delta}^{\log(1-\delta)} [\sigma'(e^x) + 1] \, e^x \, e^{\alpha x} \, dx.$$

Thus the restriction of G to the imaginary axis is the Fourier transform of a compactly supported function, hence must be rapidly decreasing. For the function (43) this is the case only if it vanishes identically, i.e. if $\sigma' = -1$ and, hence, $\epsilon = 0$. Hence for $0 < |\epsilon| < \epsilon_0$, $c_0(\alpha)$ always has a zero and U is not empty. $\square$

In fact, the author does not know of any example, other than the ball with $\rho(z) = 1 - |z|^2$, of a smoothly bounded strictly pseudoconvex domain Ω with defining function ρ for which the pole-set U of K_α would be empty.

7.4 Accumulating poles. As promised, we now exhibit an example of a "logarithmic" weight v on the unit disc $\mathbf{D}$ for which the pole-set U of K_{v^α} consists of isolated points in $\mathbf{C} \setminus \mathbf{Z}_-$, but not in $\mathbf{C}$.

To this end, take $v(z) = \phi(|z|^2)$ where

$$\phi(t) = (1 - t) + (1 - t)^2 \log(1 - t).$$

Making the change of variable $t = 1 - e^{-s}$ and expanding $(1 - e^{-s})^k$ via the binomial theorem, we again have

$$K_{v^\alpha}(x, y) = \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(x\overline{y})^k}{c_k(\alpha)}$$

where, for $\operatorname{Re} \alpha > -1$,

$$
\begin{aligned}
c_k(\alpha) : &= \int_0^1 \phi(t)^\alpha \, t^k \, dt \\
&= \int_0^\infty [e^{-s}(1 - se^{-s})]^\alpha \, (1 - e^{-s})^k \, e^{-s} \, ds \\
&= \sum_{j=0}^{k} \frac{(-k)_j}{j!} \int_0^\infty e^{-(j+\alpha+1)s} (1 - se^{-s})^\alpha \, ds.
\end{aligned}
$$

(Here $(\nu)_j := \nu(\nu + 1) \ldots (\nu + j - 1)$ is the Pochhammer symbol.) Since $|se^{-s}| \le e^{-1} < 1$ for $s \ge 0$, we can again expand $(1 - se^{-s})^\alpha$ by the binomial formula and integrate term by term. This yields

$$c_k(\alpha) = \sum_{j=0}^{k} \frac{(-k)_j}{j!} \sum_{l=0}^{\infty} \frac{(-\alpha)_l}{(j + l + \alpha + 1)^{l+1}}.$$

The series on the right-hand side converges for any $\alpha \in \mathbf{C} \setminus \mathbf{Z}_-$, and gives thus the meromorphic continuation of $c_k(\alpha)$ to the entire complex plane, with poles of order m at $\alpha = -m$, $m = 1, 2, \ldots$, of strength $(m)_{m-1} > 0$. It follows that for each k, $c_k(\alpha) \to -\infty$ as $\alpha \nearrow -1$, but $c_k(\alpha) \to +\infty$ as $\alpha \searrow -2$; thus by continuity, each $c_k(\alpha)$ has a zero — say, α_k — in the interval $(-2, -1)$. We claim that $\alpha_{k_m} \to -2$ for some sequence $k_m \to \infty$; thus the poles of K_{v^α} accumulate at $\alpha = -2$.

To see this write $\phi(t) = (1 - t)\psi(t)$ with $\psi(t) = 1 + (1 - t) \log(1 - t)$; then $1 \ge \psi(t) \ge 1 - \frac{1}{e}$ on $[0, 1]$, $\psi(0) = \psi(1) = 1$, $\psi'(t) = \log \frac{1}{1-t} - 1$, $\psi''(t) = \frac{1}{1-t}$. Write $\phi^\alpha t^k = (1 - t)^\alpha \cdot (\psi^\alpha t^k)$ and perform two integrations by parts, integrating $(1-t)^\alpha$ and differentiating $\psi^\alpha t^k$, in the integral defining $c_k(\alpha)$. One of the resulting terms contains $(1 - t)^{\alpha+2} \psi''(t) = (1 - t)^{\alpha+1}$; to this term only, apply integration by parts one more time. The final outcome is the following formula for the function $f_k(\alpha) := (\alpha + 1)(\alpha + 2)c_k(\alpha)$ valid for $k \ge 2$ and $\operatorname{Re} \alpha > -3$:

(44)
$$
\begin{aligned}
f_k(\alpha) = (\alpha + 2) \int_0^1 (1 - t)^{\alpha+2} [\underbrace{\alpha(\alpha - 1)\psi^{\alpha-2}\psi'^2 t^k}_{\text{I}} + \underbrace{2\alpha\psi^{\alpha-1}\psi' k t^{k-1}}_{\text{II}} \\
+ \underbrace{\psi^\alpha k(k-1) t^{k-2}}_{\text{III}}] \, dt \\
+ \alpha \int_0^1 (1 - t)^{\alpha+2} [\underbrace{(\alpha - 1)\psi^{\alpha-2}\psi' t^k}_{\text{IV}} + \underbrace{\psi^{\alpha-1} k t^{k-1}}_{\text{V}}] \, dt.
\end{aligned}
$$

Set $g_k(\alpha) := f_k(\frac{\alpha+2}{k} - 2)$. We now show that

$$\text{(45)} \qquad\qquad g_k(\alpha) \to \alpha \quad \text{as } k \to \infty$$

uniformly in a neighbourhood of the origin. Indeed, the contribution from the term I to the integral tends to zero as $k \to \infty$ by the LDCT (Lebesgue Dominated Convergence Theorem). The term II gives a contribution of order $\log k$. For III, pulling out the factor of k leaves an integral which the change of variable $t^{k-1} =: s$ and LDCT show to tend to 1. Altogether, we thus see that the first two lines in (44) tend to $\alpha + 2$. The contribution from IV again tends to zero by LDCT, and that from V tends to 1 by the same trick (change of variable $t^{1/k} =: s$) as in III. Thus the third line in (44) tends to -2, and (45) follows. Now by Rouché's theorem, there must be sequences $k_m \to \infty$ and $\beta_{k_m} \to 0$ such that $g_{k_m}(\beta_{k_m}) = 0$. Hence $\alpha_{k_m} = \frac{\beta_{k_m}+2}{k_m} - 2 \to -2$, proving the claim.

The situation is completely similar in all intervals $(-2m, -2m+1)$, $m = 1, 2, \dots$.

7.5 Wilder weights. If one allows weights v having logarithmic singularities also in the leading term, the analytic continuation of K_{v^α} seems to get much more complicated. For instance, on the disc with the weight $v(z) = \phi(|z|^2)$ where

$$\phi(t) = (1 - t) \log \frac{1}{1 - t},$$

a similar computation as in the preceding subsection produces

$$c_k(\alpha) = \sum_{j=0}^{k} \frac{(-k)_j}{j!} \frac{\Gamma(\alpha+1)}{(j+\alpha+1)^{\alpha+1}}.$$

Thus already

$$K_{v^\alpha}(0, 0) = \frac{1}{c_0(\alpha)} = \frac{(\alpha+1)^{\alpha+1}}{\Gamma(\alpha+1)}$$

has a logarithmic branch-point at $\alpha = -1$, and $K_{v^\alpha}(x, y)$ for general $x, y \in \mathbf{D}$ has singularities of this type at all $\alpha = -m$, $m = 1, 2, 3, \dots$.

On the other hand, taking an even worse weight

$$v(z) = \log \frac{1}{1 - |z|^2}, \qquad z \in \mathbf{D},$$

leads to much nicer meromorphic functions

$$c_k(\alpha) = \Gamma(\alpha+1) \sum_{j=0}^{k} \frac{(-k)_j}{j!(j+1)^{\alpha+1}}.$$

The general picture is thus a bit unclear.

The author does not know if there is a weight v for which $\operatorname{Re}\alpha = -1$ would be a natural boundary for K_{v^α}.

7.6 On biholomorphic invariance. Let $\phi : \Omega \to \Omega'$ be a biholomorphism between smoothly bounded strictly pseudoconvex domains and ρ' a defining function for Ω'. By Fefferman's theorem [14], ϕ extends smoothly to the boundary, so $\rho := \rho' \circ \phi$ is a defining function for Ω. By the transformation formula (37),

$$(46) \qquad K_{\rho^\alpha}^\Omega(x, y) = J_\phi(x)\, \overline{J_\phi(y)}\, K_{\rho'^\alpha}^{\Omega'}(\phi(x), \phi(y)),$$

for all $\alpha > -1$. It follows that $K_{\rho^\alpha}^\Omega$ and $K_{\rho'^\alpha}^{\Omega'}$ have the same pole-sets, and their analytic continuations in α are still linked by (46).

In this sense, K_{ρ^α} is invariant under biholomorphisms; that is, it is invariant under biholomorphic transformations of the whole pair (Ω, ρ). We have seen that the pole-set depends on the choice of ρ heavily; thus, in order to have some sort of biholomorphic invariance under transformations of Ω alone, one is left with the task of associating the defining function ρ to Ω in some "canonical" (i.e. biholomorphically invariant) manner. Unfortunately, it is well known that this is impossible (cf. [24], Theorem 2). This is why it is natural to study also the "logarithmic" weights in Section 4, since there are many biholomorphically invariant objects associated to Ω (like u_Ω, K_Ω, S_Ω), which, however, have that kind of logarithmic singularities at the boundary.

7.7 Bell's formula. Analogously to (23), it is possible to derive a similar (in fact, even simpler) recurrence formula also for the Toeplitz operators on the Bergman space $A_0^2 = L_{\mathrm{hol}}^2(\Omega)$, defined by

$$\mathbf{T}_\phi f := \mathbf{\Pi}(\phi f), \qquad f \in A_0^2, \ \phi \in L^\infty(\Omega),$$

where $\mathbf{\Pi} : L^2(\Omega) \to A_0^2$ is the orthogonal projection (the Bergman projection). Namely, fix some functions $\phi_j \in C^\infty(\overline{\Omega})$, $j = 1, 2, \ldots, n$, such that $\phi_j \geq 0$, $\sum_j \phi_j = 1$ near $\partial\Omega$, and $\partial_j \rho \neq 0$ on the support of ϕ_j. Then for $\alpha > -1$ and $f \in C^\infty(\overline{\Omega})$,

$$(47) \qquad \mathbf{\Pi}\rho^\alpha f = \mathbf{\Pi}\rho^{\alpha+1}\left[wf - \frac{1}{\alpha+1} Lf \right],$$

where

$$w := \frac{1 - \sum_j \phi_j}{\rho}$$

vanishes near $\partial\Omega$, and L is the first-order differential operator

$$Lf := \sum_j \partial_j \left(\frac{\phi_j}{\partial_j \rho} f \right).$$

See [12], p. 1430; the main idea is, however, due to Bell [2]. From (47) one immediately sees that $\frac{1}{\Gamma(\alpha+1)}\mathbf{\Pi}\rho^\alpha f$ extends to a holomorphic function of α on all of $\mathbf{C}$, for any $f \in C^\infty(\overline{\Omega})$. The Bergman space Toeplitz operators are related to the generalized Toeplitz operators on the Hardy space via

$$\gamma \mathbf{T}_\phi \mathbf{K} = T_\Lambda^{-1} T_{\mathbf{K}^* \phi \mathbf{K}}$$

(see [13], formula (42)). Since there is also a formula parallel to (19),

$$K_{0,x} = \mathbf{T}_{\rho^\alpha} K_{\alpha,x}, \qquad \text{for all } x \in \Omega, \ \alpha > -1,$$

one could in principle try to base the proof of Theorem 1 on the identity (47) instead of Proposition 10, thus working directly with the generalized Toeplitz, rather than pseudodifferential, operators. What becomes technically troublesome, however, is defining the notion of holomorphy for families of generalized Toeplitz operators[1],

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[1]One possibility being the definition (3.17)–(3.18) in Guillemin [21].

and establishing the corresponding analogues of Proposition 5 and Corollary 6. (For instance, if $T_{P(z)}$ is a holomorphic family of generalized Toeplitz operators, then it is not clear whether necessarily $T_{P(z)} = T_{Q(z)}$ for a holomorphic family $Q(z)$ commuting with Π.) Working on the level of ΨDOs eliminates these difficulties.

References

[1] T.N. Bailey, M.G. Eastwood, C.R. Graham: *Invariant theory for conformal and CR geometry,* Ann. Math. **139** (1994), 491-552.

[2] S.R. Bell: *Biholomorphic mappings and the $\overline{\partial}$-problem,* Ann. Math. **114** (1981), 103–113.

[3] H.P. Boas: *The Lu Qi-Keng conjecture fails generically,* Proc. Amer. Math. Soc. **124** (1996), 2021–2027.

[4] D. Borthwick: *Upper and lower bounds on resonances for manifolds hyperbolic near infinity,* Comm. Partial Diff. Eqs. **33** (2008), 1507–1539.

[5] D. Borthwick, P. Perry: *Scattering poles for asymptotically hyperbolic manifolds,* Trans. Amer. Math. Soc. **354** (2001), 1215–1231.

[6] L. Boutet de Monvel: *Comportement d'un opérateur pseudo-différentiel sur une variété à bord II,* J. d'analyse Math. **17** (1966), 255–304.

[7] L. Boutet de Monvel: *Boundary problems for pseudo-differential operators,* Acta Math. **126** (1971), 11–51.

[8] L. Boutet de Monvel: *On the index of Toeplitz operators in several complex variables,* Invent. Math. **50** (1979), 249–272.

[9] L. Boutet de Monvel, V. Guillemin, *The spectral theory of Toeplitz operators,* Ann. Math. Studies 99, Princeton University Press, Princeton, 1981.

[10] L. Boutet de Monvel, J. Sjöstrand: *Sur la singularité des noyaux de Bergman et de Szegö,* Astérisque **34–35** (1976), 123–164.

[11] B. Bucicovschi: *An extension of the work of V. Guillemin on complex powers and zeta functions or elliptic pseudodifferential operators,* Proc. Amer. Math. Soc. **127** (1999), 3081–3090.

[12] M. Engliš: *Toeplitz operators and weighted Bergman kernels,* J. Funct. Anal. **255** (2008), 1419–1457.

[13] M. Engliš: *Weighted Bergman kernels for logarithmic weights,* Pure Appl. Math. Quarterly **6** (2010), 781–813.

[14] C. Fefferman: *The Bergman kernel and biholomorphic mappings of pseudoconvex domains,* Inv. Math. **26** (1974), 1–65.

[15] C. Fefferman: *Parabolic invariant theory in complex analysis,* Adv. Math. **31** (1979), 131–262.

[16] I.C. Gohberg, M.G. Krein, *Introduction to the theory of linear nonselfadjoint operators,* Translations of Mathematical Monographs 18, Amer. Math. Soc., Providence, 1969.

[17] C.R. Graham: *Scalar boundary invariants and the Bergman kernel,* Complex Analysis II (College Park, 1985/86), Lecture Notes in Math. 1276, Springer, Berlin, 1987, pp. 108–135.

[18] G. Grubb: *Trace defect formulas and zeta values for boundary problems,* Traces in number theory, geometry and quantum fields (Bonn, 2005), pp. 137–153, Aspects Math. E38, Friedr. Viewegh, Weisbaden, 2008.

[19] C. Guillarmou: *Meromorphic properties of the resolvent on asymptotically hyperbolic manifolds,* Duke Math. J. **129** (2005), 1–37.

[20] C. Guillarmou: *Resonances and scattering poles on asymptotically hyperbolic manifolds,* Math. Res. Lett. **12** (2005), 103–119.

[21] V. Guillemin: *A new proof of Weyl's formula on the asymptotic distribution of eigenvalues,* Adv. Math. **55** (1985), 131–160.

[22] L. Guillopé, M. Zworski: *Scattering asymptotics for Riemann surfaces,* Ann. Math. **145** (1997), 597–660.

[23] K. Hirachi: *Invariant theory of the Bergman kernel of strictly pseudoconvex domains,* Sugaku Expositions **17** (2004), 151–169.

[24] K. Hirachi, G. Komatsu: *Local Sobolev-Bergman kernels of strictly pseudoconvex domains,* Analysis and geometry in several complex variables (Katata, 1997), pp. 63–96, Trends Math., Birkhauser, Boston, 1999.

[25] K. Hirachi, G. Komatsu, N. Nakazawa: *CR invariants of weight five in the Bergman kernel,* Adv. Math. **143** (1999), 185–250.

[26] T. Kato, *Perturbation theory for linear operators,* Springer-Verlag, New York, 1966.

[27] M. Kontsevich, S. Vishik: *Determinants of elliptic pseudo-differential operators,* preprint (1994), arXiv:math/9404046.

[28] J. Lee, R. Melrose: *Boundary behaviour of the complex Monge-Ampére equation,* Acta Math. **148** (1982), 159–192.

[29] M. Lesch: *On the noncommutative residue for pseudodifferential operators with log-polyhomo-geneous symbols,* Ann. Global Anal. Geom. **17** (1999), 151–187.

[30] M. Lesch: *Pseudodifferential Operators and Regularized Traces,* Proceedings of the Conference on Motives, Quantum Field Theory and Pseudodifferential Operators (Boston, 2008), Clay Mathematics Proceedings, to appear; arXiv:0901.1689.

[31] J.-L. Lions, E. Magenes, *Problèmes aux limites non homogènes et applications,* vol. 1, Dunod, Paris, 1968.

[32] M.-F. Ouedraogo, S. Paycha: *The Campbell-Hausdorff formula for pseudodifferential operators revisited. Applications to determinants,* preprint (2007), arXiv:math/0701076.

[33] S. Paycha: *Prologomon to renormalization,* preprint (2009), available from http://math.univ-bpclermont.fr/~paycha/book.ps.

[34] S. Paycha, S. Scott: *A Laurent expansion for regularized integrals of holomorphic symbols,* Geom. Funct. Anal. **17** (2007), 491–536.

[35] E. Schrohe: *Complex powers of elliptic pseudodifferential operators,* Integ. Eqs. Oper. Theory **9** (1986), 337–354.

[36] R.T. Seeley: *Complex powers of an elliptic operator,* Singular Integrals, Proc. Symp. Pure Math. X, AMS, Providence, 1967, pp. 288–307.

[37] M.A. Shubin, *Pseudodifferential operators and spectral theory,* Springer-Verlag, Berlin, 2001.

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