

**A Note on Gauge Systems  
from the Point of View  
of Lie Algebroids**

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# A note on gauge systems from the point of view of Lie algebroids

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ABSTRACT. In the context of the variational bi-complex, we re-explain that irreducible gauge systems define a particular example of a Lie algebroid. This is used to review some recent and not so recent results on gauge, global and asymptotic symmetries.

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# 1 Introduction

Gauge systems feature prominently in theoretical physics because the four known fundamental interactions, electromagnetism, the weak and strong nuclear forces, general relativity, and various unifying models such as string or higher spin theories, are described by theories of this type. It is therefore of interest to study the mathematical structure of such systems.

More concretely, by gauge systems we mean systems of under-determined partial differential equations deriving from variational principles. In a first approximation, one often replaces the fields, i.e., the dependent variables, by coordinates  $\phi^i$  on some finite dimensional manifold and forgets about the independent variables. For instance, the action functional then reduces to an ordinary function  $S_0(\phi^i)$ .

When applied to such a finite-dimensional toy model, the algebraic structure underlying the Batalin-Vilkovisky (BV) construction as reviewed for instance in [1] involves formulas that are reminiscent of those that occur in the context of Lie algebroids. The general picture is well-known: the base space is the space of solutions to the Euler-Lagrange equations, the algebra is the algebra of field dependent gauge parameters, their image under the anchor are the gauge symmetries; the latter form an integrable distribution and partition solution space by gauge orbits. More precisely, let us denote by  $R_\alpha^i \partial / \partial \phi^i$  an irreducible generating set of gauge symmetries, i.e., a set of vector fields such that

$$R_\alpha^i \frac{\partial S_0}{\partial \phi^i} = 0, \quad N^i \frac{\partial S_0}{\partial \phi^i} = 0 \implies N^i \approx f^\alpha R_\alpha^i,$$

for some functions  $f^\alpha$ . We use Dirac's notation for a function that vanishes when pulled back to the surface  $\Sigma$  defined by  $\partial S_0 / \partial \phi^i = 0$ ,  $g \approx 0$ , and say that  $g$  vanishes weakly or vanishes on-shell. It then follows that the vector fields  $R_\alpha^i \partial / \partial \phi^i$  are in involution on-shell. Furthermore, on-shell, they determine structure functions and an associated Lie algebroid involving the algebra of field dependent gauge parameters  $f^\alpha$  and the anchor  $f^\alpha R_\alpha^i \partial / \partial \phi^i$ . In particular for instance, the associated "longitudinal" differential  $\gamma$  coincides, up to notation, with the differential occurring in the local description of a Lie algebroid as reviewed for instance in section 2.1 of [2].

The remaining part of the BV construction consists in getting an off-shell description of this differential by using a Koszul-Tate resolution with additional generators, the anti-fields. In the variational case, the off-shell differential can then be shown to be canonically generated through a generator  $S$  in terms of a suitable antibracket.

What makes the finite-dimensional toy model uninteresting per se, at least locally, is that under standard regularity assumption one can choose local coordinates that trivialize the whole construction. This is the content of the abelianization theorem.

The formal extension to field theories proceeds by assuming that the index  $i$  includes the independent variables  $x^\mu$  and, at the same time, summations over  $i$  include integrations

over  $x^\mu$ , this is the DeWitt notation, see e.g. [3]. The danger of this approach is that one easily forgets about derivatives, and it is precisely the derivatives that make the whole construction non trivial, even when working in a local coordinate system.

In the present note, we re-explain how irreducible gauge field theories define a particular Lie algebroid. For concreteness, we choose in this note to control the functional aspects of the problem by working in the framework of the variational bi-complex. The last part of the note is devoted to a summary of results that we have derived in this context.

Other approaches realizing the general picture are of course also possible. In particular, in the context of asymptotic symmetries one deals with concrete subspaces of solutions determined by some fall-off conditions. In the conclusion, we re-interpret some of our results on asymptotic symmetries from the perspective of Lie algebroids.

## 2 Generalities

In this section, we quickly review basic definitions and results on an algebraic approach to symmetries. More details and proofs can be found for instance in [4, 5, 6, 7] and references therein.

### 2.1 Jet-bundles and Euler-Lagrange derivatives

Consider a fiber bundle  $E$  with base space  $M$ . In the following, we restrict ourselves to local coordinates  $x^\mu$  on  $M$  and  $\phi^i$  on the fiber  $C$ . Coordinates on the associated jet-bundle  $\mathcal{J}^k$  of order  $k$  are denoted by  $x^\mu, \phi_{(\mu)}^i$ . Here  $(\mu)$  stands for an unordered index  $\mu_1 \dots \mu_l$ , with  $l \leq k$ . For such an index,  $|\mu| = l$ . The total derivative is the operator

$$\partial_\nu = \frac{\partial}{\partial x^\nu} + \phi_{(\mu)\nu}^i \frac{\partial}{\partial \phi_{(\mu)}^i}, \quad (2.1)$$

where the summation conventions for repeated indices is used. A local function is a smooth functions on  $\mathcal{J}^k$  for some finite  $k$ . The space of local functions is denoted by  $Loc(E)$ .

If  $(-\partial)_{(\mu)} = (-)^{|\mu|} \partial_{(\mu)}$ , the Euler-Lagrange derivative of a local function  $f$  is defined by

$$\frac{\delta f}{\delta \phi^i} = (-\partial)_{(\mu)} \frac{\partial f}{\partial \phi_{(\mu)}^i}. \quad (2.2)$$

The adjoint of a total differential operator  $O = O^{(\mu)} \partial_{(\mu)}$  is  $O^\dagger = (-\partial)_{(\mu)} (O^{(\mu)})$  so that  $(O^\dagger)^\dagger = O$ . For a collection of local functions  $P^a$ , the Fréchet derivative is the matrix-valued total differential operator defined by

$$D_P^a{}_j = \frac{\partial P^a}{\partial \phi_{(\nu)}^j} \partial_{(\nu)}. \quad (2.3)$$

Note also that the Fréchet derivative can be defined for a collection of total differential operators  $O^a = O^{a(\mu)}\partial_{(\mu)}$  through  $D_{O^a} \equiv D_{O^{a(\mu)}} \circ \partial_{(\mu)}$ .

## 2.2 Stationary surface

Equations of motion are partial differential equations of the form  $E_a[\phi] = 0$  where  $E_a$  are local functions that vanish when the fields and their derivatives are put to zero. The equations of motion  $E_a[\phi] = 0$  are variational if the range of  $a$  and  $i$  are the same and if there exists a local function  $L$  called Lagrangian such that

$$E_i = \frac{\delta L}{\delta \phi^i}. \quad (2.4)$$

This is the case if and only if

$$D_{E_{ij}} = (D_{E_{ji}})^\dagger. \quad (2.5)$$

The ‘‘stationary’’ surface  $\Sigma$  is defined in the jet-bundles by the equations of motion and their total derivatives,

$$\partial_{(\mu)} E_a = 0. \quad (2.6)$$

Under appropriate regularity conditions (see e.g. [1]) which we assume to be fulfilled,  $f \approx 0$  if and only if there exists local functions  $k^{a(\mu)}$  such that  $f = k^{a(\mu)}\partial_{(\mu)} E_a$ . The space  $Loc(\Sigma)$  of local functions on  $\Sigma$  can then be identified with  $Loc(E)/I$  where  $I$  is the ideal of local functions vanishing on  $\Sigma$ . The associated space of local forms on  $\Sigma$  is denoted by  $\Omega_\Sigma$ .

## 2.3 Horizontal complex and prolongation of generalized vector fields

The horizontal complex is the Grassmann algebra generated by the odd elements  $dx^\mu$  with coefficients that are local functions,  $\Omega = Loc(E) \otimes \wedge(dx^\mu)$ . The horizontal differential is  $d_H = dx^\mu \partial_\mu$ . A generalized vector field is a vector field of the form  $X = P^\mu \frac{\partial}{\partial x^\mu} + R^i \frac{\partial}{\partial \phi^i}$ , with  $P^\mu, R^i$  local functions. Its prolongation to horizontal forms is defined by

$$pr X = \partial_{(\mu)} Q^i \frac{\partial}{\partial \phi^i_{(\mu)}} + P^\mu \partial_\mu + d_H P^\mu \frac{\partial}{\partial dx^\mu}, \quad Q^i = R^i - P^\mu \partial_\mu \phi^i, \quad (2.7)$$

in such a way as to commute with the horizontal differential

$$[pr X, d_H] = 0. \quad (2.8)$$

The horizontal complex pulled back to the stationary surface is denoted by  $\Omega_\Sigma$ .

## 2.4 Local functionals

The space of local functionals  $\mathcal{F}$  is defined by  $\mathcal{F} = H^n(d_H, \Omega)$ . A local functional is thus an equivalence class  $[\mathcal{L}]$ ,  $\mathcal{L} = Ld^n x$  where  $L \sim L + \partial_\mu k^\mu$ , with  $L, k^\mu$  local functions, i.e., a Lagrangian  $L$  up to a total divergence. The property

$$\frac{\delta L}{\delta \phi^i} = 0 \iff L = \partial_\mu k^\mu, \quad (2.9)$$

allows one to characterize local functionals as equivalence classes of Lagrangians with identical Euler-Lagrange derivatives. The action is the distinguished local functional  $S_0 = [\mathcal{L}]$  whose associated Euler-Lagrange derivatives define the equations of motion.

## 2.5 Equations of motion and variational symmetries

A generalized vector field  $X$  defines an equations of motion symmetry if

$$pr X E_a \approx 0. \quad (2.10)$$

A generalized vector field  $X$  defines a variational symmetry of the action  $[\mathcal{L}]$  if

$$pr X \mathcal{L} = d_H k. \quad (2.11)$$

If  $Q^i = 0$ ,  $X$  is both an equations of motion and a variational symmetry for all  $P^\mu$ . We will thus restrict ourselves in the following to generalized vector fields in evolutionary form,  $Q = Q^i \frac{\partial}{\partial \phi^i}$ , with prolongation

$$\delta_Q = \partial_{(\mu)} Q^i \frac{\partial}{\partial \phi^i_{(\mu)}}. \quad (2.12)$$

The following formulae which can be derived for instance from Eq. (6.42) and Eq. (6.43) of [8], are useful in the following:

$$[\delta_Q, \frac{\delta}{\delta \phi^j}] = -(D_{Q^i} j)^\dagger \circ \frac{\delta}{\delta \phi^i}, \quad (2.13)$$

$$\delta_{Q_1} (D_{Q_2}^i j)^\dagger = (D_{\delta_{Q_1} Q_2}^i j)^\dagger - (D_{Q_2}^i k \circ D_{Q_1}^k j)^\dagger. \quad (2.14)$$

By applying an Euler-Lagrange derivative to  $\delta_Q L = \partial_\mu k^\mu$ , an evolutionary vector field defines a variational symmetry if and only if

$$\delta_Q \frac{\delta L}{\delta \phi^j} = -(D_{Q^i} j)^\dagger [\frac{\delta L}{\delta \phi^i}]. \quad (2.15)$$

It follows that every variational symmetry is an equations of motion symmetry.

Evolutionary vector fields (*EV*), equations of motion symmetries (*MS*) and variational symmetries (*VS*) are Lie algebras with bracket

$$[Q_1, Q_2]^i = \delta_{Q_1} Q_2^i - \delta_{Q_2} Q_1^i, \quad [\delta_{Q_1}, \delta_{Q_2}] = \delta_{[Q_1, Q_2]}. \quad (2.16)$$

## 2.6 On-shell symmetries

Evolutionary vector fields such that  $Q^i \approx 0$  define equations of motion symmetries. Such equations of motion symmetries are considered trivial. They form a Lie ideal. Proper equations of motion symmetries are defined as equivalence classes of equations of motion symmetries modulo trivial ones. They restrict to well defined vector fields on  $\Sigma$ . We denote the Lie algebra of proper equations of motion symmetries by  $PMS$ .

Similarly, variational symmetries such that  $Q^i \approx 0$  form an ideal in the Lie algebra of variational symmetries.

## 2.7 Generalized conservation laws

Generalized conservation laws correspond to the cohomology spaces  $H^{n-k}(d_H, \Omega_\Sigma)$  with  $k \geq 1$  defined by

$$H^{n-k}(d_H, \Omega_\Sigma) \ni [\omega^{n-k}] \iff \begin{cases} d_H \omega^{n-k} \approx 0, \\ \omega^{n-k} \sim \omega^{n-k} + d_H \eta^{n-k-1} + t^{n-k}, \quad t^{n-k} \approx 0. \end{cases} \quad (2.17)$$

# 3 Gauge and global symmetries

## 3.1 Noether operators

A Noether operator is a total differential operator  $N^a \equiv N^{a(\mu)} \partial_{(\mu)}$  such that

$$N^a[E_a] = 0. \quad (3.1)$$

The linear space of Noether operators (NO) is a left module over the associative algebra of total differential operators.

A set of Noether operators  $R_\alpha^{\dagger a}$  is a generating set<sup>1</sup> if every Noether operator  $N^a$  can be written in terms of the generating set on-shell, i.e., if there exists operators  $O^\alpha \equiv O^{\alpha(\mu)} \partial_{(\mu)}$  such that

$$N^a \approx O^\alpha \circ R_\alpha^{\dagger a}. \quad (3.2)$$

We assume here for simplicity of the arguments below that the generating set is irreducible, i.e., that for all operators  $Z^\alpha$ ,

$$Z^\alpha \circ R_\alpha^{\dagger a} \approx 0 \implies Z^\alpha \approx 0. \quad (3.3)$$

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<sup>1</sup>To agree with standard usage, the generating set is usually expressed in terms of adjoints of some operators  $R_\alpha^a$ .

In the rest of this section, we concentrate on the case of variational equations associated with an action  $S_0 = [\mathcal{L}]$ . The associative algebra  $TDO$  of total differential operators is a Lie module over  $VS$  under the action of  $\delta_Q$  with the Leibniz rule

$$\delta_Q(O_1 \circ O_2) = \delta_Q O_1 \circ O_2 + O_1 \circ \delta_Q O_2. \quad (3.4)$$

**Proposition 3.1.** *Noether operators are a module over  $VS$ ,*

$$(Q \cdot N)^i = \delta_Q N^i - N^j \circ (D_Q^j)_i^\dagger. \quad (3.5)$$

**Proof:** Applying a variational symmetry to a Noether identity gives

$$0 = \delta_Q \left( N^i \left[ \frac{\delta L_0}{\delta \phi^i} \right] \right) = \delta_Q (N^{i(\mu)}) \partial_{(\mu)} \frac{\delta L_0}{\delta \phi^i} - (N^i \circ (D_Q^j)_i^\dagger) \left[ \frac{\delta L_0}{\delta \phi^j} \right],$$

by using (2.15). This implies that the RHS of (3.5) is a Noether operator. That

$$Q_1 \cdot (Q_2 \cdot N) - Q_2 \cdot (Q_1 \cdot N) = [Q_1, Q_2] \cdot N \quad (3.6)$$

follows from a straightforward computation using (2.14).  $\square$

It also follows directly from (3.5) that

$$Q \cdot (O \circ N) = (\delta_Q O) \circ N + O \circ (Q \cdot N). \quad (3.7)$$

### 3.2 Gauge symmetries

Standard integrations by parts show that there is a linear map  $\rho$  from the space of Noether operators to the space of variational symmetries: if  $N^i$  is a Noether operator, the characteristic of the associated variational symmetry is  $\rho(N)^i = N^{\dagger i}(1)$ . Note in particular that  $\rho(N \circ D_Q^\dagger) = \delta_{\rho(N)} Q$ .

The space of gauge symmetries  $GS$  is defined as the subspace  $\text{Im } \rho \subset VS$ . It is a Lie ideal in the space of variational symmetries. This follows from the crucial property

$$\rho(Q \cdot N) = [Q, \rho(N)]. \quad (3.8)$$

Another property of  $\rho$  which can be proved by using again formula Eq. (6.43) of [8] is

$$D_{\rho(N)}^\dagger = D_N^\dagger. \quad (3.9)$$

One then can use  $\rho$  to define a bilinear map on Noether operators through  $N_1 \star N_2 = \rho(N_1) \cdot N_2$ . Even though this map is not skew-symmetric, its image under  $\rho$  is due to (3.8). Furthermore, as a consequence of (3.6), it satisfies  $N_1 \star (N_2 \star N_3) - N_2 \star (N_1 \star N_3) - (N_1 \star N_2) \star N_3 = 0$  which is mapped to the Jacobi identity for gauge symmetries when applying  $\rho$ .



### 3.3 Global symmetries

By definition, the quotient Lie algebra  $VS/GS$  of variational symmetries modulo gauge symmetries is the Lie algebra of global symmetries.

### 3.4 Proper gauge symmetries

Trivial total differential operators or Noether operators are defined by operators whose coefficients vanish on-shell. Multiplication of a Noether operator by a trivial operator gives a trivial Noether operator. Trivial gauge symmetries are variational symmetries that lie in the image of trivial Noether operators. They form an ideal in the Lie algebra of gauge symmetries. Proper total differential operators, Noether operators, gauge symmetries are defined as total differential operators, Noether operators, gauge symmetries modulo trivial ones.

### 3.5 Gauge algebroid

Proper gauge symmetries are generated by  $\rho(O^\alpha \circ R_\alpha^{i\dagger})$  with the equivalence relation  $O^\alpha \sim O^\alpha + t^\alpha$  and where  $TDO \ni t^\alpha \approx 0$ . Let us introduce the notations  $\rho(O^\alpha \circ R_\alpha^{i\dagger}) = R_\alpha^i(f^\alpha) = R_f^i$  where  $f^\alpha = O^{\dagger\alpha}(1)$ , and also  $\delta_f = \delta_{R_f}$ . Proper gauge symmetries are thus also generated by variational symmetries with characteristic  $R_\alpha^i(f^\alpha)$  where  $f^\alpha \in Loc(\Sigma)$ . Note that irreducibility of  $R_\alpha^{i\dagger}$  can easily be shown to be equivalent to the statement that if  $R_\alpha^i(O^\alpha(g)) \approx 0$  for all  $g \in Loc(E)$  then  $O^\alpha \approx 0$ . The property that  $R_\alpha^{i\dagger}$  is a generating set is equivalent to the statement that any family of variational symmetries that depends linearly and homogeneously on an arbitrary local function  $f$  and its derivatives,  $G^i(f) = G^{i(\mu)}\partial_{(\mu)}f$  and  $\delta_G L = \partial_\mu k^\mu(f)$  can be written as  $G^i(f) \approx R_\alpha^i(O^\alpha(f))$  for some  $O^\alpha \in TDO$ .

Since  $[R_{f_1}, R_{f_2}]$  defines a variational symmetry, one can easily prove from the generating property that

$$[R_{f_1}, R_{f_2}]^i \approx R_\gamma^i(C_{\alpha\beta}^\gamma(f_1^\alpha, f_2^\beta) + \delta_{f_1} f_2^\gamma - \delta_{f_2} f_1^\gamma), \quad (3.10)$$

where  $C_{\alpha\beta}^\gamma(f_1^\alpha, f_2^\beta) = C_{\alpha\beta}^{\gamma(\mu)(\nu)}\partial_{(\mu)}f_1^\alpha\partial_{(\nu)}f_2^\beta$  are bi-differential operators that are skew-symmetric,  $C_{\alpha\beta}^\gamma(f_1^\alpha, f_2^\beta) = -C_{\beta\alpha}^\gamma(f_2^\beta, f_1^\alpha)$ . Introducing a linear space spanned by  $e_\alpha$  associated with the generating set of Noether operators  $R_\alpha^{i\dagger}$  and defining  $A$  as the linear space with elements  $f = f^\alpha e_\alpha$  where  $f^\alpha \in Loc(\Sigma)$ ,  $A$  is a Lie algebra with bracket

$$[f_1, f_2]_A = (C_{\alpha\beta}^\gamma(f_1^\alpha, f_2^\beta) + \delta_{f_1} f_2^\gamma - \delta_{f_2} f_1^\gamma)e_\gamma. \quad (3.11)$$

Indeed, the Jacobi identity for the bracket  $[\cdot, \cdot]_A$  is a direct consequence of the Jacobi identity for the bracket of evolutionary vector fields applied to  $R_{f_1}, R_{f_2}, R_{f_3}$  and the irreducibility of the generating set.

To an irreducible gauge theory and a choice of generating set  $R_\alpha^{\dagger i}$ , one can thus associate the Lie algebroid with algebra  $A$  as a vector bundle over the stationary surface  $\Sigma$  with anchor the map  $a(f) = \delta_f$ . For want of a better name, one may call this the gauge algebroid.

Up to details related to the treatment in the context of the variational bi-complex, there is of course no claim of originality. Indeed, in some way or the other, this is known to most people familiar with the Batalin-Vilkovisky construction, see for instance [9]. Related considerations have appeared for instance in [10]. Note that the off-shell description gives rise to an sh-Lie algebroid, while  $L$ -stage reducible gauge theories are  $L$ -Lie algebroids. This is most transparent in the antifield formalism to which we now turn.

## 4 BV description

Both in the variational and the non-variational case, a description with antifields and ghosts originating from the Batalin-Vilkovisky approach [11, 12, 13, 14, 15] to the quantization of Lagrangian gauge field theories turns out to be useful.

Various elements of the construction appear in [16, 17, 18, 19] and are summarized in [1]. The non-variational case has been studied in [20]. Aspects related to locality and jet-bundles are treated in [21, 22, 23, 24, 8, 25].

### 4.1 Homological resolution of on-shell functions

For an irreducible set of Noether operators, the fiber is extended to include the “antifields”  $\phi_a^*$  (even) and  $C_\alpha^*$  (odd), of resolution degrees 1 and 2 respectively with all other variables of degree 0. We denote the space of local functions on this extended space by  $Loc(E^{AF})$ . The homology of the (evolutionary) homological vector field

$$\delta = \partial_{(\mu)} R_\alpha^{\dagger a} [\phi_a^*] \frac{\partial}{\partial C_{\alpha(\mu)}^*} + \partial_{(\mu)} E_a \frac{\partial}{\partial \phi_{a(\mu)}^*}, \quad \delta^2 = 0, \quad (4.1)$$

is

$$H_k(\delta, Loc(E^{AF})) = \begin{cases} 0 & \text{for } k > 0 \\ C^\infty(\Sigma) & \text{for } k = 0. \end{cases} \quad (4.2)$$

It follows that

**Proposition 4.1.** *The Lie algebra PMS of proper equations of motion symmetries is isomorphic to  $H_0([\delta, \cdot])$ , the adjoint cohomology of  $\delta$  in the space of evolutionary vector fields acting on  $Loc(E^{AF})$  in resolution degree 0 equipped with the induced Lie bracket for evolutionary vector fields.*

Furthermore,  $H_k([\delta, \cdot], EV_{E^{AF}}) = 0, k \geq 1$ .

## 4.2 Longitudinal differential

Consider a subset of equations of motion symmetries  $\delta_A$  with characteristic  $Q_A$  that are integrable on-shell,

$$[\delta_A, \delta_B] \approx f_{AB}^C \delta_C, \quad (4.3)$$

where  $f_{BC}^A$  are local functions.

Consider the pure ghost number, i.e., the degree for which  $C^A$  are Grassmann odd generators of degree 1 with all other variables in degree 0. On the space  $Loc(\Sigma) \otimes \wedge(C^A)$ , the associated homological vector field (“longitudinal differential”) is

$$\gamma = C^A \delta_A - \frac{1}{2} C^A C^B f_{AB}^C \frac{\partial}{\partial C^C}, \quad \gamma^2 \approx 0. \quad (4.4)$$

## 4.3 Homological perturbation theory

Consider the space  $Loc(E^{AF}) \otimes \wedge(C^A)$  with total degree (“ghost number”) the pure ghost number minus the resolution degree. The main theorem on the off-shell description of the longitudinal differential and its cohomology says that perturbatively in the resolution degree, there exists a differential  $s$  (“BRST differential”) on this space

$$s = \delta + \gamma + s_1 + \dots, \quad s^2 = 0, \quad (4.5)$$

such that

$$H^k(s, Loc(E^{AF}) \otimes \wedge(C^A)) = \begin{cases} 0 & \text{for } k < 0 \\ H^k(\gamma, Loc(\Sigma \otimes \wedge(C^A))) & \text{for } k \geq 0. \end{cases} \quad (4.6)$$

## 4.4 Longitudinal differential for proper gauge symmetries

For proper gauge symmetries associated to the generating set  $R_\alpha^{i\dagger}$  we extend the fiber by odd generators  $C^\alpha$  “ghosts” and the associated longitudinal differential can be written as

$$\gamma = \partial_{(\rho)}(R_\alpha^i(C^\alpha)) \frac{\partial}{\partial \phi_{(\rho)}^i} - \frac{1}{2} \partial_{(\rho)}(C_{\alpha\beta}^\gamma(C^\alpha, C^\beta)) \frac{\partial}{\partial C_{(\rho)}^\gamma}, \quad (4.7)$$

with  $C_{\alpha\beta}^\gamma(f_1^\alpha, f_2^\beta) = C_{\alpha\beta}^{\gamma(\mu)(\nu)} \partial_{(\mu)} f_1^\alpha \partial_{(\nu)} f_2^\beta$  total bi-differential skew-symmetric operators. This differential is of course just the standard Lie algebroid differential in the particular case of the gauge algebroid.

## 4.5 Master action

In the extended fiber with ghosts and antifields,  $C^\alpha$  are of ghost number 1,  $\phi_i^*$  of ghost number  $-1$  and  $C_\alpha^*$  of ghost number  $-2$ . All other variables are of ghost number 0.

Let  $z^a = (\phi^i, C^\alpha)$ . There is an odd graded Lie algebra structure “antibracket” on the space of local functionals  $[\mathcal{A} = ad^m x]$  defined by

$$\begin{aligned} (\cdot, \cdot) : \mathcal{F}^{g_1} \times \mathcal{F}^{g_2} &\rightarrow \mathcal{F}^{g_1+g_2+1}, \\ ([\mathcal{A}_1], [\mathcal{A}_2]) &= \left[ \left( \frac{\delta^R a_1}{\delta z^a} \frac{\delta^L a_2}{\delta z_a^*} - (z \leftrightarrow z^*) \right) d^m x \right]. \end{aligned} \quad (4.8)$$

The evolutionary vector field associated with a functional  $\mathcal{A}$  is then

$$(\mathcal{A}, \cdot)_{alt} = \left( \partial_{(\mu)} \frac{\delta^R a}{\delta z^a} \frac{\partial^L}{\partial z_{a(\mu)}^*} - (z \leftrightarrow z^*) \right). \quad (4.9)$$

In the variational case, the BRST differential  $s$  is canonically generated by a master action  $S$  of ghost number 0,

$$\begin{aligned} s &= (S, \cdot)_{alt}, \quad \frac{1}{2}(S, S) = 0, \\ S &= \left[ (L + \phi_i^* R_\alpha^i(C^\alpha) + \frac{1}{2} C_\gamma^* f_{\alpha\beta}^\gamma(C^\alpha, C^\beta) + \dots) d^m x \right]. \end{aligned} \quad (4.10)$$

## 4.6 Local BRST cohomology

The cohomology space  $H^*(s, \mathcal{F})$  of the BRST differential in the space of local functionals is an odd graded Lie algebra for the antibracket induced in cohomology. Under suitable assumptions, one can prove the following results for irreducible gauge theories considered here:

1.  $H^g(s, \mathcal{F}) \cong H^{n+g}(d_H, \Omega_\Sigma) = 0$  for  $g \leq -3$ .
2.  $H^{-2}(s, \mathcal{F}) \cong H^{n-2}(d_H, \Omega_\Sigma)$  is isomorphic to the space of equivalence classes of reducibility parameters  $[f^\alpha]$ , where  $f^\alpha$  are collections of local functions such that  $R_\alpha^i(f^\alpha) \approx 0$  with  $f^\alpha \sim f^\alpha + t^\alpha$  and where  $t^\alpha \approx 0$ .
3.  $H^{-1}(s, \mathcal{F}) \cong H^{n-1}(d_H, \Omega_\Sigma)$  is isomorphic to the space of global symmetries.
4. Every variational symmetry with weakly vanishing characteristic is a gauge symmetry and thus trivial as a global symmetry. It follows that global symmetries are a sub-Lie algebra of proper equations of motion symmetries,  $VS/GS \subset PMS$ .

Furthermore, up to a suspension, the antibracket induced in  $H^{-1}(s, \mathcal{F})$  coincides with the Lie bracket of global symmetries. The Lie bracket induced in the space of equivalence classes of conserved currents  $H^{n-1}(d_H, \Omega_\Sigma)$  is defined by

$$[[j_1], [j_2]] = [-\delta_{Q_1} j_2], \quad (4.11)$$

where  $Q_1$  is the variational symmetry associated with  $j_1$ . Together with item 3 above, this provides a complete and generalized version of Noether's first theorem for irreducible gauge theories.

More generally, via the antibracket induced in cohomology,  $H^g(s, \mathcal{F})$  is a module over the Lie algebra of global symmetries.

In addition, when  $[S^{(1)}] \in H^0(s, \mathcal{F})$ , there is a derived (even) Lie bracket in  $H^{-2}(s, \mathcal{F})$  defined by

$$[[\mathcal{A}^{-2}], [\mathcal{B}^{-2}]] = [(\mathcal{A}^{-2}, (S^{(1)}, \mathcal{B}^{-2}))]. \quad (4.12)$$

Through the isomorphism, it also induces a Lie algebra structure in  $H^{n-2}(d_H, \Omega_\Sigma)$ .

## 5 Discussion

From the definition of reducibility parameters in item 2 above and the perspective of the present note, it follows that this space is precisely the kernel of the anchor  $a$ . Furthermore, reducibility parameters at a particular solution have also been considered. From the point of view of Lie algebroids, they correspond to the isotropy Lie algebra at a given point. They are related to the reducibility parameters associated with the linearized gauge theory around this solution. Together with the associated generalized conservation laws, they have important physical applications. In gravity for instance, they are the Killing vectors of the solution and the associated conservation laws, also called surface charges, are related for instance to the ADM energy-momentum. In the discussion of integrability of these surface charges, paths in solution and gauge parameter spaces have been considered [26, 27]. It should prove most instructive to try to understand better the relation to the Lie algebroid paths and integrability discussed for instance in [2].

In the context of asymptotic symmetries, one does not work in the framework of the variational bi-complex but one restricts oneself to concrete and physically relevant subspaces of solutions. The claim is the following:

*From the point of view of Lie algebroids, the results of [28, 29] on asymptotically anti-de Sitter space-times in three dimensions at spatial infinity or asymptotically flat spacetimes in three or four dimensions at null infinity can be interpreted the sense that the associated gauge algebroid reduces to an action Lie algebroid for the Virasoro algebra in the former case and a suitable contraction or extension thereof in the latter two.*

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