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MORITA CLASSES OF MICRODIFFERENTIAL ALGEBROIDS

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Abstract. Following Kashiwara, any complex contact manifold X can be canonically quantized. This means that X is endowed with a canonical microdifferential algebroid – a linear stack locally equivalent to an algebra of microdifferential operators.

In this paper, we prove that Morita (resp. equivalence) classes of microdifferential algebroids on X are classified by $H^2(Y, \mathbb{C}^\times)$, for Y the symplectification of X. We also show that any stack locally equivalent to a stack of microdifferential modules is globally equivalent to the stack of modules over a microdifferential algebroid. To obtain these results we use techniques of microlocal calculus, non commutative cohomology and Morita theory for linear stacks.

Introduction

Let X be a complex contact manifold. By Darboux theorem, a local model of X is an open subset of the projective cotangent bundle P^*M of a complex manifold M. Let $\mathcal{E}_{P^*M}$ be the sheaf of microdifferential operators on P^*M. A microdifferential algebra ($\mathcal{E}$-algebra, for short) on X is a sheaf of $\mathbb{C}$-algebras locally isomorphic to $\mathcal{E}_{P^*M}$.

In the strict sense, to quantize X means to endow it with an $\mathcal{E}$-algebra. This might not be possible in general. However, Kashiwara [16] proved that X is endowed with a canonical $\mathcal{E}$-algebroid E_X. This means the following. To an algebra A one associates the linear category with one object and elements of A as its endomorphisms. Similarly, to a sheaf of algebras on X one associates a linear stack. An $\mathcal{E}$-algebroid on X is a $\mathbb{C}$-linear stack locally equivalent to one associated with an $\mathcal{E}$-algebra.

Having to deal with an algebroid instead of an algebra is not very limiting. For example, one can consider the stack of modules $\mathsf{Mod}(\mathsf{E}_X)$ and

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in particular regular holonomic modules attached to Lagrangian subvarieties (see [16, 12] and [11] for the involutive case).

The algebroid E_X is endowed with an anti-involution, corresponding to the formal adjoint of microdifferential operators. Moreover, the graded algebroid associated to its order filtration is trivial. It is shown in [30] that E_X is unique among such $\mathcal{E}$-algebroids.

In this paper, we consider $\mathcal{E}$-algebroids with no extra structures, hence including twisted quantizations of X, i.e. filtered $\mathcal{E}$-algebroids whose associated graded algebroid is non trivial (see [32]). In fact, even more generally, we consider stacks of twisted $\mathcal{E}$-modules, i.e. stacks locally equivalent to a stack of modules over an $\mathcal{E}$-algebra.

In Theorems 5.2.3 and 5.4.3, and Corollary 5.4.2 below, we prove the following classification results.

(i) Two $\mathcal{E}$-algebroids are equivalent if and only if they are Morita equivalent, i.e. their stacks of modules are equivalent.
(ii) Any stack of twisted $\mathcal{E}$-modules is globally equivalent to the stack of modules over an $\mathcal{E}$-algebroid.
(iii) The set of equivalence classes (resp. Morita classes) of $\mathcal{E}$-algebroids is canonically isomorphic to $H^2(Y;\mathbb{C}_Y^\times)$, for Y the symplectification of X.
(iv) The group of invertible $\mathcal{E}$-bimodules is isomorphic to $H^1(Y;\mathbb{C}_Y^\times)$.

Moreover, we give an explicit geometric realization of the isomorphisms in (iii) and (iv).

To obtain our results, we use techniques of microlocal calculus, non commutative cohomology and Morita theory for linear stacks.

Recall that cohomology with values in non commutative groups is used in [2] to classify $\mathcal{E}$-algebras, and cohomology with values in 2-groups is used in [31, 32] for the classification of algebroids.

Concerning Morita theory, it is developed in [27, 29] for linear categories. The case of stacks of modules over sheaves of algebras is discussed in [20] (see also [10]).

For symplectic manifolds, or more generally for Poisson manifolds, some results related to ours appeared in the literature.

The existence of a canonical deformation quantization algebroid on a complex symplectic manifold is proved in [33] (see also [24]). The general theory of deformation quantization modules is developed in [21].

On a complex symplectic manifold, deformation quantization algebroids with anti-involution and trivial graded have been classified up to equivalence in [31] (see also [4, 5] for the possibly twisted case).

Morita-type results for deformation quantization algebras are obtained in [6, 8, 7] for real Poisson manifolds, and in [36] in the algebraic setting.

Convention. In this text, when dealing with categories and stacks, we will not mention any smallness condition (with respect to a given universe), leaving to the reader the task to make it precise when necessary.

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1. Non commutative cohomology

We are interested in classifying $\mathcal{E}$-algebroids and stacks of $\mathcal{E}$-modules. Thanks to the existence of a canonical $\mathcal{E}$-algebroid, this amounts to classify stacks locally equivalent to a given one. To this end, we recall here some techniques of cohomology with values in a stack of 2-groups. References are made to [3] and to [13] for the strictly commutative case (see also [1] for an explicit description in terms of crossed modules). We follow the presentation of [31].

Let X be a topological space (or a site).

1.1. **Stacks.** A prestack C on X is a lax presheaf of categories. Lax in the sense that for a chain of three open subsets $W \subset V \subset U$ the restriction functor $\cdot|_W \colon \mathsf{C}(U) \to \mathsf{C}(W)$ coincides with the composition $\mathsf{C}(U) \xrightarrow{\cdot|_V} \mathsf{C}(V) \xrightarrow{\cdot|_W} \mathsf{C}(W)$ only up to an invertible transformation (such transformations satisfying a natural cocycle condition for chains of four open subsets).

For $\gamma, \gamma' \in \mathsf{C}(U)$, denote by $\mathcal{H}om_{\mathsf{C}}(\gamma, \gamma')$ the presheaf on U given by $U \supset V \mapsto \mathrm{Hom}_{\mathsf{C}(V)}(\gamma|_V, \gamma'|_V)$. One says that C is a separated prestack if $\mathcal{H}om_{\mathsf{C}}(\gamma, \gamma')$ is a sheaf for any γ, γ'. A stack on X is a separated prestack satisfying a natural descent condition, analogue to that for sheaves.

Given a stack C, we denote by $\pi_0(\mathsf{C})$ the sheaf associated to the presheaf $X \supset U \mapsto \{$isomorphism classes of objects in $\mathsf{C}(U)\}$.

Let $\varphi \colon Y \to X$ be a continuous map (or a morphism of sites). For D a stack on Y and C a stack on X, we denote by $\varphi_*\mathsf{D}$ and $\varphi^{-1}\mathsf{C}$ the stack-theoretical direct and inverse image, respectively. Recall that $\varphi^{-1}\mathsf{C}$ is the stack on Y associated to the separated prestack $\varphi^+\mathsf{C}$, defined on an open subset $V \subset Y$ by the category

$$\mathrm{Ob}(\varphi^+\mathsf{C}(V)) = \bigsqcup_{\substack{U \supset \varphi(V) \\ U \text{ open}}} \mathrm{Ob}(\mathsf{C}(U)),$$

$$\mathrm{Hom}_{\varphi^+\mathsf{C}(V)}(\gamma_U, \gamma_{U'}) = \Gamma(V, \varphi^{-1}\mathcal{H}om_{\mathsf{C}}(\gamma_U|_{U\cap U'}, \gamma_{U'}|_{U\cap U'})).$$

One checks that there is a natural equivalence (in fact, a 2-adjunction)

$$\varphi_*\mathsf{Fct}(\varphi^{-1}\mathsf{C}, \mathsf{D}) \approx \mathsf{Fct}(\mathsf{C}, \varphi_*\mathsf{D}). \tag{1.1.1}$$

Hence there are adjunction functors

$$\mathsf{C} \to \varphi_*\varphi^{-1}\mathsf{C}, \qquad \varphi^{-1}\varphi_*\mathsf{D} \to \mathsf{D}.$$

By using the left-hand side functor, one gets an isomorphsim of sheaves

$$\varphi^{-1}\pi_0(\mathsf{C}) \xrightarrow{\sim} \pi_0(\varphi^{-1}\mathsf{C}). \tag{1.1.2}$$

1.2. **Stacks of 2-groups.** Let C be stack on X. Denote by $\mathsf{Aut}(\mathsf{C})$ the stack whose objects are auto-equivalences, and whose morphisms are *invertible* transformations. Proposition A.1.1 for $\mathsf{C}_i = \mathsf{C}'_i = \mathsf{C}|_{U_i}$ describes how to patch objects and morphisms of $\mathsf{Aut}(\mathsf{C})$. For $\mathcal{U} = \{U_i\}_{i\in I}$ an open cover of X, set

$$U_{ij} = U_i \cap U_j, \quad U_{ijk} = U_i \cap U_j \cap U_k, \quad \text{etc.}$$

With notations as in Proposition A.1.1, let $H^1(\mathcal{U}; \mathsf{Aut}(\mathsf{C}))$ be the pointed set of equivalence classes of pairs $(\mathsf{f}_{ij}, \mathsf{a}_{ijk})_{ijk\in I}$ satisfying the cocycle condition (A.1.1), modulo the coboundary relation described by (A.1.2). One sets

$$H^1(X; \mathsf{Aut}(\mathsf{C})) = \varinjlim_{\mathcal{U}} H^1(\mathcal{U}; \mathsf{Aut}(\mathsf{C})). \tag{1.2.1}$$

By Proposition A.1.1, it follows

Corollary 1.2.1. *The pointed set $H^1(X; \mathsf{Aut}(\mathsf{C}))$ is in bijection with the pointed set of equivalence classes of stacks locally equivalent to C.*

Let us recall how to make the construction (1.2.1) functorial.

A 2-group is a category endowed with a group structure both on objects and on morphisms. More precisely, a category G is a 2-group if it is a groupoid (i.e. all morphisms are invertible) and it has a structure $(\mathsf{G}, \otimes, \mathbf{1})$ of monoidal category (i.e. endowed with the categorical analogue of a unital product) which is rigid (i.e. each object admits the categorical analogue of an inverse with respect to $\otimes$). Functors of 2-groups and transformations between them are monoidal functors and monoidal transformations.

A stack of 2-groups is a stack G whose sections $\mathsf{G}(U)$ are 2-groups, whose restrictions are functors of 2-groups and whose transformations between restriction functors are monoidal. Functors of stacks of 2-groups are functors of monoidal stacks.

Recall that one sets $\pi_1(\mathsf{G}) = \mathcal{H}om_{\mathsf{G}}(\mathbf{1}, \mathbf{1})$. This and $\pi_0(\mathsf{G})$ are sheaves of groups, the former being necessarily commutative. Any functor of stacks of 2-groups induces a group morphism at the level of π_1 and π_0.

Example 1.2.2. For $\mathcal{G}$ a sheaf of groups, denote by $\mathcal{G}[0]$ the stack obtained by enriching $\mathcal{G}$ with identity arrows, and by $\mathcal{G}[1]$ the stack of right $\mathcal{G}$-torsors. Then $\mathcal{G}[0]$ is a stack of 2-groups, and $\mathcal{G}[1]$ is a stack of 2-groups if and only if $\mathcal{G}$ is commutative.

Another example of stack of 2-groups is given by $\mathsf{Aut}(\mathsf{C})$ for C a stack. Let G be a stack of 2-groups and $\mathcal{U}$ an open cover of X. One can extend as follows the construction (1.2.1), where one should read "$\otimes$" instead of "$\circ$" in all diagrams in Appendix A.1.

A 1-cocycle with values in G is a pair $(\mathsf{f}_{ij},\, \mathsf{a}_{ijk})_{ijk\in I}$ with $\mathsf{f}_{ij} \in \mathsf{G}(U_{ij})$ and $\mathsf{a}_{ijk} \in \mathrm{Hom}_{\mathsf{G}}(\mathsf{f}_{ik}, \mathsf{f}_{ij} \otimes \mathsf{f}_{jk})$ satisfying (A.1.1). Two such 1-cocycles $(\mathsf{f}_{ij},\, \mathsf{a}_{ijk})_{ijk\in I}$ and $(\mathsf{f}'_{ij},\, \mathsf{a}'_{ijk})_{ijk\in I}$ are cohomologous if there is a pair $(\mathsf{g}_i,\, \mathsf{b}_{ij})_{ij\in I}$ with $\mathsf{g}_i \in \mathsf{G}(U_i)$ and $\mathsf{b}_{ij} \in \mathrm{Hom}_{\mathsf{G}}(\mathsf{f}'_{ij} \otimes \mathsf{g}_j, \mathsf{g}_i \otimes \mathsf{f}_{ij})$ satisfying (A.1.2).

The first cohomology pointed set of G on X is given by

$$H^1(X;\mathsf{G}) = \varinjlim_{\mathcal{U}} H^1(\mathcal{U};\mathsf{G}),$$

where $H^1(\mathcal{U};\mathsf{G})$ denotes the pointed set of equivalence classes of 1-cocycles on $\mathcal{U}$, modulo the relation of being cohomologous. One can also define cohomology in degree 0 and -1. This construction is functorial in the sense that short exact sequences of 2-groups induce long exact cohomology sequences (in a sense to be made precise). In particular, equivalent 2-groups have isomorphic cohomologies.

With the notations as in Example 1.2.2 one has

$$H^1(X;\mathcal{G}[i]) \simeq H^{1+i}(X;\mathcal{G}) \quad \text{for } i = 0, 1, \tag{1.2.2}$$

where the pointed set $H^1(X;\mathcal{G})$ is defined by Cech cohomology and $H^2(X;\mathcal{G})$ is considered only for $\mathcal{G}$ abelian.

1.3. **Crossed modules.** A crossed module is the data

$$\mathcal{G}^\bullet = (\mathcal{G}^{-1} \xrightarrow{d} \mathcal{G}^0,\ \delta)$$

of a complex of sheaves of groups and of a left action δ of $\mathcal{G}^0$ on $\mathcal{G}^{-1}$ such that for any $f \in \mathcal{G}^0$ and $a \in \mathcal{G}^{-1}$

$$d \circ \delta(f) = \mathrm{Ad}(f) \circ d, \qquad \delta\big(d(a)\big) = \mathrm{Ad}(a),$$

where $\mathrm{Ad}(a)(b) = aba^{-1}$. A morphism of crossed modules is a morphism of complexes of sheaves of groups compatible with the left actions.

There is a functorial way of associating to a crossed module a stack of 2-groups as follows. For $\mathcal{G}^\bullet$ a crossed module one denotes by $[\mathcal{G}^\bullet]$ the stack of 2-groups associated to the separated prestack whose objects on $U \subset X$ are sections $f \in \mathcal{G}^0(U)$ and whose morphisms $f \to f'$ are sections $a \in \mathcal{G}^{-1}(U)$ satisfying $f' = d(a)f$. Then $[\mathcal{G}^\bullet]$ is a stack of 2-groups, with monoidal structure given by $f \otimes g = fg$ at the level of objects and by $a \otimes b = a\delta(f)(b)$ at the level of morphisms, for $a\colon f \to f'$ and $b\colon g \to g'$.

One checks that there are isomorphisms of groups

$$\pi_i([\mathcal{G}^\bullet]) \simeq H^{-i}(\mathcal{G}^\bullet), \qquad i = 0,\, 1$$

and, with the notations and conventions as in Example 1.2.2, equivalences of stacks of 2-groups

$$\big[\mathcal{G}[i]\big] \approx \mathcal{G}[i], \qquad i = 0,\, 1.$$

1.4. **Strictly abelian crossed modules.** Denote by $\mathsf{D}^{[-1,0]}(\mathbb{Z}_X)$ the full subcategory of the derived category of sheaves of abelian groups whose objects have cohomology concentrated in degree $[-1,0]$. Consider a complex of abelian groups $\mathcal{F}^\bullet \in \mathsf{C}^{[-1,0]}(\mathbb{Z}_X)$ as a crossed module with trivial left action. Then the functor $\mathcal{F}^\bullet \mapsto [\mathcal{F}^\bullet]$ factorizes through $\mathsf{D}^{[-1,0]}(\mathbb{Z}_X)$ and one has

$$H^1(X;\mathcal{F}^\bullet) = H^1(X;[\mathcal{F}^\bullet]). \tag{1.4.1}$$

Let $\psi\colon X \to Y$ be a continuous map (or a morphism of sites). The inverse and direct image of stacks of 2-groups are again stacks of 2-groups, and one has

$$\psi^{-1}[\mathcal{G}^\bullet] \approx [\psi^{-1}\mathcal{G}^\bullet] \qquad \psi_*[\mathcal{F}^\bullet] \approx [\tau_{\leq 0} R\psi_*\mathcal{F}^\bullet], \tag{1.4.2}$$

where $\tau_{\leq 0}$ is the truncation functor. In particular, for a commutative sheaf of groups $\mathcal{F}$, one gets

$$\pi_i(\psi_*(\mathcal{F}[1])) \simeq R^{1-i}\psi_*\mathcal{F}, \qquad i = 0,\, 1. \tag{1.4.3}$$

2. Algebroids

Mitchell [28] showed how algebras can be replaced by linear categories. Similarly, sheaves of algebras can be replaced by linear stacks. An algebroid is a linear stack locally equivalent to an algebra. This notion, already implicit in [16], was introduced in [24] and developed in [11] (see also [21, §2.1] and [9]). It is the linear analogue of the notion of gerbe from [14]: an algebroid is to a gerbe as an algebra is to a group.

Let X be a topological space (or a site), and $\mathcal{R}$ a sheaf of commutative rings on X.

2.1. **Linear stacks.** A stack C on X is called $\mathcal{R}$-linear ($\mathcal{R}$-stack, for short) if for any $\gamma, \gamma' \in \mathsf{C}(U)$ the sheaf $\mathcal{H}om_{\mathsf{C}}(\gamma,\gamma')$ is endowed with an $\mathcal{R}|_U$-module structure compatible with composition. In particular, $\mathcal{E}nd_{\mathsf{C}}(\gamma)$ has an $\mathcal{R}|_U$-algebra structure with product given by composition. A functor between $\mathcal{R}$-linear stacks is called $\mathcal{R}$-linear ($\mathcal{R}$-functor, for short) if it is $\mathcal{R}$-linear at the level of morphisms, while no linearity conditions are required on transformations.

One says that two $\mathcal{R}$-stacks are equivalent if they are equivalent through an $\mathcal{R}$-functor. This implies that the quasi-inverse is also an $\mathcal{R}$-functor. We denote by $\approx_{\mathcal{R}}$ this equivalence relation.

The center $Z(\mathsf{C})$ of an $\mathcal{R}$-stack C is the sheaf of endo-transformations of the identity functor id_{C}. It has a natural structure of sheaf of commutative $\mathcal{R}$-algebras. Note that a stack C is $\mathcal{R}$-linear if and only if it is $\mathbb{Z}$-linear and its center is an $\mathcal{R}$-algebra.

If C is an $\mathcal{R}$-stack, then its opposite stack C^{op} is again an $\mathcal{R}$-linear. For D another $\mathcal{R}$-stack, denote by $\mathsf{Fct}_{\mathcal{R}}(\mathsf{C},\mathsf{D})$ the $\mathcal{R}$-stack whose objects are $\mathcal{R}$-functors and whose morphisms are transformations. The tensor product $\mathsf{C} \otimes_{\mathcal{R}} \mathsf{D}$ is the $\mathcal{R}$-stack associated with the prestack $U \mapsto \mathsf{C}(U) \otimes_{\mathcal{R}(U)} \mathsf{D}(U)$ whose objects are pairs in $\mathsf{C}(U) \times \mathsf{D}(U)$, with morphisms

$$\operatorname{Hom}_{\mathsf{C}(U) \otimes_{\mathcal{R}(U)} \mathsf{D}(U)}((\gamma, \delta), (\gamma', \delta')) = \operatorname{Hom}_{\mathsf{C}(U)}(\gamma, \gamma') \otimes_{\mathcal{R}(U)} \operatorname{Hom}_{\mathsf{D}(U)}(\delta, \delta').$$

Lemma 2.1.1. *If $\mathcal{R}$ is an $\mathcal{S}$-algebra and E an $\mathcal{S}$-stack, then*

$$\mathsf{Fct}_{\mathcal{S}}(\mathsf{C} \otimes_{\mathcal{R}} \mathsf{D}, \mathsf{E}) \approx_{\mathcal{R}} \mathsf{Fct}_{\mathcal{R}}(\mathsf{C}, \mathsf{Fct}_{\mathcal{S}}(\mathsf{D}, \mathsf{E})).$$

(This is in fact a 2-adjunction.)

Let $\varphi\colon Y \to X$ be a continuous map (or a morphism of sites). Then $\varphi^{-1}\mathsf{C}$ is $\varphi^{-1}\mathcal{R}$-linear and there is a $\varphi^{-1}\mathcal{R}$-equivalence

$$\varphi^{-1}(\mathsf{C} \otimes_{\mathcal{R}} \mathsf{D}) \approx \varphi^{-1}\mathsf{C} \otimes_{\varphi^{-1}\mathcal{R}} \varphi^{-1}\mathsf{D}.$$

If E is a $\varphi^{-1}\mathcal{R}$-stack, then $\varphi_*\mathsf{E}$ is $\mathcal{R}$-linear and there is an $\mathcal{R}$-functor

$$\varphi_*\mathsf{E} \otimes_{\mathcal{R}} \varphi_*\mathsf{F} \to \varphi_*(\mathsf{E} \otimes_{\varphi^{-1}\mathcal{R}} \mathsf{F}). \tag{2.1.1}$$

2.2. **Modules over a linear stack.** Denote by $\operatorname{Mod}(\mathcal{R})$ the category $\mathcal{R}$-modules and by $\mathsf{Mod}(\mathcal{R})$ the corresponding $\mathcal{R}$-stack given by $U \mapsto \operatorname{Mod}(\mathcal{R}|_U)$

For C an $\mathcal{R}$-stack, the stack of C-modules is defined by

$$\mathsf{Mod}(\mathsf{C}) = \mathsf{Fct}_{\mathcal{R}}(\mathsf{C}, \mathsf{Mod}(\mathcal{R})). \tag{2.2.1}$$

(It follows from Lemma 2.3.5 that this definition does not depend on the base ring. See also Lemma 3.1.6.)

The contravariant 2-functor $\mathsf{Mod}(\cdot)$ is defined as follows. On objects, it is given by (2.2.1). Consider the diagram

$$\begin{tikzcd} \mathsf{C} \arrow[r, bend left, "\mathsf{f}"] \arrow[r, bend right, "\mathsf{f}'"'] \arrow[r, phantom, "\Downarrow \mathsf{d}"] & \mathsf{D} \arrow[r, "\mathcal{N}"] & \mathsf{Mod}(\mathcal{R}) \end{tikzcd}.$$

To an $\mathcal{R}$-functor $\mathsf{f}\colon \mathsf{C} \to \mathsf{D}$ one associates the $\mathcal{R}$-functor

$$\mathsf{Mod}(\mathsf{f})\colon \mathsf{Mod}(\mathsf{D}) \to \mathsf{Mod}(\mathsf{C}), \qquad \mathcal{N} \mapsto \mathcal{N} \circ \mathsf{f},$$

and to a transformation $\mathsf{d}\colon \mathsf{f} \Rightarrow \mathsf{f}'$ one associates the transformation,

$$\mathsf{Mod}(\mathsf{d})\colon \mathsf{Mod}(\mathsf{f}) \Rightarrow \mathsf{Mod}(\mathsf{f}'),$$

such that $\mathsf{Mod}(\mathsf{d})(\mathcal{N}) = \mathrm{id}_{\mathcal{N}} \bullet \mathsf{d}$ is the morphism associated to $\mathcal{N} \in \mathsf{Mod}(\mathsf{D})$, where $\bullet$ denotes the horizontal composition of transformations. In other words, for $\gamma \in \mathsf{C}$ one has $\mathsf{Mod}(\mathsf{d})(\mathcal{N})(\gamma) = \mathcal{N}(\mathsf{d}(\gamma))$ as morphisms from $\mathcal{N}(\mathsf{f}(\gamma))$ to $\mathcal{N}(\mathsf{f}'(\gamma))$ in $\mathsf{Mod}(\mathcal{R})$. We use the notations

$${}_{\mathsf{f}}(\cdot) = \mathsf{Mod}(\mathsf{f}), \quad \mathsf{d} = \mathsf{Mod}(\mathsf{d}). \tag{2.2.2}$$

2.3. **Algebras as stacks.** Let $\mathcal{A}$ be a sheaf of $\mathcal{R}$-algebras. Denote by $\mathcal{A}^{\mathrm{op}}$ the opposite algebra and by $\mathsf{Mod}(\mathcal{A})$ the $\mathcal{R}$-stack of left $\mathcal{A}$-modules.

Denote by $\mathcal{A}^+$ the full substack of $\mathsf{Mod}(\mathcal{A}^{\mathrm{op}})$ whose objects are locally free right $\mathcal{A}$-modules of rank one. For any $\mathcal{N} \in \mathcal{A}^+(U)$ there is an $\mathcal{R}|_U$-algebra isomorphism $\mathcal{E}nd_{\mathcal{A}^+}(\mathcal{N}) \simeq \mathcal{A}|_U$. Note that the stack $\mathcal{A}^+$ has a canonical global object given by $\mathcal{A}$ itself with its structure of right $\mathcal{A}$-module. In particular, the sheaf $\pi_0(\mathcal{A}^+)$ is a singleton.

For $f\colon \mathcal{A} \to \mathcal{B}$ an $\mathcal{R}$-algebra morphism, denote by $f^+\colon \mathcal{A}^+ \to \mathcal{B}^+$ the $\mathcal{R}$-functor induced by the extension of scalars $(\cdot) \otimes_{\mathcal{A}} \mathcal{B}$. We thus have a functor between the stack of $\mathcal{R}$-algebras and that of $\mathcal{R}$-stacks

$$(\cdot)^+ \colon \mathcal{R}\text{-}\mathsf{Alg}_X \to \mathcal{R}\text{-}\mathsf{Stk}_X.$$

Remark 2.3.1. Let $\mathcal{A}^{\text{“}+\text{”}}$ be the separated prestack $U \mapsto \mathcal{A}(U)^+$, where $\mathcal{A}(U)^+$ denotes the $\mathcal{R}(U)$-category with one object and sections of $\mathcal{A}(U)$ as its endomorphisms. By Yoneda lemma (see §3.1), the stack associated to $\mathcal{A}^{\text{“}+\text{”}}$ is $\mathcal{R}$-equivalent to $\mathcal{A}^+$.

The stack $\mathcal{R}\text{-}\mathsf{Stk}_X$ is naturally upgraded to a 2-stack by considering transformations of functors. By enriching $\mathcal{R}\text{-}\mathsf{Alg}_X$ with identity transformations, the functor $(\cdot)^+$ upgrades to a 2-functor. With the terminology of 2-stacks, one has

Lemma 2.3.2. *The 2-functor $(\cdot)^+$ is faithful and locally full.*

Here, locally full means that for any two $\mathcal{R}$-algebras $\mathcal{A}$ and $\mathcal{B}$ on $U \subset X$ and any $\mathcal{R}$-functor $\mathsf{f}\colon \mathcal{A}^+ \to \mathcal{B}^+$ there exist a cover $\mathcal{U} = \{U_i\}_{i \in I}$ of U and morphisms of $\mathcal{R}$-algebras $f_i\colon \mathcal{A}|_{U_i} \to \mathcal{B}|_{U_i}$ such that $\mathsf{f}|_{U_i} \simeq f_i^+$.

Proof. By Remark 2.3.1, the 2-functor $(\cdot)^+$ is the composition of the 2-functor $(\cdot)^{\text{“}+\text{”}}$, which is full and faithful, and of the "associated stack" 2-functor $(\cdot)^\dagger$, which is faithful and locally full when restricted to separated prestacks. □

Definition 2.3.3. One says that an $\mathcal{R}$-stack C is equivalent to an $\mathcal{R}$-algebra $\mathcal{A}$ if $\mathsf{C} \approx_{\mathcal{R}} \mathcal{A}^+$.

In Proposition 2.6.2 we characterize the condition of equivalence between algebras.

Recall that a stack C is non empty if it has at least one global object, and it is locally connected by isomorphisms if any two objects $\gamma, \gamma' \in \mathsf{C}(U)$ are locally isomorphic. If C is $\mathcal{R}$-linear, this amounts to ask that the sheaf $\mathcal{H}om_{\mathsf{C}}(\gamma, \gamma')$ is a locally free $\mathcal{E}nd_{\mathsf{C}}(\gamma')$-module of rank one.

Lemma 2.3.4. *An $\mathcal{R}$-stack C is equivalent to an $\mathcal{R}$-algebra if and only if it is non empty and locally connected by isomorphisms*

Proof. One implication is clear. Suppose that C is non empty and let $\gamma \in \mathsf{C}(X)$. Then the fully faithful functor $\mathcal{E}nd_{\mathsf{C}}(\gamma)^+ \to \mathsf{C}$ is an equivalence if and only if C is locally connected by isomorphisms. □

Let C be an $\mathcal{R}$-stack. For $\mathcal{N} \in \mathcal{R}^+$ and $\gamma \in \mathsf{C}$, one defines $\mathcal{N} \otimes_{\mathcal{R}} \gamma \in \mathsf{C}$ as the representative of $\mathcal{N} \otimes_{\mathcal{R}} \mathcal{H}om_{\mathsf{C}}(\cdot, \gamma) \in \mathsf{Mod}(\mathsf{C}^{\mathrm{op}})$. Then one has $\mathcal{R}$-equivalences

$$\begin{aligned} \mathcal{R}^+ \otimes_{\mathcal{R}} \mathsf{C} \approx_{\mathcal{R}} \mathsf{C}, &\qquad (\mathcal{N}, \gamma) \mapsto \mathcal{N} \otimes_{\mathcal{R}} \gamma, \\ \mathsf{C} \approx_{\mathcal{R}} \mathsf{Fct}_{\mathcal{R}}(\mathcal{R}^+, \mathsf{C}), &\qquad \gamma \mapsto (\cdot) \otimes_{\mathcal{R}} \gamma. \end{aligned}$$

Lemma 2.3.5. *The definition* (2.2.1) *of stack of* C*-modules does not depend on the base ring* $\mathcal{R}$*.*

Proof. Let $\mathcal{R}$ be an $\mathcal{S}$-algebra. It follows from Lemma 2.1.1 for $\mathsf{D} = \mathcal{R}^+$ and $\mathsf{E} = \mathsf{Mod}(\mathcal{S})$ that

$$\mathsf{Fct}_{\mathcal{S}}(\mathsf{C}, \mathsf{Mod}(\mathcal{S})) \approx_{\mathcal{R}} \mathsf{Fct}_{\mathcal{R}}(\mathsf{C}, \mathsf{Mod}(\mathcal{R})),$$

where we use the equivalence $\mathsf{Fct}_{\mathcal{S}}(\mathcal{R}^+, \mathsf{Mod}(\mathcal{S})) \approx_{\mathcal{R}} \mathsf{Mod}(\mathcal{R})$. $\square$

2.4. **Compatibility.** Let $\mathcal{A}$ and $\mathcal{B}$ be two $\mathcal{R}$-algebras, and $\varphi\colon Y \to X$ a continuous map (or a morphism of sites). There are an $\mathcal{R}$-algebra isomorphism

$$Z(\mathcal{A}) \xrightarrow{\sim} Z(\mathcal{A}^+), \qquad a \mapsto (\mathcal{N} \to \mathcal{N} \colon n \mapsto an),$$

and $\mathcal{R}$-equivalences

$$\begin{aligned} (\mathcal{A}^+)^{\mathrm{op}} \approx_{\mathcal{R}} (\mathcal{A}^{\mathrm{op}})^+, &\qquad \mathcal{N} \mapsto \mathcal{H}om_{\mathcal{A}^{\mathrm{op}}}(\mathcal{N}, \mathcal{A}), \\ \mathsf{Mod}(\mathcal{A}) \approx_{\mathcal{R}} \mathsf{Mod}(\mathcal{A}^+), &\qquad \mathcal{M} \mapsto (\cdot) \otimes_{\mathcal{A}} \mathcal{M}, \\ \mathcal{A}^+ \otimes_{\mathcal{R}} \mathcal{B}^+ \approx_{\mathcal{R}} (\mathcal{A} \otimes_{\mathcal{R}} \mathcal{B})^+, &\qquad (\mathcal{N}, \mathcal{Q}) \mapsto \mathcal{N} \otimes_{\mathcal{R}} \mathcal{Q} \\ \varphi^{-1}\mathcal{A}^+ \approx_{\mathcal{R}} (\varphi^{-1}\mathcal{A})^+, &\qquad \mathcal{N} \mapsto \varphi^{-1}\mathcal{N}. \end{aligned}$$

For $f, f' \colon \mathcal{A} \to \mathcal{B}$ two $\mathcal{R}$-algebra morphisms, the sections on $U \subset X$ of the sheaf $\mathcal{H}om_{\mathsf{Fct}_{\mathcal{R}}(\mathcal{A}^+, \mathcal{B}^+)}(f^+, f'^+)$ are given by

$$\{b \in \mathcal{B}(U) \colon bf(a) = f'(a)b \text{ for each } a \in \mathcal{A}(V) \text{ and } V \subset U\}, \tag{2.4.1}$$

with composition of transformations given by the product in $\mathcal{B}$.

For $\mathcal{N}$ a left $\mathcal{B}$-module, denote by ${}_f\mathcal{N}$ the associated left $\mathcal{A}$-module. With notations (2.2.2), one has

$${}_{f^+}\mathcal{N} = {}_f\mathcal{N}, \qquad b(\mathcal{N}) \colon {}_f\mathcal{N} \to {}_{f'}\mathcal{N} \colon n \mapsto bn. \tag{2.4.2}$$

2.5. **Algebroids.** Recall from Lemma 2.3.4 that an $\mathcal{R}$-stack is equivalent to an $\mathcal{R}$-algebra if and only if it is non empty and is locally connected by isomorphisms.

Definition 2.5.1. An $\mathcal{R}$-algebroid is an $\mathcal{R}$-stack which is locally non empty and locally connected by isomorphisms.

In other words, an $\mathcal{R}$-algebroid is an $\mathcal{R}$-stack A which is locally equivalent to an algebra. It is globally an algebra if and only if it has a global object[1].

The stack $\mathsf{Mod}(\mathsf{A})$ is an example of stack of twisted sheaves, i.e. it is a stack locally equivalent to a stack of modules over an algebra (see [20, 10]). A cocyclic description of algebroids and of their modules is recalled in Appendix A.2 and A.3.

Note that the existence of an $\mathcal{R}$-functor $\mathcal{R}^+ \to \mathsf{A}$ is equivalent to the existence of a global object for A. In this case there is a forgetful functor

$$\mathsf{Mod}(\mathsf{A}) \to \mathsf{Mod}(\mathcal{R}).$$

Lemma 2.5.2. *An $\mathcal{R}$-stack C is an algebroid if and only if $\pi_0(\mathsf{C})$ is a singleton.*

It follows from (1.1.2) that inverse images of algebroids are algebroids.

Let C be an $\mathcal{R}$-stack. Then for any $\mathcal{R}$-algebroid A one has

$$\pi_0(\mathsf{A} \otimes_{\mathcal{R}} \mathsf{C}) \simeq \pi_0(\mathsf{C}).$$

In particular, the tensor product of algebroids is an algebroid.

Definition 2.5.3. (i) Let $\mathcal{A}$ be an $\mathcal{R}$-algebra. An $\mathcal{R}$-twisted form of $\mathcal{A}$ is an $\mathcal{R}$-algebroid locally $\mathcal{R}$-equivalent to $\mathcal{A}$.
(ii) An invertible $\mathcal{R}$-algebroid is an $\mathcal{R}$-twisted form of $\mathcal{R}$.

Note that any $\mathcal{R}$-functor between invertible $\mathcal{R}$-algebroids is an equivalence, since it is locally isomorphic to the identity of $\mathcal{R}^+$.

If C is an invertible $\mathcal{R}$-algebroid, then $\mathcal{R} \xrightarrow{\sim} Z(\mathsf{C})$ and for any $\mathcal{R}$-stack D there is an $\mathcal{R}$-equivalence

$$\mathsf{C}^{\mathrm{op}} \otimes_{\mathcal{R}} \mathsf{D} \approx_{\mathcal{R}} \mathsf{Fct}_{\mathcal{R}}(\mathsf{C}, \mathsf{D}), \quad (\gamma, \delta) \mapsto \mathcal{H}om_{\mathsf{C}}(\gamma, \cdot) \otimes_{\mathcal{R}} \delta.$$

In particular, the set of $\mathcal{R}$-equivalence classes of invertible $\mathcal{R}$-algebroids is a group, with multiplication given by $\otimes_{\mathcal{R}}$ and inverse given by $(\cdot)^{\mathrm{op}}$.

By Corollary 1.2.1, the cohomology $H^1(X; \mathsf{Aut}_{\mathcal{R}}(\mathcal{A}^+))$ classifies $\mathcal{R}$-equivalence classes of $\mathcal{R}$-twisted forms of $\mathcal{A}$. In terms of crossed modules, one has

$$\mathsf{Aut}_{\mathcal{R}}(\mathcal{A}^+) \approx [(\mathcal{A}^\times \xrightarrow{\mathrm{Ad}} \mathcal{A}ut_{\mathcal{R}\text{-}\mathsf{Alg}_X}(\mathcal{A}),\ \delta)],$$

where $\delta(f)(a) = f(a)$. In particular, $\mathsf{Aut}_{\mathcal{R}}(\mathcal{R}^+) \approx \mathcal{R}^\times[1]$ and (1.2.2) implies

Lemma 2.5.4. *The group of $\mathcal{R}$-equivalence classes of invertible $\mathcal{R}$-algebroids is isomorphic to $H^2(X; \mathcal{R}^\times)$.*

---

[1] If the category $\mathsf{A}(U)$ has a zero objects for $U \subset X$, then $\mathsf{A}|_U \approx 0^+$, where 0 denotes the ring with $1 = 0$. In particular, except for the case 0^+, algebroids are not stacks of additive categories.

2.6. **Inner forms.** Let $\mathcal{A}$ be a central $\mathcal{R}$-algebra, i.e. $Z(\mathcal{A}) = \mathcal{R}$. (If $\mathcal{A}$ is not central, the following discussion still holds by replacing $\mathcal{R}$ with $Z(\mathcal{A})$.)

Denote by $\mathcal{I}nn\,(\mathcal{A})$ the sheaf of inner automorphisms of $\mathcal{A}$, i.e. automorphisms locally of the form $\mathrm{Ad}(a)$ for some $a \in \mathcal{A}^\times$. Recall that an $\mathcal{R}$-algebra $\mathcal{B}$ is called an inner form of $\mathcal{A}$ if there exists an open cover $\{U_i\}_{i\in I}$ of X and ring isomorphisms $f_i\colon \mathcal{A}|_{U_i} \to \mathcal{B}|_{U_i}$ such that $f_j^{-1} f_i \in \mathcal{I}nn\,(\mathcal{A}|_{U_{ij}})$.

Examples of inner forms are given by Azumaya algebras and rings of twisted differential operators (see for example [10] for more details).

Let $\mathcal{B}$ be an $\mathcal{R}$-algebra. Denote by $\mathsf{E}_{\mathcal{A},\mathcal{B}} \subset \mathsf{Fct}_{\mathcal{R}}(\mathcal{A}^+, \mathcal{B}^+)$ the full substack of $\mathcal{R}$-equivalences. Note that $\mathsf{E}^{\mathrm{op}}_{\mathcal{A},\mathcal{B}} \approx_{\mathcal{R}} \mathsf{E}_{\mathcal{B},\mathcal{A}}$.

Lemma 2.6.1. *$\mathcal{B}$ is an inner form of $\mathcal{A}$ if and only if $\mathsf{E}_{\mathcal{A},\mathcal{B}}$ is an $\mathcal{R}$-algebroid.*

Proof. Since $\mathcal{R}$-equivalences $\mathcal{A}^+ \xrightarrow{\approx} \mathcal{B}^+$ are locally given by $\mathcal{R}$-algebra isomorphisms $\mathcal{A} \xrightarrow{\sim} \mathcal{B}$, it follows that $\mathsf{E}_{\mathcal{A},\mathcal{B}}$ is locally non empty if and only if $\mathcal{B}$ is locally isomorphic to $\mathcal{A}$.

Let $f, f'\colon \mathcal{A} \to \mathcal{B}$ be $\mathcal{R}$-algebra isomorphisms. By (2.4.1), the invertible transformations from f^+ to f'^+ are given by

$$\{a \in \mathcal{A}^\times\colon f^{-1}f' = \mathrm{Ad}(a)\},$$

hence $\mathsf{E}_{\mathcal{A},\mathcal{B}}$ is an $\mathcal{R}$-algebroid if and only if $\mathcal{B}$ is an inner form of $\mathcal{A}$. □

Since $\mathcal{E}nd_{\mathsf{E}_{\mathcal{A},\mathcal{B}}}(f^+) = \mathcal{R}$, if $\mathcal{B}$ is an inner form of $\mathcal{A}$ it follows that $\mathsf{E}_{\mathcal{A},\mathcal{B}}$ is an invertible $\mathcal{R}$-algebroid and $\mathsf{E}_{\mathcal{A},\mathcal{B}} \otimes_{\mathcal{R}} \mathcal{A}^+ \approx_{\mathcal{R}} \mathcal{B}^+$. In particular, one gets an equivalence of stacks of 2-groups $\mathsf{Aut}_{\mathcal{R}}(\mathcal{A}^+) \approx \mathsf{Aut}_{\mathcal{R}}(\mathcal{B}^+)$.

Consider the non abelian exact sequence

$$H^1(X;\mathcal{A}^\times) \xrightarrow{b} H^1(X;\mathcal{I}nn\,(\mathcal{A})) \xrightarrow{c} H^2(X;\mathcal{R}^\times)$$

induced by the short exact sequence

$$1 \to \mathcal{R}^\times \to \mathcal{A}^\times \xrightarrow{\mathrm{Ad}} \mathcal{I}nn\,(\mathcal{A}) \to 1.$$

For $\mathcal{B}$ an inner form of $\mathcal{A}$ and $\mathcal{P}$ a locally free $\mathcal{A}^{\mathrm{op}}$-module of rank one, denote by $[\mathcal{B}]$ and $[\mathcal{P}]$ the associated cohomology classes in $H^1(X;\mathcal{I}nn\,(\mathcal{A}))$ and $H^1(X;\mathcal{A}^\times)$ respectively. Then $b[\mathcal{P}] = [\mathcal{E}nd_{\mathcal{A}^{\mathrm{op}}}(\mathcal{P})]$ and $c([\mathcal{B}]) = [\mathsf{E}_{\mathcal{A},\mathcal{B}}]$.

Proposition 2.6.2. *The following conditions are equivalent.*

(i) *The stacks $\mathcal{A}^+$ and $\mathcal{B}^+$ are $\mathcal{R}$-equivalent.*
(ii) *There exists a locally free $\mathcal{A}^{\mathrm{op}}$-module $\mathcal{P}$ of rank one such that $\mathcal{B} \simeq \mathcal{E}nd_{\mathcal{A}^{\mathrm{op}}}(\mathcal{P})$.*
(iii) *$\mathcal{B}$ is an inner form of $\mathcal{A}$ and $c([\mathcal{B}]) = 1$.*

Proof. (i)⇒(ii)[2] Let $\mathsf{g}\colon \mathcal{B}^+ \to \mathcal{A}^+$ be an $\mathcal{R}$-equivalence. Recall that $\mathcal{B}^+ \subset \mathsf{Mod}(\mathcal{B}^{\mathrm{op}})$ is the substack of locally free modules of rank one. Let β be the canonical global object of $\mathcal{B}^+$, and set $\mathcal{P} = \mathsf{g}(\beta)$. Then $\mathcal{B}$ is isomorphic to $\mathcal{E}nd_{\mathcal{A}^{\mathrm{op}}}(\mathcal{P})$.

(ii)⇒(iii) $\mathcal{B}$ is clearly an inner form of $\mathcal{A}$ and $\mathcal{P}$ has a structure of $\mathcal{A}^{\mathrm{op}} \otimes_{\mathcal{R}} \mathcal{B}$-module by the isomorphism $\mathcal{B} \xrightarrow{\sim} \mathcal{E}nd_{\mathcal{A}^{\mathrm{op}}}(\mathcal{P})$. Then $(\cdot) \otimes_{\mathcal{B}} \mathcal{P}$ gives a global object of $\mathsf{E}_{\mathcal{B},\mathcal{A}}$ and $c([\mathcal{B}]) = [\mathsf{E}^{\mathrm{op}}_{\mathcal{B},\mathcal{A}}] = 1$.

(i)⇐(iii) By Lemma 2.6.1 follows that $c([\mathcal{B}]) = 1$ if and only if $\mathsf{E}_{\mathcal{A},\mathcal{B}}$ has a global object.

□

3. Morita theory for linear stacks

Morita theory classically deals with modules over algebras. It is extended to modules over linear categories in [27, 29] and to stacks of modules over sheaves of algebras in [20, Chapter 19] (see also [10]). Here, we summarize these extensions by considering stacks of modules over linear stacks, and in particular over algebroids.

Let X be a topological space (or a site), and $\mathcal{R}$ a sheaf of commutative rings on X.

3.1. **Yoneda embedding.** Recall that a category is called (co)complete if it admits small (co)limits. A prestack C on X is called (co)complete if the categories $\mathsf{C}(U)$ are (co)complete for each $U \subset X$, and the restriction functors commute with (co)limits.

A prestack C on X is called a proper stack (see [19, 34]) if it is separated, cocomplete, and if for each inclusion of open subsets $v\colon V \hookrightarrow U$, the restriction functors $\mathsf{C}(v) = (\cdot)|_V$ admits a fully faithful left adjoint

$$v_!\colon \mathsf{C}(V) \to \mathsf{C}(U),$$

called zero-extension, such that for a diagram of open inclusions

$$\begin{tikzcd} V \cap W \arrow[r, "v'"] \arrow[d, "w'"'] & W \arrow[d, "w"] \\ V \arrow[r, "v"] & U, \end{tikzcd}$$

the natural transformation $v'_! \circ \mathsf{C}(w') \to \mathsf{C}(w) \circ v_!$ is an isomorphism.

Lemma 3.1.1. *For $\gamma \in \mathsf{C}(V)$ and $\gamma' \in \mathsf{C}(U)$ there is an isomorphism of $\mathcal{R}|_U$-modules*

$$v_* \mathcal{H}om_{\mathsf{C}|_V}(\gamma, \gamma'|_V) \simeq \mathcal{H}om_{\mathsf{C}|_U}(v_!\gamma, \gamma').$$

Recall that proper stacks are stacks.

[2]The equivalence between (i) and (ii) can also be deduced from Corollary 3.3.8.

Lemma 3.1.2. *For any $\mathcal{R}$-stack C, the $\mathcal{R}$-stack $\mathsf{Mod}(\mathsf{C})$ is proper and complete.*

Proof. Recall first that $\mathsf{Mod}(\mathcal{R})$ is complete and cocomplete. It is also proper. In fact, for $v\colon V \hookrightarrow U$ an open inclusion, the restriction functor of $\mathsf{Mod}(\mathcal{R})$ coincides with the sheaf-theoretical pull-back v^{-1}. This admits the direct image functor v_* as left adjoint, and the zero-extension functor $v_!$ as a right adjoint.

The statement follows, as $\mathsf{Mod}(\mathsf{C}) = \mathsf{Fct}_{\mathcal{R}}(\mathsf{C}, \mathsf{Mod}(\mathcal{R}))$ inherits the properties and structures of $\mathsf{Mod}(\mathcal{R})$. For example, for $v\colon V \hookrightarrow U$ an inclusion of open subsets, the functor $v_!\colon \mathsf{Mod}(\mathsf{C}|_V) \to \mathsf{Mod}(\mathsf{C}|_U)$ is given by $(v_!\mathcal{M})(\gamma) = u_!(\mathcal{M}(\gamma|_{V\cap W}))$, where $\mathcal{M}\colon \mathsf{C}|_V \to \mathsf{Mod}(\mathcal{R}|_V)$ is a $\mathsf{C}|_V$-module, $W \subset U$ is an open subset, $\gamma \in \mathsf{C}(W)$, and $u\colon V \cap W \to U$ is the embedding. $\square$

Let C be an $\mathcal{R}$-stack. The (linear) Yoneda embedding is the full and faithful $\mathcal{R}$-functor

$$\mathsf{Y}_{\mathsf{C}}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Mod}(\mathsf{C}), \quad \gamma \mapsto \mathcal{H}om_{\mathsf{C}}(\gamma, \cdot) \tag{3.1.1}$$

whose essential image are the functors $\mathsf{C} \to \mathsf{Mod}(\mathcal{R})$ which are representable. In analogy with the case $\mathsf{C} = \mathcal{A}^+$ for $\mathcal{A}$ and $\mathcal{R}$-algebra, a module $\mathcal{M} \in \mathsf{Mod}(\mathsf{C})$ which is representable is called locally free of rank one.

As in the classical case, the full faithfulness of (3.1.1) follows from

Lemma 3.1.3. *For $\mathcal{M} \in \mathsf{Mod}(\mathsf{C})(U)$ there is an isomorphism of $\mathsf{C}|_U$-modules*

$$\mathcal{M}(\cdot) \simeq \mathcal{H}om_{\mathsf{Mod}(\mathsf{C})}(\mathsf{Y}_{\mathsf{C}}(\cdot), \mathcal{M}). \tag{3.1.2}$$

Denote by C/X the fibered category associated with C. Recall that objects of C/X are pairs (u,γ) with $u\colon U \hookrightarrow X$ an open inclusion, and $\gamma \in \mathsf{C}(U)$. Morphisms $\mathsf{a}\colon (u,\gamma) \to (u',\gamma')$ are defined only if $U' \subset U$, and in that case are given by morphisms $\gamma|_{U'} \to \gamma'$ in $\mathsf{C}(U')$. For $\mathsf{a}'\colon (u',\gamma') \to (u'',\gamma'')$ another morphism, the composition[3] is given by $\mathsf{a}' \circ \mathsf{a}|_{U''}$. A functor of stacks $\mathsf{f}\colon \mathsf{C} \to \mathsf{D}$ naturally induces a functor $\mathsf{f}/X\colon \mathsf{C}/X \to \mathsf{D}/X$.

For $\mathcal{M}\colon \mathsf{C} \to \mathsf{Mod}(\mathcal{R})$ an object in $\mathsf{Mod}(\mathsf{C})$, denote for short by $\mathsf{C}^{\mathrm{op}}_{\mathcal{M}}$ the comma category $(\mathsf{C}/X^{\mathrm{op}})_{\mathcal{M}/X}$.By (3.1.2), objects of $\mathsf{C}^{\mathrm{op}}_{\mathcal{M}}$ are triples (u,γ,m) with $u\colon U \hookrightarrow X$ an open inclusion, $\gamma \in \mathsf{C}^{\mathrm{op}}(U)$, and $m \in \Gamma(U;\mathcal{M}(\gamma))$.

Lemma 3.1.4. *For $\mathcal{M},\mathcal{N} \in \mathsf{Mod}(\mathsf{C})$ there is an isomorphism in $\mathsf{Mod}(\mathcal{R})$*

$$\mathcal{H}om_{\mathsf{Mod}(\mathsf{C})}(\mathcal{M},\mathcal{N}) \simeq \varprojlim_{(u,\gamma,m)\in\mathsf{C}^{\mathrm{op}}_{\mathcal{M}}} u_*\mathcal{N}(\gamma).$$

[3]Here we denote for short by $\mathsf{a}|_{U''}$ the composite $\gamma|_{U''} \underset{\sim}{\leftarrow} \gamma|_{U'}|_{U''} \xrightarrow{\mathsf{a}|_{U''}} \gamma'|_{U''}$.

Proof. For any open subset $V \subset X$, one has the isomorphism

$$\mathrm{Hom}_{\mathsf{Mod}(\mathsf{C}|_V)}(\mathcal{M}|_V, \mathcal{N}|_V) \xrightarrow{\sim} \varprojlim_{(u,\gamma,m)\in(\mathsf{C}|_V^{\mathrm{op}})_{\mathcal{M}|_V}} \mathit{\Gamma}(U;\mathcal{N}(\gamma)),$$

associating to $\mathsf{f}\colon \mathcal{M}|_V \to \mathcal{N}|_V$ the family $\{\mathsf{f}(\gamma)(m))\}_{(u,\gamma,m)}$. $\square$

Lemma 3.1.5. *For* $\mathcal{M} \in \mathsf{Mod}(\mathsf{C})$ *there is an isomorphism in* $\mathsf{Mod}(\mathsf{C})$

$$\mathcal{M} \simeq \varinjlim_{(u,\gamma,m)\in\mathsf{C}^{\mathrm{op}}_{\mathcal{M}}} u_!\mathsf{Y}_{\mathsf{C}}(\gamma).$$

Proof. This follows from the fact that for any $\mathcal{N} \in \mathsf{Mod}(\mathsf{C})$ there are $\mathcal{R}$-module isomorphisms

$$\begin{aligned}
\mathcal{H}om_{\mathsf{Mod}(\mathsf{C})}(\mathcal{M},\mathcal{N}) &\simeq \varprojlim_{(u,\gamma,m)\in\mathsf{C}^{\mathrm{op}}_{\mathcal{M}}} u_*\mathcal{N}(\gamma)\\
&\simeq \varprojlim_{(u,\gamma,m)\in\mathsf{C}^{\mathrm{op}}_{\mathcal{M}}} u_*\mathcal{H}om_{\mathsf{Mod}(\mathsf{C}|_U)}(\mathsf{Y}_{\mathsf{C}}(\gamma),\mathcal{N}|_U)\\
&\simeq \varprojlim_{(u,\gamma,m)\in\mathsf{C}^{\mathrm{op}}_{\mathcal{M}}} \mathcal{H}om_{\mathsf{Mod}(\mathsf{C})}(u_!\mathsf{Y}_{\mathsf{C}}(\gamma),\mathcal{N})\\
&\simeq \mathcal{H}om_{\mathsf{Mod}(\mathsf{C})}(\varinjlim_{(u,\gamma,m)\in\mathsf{C}^{\mathrm{op}}_{\mathcal{M}}} u_!\mathsf{Y}_{\mathsf{C}}(\gamma),\mathcal{N}).
\end{aligned}$$

Here, the first isomorphism follows from Lemma 3.1.4, the second isomorphism follows from (3.1.2), and the third isomorphism follows from Lemma 3.1.1. $\square$

Lemma 3.1.6. *For* C *an* $\mathcal{R}$*-stack, there is a natural isomorphism of* $\mathcal{R}$*-algebras* $Z(\mathsf{Mod}(\mathsf{C})) \simeq Z(\mathsf{C})$*.*

Proof. The Yoneda embedding induces by adjunction a morphism of $\mathcal{R}$-algebras $Z(\mathsf{Mod}(\mathsf{C})) \to Z(\mathsf{C}^{\mathrm{op}})$. Its inverse associates to $\mathsf{c} \in Z(\mathsf{C}^{\mathrm{op}})$ the endo-transformation $\tilde{\mathsf{c}}$ of $\mathrm{id}_{\mathsf{Mod}(\mathsf{C})}$ given by $\tilde{\mathsf{c}}(\mathcal{M}) = \varinjlim_{(u,\gamma,m)\in\mathsf{C}^{\mathrm{op}}_{\mathcal{M}}} \mathsf{Y}_{\mathsf{C}}(\mathsf{c}(\gamma))$. $\square$

3.2. **Operations via Kan extensions.** For non linear categories, the following result is known as Kan extension (see for example [27, pag. 106] or [20, Prop. 2.7.1]).

Theorem 3.2.1. *For* $\mathcal{N} \in \mathsf{Mod}(\mathsf{C})$ *consider the diagram*

$$\begin{tikzcd}
\mathsf{Mod}(\mathsf{C}^{\mathrm{op}}) \arrow[dr, dotted, "\mathsf{t}_{\mathcal{N}}"'] & \mathsf{C} \arrow[l, hook', "\mathsf{Y}_{\mathsf{C}^{\mathrm{op}}}"'] \arrow[r, hook, "\mathsf{Y}_{\mathsf{C}}^{\mathrm{op}}"] \arrow[d, "\mathcal{N}"'] & \mathsf{Mod}(\mathsf{C})^{\mathrm{op}} \arrow[dl, dotted, "\mathsf{h}_{\mathcal{N}}"] \\
& \mathsf{Mod}(\mathcal{R}). &
\end{tikzcd}$$

(i) *There exists a unique* $\mathcal{R}$*-functor* $\mathsf{t}_{\mathcal{N}}$ *(up to unique isomorphism) commuting with colimits and zero-extensions, and making the left hand side of the diagram (quasi)-commute.*

(ii) *The functor* $\mathsf{h}_{\mathcal{N}} = \mathcal{H}om_{\mathsf{Mod}(\mathsf{C})}(\cdot, \mathcal{N})$ *is the only* $\mathcal{R}$*-functor (up to unique isomorphism) commuting with limits and making the right hand side of the diagram (quasi)-commute.*

Sketch of proof. (i) For $\mathcal{P} \in \mathsf{Mod}(\mathsf{C}^{\mathrm{op}})$ one has

$$\begin{aligned}\mathsf{t}_{\mathcal{N}}(\mathcal{P}) &\simeq \mathsf{t}_{\mathcal{N}}(\varinjlim_{(u,\gamma,p)\in \mathsf{C}_{\mathcal{P}}} u_! \mathsf{Y}_{\mathsf{C}^{\mathrm{op}}}(\gamma)) \\ &\simeq \varinjlim_{(u,\gamma,p)\in \mathsf{C}_{\mathcal{P}}} u_! \mathsf{t}_{\mathcal{N}}(\mathsf{Y}_{\mathsf{C}^{\mathrm{op}}}(\gamma)) \simeq \varinjlim_{(u,\gamma,p)\in \mathsf{C}_{\mathcal{P}}} u_! \mathcal{N}(\gamma).\end{aligned}$$

(ii) Similarly, for $\mathcal{M} \in \mathsf{Mod}(\mathsf{C})$ one has

$$\begin{aligned}\mathsf{h}_{\mathcal{N}}(\mathcal{M}) &\simeq \mathsf{h}_{\mathcal{N}}(\varinjlim_{(u,\gamma,m)\in \mathsf{C}_{\mathcal{M}}^{\mathrm{op}}} u_! \mathsf{Y}_{\mathsf{C}}(\gamma)) \\ &\simeq \varprojlim_{(u,\gamma,m)\in \mathsf{C}_{\mathcal{M}}^{\mathrm{op}}} u_* \mathsf{h}_{\mathcal{N}}(\mathsf{Y}_{\mathsf{C}}(\gamma)) \simeq \varprojlim_{(u,\gamma,m)\in \mathsf{C}_{\mathcal{M}}^{\mathrm{op}}} v_* \mathcal{N}(\gamma),\end{aligned}$$

so that $\mathsf{h}_{\mathcal{N}} = \mathcal{H}om_{\mathsf{Mod}(\mathsf{C})}(\cdot, \mathcal{N})$ by Lemma 3.1.4. □

As for modules over a ring, we will often use the short hand notation

$$\mathcal{H}om_{\mathsf{C}}(\cdot, \cdot) = \mathcal{H}om_{\mathsf{Mod}(\mathsf{C})}(\cdot, \cdot).$$

Notation 3.2.2. We denote by

$$\begin{aligned}\mathcal{H}om_{\mathsf{C}} &\colon \mathsf{Mod}(\mathsf{C} \otimes_{\mathcal{R}} \mathsf{D}^{\mathrm{op}})^{\mathrm{op}} \otimes_{\mathcal{R}} \mathsf{Mod}(\mathsf{C} \otimes_{\mathcal{R}} \mathsf{E}) \to \mathsf{Mod}(\mathsf{D} \otimes_{\mathcal{R}} \mathsf{E}), \\ \otimes_{\mathsf{C}} &\colon \mathsf{Mod}(\mathsf{C}^{\mathrm{op}} \otimes_{\mathcal{R}} \mathsf{D}) \otimes_{\mathcal{R}} \mathsf{Mod}(\mathsf{C} \otimes_{\mathcal{R}} \mathsf{E}) \to \mathsf{Mod}(\mathsf{D} \otimes_{\mathcal{R}} \mathsf{E})\end{aligned}$$

the $\mathcal{R}$-functors obtained by picking up operators from the $\mathcal{R}$-functors

$$\begin{aligned}\mathcal{H}om_{\mathsf{C}} &\colon \mathsf{Mod}(\mathsf{C})^{\mathrm{op}} \otimes_{\mathcal{R}} \mathsf{Mod}(\mathsf{C}) \to \mathsf{Mod}(\mathcal{R}), \\ \mathsf{t} &\colon \mathsf{Mod}(\mathsf{C}^{\mathrm{op}}) \otimes_{\mathcal{R}} \mathsf{Mod}(\mathsf{C}) \to \mathsf{Mod}(\mathcal{R}), \quad (\mathcal{P}, \mathcal{N}) \mapsto \mathsf{t}_{\mathcal{N}}(\mathcal{P}).\end{aligned}$$

For $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ three $\mathcal{R}$-algebra, and $\mathsf{C} = \mathcal{A}^+$, $\mathsf{D} = \mathcal{B}^+$, $\mathsf{E} = \mathcal{C}^+$, the functor $\otimes_{\mathcal{A}^+}$ is isomorphic to the usual tensor product of modules $\otimes_{\mathcal{A}}$. For example, for $\mathcal{N} \in \mathsf{Mod}(\mathcal{A})$ and $\mathcal{P} \in \mathsf{Mod}(\mathcal{A}^{\mathrm{op}})$, the isomorphism

$$\mathcal{P} \otimes_{\mathcal{A}} \mathcal{N} = \mathsf{t}_{\mathcal{N}}(\mathcal{P}) \simeq \varinjlim_{u\colon U\subset X,\ p\in\mathcal{P}(U)} \mathcal{N},$$

amounts to present $\mathcal{P} \otimes_{\mathcal{A}} \mathcal{N}$ as a quotient of $\bigoplus_{u\colon U\subset X,\ p\in\mathcal{P}(U)} u_!(\mathcal{N}|_U)$.

Most of the formulas concerning the usual hom-functor and tensor product hold. For example,

Lemma 3.2.3. *For* $\mathcal{M} \in \mathsf{Mod}(\mathsf{C}^{\mathrm{op}} \otimes_{\mathcal{R}} \mathsf{D})$, $\mathcal{N} \in \mathsf{Mod}(\mathsf{C} \otimes_{\mathcal{R}} \mathsf{E})$, *and* $\mathcal{P} \in \mathsf{Mod}(\mathsf{D} \otimes_{\mathcal{R}} \mathsf{F})$, *there is an isomorphism in* $\mathsf{Mod}(\mathsf{E}^{\mathrm{op}} \otimes_{\mathcal{R}} \mathsf{F})$

$$\mathcal{H}om_{\mathsf{D}}(\mathcal{M} \otimes_{\mathsf{C}} \mathcal{N}, \mathcal{P}) \simeq \mathcal{H}om_{\mathsf{C}}(\mathcal{N}, \mathcal{H}om_{\mathsf{D}}(\mathcal{M}, \mathcal{P})).$$

Proof. One checks that in $\mathsf{Mod}(\mathsf{E} \otimes_{\mathcal{R}} \mathsf{D})$
$$\mathcal{M} \otimes_{\mathsf{C}} \mathcal{N} \simeq \varinjlim_{(u,\mathsf{f},n)\in \mathsf{Fct}_{\mathcal{R}}(\mathsf{E},\mathsf{C}^{\mathrm{op}})_{\mathcal{N}}} u_!\mathcal{M} \circ \mathsf{f},$$
where the comma category is defined via the functor $\mathsf{Fct}_{\mathcal{R}}(\mathsf{E}, \mathsf{Y}_{\mathsf{C}})$. Then, both terms in the statement are isomorphic to
$$\varprojlim_{(u,\mathsf{f},n)\in \mathsf{Fct}_{\mathcal{R}}(\mathsf{E},\mathsf{C}^{\mathrm{op}})_{\mathcal{N}}} u_*\mathcal{H}om_{\mathsf{D}}(\mathcal{M} \circ \mathsf{f}, \mathcal{P}|_U).$$
□

3.3. **Morita equivalence.** Let us discuss how classical Morita theory extends to linear stacks.

Lemma 3.3.1. *An $\mathcal{R}$-functor $\mathsf{h}\colon \mathsf{Mod}(\mathsf{C}) \to \mathsf{Mod}(\mathsf{D})$ commutes with colimits and zero-extensions (resp. limits and extensions) if and only if it admits a right (resp. left) adjoint.*

Proof. Assume that h commutes with colimits and zero-extensions. Set $\mathcal{P} = \mathsf{h} \circ \mathsf{Y}_{\mathsf{C}} \in \mathsf{Mod}(\mathsf{C}^{\mathrm{op}} \otimes_{\mathcal{R}} \mathsf{D})$. For $\mathcal{M} \in \mathsf{Mod}(\mathsf{C})$ one has
$$\mathsf{h}(\mathcal{M}) \simeq \mathsf{h}\big(\varinjlim_{(u,\gamma,m)\in \mathsf{C}^{\mathrm{op}}_{\mathcal{M}}} u_!\mathsf{Y}_{\mathsf{C}}(\gamma)\big) \simeq \varinjlim_{(u,\gamma,m)\in \mathsf{C}^{\mathrm{op}}_{\mathcal{M}}} u_!\mathcal{P}(\gamma) \simeq \mathcal{P} \otimes_{\mathsf{C}} \mathcal{M}.$$
Hence $\mathsf{h} \simeq \mathcal{P}\otimes_{\mathsf{C}}(\cdot)$ admits $\mathcal{H}om_{\mathsf{D}}(\mathcal{P}, \cdot)$ as a right adjoint by Lemma 3.2.3. The converse implication is obvious, and the dual statement is similar.
□

Denote by $\mathsf{Fct}^r_{\mathcal{R}}(\mathsf{Mod}(\mathsf{C}), \mathsf{Mod}(\mathsf{D}))$ the stack of $\mathcal{R}$-functors admitting a right adjoint.

Theorem 3.3.2. (i) *The functor*
$$\mathsf{Mod}(\mathsf{C}^{\mathrm{op}} \otimes_{\mathcal{R}} \mathsf{D}) \to \mathsf{Fct}^r_{\mathcal{R}}(\mathsf{Mod}(\mathsf{C}), \mathsf{Mod}(\mathsf{D})), \quad \mathcal{P} \mapsto \mathcal{P} \otimes_{\mathsf{C}} (\cdot),$$
is an $\mathcal{R}$-equivalence.

(ii) *For $\mathcal{P} \in \mathsf{Mod}(\mathsf{C}^{\mathrm{op}} \otimes_{\mathcal{R}} \mathsf{D})$ and $\mathcal{Q} \in \mathsf{Mod}(\mathsf{D}^{\mathrm{op}} \otimes_{\mathcal{R}} \mathsf{E})$ one has*
$$(\mathcal{Q} \otimes_{\mathsf{D}} \mathcal{P}) \otimes_{\mathsf{C}} (\cdot) \simeq (\mathcal{Q} \otimes_{\mathsf{D}} (\cdot)) \circ (\mathcal{P} \otimes_{\mathsf{C}} (\cdot)).$$

Proof. (i) By uniqueness of the Kan extension, a quasi-inverse is given by $\mathsf{h} \mapsto \mathsf{h} \circ \mathsf{Y}_{\mathsf{C}}$.

(ii) also follows from uniqueness of Kan extension. □

Remark 3.3.3. Denoting by $\mathsf{Fct}^l_{\mathcal{R}}(\mathsf{Mod}(\mathsf{C}), \mathsf{Mod}(\mathsf{D}))$ the stack of $\mathcal{R}$-functors admitting a left adjoint, one similarly gets an $\mathcal{R}$-equivalence
$$\mathsf{Mod}(\mathsf{C}^{\mathrm{op}} \otimes_{\mathcal{R}} \mathsf{D}) \to \mathsf{Fct}^l_{\mathcal{R}}(\mathsf{Mod}(\mathsf{D}), \mathsf{Mod}(\mathsf{C}))^{\mathrm{op}}, \quad \mathcal{P} \mapsto \mathcal{H}om_{\mathsf{D}}(\mathcal{P}, \cdot),$$
and the corresponding commutative diagram as in Theorem 3.3.2 (ii). These constructions are interchanged by the $\mathcal{R}$-equivalence
$$\mathsf{Fct}^r_{\mathcal{R}}(\mathsf{Mod}(\mathsf{C}), \mathsf{Mod}(\mathsf{D})) \approx_{\mathcal{R}} \mathsf{Fct}^l_{\mathcal{R}}(\mathsf{Mod}(\mathsf{D}), \mathsf{Mod}(\mathsf{C}))^{\mathrm{op}}$$
sending a functor to its adjoint.

We use the notation

$$\mathsf{C} \in \mathsf{Mod}(\mathsf{C}^{\mathrm{op}} \otimes_{\mathcal{R}} \mathsf{C}) \tag{3.3.1}$$

for the canonical object $\mathcal{H}om_{\mathsf{C}}(\cdot,\cdot)$. This corresponds to the Yoneda embedding Y_{C} via the equivalence induced by Lemma 2.1.1

$$\mathsf{Fct}_{\mathcal{R}}(\mathsf{C}^{\mathrm{op}} \otimes_{\mathcal{R}} \mathsf{C}, \mathsf{Mod}(\mathcal{R})) \approx_{\mathcal{R}} \mathsf{Fct}_{\mathcal{R}}(\mathsf{C}^{\mathrm{op}}, \mathsf{Mod}(\mathsf{C})).$$

If $\mathsf{C} = \mathcal{A}^+$, the object $\mathcal{A}^+ \in \mathsf{Mod}(\mathcal{A}^{\mathrm{op}} \otimes_{\mathcal{R}} \mathcal{A})$ coincides with $\mathcal{A}$, considered as a bimodule over itself. If C is an invertible $\mathcal{R}$-algebroid, then $\mathsf{C}^{\mathrm{op}} \otimes_{\mathcal{R}} \mathsf{C} \approx_{\mathcal{R}} \mathcal{R}^+$ and C is isomorphic to $\mathcal{R}$ as a bimodule over itself.

Note that, by Lemma 3.1.3 the functor $\mathcal{H}om_{\mathsf{C}}(\mathsf{C},\cdot)$, and hence $\mathsf{C} \otimes_{\mathsf{C}} (\cdot)$, is isomorphic to the identity.

Definition 3.3.4. (i) One says that $\mathcal{Q} \in \mathsf{Mod}(\mathsf{D}^{\mathrm{op}} \otimes_{\mathcal{R}} \mathsf{C})$ is an inverse of $\mathcal{P} \in \mathsf{Mod}(\mathsf{C}^{\mathrm{op}} \otimes_{\mathcal{R}} \mathsf{D})$ if there are isomorphisms of $\mathsf{C} \otimes_{\mathcal{R}} \mathsf{C}^{\mathrm{op}}$- and $\mathsf{D} \otimes_{\mathcal{R}} \mathsf{D}^{\mathrm{op}}$-modules, respectively,

$$\mathcal{Q} \otimes_{\mathsf{D}} \mathcal{P} \simeq \mathsf{C}, \quad \mathcal{P} \otimes_{\mathsf{C}} \mathcal{Q} \simeq \mathsf{D}.$$

(ii) An object $\mathcal{P} \in \mathsf{Mod}(\mathsf{C}^{\mathrm{op}} \otimes_{\mathcal{R}} \mathsf{D})$ is called invertible if it has an inverse.

One proves (see e.g. [20, §19.5]) that $\mathcal{P}$ is invertible if and only if one of the following equivalent conditions is satisfied

(i) $\mathcal{H}om_{\mathsf{C}^{\mathrm{op}}}(\mathcal{P}, \mathsf{C})$ is an inverse of $\mathcal{P}$;
(ii) the functor $\mathcal{P} \otimes_{\mathsf{C}} (\cdot)\colon \mathsf{Mod}(\mathsf{C}) \to \mathsf{Mod}(\mathsf{D})$ is an $\mathcal{R}$-equivalence.
(iii) the functor $\mathcal{H}om_{\mathsf{C}^{\mathrm{op}}}(\mathcal{P}, \cdot)\colon \mathsf{Mod}(\mathsf{C}^{\mathrm{op}}) \to \mathsf{Mod}(\mathsf{D}^{\mathrm{op}})$ is an $\mathcal{R}$-equivalence.

For any $\mathcal{R}$-functor $\mathsf{f}\colon \mathsf{C} \to \mathsf{C}'$, denote by $\mathsf{End}_{\mathsf{C}}(\mathsf{f})$ the $\mathcal{R}$-stack associated to the separated prestack whose objects on $U \subset X$ are those of $\mathsf{C}(U)$ and $\mathrm{Hom}(\gamma, \gamma') = \mathrm{Hom}_{\mathsf{C}'(U)}(\mathsf{f}(\gamma), \mathsf{f}(\gamma'))$. Then, considering $\mathsf{C} \in \mathsf{Mod}(\mathsf{C}^{\mathrm{op}} \otimes_{\mathcal{R}} \mathsf{C})$ as a functor $\mathsf{C}^{\mathrm{op}} \to \mathsf{Mod}(\mathsf{C})$, one has $\mathsf{C} \approx_{\mathcal{R}} \mathsf{End}_{\mathsf{C}^{\mathrm{op}}}(\mathsf{C})$ by (3.1.1). Moreover, considering $\mathcal{P} \in \mathsf{Mod}(\mathsf{C}^{\mathrm{op}} \otimes_{\mathcal{R}} \mathsf{D})$ as a functor $\mathsf{C}^{\mathrm{op}} \to \mathsf{Mod}(\mathsf{D})$, the condition of $\mathcal{P}$ being invertible is further equivalent to

(iv) $\mathcal{P}$ is a faithfully flat[4] C^{op}-module locally of finite presentation[5] and $\mathsf{D} \approx_{\mathcal{R}} \mathsf{End}_{\mathsf{C}^{\mathrm{op}}}(\mathcal{P})$;
(v) $\mathcal{P}$ is C^{op}-progenerator[6] locally of finite type and $\mathsf{D} \approx_{\mathcal{R}} \mathsf{End}_{\mathsf{C}^{\mathrm{op}}}(\mathcal{P})$.

By reversing the role of C and D, one gets dual equivalent conditions.

Given an $\mathcal{R}$-functor $\mathsf{h}\colon \mathsf{Mod}(\mathsf{C}) \to \mathsf{Mod}(\mathsf{D})$, we will use the same notation h for the induced $\mathcal{R}$-functor, obtained by picking up operators,

$$\mathsf{Mod}(\mathsf{C}^{\mathrm{op}} \otimes_{\mathcal{R}} \mathsf{C}) \to \mathsf{Mod}(\mathsf{C}^{\mathrm{op}} \otimes_{\mathcal{R}} \mathsf{D}).$$

[4] $\mathcal{P}$ is a faithfully flat C^{op}-module if the functor $\mathcal{P} \otimes_{\mathsf{C}} (\cdot)$ is faithful and exact.

[5] $\mathcal{P}$ is a C^{op}-module of finite presentation if the functor $\mathcal{H}om_{\mathsf{C}^{\mathrm{op}}}(\mathcal{P}, \cdot)$ commutes with small filtrant colimits.

[6] $\mathcal{P}$ is C^{op}-progenerator if the functor $\mathcal{H}om_{\mathsf{C}^{\mathrm{op}}}(\mathcal{P}, \cdot)$ is faithful and exact.

Corollary 3.3.5 (Morita)**.** *An $\mathcal{R}$-functor* $\mathsf{h}\colon \mathsf{Mod}(\mathsf{C}) \to \mathsf{Mod}(\mathsf{D})$ *is an equivalence if and only if* $\mathcal{P} = \mathsf{h}(\mathsf{C})$ *is an invertible* $(\mathsf{C}^{\mathrm{op}} \otimes_{\mathcal{R}} \mathsf{D})$*-module. Moreover, one has* $\mathsf{h} \simeq \mathcal{P} \otimes_{\mathsf{C}} (\cdot)$*.*

Definition 3.3.6. Two stacks C and D are Morita $\mathcal{R}$-equivalent if their stacks of modules $\mathsf{Mod}(\mathsf{C})$ and $\mathsf{Mod}(\mathsf{D})$ are $\mathcal{R}$-equivalent.

Hence C and D are Morita $\mathcal{R}$-equivalent if and only if there exists an invertible $(\mathsf{C}^{\mathrm{op}} \otimes_{\mathcal{R}} \mathsf{D})$-module.

Let us say that $\mathcal{P} \in \mathsf{Mod}(\mathsf{C}^{\mathrm{op}} \otimes_{\mathcal{R}} \mathsf{D})$ is locally free of rank one over C^{op} if for any $\delta \in \mathsf{D}$ the C^{op}-module $\mathcal{P}(\delta)$ is locally free of rank one, that is to say, the functor $\mathcal{P}(\delta)\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Mod}(\mathcal{R})$ is representable.

Recall from (2.2.2) that ${}_{\mathsf{f}}(\cdot)\colon \mathsf{Mod}(\mathsf{C}) \to \mathsf{Mod}(\mathsf{D})$ denotes the functor associated to an $\mathcal{R}$-functor $\mathsf{f}\colon \mathsf{D} \to \mathsf{C}$.

Proposition 3.3.7. *The $\mathcal{R}$-functor*

$$\mathsf{Fct}_{\mathcal{R}}(\mathsf{D}, \mathsf{C}) \to \mathsf{Mod}(\mathsf{C}^{\mathrm{op}} \otimes_{\mathcal{R}} \mathsf{D}), \quad \mathsf{f} \mapsto {}_{\mathsf{f}}\mathsf{C} \tag{3.3.2}$$

is fully faithful and induces an equivalence with the full substack of locally free modules of rank one over C^{op}*.*

Proof. (i) The functor in the statement equals $\mathsf{Y}_{\mathsf{C}^{\mathrm{op}}} \circ \cdot$. This is fully faithful, since $\mathsf{Y}_{\mathsf{C}^{\mathrm{op}}}$ is fully faithful.

(ii) Assume that $\mathcal{P} \in \mathsf{Mod}(\mathsf{C}^{\mathrm{op}} \otimes_{\mathcal{R}} \mathsf{D})$ is a locally free module of rank one over C^{op}. Then $\mathcal{P} \simeq {}_{\mathsf{f}}\mathsf{C}$, where $\mathsf{f}\colon \mathsf{D} \to \mathsf{C}$ is the functor associating to $\delta \in \mathsf{D}$ the representative of $\mathcal{P}(\delta)$. ☐

Corollary 3.3.8. *Two stacks* C *and* D *are $\mathcal{R}$-equivalent if and only if there exists* $\mathcal{P} \in \mathsf{Mod}(\mathsf{C}^{\mathrm{op}} \otimes_{\mathcal{R}} \mathsf{D})$ *which is invertible and locally free of rank one over* C^{op}*.*

In particular, two algebroids A and B are $\mathcal{R}$-equivalent if and only if there exists an invertible $(\mathsf{A}^{\mathrm{op}} \otimes_{\mathcal{R}} \mathsf{B})$-module $\mathcal{P}$ which is locally free of rank one over A^{op}. These conditions on $\mathcal{P}$ are equivalent to the condition that $\mathcal{P}$ is bi-invertible in the sense of [21, Corollary 2.1.10].

Remark 3.3.9. If $\mathsf{C} \approx_{\mathcal{R}} \mathcal{A}^+$ and $\mathsf{D} \approx_{\mathcal{R}} \mathcal{B}^+$, the functor $\mathcal{B}^+ \to \mathcal{A}^+$ associated to an $\mathcal{A}^{\mathrm{op}} \otimes_{\mathcal{R}} \mathcal{B}$-module $\mathcal{P}$ locally free of rank one over $\mathcal{A}^{\mathrm{op}}$ is $\mathsf{f} = (\cdot) \otimes_{\mathcal{B}} \mathcal{P}$. Note that any local isomorphism $h\colon \mathcal{A} \xrightarrow{\sim} \mathcal{P}$ of right $\mathcal{A}$-modules defines a local $\mathcal{R}$-algebra morphism (isomorphism if $\mathcal{P}$ invertible)

$$f\colon \mathcal{B} \to \mathcal{E}nd_{\mathcal{A}^{\mathrm{op}}}(\mathcal{P}) \xrightarrow{\mathrm{Ad}(h^{-1})} \mathcal{E}nd_{\mathcal{A}^{\mathrm{op}}}(\mathcal{A}) \simeq \mathcal{A}, \tag{3.3.3}$$

(the first arrow is induced by the $\mathcal{B}$-module structure of $\mathcal{P}$), for which $h\colon {}_f\mathcal{A} \xrightarrow{\sim} \mathcal{P}$ is an isomorphism of $\mathcal{A}^{\mathrm{op}} \otimes_{\mathcal{R}} \mathcal{B}$-modules and $\mathsf{f} \simeq f^+$. If h is given by $a \mapsto ua$ for a local generator u of the right $\mathcal{A}$-modules $\mathcal{P}$, then $f(b) = a$ for a such that $ua = bu$.

3.4. **Picard good stacks.** We will use the notation

$$\mathsf{C}^e = \mathsf{C}^{\mathrm{op}} \otimes_{\mathcal{R}} \mathsf{C}.$$

Denote by $\mathsf{Inv}(\mathsf{C}^e)$ the substack of $\mathsf{Mod}(\mathsf{C}^e)$ whose objects are invertible C^e-modules and whose morphisms are only those morphisms which are invertible. Then $\otimes_{\mathsf{C}}$ induces on $\mathsf{Inv}(\mathsf{C}^e)$ a natural structure of stack of 2-groups, and (3.3.2) gives a fully faithful functor of stacks of 2-groups

$$\mathsf{Aut}_{\mathcal{R}}(\mathsf{C})_{\mathrm{op}} \hookrightarrow \mathsf{Inv}(\mathsf{C}^e), \quad \mathsf{f} \mapsto {}_{\mathsf{f}}\mathsf{C}. \tag{3.4.1}$$

Here, for G a stack of 2-groups, G_{op} denotes the stack of 2-groups with the same groupoid structure as G and with reversed monoidal structure.

Definition 3.4.1. An $\mathcal{R}$-stack C is Picard good if (3.4.1) is an equivalence.

By Proposition 3.3.7, it follows that C is Picard good if and only if all invertible C^e-modules are locally free of rank one over C^{op}.

An $\mathcal{R}$-algebra $\mathcal{A}$ is Picard good if it is so as an $\mathcal{R}$-stack, hence if and only if all invertible $\mathcal{A}^e$-modules are locally free as right (or, equivalently, left) $\mathcal{A}$-modules. Since invertible bimodules are projective as right (or left) modules, it follows that examples of Picard good rings are projective-free rings, and in particular local rings. Note however that Picard-good does not imply projective-free (see Remark 4.3.3).

Since the condition of being Picard good is local, an algebroid is Picard good if and only if so are the algebras that locally represent it.

By Corollary 3.3.5, there is an equivalence of stacks of 2-groups

$$\mathsf{Inv}(\mathsf{C}^e) \xrightarrow{\approx} \mathsf{Aut}_{\mathcal{R}}(\mathsf{Mod}(\mathsf{C})), \quad \mathcal{P} \mapsto \mathcal{P} \otimes_{\mathsf{C}} (\cdot).$$

We thus have a (quasi-)commutative diagram

$$\begin{tikzcd} \mathsf{Inv}(\mathsf{C}^e) \arrow[rr, "\approx"] & & \mathsf{Aut}_{\mathcal{R}}(\mathsf{Mod}(\mathsf{C})) \\ & \mathsf{Aut}_{\mathcal{R}}(\mathsf{C})_{\mathrm{op}}, \arrow[ul, hook] \arrow[ur, hook, "m"'] & \end{tikzcd} \tag{3.4.2}$$

where m is induced by the functor $\mathsf{Mod}(\cdot)$. It follows that C is Picard good if and only if m is an equivalence.

Proposition 3.4.2. *Let C be a Picard good $\mathcal{R}$-stack.*

(i) *Let D be an $\mathcal{R}$-stack locally equivalent to C. Then C and D are Morita $\mathcal{R}$-equivalent if and only if they are $\mathcal{R}$-equivalent.*

(ii) *Let M be an $\mathcal{R}$-stack locally $\mathcal{R}$-equivalent to $\mathsf{Mod}(\mathsf{C})$. Then $\mathsf{M} \approx_{\mathcal{R}} \mathsf{Mod}(\mathsf{D})$ for an $\mathcal{R}$-stack D locally $\mathcal{R}$-equivalent to C.*

Proof. (i) Let $\mathsf{Equiv}_{\mathcal{R}}(\cdot,\cdot)$ denote the stack of $\mathcal{R}$-equivalences, with invertible transformations as morphisms. Consider the functor

$$\mathsf{Equiv}_{\mathcal{R}}(\mathsf{C},\mathsf{D}) \to \mathsf{Equiv}_{\mathcal{R}}(\mathsf{Mod}(\mathsf{D}),\mathsf{Mod}(\mathsf{C}))$$

induced by the 2-functor $\mathsf{Mod}(\cdot)$. Since D is locally equivalent to C, this locally reduces to the functor m as in (3.4.2). It follows that this is locally, hence globally, an equivalence.

(ii) Let $\mathsf{E} \subset \mathsf{M}$ be the full substack of objects $\mathcal{P}$ with the property that for any local $\mathcal{R}$-equivalence $\mathsf{h}\colon \mathsf{M} \xrightarrow{\approx} \mathsf{Mod}(\mathsf{C})$, the C-module $\mathsf{h}(\mathcal{P})$ is locally free of rank one. Since C is Picard good, the $\mathcal{R}$-stack E is well defined and locally $\mathcal{R}$-equivalent to C^{op}. Set $\mathsf{D} = \mathsf{E}^{\mathrm{op}}$. Then the $\mathcal{R}$-functor

$$\mathsf{M} \to \mathsf{Mod}(\mathsf{D}), \quad \mathcal{N} \mapsto \mathcal{H}om_{\mathsf{M}}(\cdot, \mathcal{N})$$

is locally, hence globally, an equivalence. $\square$

If C is an invertible $\mathcal{R}$-algebroid, then it is Picard good if and only if $\mathcal{R}$ is, and one has equivalences of stacks of 2-groups

$$\mathcal{R}^{\times}[1] \xrightarrow{\approx} \mathsf{Inv}(\mathcal{R}) \approx \mathsf{Inv}(\mathsf{C}^{e}), \quad \mathcal{P} \mapsto \mathcal{R} \times_{\mathcal{R}^{\times}} \mathcal{P}. \tag{3.4.3}$$

(Recall that $\mathcal{R}^{\times}[1]$ denotes the stack of $\mathcal{R}^{\times}$-torsors.) Moreover, in this situation the stack D in (ii) above is $\mathcal{R}$-equivalent to the full substack of $\mathsf{Fct}_{\mathcal{R}}(\mathsf{M}, \mathsf{Mod}(\mathcal{R}))$ whose objects are equivalences.

Examples of stacks as in Proposition 3.4.2 (ii) arise from deformations of categories of modules as discussed in [26]. In particular, Proposition 3.4.2 applies when C is (equivalent to) the structure sheaf of a ringed space. We thus recover results of [25].

4. Microdifferential operators

We collect here some results from the theory of microdifferential operators of [35] (see also [15, 17]). The statements about the automorphisms of the sheaf of microdifferential operators are well known. Since we lack a reference for the proofs, we give them here.

4.1. **Microdifferential operators.** Let M be an n-dimensional complex manifold, T^*M its cotangent bundle and $\dot{T}^*M \subset T^*M$ the open subset obtained by removing the zero-section.

Denote by $\mathcal{E}_{\dot{T}^*M}$ the sheaf of microdifferential operators on $\dot{T}^*M$ (see [35, 17]). Recall that $\mathcal{E}_{\dot{T}^*M}$ is a sheaf of central $\mathbb{C}$-algebras endowed with a $\mathbb{Z}$-filtration by the order of the operators, and one has

$$\mathcal{G}r\, \mathcal{E}_{\dot{T}^*M} \simeq \bigoplus_{m \in \mathbb{Z}} \mathcal{O}_{\dot{T}^*M}(m),$$

where $\mathcal{O}_{\dot{T}^*M}(m)$ is the subsheaf of $\mathcal{O}_{\dot{T}^*M}$ of holomorphic functions homogeneous of degree m.

For $\lambda \in \mathbb{C}$, denote by $\mathcal{E}_{\dot{T}^*M}(\lambda)$ the sheaf of microdifferential operators of order at most λ, and set

$$\mathcal{E}^{[\lambda]}_{\dot{T}^*M} = \bigcup_{n \in \mathbb{Z}} \mathcal{E}_{\dot{T}^*M}(\lambda + n),$$

where $[\lambda]$ is the class of λ in $\mathbb{C}/\mathbb{Z}$. Note that $\mathcal{E}^{[\lambda]}_{\dot{T}^*M}$ is a bimodule over $\mathcal{E}_{\dot{T}^*M} = \mathcal{E}^{[0]}_{\dot{T}^*M}$.

In a local coordinate system (x) on M, with associated symplectic coordinates $(x;\xi)$ on $\dot{T}^*M$, a section $P \in \Gamma(V;\mathcal{E}_{\dot{T}^*M}(\lambda))$ is determined by its total symbol, which is a formal series

$$\operatorname{tot}(P) = \sum_{j=0}^{+\infty} p_{\lambda-j}(x,\xi)$$

with $p_{\lambda-j} \in \mathcal{O}_{\dot{T}^*M}(V)$ homogeneous of degree $\lambda - j$, satisfying suitable growth conditions in j. If Q is a section of $\mathcal{E}_{\dot{T}^*M}(\mu)$, then $PQ \in \mathcal{E}_{\dot{T}^*M}(\lambda+\mu)$ has total symbol given by the Leibniz formula

$$\operatorname{tot}(PQ) = \sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!}\partial_\xi^\alpha \operatorname{tot}(P)\partial_x^\alpha \operatorname{tot}(Q).$$

Denote by

$$\sigma_\lambda\colon \mathcal{E}_{\dot{T}^*M}(\lambda) \to \mathcal{O}_{\dot{T}^*M}(\lambda) \quad \text{and} \quad \sigma\colon \mathcal{E}^{[\lambda]}_{\dot{T}^*M} \to \mathcal{O}_{\dot{T}^*M}$$

the symbol of order λ and the principal symbol, respectively, where $\sigma(P) = \sigma_\lambda(P)$ for $P \in \mathcal{E}_{\dot{T}^*M}(\lambda) \setminus \mathcal{E}_{\dot{T}^*M}(\lambda-1)$. Note that for any $P \in \mathcal{E}_{\dot{T}^*M}(\lambda)$ and $Q \in \mathcal{E}_{\dot{T}^*M}(\mu)$ one has

$$\sigma_{\lambda+\mu}(PQ) = \sigma_\lambda(P)\sigma_\mu(Q).$$

Recall that a microdifferential operator is invertible at $p \in \dot{T}^*M$ if and only if its principal symbol does not vanish at p.

4.2. **Endomorphisms of $\mathcal{E}_{\dot{T}^*M}$.**

Lemma 4.2.1. *Any $\mathbb{C}$-algebra automorphism of $\mathcal{E}_{\dot{T}^*M}$ is filtered and symbol preserving.*

Proof. Define the spectrum of $P \in \mathcal{E}_{\dot{T}^*M}(V)$ as the set-valued function

$$U \ni p \mapsto \{a \in \mathbb{C}\colon a - P \text{ is not invertible at } p\} \subset \mathbb{C}.$$

One checks that the spectrum of P is singleton-valued if and only if $P \in \mathcal{E}_{\dot{T}^*M}(0)$ and its symbol $\sigma_0(P)$ is not locally constant, and in this case the only value of the spectrum of P at p is $\sigma_0(P)(p)$.

Let f be a $\mathbb{C}$-algebra automorphism of $\mathcal{E}_{\dot{T}^*M}$. As the spectra of P and of $f(P)$ coincide, it follows that f induces a symbol preserving isomorphism on $\mathcal{E}_{\dot{T}^*M}(0) \setminus \sigma_0^{-1}(\mathbb{C}_{\dot{T}^*M})$.

Set for short

$$\mathcal{E}_m = \mathcal{E}_{\dot{T}^*M}(m) \setminus \mathcal{E}_{\dot{T}^*M}(m-1).$$

If $P \in \mathcal{E}_0$ has a locally constant symbol, for any $Q \in \mathcal{E}_{\dot{T}^*M}(0) \setminus \sigma_0^{-1}(\mathbb{C}_{\dot{T}^*M})$ one has

$$\sigma_0(P)\sigma_0(Q) = \sigma_0(PQ) = \sigma_0(f(PQ)) = \sigma(f(P))\sigma_0(f(Q)) = \sigma(f(P))\sigma_0(Q)$$

so that also $f(P)$ belongs to $\mathcal{E}_{\dot{T}^*M}(0)$ and has a locally constant symbol. Thus f induces a symbol preserving isomorphism on $\mathcal{E}_0$.

Pick an operator $D \in \mathcal{E}_1$ invertible at p, and let d be the degree of $f(D)$. Then $f(D)^m$ is an invertible operator of order dm and one has

$$f(\mathcal{E}_m) = f(D^m\mathcal{E}_0) = f(D)^m f(\mathcal{E}_0) = f(D)^m\mathcal{E}_0 = \mathcal{E}_{dm}.$$

Since f is an automorphism of $\mathcal{E}_{\dot{T}^*M} \setminus \{0\} = \bigsqcup_{m\in\mathbb{Z}} \mathcal{E}_m$, it follows that $d = \pm 1$. Thus f either preserves or reverses the order. Note that an operator P with $\sigma(P)(p) = 0$ has spectrum equal to $\mathbb{C}$ at p if and only if it has positive order. Hence f preserves the order.

We have proved that f is filtered and preserves the symbol of operators in $\mathcal{E}_0$. As $\mathcal{E}_m = D^m\mathcal{E}_0$, to show that f is symbol preserving it is enough to check that $\sigma_1(D) = \sigma_1(f(D))$.

Let $(x;\xi)$ be a local system of symplectic coordinates at p. Identifying x_i with the operator in $\mathcal{E}_0$ whose total symbol is x_i, one has

$$\begin{aligned}\partial_{\xi_i}\sigma_1(D) &= \{x_i, \sigma_1(D)\} = \{\sigma_0(x_i), \sigma_1(D)\} = \sigma_0([x_i, D]) \\ &= \sigma_0(f([x_i, D])) = \sigma_0([f(x_i), f(D)]) = \{\sigma_0(f(x_i)), \sigma_1(f(D))\} \\ &= \{x_i, \sigma_1(f(D))\} = \partial_{\xi_i}\sigma_1 f((D)), \quad \text{for } i = 1, \dots, n,\end{aligned}$$

so that

$$\sigma_1(D) = \sigma_1(f(D)) + \varphi(x),$$

and one takes the homogeneous component of degree 1. □

Proposition 4.2.2. *Any $\mathbb{C}$-algebra automorphism of $\mathcal{E}_{\dot{T}^*M}$ is locally of the form* $\mathrm{Ad}(P)$ *for some $\lambda \in \mathbb{C}$ and some invertible $P \in \mathcal{E}_{\dot{T}^*M}(\lambda)$.*

Proof. Identify $\dot{T}^*M\times\dot{T}^*M$ to an open subset of $T^*(M\times M)$. Let (x) be a system of local coordinates on M, and denote by (x, y) the coordinates on $M \times M$. For $Q \in \mathcal{E}_{\dot{T}^*M}$, denote by Q_x and Q_y its pull-back to $\mathcal{E}_{\dot{T}^*M\times\dot{T}^*M}$ by the first and second projection, respectively.

Let $f\colon \mathcal{E}_{\dot{T}^*M} \to \mathcal{E}_{\dot{T}^*M}$ be a $\mathbb{C}$-algebra automorphism. By Lemma 4.2.1, f is filtered and symbol preserving. Denote by $\mathcal{L}$ the $\mathcal{E}_{\dot{T}^*M\times\dot{T}^*M}$-module with one generator u and relations

$$\big(x_i - f(y_i)\big)\, u = \big(\partial_{x_i} - f(\partial_{y_i})\big)\, u = 0, \quad \text{for } i = 1, \dots, n.$$

Then the image $f(Q)$ of $Q \in \mathcal{E}_{\dot{T}^*M}$ is characterized by the relation

$$f(Q)_y\, u = Q^*_x\, u \quad \text{in } \mathcal{L}, \tag{4.2.1}$$

where Q^* denotes the adjoint operator, and $(\mathcal{L}, u)$ is a simple module along the conormal bundle of the diagonal Δ in $T^*(M \times M)$ (see [17]). Denote by $\mathcal{C}_\Delta$ the sheaf of complex microfunctions along the conormal bundle of Δ. By [17, Theorem 8.21], there exists $\lambda \in \mathbb{C}$ and an isomorphism

$$\varphi\colon \mathcal{E}^{[\lambda]}_{\dot{T}^*M\times\dot{T}^*M} \otimes_{\mathcal{E}_{\dot{T}^*M\times\dot{T}^*M}} \mathcal{C}_\Delta \xrightarrow{\sim} \mathcal{L},$$

so that $\varphi(P_y \otimes \delta_\Delta) = u$ for some invertible $P \in \mathcal{E}_{\dot{T}^*M}(\lambda)$. One then has

$$P_y Q_y P_y^{-1} u = P_y Q_y P_y^{-1} \varphi(P_y \otimes \delta_\Delta) = \varphi(P_y Q_y \otimes \delta_\Delta) = \varphi(Q_x^* P_x^* \otimes \delta_\Delta)$$
$$= Q_x^* \varphi(P_x^* \otimes \delta_\Delta) = Q_x^* \varphi(P_y \otimes \delta_\Delta) = Q_x^* u.$$

It follows by (4.2.1) that one has $f = \mathrm{Ad}(P)$. □

4.3. **Invertible $\mathcal{E}$-bimodules.** Denote by P^*M the projective cotangent bundle of M and by $\gamma\colon \dot{T}^*M \to P^*M$ the projection. Set

$$\mathcal{E}_{P^*M} = \gamma_* \mathcal{E}_{\dot{T}^*M}.$$

This is a sheaf of $\mathbb{C}$-algebras endowed with a $\mathbb{Z}$-filtration such that $\mathcal{G}r\, \mathcal{E}_{P^*M} \simeq \bigoplus_{m\in\mathbb{Z}} \mathcal{O}_{P^*M}(m)$, where one sets $\mathcal{O}_{P^*M}(m) = \gamma_* \mathcal{O}_{\dot{T}^*M}(m)$. Note that $\mathcal{E}_{\dot{T}^*M}$ is constant along the fibers of γ. Since these are connected, the adjunction morphism gives an isomorphism

$$\gamma^{-1}\mathcal{E}_{P^*M} \xrightarrow{\sim} \mathcal{E}_{\dot{T}^*M}.$$

Lemma 4.3.1. *Let $Z \subset \dot{T}^*M$ be a closed conic analytic subset. Then*

$$H^j \mathrm{R}\Gamma_Z \mathcal{E}_{\dot{T}^*M} = 0 \qquad \textit{for } j < \operatorname{codim}_{\dot{T}^*M} Z.$$

Proof. Setting $W = \gamma(Z)$, we have $\mathrm{R}\Gamma_Z \mathcal{E}_{\dot{T}^*M} \simeq \gamma^{-1}\mathrm{R}\Gamma_W \mathcal{E}_{P^*M}$. We thus have to show that $H^j \mathrm{R}\Gamma_W \mathcal{E}_{P^*M} = 0$ for $j < \operatorname{codim}_{P^*M} W$. Identify $\mathcal{E}_{P^*M}$ with the sheaf $\mathcal{C}_\Delta$ of complex microfunctions along the conormal bundle of the diagonal in $P^* = P^*(M \times M)$. By quantized contact transformations, $\mathcal{C}_\Delta$ can further be identified with the sheaf of complex microfunctions $\mathcal{C}_S$ along the conormal bundle to a hypersurface $S \subset P^*$. One has $\mathcal{C}_S \simeq \mathcal{O}_S \oplus H^1_{[S]}\mathcal{O}_{P^*} \simeq \mathcal{O}_S^{\oplus\mathbb{Z}}$. Hence $H^j \mathrm{R}\Gamma_W \mathcal{C}_S = 0$ for $j < \operatorname{codim}_S W$. □

Proposition 4.3.2. *Let $\mathcal{M}$ be a coherent torsion-free $\mathcal{E}_{\dot{T}^*M}$-module. Then $\mathcal{M}$ is locally free outside a closed conic analytic 2-codimensional subset.*

Proof. We will reduce to the analogue statement for $\mathcal{O}$-modules, which is well-known (see [23, Corollary 5.15]).

Set for short $\mathcal{E} = \mathcal{E}_{\dot{T}^*M}$, $\mathcal{E}(0) = \mathcal{E}_{\dot{T}^*M}(0)$ and $\mathcal{O}(0) = \mathcal{O}_{\dot{T}^*M}(0)$. A coherent $\mathcal{E}(0)$-submodule $\mathcal{L} \subset \mathcal{M}$ such that $\mathcal{E}\mathcal{L} = \mathcal{M}$ is called a lattice.

(a) $\mathcal{M}$ has a torsion-free lattice $\mathcal{L}$. In fact, let $\mathcal{F}$ be a lattice in $\mathcal{M}^* = \mathcal{H}om_{\mathcal{E}}(\mathcal{M}, \mathcal{E})$. Then $\mathcal{F}^* = \mathcal{H}om_{\mathcal{E}(0)}(\mathcal{F}, \mathcal{E}(0)) \subset \mathcal{H}om_{\mathcal{E}}(\mathcal{M}^*, \mathcal{E}) = \mathcal{M}^{**}$ and $\mathcal{E}\mathcal{F}^* = \mathcal{M}^{**}$, i.e. $\mathcal{F}^*$ is a lattice in $\mathcal{M}^{**}$. Then $\mathcal{L} = \mathcal{F}^* \cap \mathcal{M}$ is a lattice in $\mathcal{M}$. Since $\mathcal{F}^*$ is reflexive (that is, $\mathcal{F}^* \to (\mathcal{F}^*)^{**}$ is an isomorphism), $\mathcal{F}^*$ is torsion free, and so is its submodule $\mathcal{L}$.

(b) The coherent $\mathcal{O}(0)$-module $\overline{\mathcal{L}} = \mathcal{L}/\mathcal{L}(-1)$ is torsion-free. In fact, consider the exact sequence

$$0 \to \mathcal{E}(-1) \to \mathcal{E}(0) \xrightarrow{\sigma_0} \mathcal{O}(0) \to 0.$$

Then $\mathcal{O}(0) \otimes_{\mathcal{E}(0)} \mathcal{L} \simeq \overline{\mathcal{L}}$. Hence $(\overline{\mathcal{L}})^* = \mathcal{H}om_{\mathcal{O}(0)}(\overline{\mathcal{L}}, \mathcal{O}(0)) \simeq \mathcal{H}om_{\mathcal{O}(0)}(\mathcal{O}(0) \otimes_{\mathcal{E}(0)} \mathcal{L}, \mathcal{O}(0)) \simeq \mathcal{H}om_{\mathcal{E}(0)}(\mathcal{L}, \mathcal{O}(0))$. The exact sequence

$$0 \to \mathcal{H}om_{\mathcal{E}(0)}(\mathcal{L}, \mathcal{E}(-1)) \to \mathcal{H}om_{\mathcal{E}(0)}(\mathcal{L}, \mathcal{E}(0)) \to \mathcal{H}om_{\mathcal{E}(0)}(\mathcal{L}, \mathcal{O}(0))$$

thus reads

$$0 \to \mathcal{L}^*(-1) \to \mathcal{L}^* \to (\overline{\mathcal{L}})^*.$$

Hence $\overline{\mathcal{L}^*} \subset (\overline{\mathcal{L}})^*$. Then $\overline{\mathcal{L}} \subset \overline{\mathcal{L}^{**}} \subset (\overline{\mathcal{L}^*})^* \xrightarrow{\sim} (\overline{\mathcal{L}^*})^{***}$, so that $\overline{\mathcal{L}}$ is torsion-free.

(c) Since $\overline{\mathcal{L}}$ is torsion-free, it is locally free outside a closed conic analytic 2-codimensional subset S. Hence the same holds true for $\mathcal{L}$ by Nakayama lemma. Thus $\mathcal{M} = \mathcal{E}\mathcal{L}$ is also locally free outside S. $\square$

Remark 4.3.3. Since projective $\mathcal{E}_{\dot{T}^*M}$-modules are torsion-free, it follows that $\mathcal{E}_{\dot{T}^*M}$ is (coherent) projective-free if $\dim M = 1$. This is no more true if $\dim M > 1$.

Set

$$\mathcal{E}^e_{\dot{T}^*M} = \mathcal{E}^{\mathrm{op}}_{\dot{T}^*M} \otimes_{\mathbb{C}} \mathcal{E}_{\dot{T}^*M}.$$

Note that, for $[\lambda], [\mu] \in \mathbb{C}/\mathbb{Z}$ the morphism of $\mathcal{E}^e_{\dot{T}^*M}$-modules

$$\mathcal{E}^{[\lambda]}_{\dot{T}^*M} \otimes_{\mathcal{E}_{\dot{T}^*M}} \mathcal{E}^{[\mu]}_{\dot{T}^*M} \to \mathcal{E}^{[\lambda+\mu]}_{\dot{T}^*M}, \quad P \otimes Q \mapsto PQ$$

is an isomorphism. In particular, $\mathcal{E}^{[\lambda]}_{\dot{T}^*M}$ is an invertible $\mathcal{E}^e_{\dot{T}^*M}$-module. Moreover, if $P \in \mathcal{E}_{\dot{T}^*M}(\lambda)$ has non vanishing symbol on $V \subset \dot{T}^*M$, there is an isomorphism of $\mathcal{E}^e_V$-modules

$$(4.3.1) \qquad {}_{\mathrm{Ad}(P^{-1})}(\mathcal{E}_V) \xrightarrow{\sim} \mathcal{E}^{[\lambda]}_V, \quad Q \mapsto PQ.$$

Lemma 4.3.4. *For $[\lambda], [\mu] \in \mathbb{C}/\mathbb{Z}$, one has*

$$\mathcal{H}om_{\mathcal{E}^e_{\dot{T}^*M}}(\mathcal{E}^{[\lambda]}_{\dot{T}^*M}, \mathcal{E}^{[\mu]}_{\dot{T}^*M}) = \begin{cases} \mathbb{C}_{\dot{T}^*M} & \text{for } [\lambda] = [\mu], \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The problem is local and we take a system $(x) = (x_1, \dots, x_n)$ of local coordinates in $V \subset \dot{T}^*M$ such that ∂_1 is invertible in V. By (4.3.1)

$$\begin{aligned} \mathcal{H}om_{\mathcal{E}^e_V}(\mathcal{E}^{[\lambda]}_V, \mathcal{E}^{[\mu]}_V) &\simeq \mathcal{H}om_{\mathcal{E}^e_V}({}_{\mathrm{Ad}(\partial_1^{-\lambda})}(\mathcal{E}_V), {}_{\mathrm{Ad}(\partial_1^{-\mu})}(\mathcal{E}_V)) \\ &\simeq \{P \in \mathcal{E}_V \colon P\partial_1^{-\lambda} Q \partial_1^{\lambda} = \partial_1^{-\mu} Q \partial_1^{\mu} P,\ \forall Q \in \mathcal{E}_V\}. \end{aligned}$$

Assume that there exists $P \neq 0$ as above. Taking for Q the operators ∂_1, x_i and ∂_i, respectively, we deduce that $[P, \partial_1] = [P, x_i] = [P, \partial_i] = 0$ for $i = 2, \dots, n$. It follows that P only depends on ∂_1. Noting that $[\partial_1^\lambda, x_1] = \lambda \partial_1^{\lambda-1}$ and taking $Q = x_1$, we get

$$[x_1, P] = (\mu - \lambda) P \partial_1^{-1}.$$

Write $P = \sum_{j \le m} c_j \partial_1^j$ with $c_i \in \mathbb{C}$ and $c_m \neq 0$. Then the above equality gives $m = \mu - \lambda$ and $c_j = 0$ for $j < m$. $\square$

The following result was communicated to us by Masaki Kashiwara (refer to [22] for related results).

Theorem 4.3.5. *Any invertible $\mathcal{E}^e_{\dot{T}^*M}$-module is isomorphic to $L \otimes_{\mathbb{C}} \mathcal{E}^{[\lambda]}_{\dot{T}^*M}$, for some local system of rank one L and some locally constant $\mathbb{C}/\mathbb{Z}$-valued function $[\lambda]$.*

Proof. Set for short $\mathcal{E} = \mathcal{E}_{\dot{T}^*M}$. Let $\mathcal{P}$ be an invertible $\mathcal{E}^e$-module. It is enough to show that $\mathcal{P}$ is locally isomorphic to $\mathcal{E}^{[\lambda]}$ for some locally constant function $[\lambda]$. In fact, it will follow from Lemma 4.3.4 that $L = \mathcal{H}om_{\mathcal{E}^e}(\mathcal{E}^{[\lambda]}, \mathcal{P})$ is a local system of rank one and $L \otimes_{\mathbb{C}} \mathcal{E}^{[\lambda]} \xrightarrow{\sim} \mathcal{P}$.

(a) Since $\mathcal{P}$ is invertible, the underlying $\mathcal{E}$-module ${}_\bullet\mathcal{P}$ is projective locally of finite presentation by (iv) and (v) on page 17, and hence coherent torsion-free. By Proposition 4.3.2, ${}_\bullet\mathcal{P}$ is locally free outside a closed analytic 2-codimensional subset Z. As $\mathcal{P}$ is invertible, its rank is one.

(b) Suppose that ${}_\bullet\mathcal{P}$ is free of rank one. Then there exists $[\lambda]$ such that $\mathcal{P}^{[-\lambda]} = \mathcal{P} \otimes_{\mathcal{E}^e} \mathcal{E}^{[-\lambda]}$ admits a regular generator, i.e. a generator u of ${}_\bullet\mathcal{P}^{[-\lambda]}$ such that $Pu = uP$ for any $P \in \mathcal{E}$. Indeed, let t be a generator of ${}_\bullet\mathcal{P}$ and let $f\colon \mathcal{E} \xrightarrow{\sim} \mathcal{E}$, be the $\mathbb{C}$-algebra isomorphism as in (3.3.3): $f(P) = Q$ for Q such that $tP = Qt$. By Proposition 4.2.2, f is locally of the form $\mathrm{Ad}(P)$ for some $\lambda \in \mathbb{C}$ and $P \in \mathcal{E}(\lambda)$ with never vanishing symbol. Then $u = tP^{-1}$ is a regular generator of $\mathcal{P}^{[-\lambda]}$.

Let V be a contractible open neighborhood of a point in Z. We are left to show that if ${}_\bullet\mathcal{P}$ is locally free of rank one on $V \setminus Z$, then ${}_\bullet\mathcal{P}^{[-\lambda]}$ has a regular generator on V. It will follow that $\mathcal{P}|_V \simeq \mathcal{E}^{[\lambda]}_V$.

(c) Since local regular generators u of $\mathcal{P}^{[-\lambda]}$ are unique up to multiplicative constants, $\mathbb{C}u \subset \mathcal{P}^{[-\lambda]}$ defines a local system of rank one on $V \setminus Z$. As $V \setminus Z$ is simply connected, such local system is constant. Thus $\mathcal{P}^{[-\lambda]}$ has a regular generator u on $V \setminus Z$.

Consider the distinguished triangle

$$\mathrm{R}\varGamma_Z \mathcal{P}^{[-\lambda]} \to \mathcal{P}^{[-\lambda]} \to \mathrm{R}\varGamma_{V\setminus Z} \mathcal{P}^{[-\lambda]} \xrightarrow{+1}$$

Since $\mathcal{P}^{[-\lambda]}$ is invertible, then ${}_\bullet\mathcal{P}^{[-\lambda]}$ is flat by (vi) on page 17, so that

$$\mathrm{R}\varGamma_Z(V; \mathcal{P}^{[-\lambda]}) \simeq \mathrm{R}\varGamma(V; \mathrm{R}\varGamma_Z \mathcal{E} \otimes_{\mathcal{E}} \mathcal{P}^{[-\lambda]}).$$

By Lemma 4.3.1 one gets $H^j \mathrm{R}\varGamma_Z(V; \mathcal{P}^{[-\lambda]}) = 0$ for $j = 0, 1$. It follows that $\Gamma(V; \mathcal{P}^{[-\lambda]}) \xrightarrow{\sim} \Gamma(V \setminus Z; \mathcal{P}^{[-\lambda]})$, hence the generator u of ${}_\bullet\mathcal{P}^{[-\lambda]}$ on $V \setminus Z$ extends uniquely to V.

□

In particular, since any $\mathcal{E}^{[\lambda]}_{\dot{T}^*M}$ is a locally free right $\mathcal{E}_{\dot{T}^*M}$-module of rank one by (4.3.1), it follows that the $\mathbb{C}$-algebra $\mathcal{E}_{\dot{T}^*M}$ is Picard good.

Recall that the projection $\gamma\colon \dot{T}^*M \to P^*M$ is a principal $\mathbb{C}^\times$-bundle.

Theorem 4.3.6. *The $\mathbb{C}$-algebra $\mathcal{E}_{P^*M}$ is Picard good.*

Proof. Let us prove that any invertible $\mathcal{E}^e_{P^*M}$-module $\mathcal{P}$ is locally free of rank one as right $\mathcal{E}_{P^*M}$-module.

Since this is a local problem, we may restrict to a simply connected open subset $U \subset P^*M$, so that $\gamma^{-1}(U) \simeq U \times \mathbb{C}^\times$. The $\mathcal{E}^e_{\gamma^{-1}(U)}$-module $\gamma^{-1}\mathcal{P}$ being invertible, by Theorem 4.3.5 one gets

$$\mathcal{P} \xrightarrow{\sim} \gamma_*\gamma^{-1}\mathcal{P} \simeq \gamma_*(L \otimes_{\mathbb{C}} \mathcal{E}^{[\lambda]}_{\gamma^{-1}(U)})$$

for some $[\lambda] \in \mathbb{C}/\mathbb{Z}$ and some local system of rank one L on $\gamma^{-1}(U)$ with monodromy $e^{-2\pi i\lambda}$ on $\mathbb{C}^\times$.

By restricting to $U' \subset U$, we may assume that there exists an invertible operator D of order 1. This defines an isomorphism of right $\mathcal{E}_{U'}$-modules

$$\mathcal{E}_{U'} \xrightarrow{\sim} \gamma_*(L \otimes_{\mathbb{C}} \mathcal{E}^{[\lambda]}_{\gamma^{-1}(U')}) \quad Q \mapsto D^\lambda Q.$$

□

Note that, given a local system of rank one L and $[\lambda] \in \mathbb{C}/\mathbb{Z}$, one has $\gamma_*(L \otimes_{\mathbb{C}} \mathcal{E}^{[\lambda]}_{\dot{T}^*M}) \neq 0$ if and only if the monodromy of L along the fiber of γ is given by $e^{-2\pi i\lambda}$. In particular, $\gamma_*\mathcal{E}^{[\lambda]}_{\dot{T}^*M} = 0$ for any $[\lambda] \neq 0$.

5. Microdifferential algebroids

Here we state and prove our results on classification of $\mathcal{E}$-algebroids on a contact manifold.

5.1. **Contact manifolds.** Let X be a complex manifold of odd dimension, say $2n - 1$. Denote by $\mathcal{O}_X$ the sheaf of holomorphic functions and by Ω^1_X the sheaf of holomorphic 1-forms. A structure of (complex) contact manifold on X is the assignment of a holomorphic principal $\mathbb{C}^\times$-bundle $\gamma\colon Y \to X$, called symplectification, and of a holomorphic one-form $\alpha \in \mathit{\Gamma}(Y;\Omega^1_Y)$, called contact form, such that $\omega = d\alpha$ is symplectic (i.e. ω^n vanishes nowhere) and $i_\theta\alpha = 0$, $L_\theta\alpha = \alpha$. Here, θ denotes the infinitesimal generator of the action of $\mathbb{C}^\times$ on Y, i_θ the inner product and L_θ the Lie derivative. One may consider α as a global section of $\Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(1)$, where $\mathcal{O}_X(1)$ denotes the dual of the sheaf of sections of the line bundle $\mathbb{C} \times_{\mathbb{C}^\times} Y$.

Let M be a complex manifold of dimension n. Then P^*M has a natural contact structure given by the Liouville one-form on $\dot{T}^*M$ and by the projection $\gamma\colon \dot{T}^*M \to P^*M$. By Darboux theorem, P^*M is a local model for a contact manifold X, meaning that there are an open cover $\{U_i\}_{i\in I}$ of X and contact embeddings (i.e. embeddings preserving the contact forms) $j_i\colon U_i \hookrightarrow P^*M$ for any $i \in I$.

A fundamental result by [35] asserts that contact transformations (i.e. biholomorphisms preserving the contact forms) can be locally quantized. This means the following. Let N be another complex manifold of dimension n, $U \subset P^*M$ and $V \subset P^*N$ open subsets and $\chi\colon U \to V$ a contact transformation. Then any $x \in U$ has an open neighborhood U' such that there is a $\mathbb{C}$-algebra isomorphism $\chi^{-1}(\mathcal{E}_{P^*N}|_{\chi(U')}) \xrightarrow{\sim} \mathcal{E}_{P^*M}|_{U'}$.

Definition 5.1.1. An $\mathcal{E}$-algebra on a contact manifold X is a sheaf $\mathcal{A}$ of $\mathbb{C}$-algebras such that there are an open cover $\{U_i\}_{i\in I}$ of X, contact embeddings $j_i\colon U_i \hookrightarrow P^*M$ and $\mathbb{C}$-algebra isomorphisms $\mathcal{A}|_{U_i} \simeq j_i^{-1}\mathcal{E}_{P^*M}$ for any $i \in I$.

Given an $\mathcal{E}$-algebra $\mathcal{A}$, the $\mathbb{C}$-algebra $\gamma^{-1}\mathcal{A}$ on Y satisfies $\gamma^{-1}\mathcal{A}|_{\gamma^{-1}(U_i)} \simeq \tilde{j}_i^{-1}\mathcal{E}_{\dot{T}^*M}$ for $\tilde{j}_i$ a lifting of $j_i\colon U_i \hookrightarrow P^*M$. Note that, from Proposition 4.2.2 it follows that for $[\lambda] \in \mathbb{C}/\mathbb{Z}$ the invertible $\gamma^{-1}\mathcal{A}^e$-module $(\gamma^{-1}\mathcal{A})^{[\lambda]}$ is well-defined.

In the strict sense, to quantize X means to endow it with an $\mathcal{E}$-algebra (see [2]). This might not be possible in general. However, as we now recall, Kashiwara [16] proved that X is endowed with a canonical $\mathcal{E}$-algebroid.

5.2. **Microdifferential algebroids.**

Definition 5.2.1.
(i) An $\mathcal{E}$-algebroid on X is a $\mathbb{C}$-algebroid A such that for every open subset $U \subset X$ and any object $\alpha \in \mathsf{A}(U)$, the $\mathbb{C}$-algebra $\mathcal{E}nd_{\mathsf{A}}(\alpha)$ is an $\mathcal{E}$-algebra on U.
(ii) A stack of twisted $\mathcal{E}$-modules on X is a $\mathbb{C}$-stack M such that there are an open cover $\{U_i\}_{i\in I}$ of X, $\mathcal{E}$-algebras $\mathcal{E}_i$ on U_i and equivalences $\mathsf{M}|_{U_i} \approx_{\mathbb{C}} \mathsf{Mod}(\mathcal{E}_i)$ for any $i \in I$.

Note that a $\mathbb{C}$-stack A is an $\mathcal{E}$-algebroid if and only if there are an open cover $\{U_i\}_{i\in I}$ of X, $\mathcal{E}$-algebras $\mathcal{E}_i$ on U_i and equivalences $\mathsf{A}|_{U_i} \approx_{\mathbb{C}} \mathcal{E}_i^+$ for any $i \in I$. In particular, $\mathsf{Mod}(\mathsf{A})$ is a stack of twisted $\mathcal{E}$-modules.

Kashiwara's construction of the canonical $\mathcal{E}$-algebroid on X was performed by patching data as explained in Appendix A.2 (see [9] for a more intrinsic construction). More precisely, he proved in [16] the existence of an open cover $\mathcal{U} = \{U_i\}_{i\in I}$ of X, of $\mathcal{E}$-algebras $\mathcal{E}_i$ on U_i, of isomorphisms of $\mathbb{C}$-algebras $f_{ij}\colon \mathcal{E}_j \to \mathcal{E}_i$ on U_{ij} and of sections $a_{ijk} \in \Gamma(U_{ijk}; \mathcal{E}_i(0)^\times)$, satisfying the cocycle condition

$$\begin{cases} f_{ij}f_{jk} = \mathrm{Ad}(a_{ijk})f_{ik}, \\ a_{ijk}a_{ikl} = f_{ij}(a_{jkl})a_{ijl}. \end{cases} \tag{5.2.1}$$

By Proposition A.2.1 (i), this implies

Theorem 5.2.2 ([16])**.** *Any complex contact manifold X is endowed with a canonical $\mathcal{E}$-algebroid E_X.*

It follows that a $\mathbb{C}$-stack on X is an $\mathcal{E}$-algebroid (resp. a stack of twisted $\mathcal{E}$-modules) if and only if it is locally $\mathbb{C}$-equivalent to E_X (resp. to $\mathsf{Mod}(\mathsf{E}_X)$). In particular, if $X = P^*M$ then E_{P^*M} is $\mathbb{C}$-equivalent to $\mathcal{E}_{P^*M}$, and $\mathcal{E}$-algebroids are $\mathbb{C}$-twisted forms of $\mathcal{E}_{P^*M}$.

Recall that an algebroid is Picard good if and only if so are the algebras that locally represent it. Hence, by Theorem 4.3.6 one gets that any $\mathcal{E}$-algebroid, and in particular E_X, is Picard good. From Proposition 3.4.2, we thus deduce the following

Theorem 5.2.3. (i) *Two $\mathcal{E}$-algebroids are $\mathbb{C}$-equivalent if and only if they are Morita equivalent.*
(ii) *Any stack of twisted $\mathcal{E}$-modules is $\mathbb{C}$-equivalent to the stack of modules over an $\mathcal{E}$-algebroid.*

To classify $\mathcal{E}$-algebroids, we thus need to compute the first cohomology with value in the stack of 2-groups $\mathsf{Aut}_{\mathbb{C}}(\mathsf{E}_X) \approx \mathsf{Inv}(\mathsf{E}^e_X)_{\mathrm{op}}$, where we set $\mathsf{E}^e_X = \mathsf{E}^{\mathrm{op}}_X \otimes_{\mathbb{C}} \mathsf{E}_X$.

5.3. **Geometry of $\gamma\colon Y \to X$.**

Lemma 5.3.1. *For M an abelian group, there is a distinguished triangle*

$$M_X \to R\gamma_* M_Y \to M_X[-1] \xrightarrow{+1}$$

Proof. As the complex $R\gamma_* M_Y$ is concentrated in degrees $[0,1]$, by truncation it is enough to prove the isomorphisms

$$H^i R\gamma_* M_Y \simeq M_X, \quad \text{for } i = 0, 1. \tag{5.3.1}$$

For $i = 0$ it is induced by the adjunction morphism $M_X \to R\gamma_* M_Y$.

Set $SY = Y/\mathbb{R}_{>0}$ and consider γ as the composite of $p\colon Y \to SY$ and $q\colon SY \to X$, which are principal bundle for the groups $\mathbb{R}_{>0}$ and S^1, respectively. Note that $Rp_* M_Y \simeq M_{SY}$, so that $R\gamma_* M_Y \simeq Rq_* M_{SY} \simeq Rq_! M_{SY}$. The infinitesimal generator θ of the action of $\mathbb{C}^\times$ on Y induces a trivialization of the relative orientation sheaf $or_{SY/X}$. Hence $q^! M_X \simeq M_{SY}[1]$. Then the isomorphism (5.3.1) for $i = 1$ is induced by the adjunction morphism $Rq_! M_{SY} \simeq Rq_! q^! M_X[-1] \to M_X[-1]$. $\square$

Let $M = \mathbb{C}^\times$. The induced long exact cohomology sequence is

$$H^1(Y;\mathbb{C}^\times) \xrightarrow{\mu_1} H^0(X;\mathbb{C}^\times) \xrightarrow{\delta} H^2(X;\mathbb{C}^\times) \xrightarrow{\gamma^\#} H^2(Y;\mathbb{C}^\times) \xrightarrow{\mu_2} H^1(X;\mathbb{C}^\times).$$

Let us describe the above sequence (see also [14, Chapitre V §3.1, 3.2]), were we use the notation $[\cdot]$ both for isomorphism and $\mathbb{C}$-equivalence classes.

For L a local systems of rank one on Y, $\mu_1([L])$ is the locally constant function on X giving the monodromy of L along the fibers of γ.

Lemma 5.3.2. (i) *There is a group isomorphism $\pi_0(\gamma_* \mathbb{C}^+_Y) \simeq \mathbb{C}^\times_X$, where the group structure on the left-hand side is induced by $\otimes_{\mathbb{C}}$.*
(ii) *For any $\mathbb{C}$-stack D on Y, the sheaf $\pi_0(\gamma_* \mathsf{D})$ is endowed with a $\mathbb{C}^\times_X$-action.*

Proof. (i) Recall that $\mathbb{C}^+_Y$ is the stack of local systems of rank one on Y and $\mathbb{C}^\times_Y[1]$ that of $\mathbb{C}^\times_Y$-torsors. Then the functor

$$\mathbb{C}^\times_Y[1] \to \mathbb{C}^+_Y, \quad \mathcal{P} \mapsto \mathbb{C} \times_{\mathbb{C}^\times} \mathcal{P}$$

defines a group isomorphism $\pi_0(\gamma_* \mathbb{C}^+_Y) \simeq \pi_0(\gamma_*(\mathbb{C}_Y[1]))$. By (1.4.3), the latter is isomorphic to $R^1\gamma_* \mathbb{C}^\times_Y$, hence to $\mathbb{C}^\times_X$ by Lemma 5.3.1.

(ii) By using (2.1.1), one gets a $\mathbb{C}$-functor

$$\gamma_*\mathbb{C}_Y^+ \otimes_{\mathbb{C}} \gamma_*\mathsf{D} \to \gamma_*\mathsf{D}, \quad (L,\delta) \mapsto L \otimes_{\mathbb{C}} \delta.$$

This defines an action of $\pi_0(\gamma_*\mathbb{C}_Y^+) \simeq \mathbb{C}_X^\times$ on $\pi_0(\gamma_*\mathsf{D})$. □

Notation 5.3.3. Let C be a $\mathbb{C}$-stack. For s a global section of $\pi_0(\mathsf{C})$, we denote by C^s the full substack of C whose objects have isomorphism class s in $\pi_0(\mathsf{C})$.

Note that C^s is a $\mathbb{C}$-algebroid, since $\pi_0(\mathsf{C}^s) = \{s\}$. It is locally $\mathbb{C}$-equivalent to the algebra $\mathcal{E}nd_{\mathsf{C}}(\gamma)$ for any local representative γ of s.

By Lemma 2.5.4, the cohomology group $H^2(X;\mathbb{C}^\times)$ classifies equivalence classes of invertible $\mathbb{C}_X$-algebroids. Then, for $m \in H^0(X;\mathbb{C}^\times) \simeq \mathit{\Gamma}(X, \pi_0(\gamma_*\mathbb{C}_Y^+))$, one has

$$\delta(m) = [(\gamma_*\mathbb{C}_Y^+)^m].$$

Here $(\gamma_*\mathbb{C}_Y^+)^m$ is identified with the $\mathbb{C}_X$-algebroid of local systems $L \in \gamma_*\mathbb{C}_Y^+$ with $\mu_1([L]) = m$. In particular, $\mathbb{C}_X^+$ is equivalent to $(\gamma_*\mathbb{C}_Y^+)^1$ via the adjunction functor $\mathbb{C}_X^+ \to \gamma_*\mathbb{C}_Y^+$ and one has a decomposition $\gamma_*\mathbb{C}_Y^+ \approx_{\mathbb{C}} \coprod_{m\in\mathbb{C}_X^\times}(\gamma_*\mathbb{C}_Y^+)^m$.

For S an invertible $\mathbb{C}_X$-algebroid, $\gamma^\#([\mathsf{S}]) = [\gamma^{-1}\mathsf{S}]$.

Proposition 5.3.4. *For T an invertible $\mathbb{C}_Y$-algebroid, $\mu_2([\mathsf{T}])$ is the class of the local systems of rank one $\mathbb{C} \times_{\mathbb{C}^\times} \pi_0(\gamma_*\mathsf{T})$.*

Proof. By Lemma 5.3.2, there is an action of $\pi_0(\gamma_*\mathbb{C}_Y^+) \simeq \mathbb{C}_X^\times$ on $\pi_0(\gamma_*\mathsf{T})$. Since $R^2\gamma_*\mathbb{C}_Y^\times = 0$, the stack $\gamma_*\mathsf{T}$ is locally $\mathbb{C}$-equivalent to $\gamma_*\mathbb{C}_Y^+$, hence $\pi_0(\gamma_*\mathsf{T})$ is a $\mathbb{C}_X^\times$-torsor. It follows that $\mathbb{C} \times_{\mathbb{C}^\times} \pi_0(\gamma_*\mathsf{T})$ is a local system of rank one on X.

Choose an open covering $\{U_i\}$ of X in such a way that T is described, by means of the Proposition A.1.1 (i), by the data $(\mathbb{C}_{V_i}^+, (\cdot)\otimes_{\mathbb{C}} M_{ji}, \mathsf{a}_{ijk})$, where $V_i = \gamma^{-1}(U_i)$ and M_{ji} are local sistem of rank one on V_{ij}. Then $\mathbb{C} \times_{\mathbb{C}^\times} \pi_0(\gamma_*\mathsf{T})$ is represented by the 1-cocycle $\{\mu_1(M_{ji})\}$ with values in $\mathbb{C}^\times$, which gives a Cech representative of the class $\mu_2([\mathsf{T}])$. □

5.4. **Classification results.** Set

$$\mathsf{E}_Y = \gamma^{-1}\mathsf{E}_X.$$

This can be described by patching the $\mathbb{C}$-algebras $\gamma^{-1}\mathcal{E}_i$ along the pull back on Y of the data (5.2.1).

Let $\mathsf{E}_Y^e = \mathsf{E}_Y^{\mathrm{op}} \otimes_{\mathbb{C}} \mathsf{E}_Y$. For $[\lambda] \in \mathbb{C}/\mathbb{Z}$, the algebroid version of the invertible bimodule $\mathcal{E}_{\dot{T}^*M}^{[\lambda]}$ is the E_Y^e-module $\mathsf{E}_Y^{[\lambda]}$ defined by

$$(\alpha,\beta) \mapsto \mathcal{E}nd_{\mathsf{E}_Y}(\beta)^{[\lambda/2]} \otimes_{\mathcal{E}nd_{\mathsf{E}_Y}(\beta)} \mathcal{H}om_{\mathsf{E}_Y}(\alpha,\beta) \otimes_{\mathcal{E}nd_{\mathsf{E}_Y}(\alpha)} \mathcal{E}nd_{\mathsf{E}_Y}(\alpha)^{[\lambda/2]}.$$

It is invertible, as being invertible is a local property.

Consider the direct image functor

$$\gamma_*\colon \gamma_*\mathsf{Mod}(\mathsf{E}^e_Y) \to \mathsf{Mod}(\mathsf{E}^e_X)$$

and recall the morphism $H^1(Y;\mathbb{C}^\times) \xrightarrow{\mu_1} H^0(X;\mathbb{C}^\times) \simeq H^0(X;\mathbb{C}/\mathbb{Z})$ from §5.3.

Theorem 5.4.1. *The functor*

$$\gamma_*\mathsf{Inv}(\mathbb{C}_Y) \to \mathsf{Inv}(\mathsf{E}^e_X), \qquad L \mapsto \gamma_*(L \otimes_{\mathbb{C}} \mathsf{E}_Y^{\mu_1(L^*)}) \tag{5.4.1}$$

is an equivalence of stacks of 2-groups.

Proof. (a) A priori, $\gamma_*(L \otimes_{\mathbb{C}} \mathsf{E}_Y^{\mu_1(L^*)})$ is an object of $\mathsf{Mod}(\mathsf{E}^e_X)$. This is locally, hence globally, invertible with inverse given by $\gamma_*(L^* \otimes_{\mathbb{C}} \mathsf{E}_Y^{\mu_1(L)})$.

(b) The sheaf $\mathbb{C}_Y$ is sent to E_X, since $\gamma_*(\mathsf{E}_Y) \simeq \mathsf{E}_X$ as E^e_X-modules. Moreover, the natural morphism

$$\gamma_*(L \otimes_{\mathbb{C}} \mathsf{E}_Y^{\mu_1(L^*)}) \otimes_{\mathsf{E}_X} \gamma_*(L' \otimes_{\mathbb{C}} \mathsf{E}_Y^{\mu_1(L'^*)}) \to \gamma_*(L \otimes_{\mathbb{C}} L'^* \otimes_{\mathbb{C}} \mathsf{E}_Y^{\mu_1(L^*)+\mu_1(L'^*)})$$

is locally, hence globally, an isomorphism. Hence (5.4.1) is monoidal.

(c) For an invertible E^e_X-module $\mathcal{P}$, define its exponential as the unique locally constant $\mathbb{C}/\mathbb{Z}$-valued function $\epsilon(\mathcal{P})$ on X such that $\gamma^{-1}\mathcal{P}$ is locally isomorphic to $\mathsf{E}_Y^{\epsilon(\mathcal{P})}$ (this is well-defined by Theorem 4.3.5.). Then $\epsilon(\gamma_*(L \otimes_{\mathbb{C}} \mathsf{E}_Y^{\mu_1(L^*)})) = \mu_1(L^*)$, and by using the Lemma 4.3.4 one gets that the functor

$$\mathcal{P} \mapsto \mathcal{H}om_{\mathsf{E}^e_Y}(\mathsf{E}_Y^{\epsilon(\mathcal{P})}, \gamma^{-1}\mathcal{P})$$

is a quasi-inverse of (5.4.1). □

Let $\mathrm{Pic}(\mathsf{E}^e_X)$ denote the set of isomorphism class of invertible E^e_X-modules, endowed with the group structure induced by $\otimes_{\mathsf{E}_X}$.

Corollary 5.4.2. *There is a group isomorphism* $\mathrm{Pic}(\mathsf{E}^e_X) \simeq H^1(Y;\mathbb{C}^\times_Y)$.

Theorem 5.4.3. *The set of $\mathbb{C}$-equivalence classes (resp. Morita classes) of $\mathcal{E}$-algebroids is canonically isomorphic, as a pointed set, to $H^2(Y;\mathbb{C}^\times_Y)$.*

Proof. Since E_X is Picard good, by Theorem 5.4.1 there is an equivalence of stacks of 2-groups

$$\mathsf{Aut}_{\mathbb{C}}(\mathsf{E}_X) \approx \gamma_*\mathsf{Inv}(\mathbb{C}_Y)_{\mathrm{op}}.$$

The right-hand term is equivalent to $\gamma_*\mathsf{Inv}(\mathbb{C}_Y)$ by the functor $L \mapsto L^*$. Since $\mathbb{C}_Y$ is Picard good, from (3.4.3) and by using (1.4.2) one gets an equivalence of stacks of 2-groups

$$\gamma_*\mathsf{Inv}(\mathbb{C}_Y) \approx [R\gamma_*\mathbb{C}^\times_Y[1]].$$

It then follows from (1.4.1) that

$$H^1(X;\mathsf{Aut}_{\mathbb{C}}(\mathsf{E}_X)) \simeq H^2(Y;\mathbb{C}^\times_Y). \tag{5.4.2}$$

□

We end by giving a geometric realization of the isomorphism (5.4.2).

First, let us explain how to twist E_Y by a local system of rank one L on X, obtaining a $\mathbb{C}$-algebroid E_Y^L on Y locally $\mathbb{C}$-equivalent to E_Y.

Choose an open covering $\{U_i\}$ of X in such a way that L is represented by a 1-cocycle $\{[\lambda_{ij}]\}$ with values in $\mathbb{C}/\mathbb{Z}$. Set $V_i = \gamma^{-1}(U_i)$ and consider the data $(\mathsf{E}_{V_i}, (\cdot)\otimes_{\mathsf{E}_{V_{ij}}} \mathsf{E}_{V_{ij}}^{[\lambda_{ij}]}, \mathsf{m}_{ijk})$, where m_{ijk} denotes the invertible transformation induced by the canonical isomorphism of $\mathsf{E}^e_{V_{ijk}}$-modules

$$\mathsf{E}_{V_{ijk}}^{[\lambda_{ij}]} \otimes_{\mathsf{E}_{V_{ijk}}} \mathsf{E}_{V_{ijk}}^{[\lambda_{jk}]} \xrightarrow{\sim} \mathsf{E}_{V_{ijk}}^{[\lambda_{ik}]}.$$

Then E_Y^L is the $\mathbb{C}$-stack on Y obtained from these data by Proposition A.1.1 (i). Note that $(\mathsf{E}_Y^L)^{\mathrm{op}} \approx_{\mathbb{C}} \mathsf{E}_Y^{L^*}$ and $\mathsf{E}_Y^L \approx_{\mathbb{C}} \mathsf{E}_Y$ if L is trivial.

Recall from Lemma 5.3.2 that $\pi_0(\gamma_*\mathsf{E}_Y^L)$ is endowed with a $\mathbb{C}_X^\times$-action, and denote by $L^\times$ the $\mathbb{C}^\times$-torsor associated to L.

Lemma 5.4.4. $\pi_0(\gamma_*\mathsf{E}_Y^L) \simeq L^\times \times_{\mathbb{C}^\times} \pi_0(\gamma_*\mathsf{E}_Y)$ *as* $\mathbb{C}^\times$*-sheaves.*

Proof. Let $\{[\lambda_{ij}]\}$ be a 1-cocycle with values in $\mathbb{C}/\mathbb{Z}$ representing L on an open covering $\{U_i\}$ of X. Then $\gamma_*\mathsf{E}_Y^L|_{U_i} \approx_{\mathbb{C}} \gamma_*\mathsf{E}_Y|_{U_i}$ and the associated glueing $\mathbb{C}$-equivalences $\gamma_*\mathsf{E}_Y|_{U_{ij}} \to \gamma_*\mathsf{E}_Y|_{U_{ij}}$ are given by $(\cdot) \otimes_{\mathsf{E}_{V_{ij}}} \mathsf{E}_{V_{ij}}^{[\lambda_{ij}]}$, where $V_i = \gamma^{-1}(U_i)$. We thus get isomorphisms of $\mathbb{C}^\times$-sheaves $\pi_0(\gamma_*\mathsf{E}_Y^L)|_{U_i} \simeq \pi_0(\gamma_*\mathsf{E}_Y)|_{U_i}$, with associated glueing automorphisms of $\pi_0(\gamma_*\mathsf{E}_Y)|_{U_{ij}}$ given by multiplication by $e^{2\pi i\lambda_{ij}}$. This follows from the commutative diagram of stacks of 2-groups

$$\begin{tikzcd}
\mathbb{C}/\mathbb{Z}_X[0] \arrow[r, "\simeq"] \arrow[d] & \mathbb{C}_X^\times[0] \arrow[d] \\
\gamma_*\mathsf{Aut}_{\mathbb{C}}(\mathsf{E}_Y) \arrow[r, "\pi_0"] & \mathcal{A}ut(\pi_0(\gamma_*\mathsf{E}_Y))[0],
\end{tikzcd}$$

where the left-hand vertical arrow is the functor $[\lambda] \mapsto (\cdot) \otimes_{\mathsf{E}_Y} \mathsf{E}_Y^{[\lambda]}$ and the right-hand one is the $\mathbb{C}^\times$-action. Hence $\pi_0(\gamma_*\mathsf{E}_Y^L)$ is isomorphic to $\pi_0(\gamma_*\mathsf{E}_Y)$ twisted by the $\mathbb{C}^\times$-torsor $L^\times$. $\square$

Let T be an invertible $\mathbb{C}_Y$-algebroid. Recall that we denote by $\mu_2(\mathsf{T})$ the local system of rank one on X associated to the $\mathbb{C}^\times$-torsor $\pi_0(\gamma_*\mathsf{T})$.

Lemma 5.4.5. $\pi_0(\gamma_*(\mathsf{T} \otimes_{\mathbb{C}} \mathsf{E}_Y^{\mu_2(\mathsf{T}^{\mathrm{op}})})) \simeq \pi_0(\gamma_*\mathsf{E}_Y)$ *as* $\mathbb{C}^\times$*-sheaves.*

Proof. By using the functor (2.1.1), one gets a morphism

$$\pi_0(\gamma_*\mathsf{T}) \times \pi_0(\gamma_*\mathsf{E}_Y^{\mu_2(\mathsf{T}^{\mathrm{op}})}) \to \pi_0(\gamma_*(\mathsf{T} \otimes_{\mathbb{C}} \mathsf{E}_Y^{\mu_2(\mathsf{T}^{\mathrm{op}})}))$$

which is $\mathbb{C}^\times$-equivariant on each term. Hence it factors through $\pi_0(\gamma_*\mathsf{T}) \times_{\mathbb{C}^\times} \pi_0(\gamma_*\mathsf{E}_Y^{\mu_2(\mathsf{T}^{\mathrm{op}})})$. By Lemma 5.4.5, this is isomorphic to $\pi_0(\gamma_*\mathsf{E}_Y)$, since

$\pi_0(\gamma_* \mathsf{T}^{\mathrm{op}})$ is isomorphic to the $\mathbb{C}^\times$-torsor opposite to $\pi_0(\gamma_* \mathsf{T})$. It follows that we have a morphism

$$\pi_0(\gamma_* \mathsf{E}_Y) \to \pi_0(\gamma_*(\mathsf{T} \otimes_{\mathbb{C}} \mathsf{E}_Y^{\mu_2(\mathsf{T}^{\mathrm{op}})}))$$

of $\mathbb{C}^\times$-sheaves, which is locally, hence globally, an isomorphism. $\square$

Corollary 5.4.6. $\pi_0(\gamma_*(\mathsf{T} \otimes_{\mathbb{C}} \mathsf{E}_Y^{\mu_2(\mathsf{T}^{\mathrm{op}})}))$ *has a canonical global section.*

Proof. The adjunction functor $\mathsf{E}_X \to \gamma_* \mathsf{E}_Y$ defines a morphism $\pi_0(\mathsf{E}_X) \to \pi_0(\gamma_* \mathsf{E}_Y)$. Since $\pi_0(\mathsf{E}_X)$ is a singleton, this gives a global section of $\pi_0(\gamma_* \mathsf{E}_Y)$, hence of $\pi_0(\gamma_*(\mathsf{T} \otimes_{\mathbb{C}} \mathsf{E}_Y^{\mu_2(\mathsf{T}^{\mathrm{op}})}))$ by Lemma 5.4.5. $\square$

Denote by *can* the canonical global section of $\pi_0(\gamma_*(\mathsf{T} \otimes_{\mathbb{C}} \mathsf{E}_Y^{\mu_2(\mathsf{T}^{\mathrm{op}})}))$. Then the inverse of the isomorphism (5.4.2) is realized as

$$[\mathsf{T}] \mapsto [(\gamma_*(\mathsf{T} \otimes_{\mathbb{C}} \mathsf{E}_Y^{\mu_2(\mathsf{T}^{\mathrm{op}})}))^{can}],$$

where $[\cdot]$ denotes the $\mathbb{C}$-equivalence class and we use the Notation 5.3.3.

If S be an invertible $\mathbb{C}_X$-algebroid, then $\mu_2(\gamma^{-1}\mathsf{S}^{\mathrm{op}})$ is trivial and the above isomorphism reduces to

$$[\gamma^{-1}\mathsf{S}] \mapsto [(\gamma_* \gamma^{-1}(\mathsf{S} \otimes_{\mathbb{C}} \mathsf{E}_X))^{can}] = [\mathsf{S} \otimes_{\mathbb{C}} \mathsf{E}_X].$$

Remark 5.4.7. Replacing E_X by an $\mathcal{E}$-algebroid in the previous construction, one gets an action of $H^2(Y; \mathbb{C}_Y^\times)$ on the set of $\mathbb{C}$-equivalence classes (resp. Morita classes) of $\mathcal{E}$-algebroids. In such a way, the latter becomes an $H^2(Y; \mathbb{C}_Y^\times)$-torsor and the canonical isomorphism (5.4.2) is obtained by choosing the $\mathbb{C}$-equivalence class of E_X as base point.

Appendix A. Cocycles

In this Appendix we recall the descent condition for stacks and detail the case of algebroids, as in [16]. This is parallel to the case of gerbes, which is discussed for example in [3].

Let X be a topological space (or a site), and $\mathcal{R}$ a sheaf of commutative rings on X. If $\mathcal{U} = \{U_i\}_{i \in I}$ is an open cover of X, we set $U_{ij} = U_i \cap U_j$, $U_{ijk} = U_i \cap U_j \cap U_k$ etc. We use the notation $\bullet$ for the horizontal composition of transformations.

A.1. Glueing of stacks.

Let us recall here how to recover $\mathcal{R}$-stacks and $\mathcal{R}$-functors from collections of local data.

Proposition A.1.1. *Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X.*

(i) *Consider the data $(\mathsf{C}_i, \mathsf{f}_{ij}, \mathsf{a}_{ijk})_{ijk \in I}$, where C_i are stacks on U_i, $\mathsf{f}_{ij} \colon \mathsf{C}_j \to \mathsf{C}_i$ are equivalences on U_{ij}, and $\mathsf{a}_{ijk} \colon \mathsf{f}_{ik} \to \mathsf{f}_{ij} \circ \mathsf{f}_{jk}$ are* *invertible transformations on U_{ijk}, such that*

(A.1.1)
$$\begin{tikzcd}
\mathsf{f}_{ij}\circ\mathsf{f}_{jk}\circ\mathsf{f}_{kl} & \mathsf{f}_{ik}\circ\mathsf{f}_{kl} \arrow[l, "\mathsf{a}_{ijk}\bullet\mathrm{id}_{\mathsf{f}_{kl}}"'] \\
\mathsf{f}_{ij}\circ\mathsf{f}_{jl} \arrow[u, "\mathrm{id}_{\mathsf{f}_{ij}}\bullet\mathsf{a}_{jkl}"] & \mathsf{f}_{il} \arrow[l, "\mathsf{a}_{ijl}"] \arrow[u, "\mathsf{a}_{ikl}"']
\end{tikzcd} \quad \textit{commutes.}$$

Then, there exists a stack C *on* X *endowed with equivalences* $\mathsf{f}_i\colon \mathsf{C}|_{U_i} \to \mathsf{C}_i$ *and invertible transformations* $\mathsf{a}_{ij}\colon \mathsf{f}_i \to \mathsf{f}_{ij}\circ\mathsf{f}_j$ *on* U_{ij}*, such that*

$$\begin{tikzcd}
\mathsf{f}_{ij}\circ\mathsf{f}_{jk}\circ\mathsf{f}_{k} & \mathsf{f}_{ij}\circ\mathsf{f}_{j} \arrow[l, "\mathrm{id}_{\mathsf{f}_{ij}}\bullet\mathsf{a}_{jk}"'] \\
\mathsf{f}_{ik}\circ\mathsf{f}_{k} \arrow[u, "\mathsf{a}_{jkl}\bullet\mathrm{id}_{\mathsf{f}_{k}}"] & \mathsf{f}_{i}. \arrow[l, "\mathsf{a}_{ik}"] \arrow[u, "\mathsf{a}_{ij}"']
\end{tikzcd} \quad \textit{commutes.}$$

The stack C *is unique up to an equivalence unique up to a unique invertible transformation.*

(ii) *Let* C *be as above, and let* C' *be associated with the data* $(\mathsf{C}'_i, \mathsf{f}'_{ij}, \mathsf{a}'_{ijk})_{ijk\in I}$. *Consider the data* $(\mathsf{g}_i, \mathsf{b}_{ij})_{ij\in I}$*, where* $\mathsf{g}_i\colon \mathsf{C}_i \to \mathsf{C}'_i$ *are functors on* U_i*, and* $\mathsf{b}_{ij}\colon \mathsf{f}'_{ij}\circ\mathsf{g}_j \to \mathsf{g}_i\circ\mathsf{f}_{ij}$ *are invertible transformations on* U_{ij}*, such that*

(A.1.2)
$$\begin{tikzcd}
\mathsf{g}_i\circ\mathsf{f}_{ij}\circ\mathsf{f}_{jk} & \mathsf{g}_i\circ\mathsf{f}_{ik} \arrow[l, "\mathrm{id}_{\mathsf{g}_i}\bullet\mathsf{a}_{ijk}"'] & \\
\mathsf{f}'_{ij}\circ\mathsf{g}_j\circ\mathsf{f}_{jk} \arrow[u, "\mathsf{b}_{ij}\bullet\mathrm{id}_{\mathsf{f}_{jk}}"] & \mathsf{f}'_{ij}\circ\mathsf{f}'_{jk}\circ\mathsf{g}_k \arrow[l, "\mathrm{id}_{\mathsf{f}'_{ij}}\bullet\mathsf{b}_{jk}"'] & \mathsf{f}'_{ik}\circ\mathsf{g}_k \arrow[l, "\mathsf{a}'_{ijk}\bullet\mathrm{id}_{\mathsf{g}_k}"'] \arrow[ul, "\mathsf{b}_{ik}"']
\end{tikzcd} \quad \textit{commutes.}$$

Then, there exists a functor $\mathsf{g}\colon \mathsf{C} \to \mathsf{C}'$ *endowed with invertible transformations* $\mathsf{b}_i\colon \mathsf{f}'_i\circ\mathsf{g} \to \mathsf{g}_i\circ\mathsf{f}_i$ *on* U_i*, such that*

$$\begin{tikzcd}
\mathsf{g}_i\circ\mathsf{f}_{ij}\circ\mathsf{f}_{j} & \mathsf{g}_i\circ\mathsf{f}_{i} \arrow[l, "\mathrm{id}_{\mathsf{g}_i}\bullet\mathsf{a}_{ij}"'] & \\
\mathsf{f}'_{ij}\circ\mathsf{g}_j\circ\mathsf{f}_{j} \arrow[u, "\mathsf{b}_{ij}\bullet\mathrm{id}_{\mathsf{f}_{j}}"] & \mathsf{f}'_{ij}\circ\mathsf{f}'_{j}\circ\mathsf{g} \arrow[l, "\mathrm{id}_{\mathsf{f}'_{ij}}\bullet\mathsf{b}_{j}"'] & \mathsf{f}'_{i}\mathsf{g} \arrow[l, "\mathsf{a}'_{ij}\bullet\mathrm{id}_{\mathsf{g}}"'] \arrow[ul, "\mathsf{b}_{i}"']
\end{tikzcd} \quad \textit{commutes.}$$

The functor $\mathsf{g}\colon \mathsf{C} \to \mathsf{C}'$ *is unique up to a unique invertible transformation.*

(iii) *Let* $\mathsf{g}\colon \mathsf{C} \to \mathsf{C}'$ *be as above, and let* $\tilde{\mathsf{g}}\colon \mathsf{C} \to \mathsf{C}'$ *be the functor associated with the data* $(\tilde{\mathsf{g}}_i, \tilde{\mathsf{b}}_{ij})_{ij\in I}$. *Consider the data* $(\mathsf{d}_i)_{i\in I}$*, where* $\mathsf{d}_i\colon \mathsf{g}_i \to \tilde{\mathsf{g}}_i$ *are transformations on* U_i *such that*

(A.1.3)
$$\begin{tikzcd}
\mathsf{g}_i\circ\mathsf{f}_{ij} \arrow[d, "\mathsf{d}_i\bullet\mathrm{id}_{\mathsf{f}_{ij}}"'] & \mathsf{f}'_{ij}\circ\mathsf{g}_j \arrow[l, "\mathsf{b}_{ij}"'] \arrow[d, "\mathrm{id}_{\mathsf{f}'_{ij}}\bullet\mathsf{d}_j"] \\
\tilde{\mathsf{g}}_i\circ\mathsf{f}_{ij} & \mathsf{f}'_{ij}\circ\tilde{\mathsf{g}}_j \arrow[l, "\tilde{\mathsf{b}}_{ij}"]
\end{tikzcd} \quad \textit{commutes.}$$

Then, there exists a unique transformation $\mathsf{d}\colon \mathsf{g} \to \tilde{\mathsf{g}}$ *such that* $\mathsf{d}|_{U_i} = \mathsf{d}_i$.

A.2. **Algebroid cocycles.** We give here a description of $\mathcal{R}$-algebroids and $\mathcal{R}$-functors between them in terms of $\mathcal{R}$-algebras and $\mathcal{R}$-algebra morphisms.

Let A be an $\mathcal{R}$-algebroid on X. By definition, there exists an open cover $\{U_i\}_{i\in I}$ of X such that $\mathsf{A}|_{U_i}$ is non-empty. For $\alpha_i \in \mathsf{A}(U_i)$ and $\mathcal{A}_i = \mathcal{E}nd_{\mathsf{A}}(\alpha_i)$, there are $\mathcal{R}$-equivalences $\mathsf{f}_i\colon \mathsf{A}|_{U_i} \to \mathcal{A}_i^+$. Choose quasi-inverses f_i^{-1} and invertible transformations $\mathrm{id} \to \mathsf{f}_j^{-1}\circ\mathsf{f}_j$. Set $\mathsf{f}_{ij} := \mathsf{f}_i \circ \mathsf{f}_j^{-1}\colon \mathcal{A}_j^+ \to \mathcal{A}_i^+$ on U_{ij}. On U_{ijk} there are invertible transformations $\mathsf{a}_{ijk}\colon \mathsf{f}_{ik} \to \mathsf{f}_{ij}\circ\mathsf{f}_{jk}$ induced by $\mathrm{id} \to \mathsf{f}_j^{-1}\circ\mathsf{f}_j$. On U_{ijkl} one checks that the diagram (A.1.1) commutes. By Proposition A.1.1 (i), the data $(\mathsf{A}_i,\ \mathsf{f}_{ij},\ \mathsf{a}_{ijk})_{i,j,k\in I}$ are enough to reconstruct A, in the sense that the stack obtained by glueing these data is $\mathcal{R}$-equivalent to A.

The $\mathcal{R}$-equivalence $\mathsf{f}_{ij}\colon \mathcal{A}_j^+ \to \mathcal{A}_i^+$ on U_{ij} is locally induced by $\mathcal{R}$-algebra isomorphisms. There thus exist an open cover $\{U_{ij}^{\alpha}\}_{\alpha\in A}$ of U_{ij} such that $\mathsf{f}_{ij} = (f_{ij}^{\alpha})^+$ on U_{ij}^{α} for $f_{ij}^{\alpha}\colon \mathcal{A}_j \to \mathcal{A}_i$ isomorphisms of $\mathcal{R}$-algebras. On triple intersections $U_{ijk}^{\alpha\beta\gamma} = U_{ij}^{\alpha}\cap U_{ik}^{\beta}\cap U_{jk}^{\gamma}$, the invertible transformations $\mathsf{a}_{ijk}\colon (f_{ik}^{\beta})^+ \to (f_{ij}^{\alpha} f_{jk}^{\gamma})^+$ are given by invertible sections $a_{ijk}^{\alpha\beta\gamma} \in \mathcal{A}_i(U_{ijk}^{\alpha\beta\gamma})$ such that $f_{ij}^{\alpha} f_{jk}^{\gamma} = \mathrm{Ad}(a_{ijk}^{\alpha\beta\gamma}) f_{ik}^{\beta}$. On quadruple intersections $U_{ijkl}^{\alpha\beta\gamma\delta\epsilon\varphi} = U_{ijk}^{\alpha\beta\gamma}\cap U_{ijl}^{\alpha\delta\epsilon}\cap U_{ikl}^{\beta\delta\varphi}\cap U_{jkl}^{\gamma\epsilon\varphi}$, the commutative diagram (A.1.1) is equivalent to the equality $a_{ijk}^{\alpha\beta\gamma} a_{ikl}^{\beta\delta\varphi} = f_{ij}^{\alpha}(a_{jkl}^{\gamma\varepsilon\varphi}) a_{ijl}^{\alpha\delta\varepsilon}$.

One can treat in the same manner $\mathcal{R}$-functors and transformations. We summarize the results in the next proposition. However, as indices of hypercovers are quite cumbersome, we will not write them explicitly anymore. Instead, we will assume that X is such that covers are cofinal among hypercovers, as is for example the case for paracompact spaces.

Proposition A.2.1. *Let* $\{U_i\}_{i\in I}$ *be a sufficiently fine open cover of* X

(i) *Any* $\mathcal{R}$*-algebroid* A *is reconstructed from the data* $(\mathcal{A}_i,\ f_{ij},\ a_{ijk})_{i,j,k\in I}$, *where* $\mathcal{A}_i$ *are* $\mathcal{R}$*-algebras on* U_i, $f_{ij}\colon \mathcal{A}_j|_{U_{ij}} \to \mathcal{A}_i|_{U_{ij}}$ *are* $\mathcal{R}$*-algebra isomorphisms, and* $a_{ijk} \in \mathcal{A}_i(U_{ijk})$ *are invertible sections, such that*

$$\text{(A.2.1)}\qquad \begin{cases} f_{ij} f_{jk} = \mathrm{Ad}(a_{ijk}) f_{ik}, & \text{in } \mathcal{H}om_{\mathcal{R}\text{-}\mathsf{Alg}}(\mathcal{A}_k, \mathcal{A}_i)(U_{ijk}),\\ a_{ijk} a_{ikl} = f_{ij}(a_{jkl}) a_{ijl}, & \text{in } \mathcal{A}_i(U_{ijkl}). \end{cases}$$

(ii) *Let* A *be as above, and let* A' *be an* $\mathcal{R}$*-algebroid constructed from the data* $(\mathcal{A}'_i,\ f'_{ij},\ a'_{ijk})_{i,j,k\in I}$. *Any* $\mathcal{R}$*-functor* $\mathsf{g}\colon \mathsf{A} \to \mathsf{A}'$ *is reconstructed from the data* $(g_i,\ b_{ij})_{i,j\in I}$, *where* $g_i\colon \mathcal{A}_i \to \mathcal{A}'_i$ *are* $\mathcal{R}$*-algebra morphisms, and* $b_{ij} \in \mathcal{A}'_i(U_{ij})$ *are invertible sections,* *such that*

$$\begin{cases} g_i f_{ij} = \mathrm{Ad}(b_{ij}) f'_{ij} g_j, & \text{in } \mathcal{H}om_{\mathcal{R}\text{-}\mathsf{Alg}}(\mathcal{A}_j, \mathcal{A}'_i)(U_{ij}), \\ g_i(a_{ijk}) b_{ik} = b_{ij} f'_{ij}(b_{jk}) a'_{ijk}, & \text{in } \mathcal{A}'_i(U_{ijk}). \end{cases} \tag{A.2.2}$$

(iii) *Let* $\mathsf{g}\colon \mathsf{A} \to \mathsf{A}'$ *be as above, and let* $\mathsf{g}'\colon \mathsf{A} \to \mathcal{A}'$ *be constructed from the data* $(f'_i, b'_{ij})_{i,j\in I}$. *Any transformation of* $\mathcal{R}$*-functors* $\mathsf{d}\colon \mathsf{g} \to \mathsf{g}'$ *is reconstructed from the data* $(d_i)_{i\in I}$, *where* $d_i \in \mathcal{A}'_i(U_i)$ *are sections such that*

$$d_i b_{ij} = b'_{ij} f'_{ij}(d_j), \quad \text{in } \mathcal{A}'_i(U_{ij}). \tag{A.2.3}$$

In particular, the families $(\mathcal{A}_i,\, f_{ij},\, a_{ijk})_{i,j,k\in I}$ and $(\mathcal{A}'_i,\, f'_{ij},\, a'_{ijk})_{i,j,k\in I}$ describe $\mathcal{R}$-equivalent stacks if and only if there exists a family $(g_i,\, b_{ij})_{i,j\in I}$ satisfying (A.2.2) with g_i isomorphisms of $\mathcal{R}$-algebras.

Viceversa, let A be as in Proposition A.2.1 (i). For $i, j \in I$ let $\mathcal{A}'_i$ be $\mathcal{R}|_{U_i}$-algebras, $g_i\colon \mathcal{A}_i \to \mathcal{A}'_i$ isomorphisms of $\mathcal{R}$-algebras and $b_{ij} \in \mathcal{A}'_i(U_{ij})$ invertible sections. Then equalities (A.2.2) define a family $(\mathcal{A}'_i,\, f'_{ij},\, a'_{ijk})_{i,j,k\in I}$ describing an $\mathcal{R}$-algebroid $\mathcal{R}$-equivalent to A.

Remark A.2.2. Note that (A.2.1) implies the relations

$$f_{ij} f_{ji} = \mathrm{Ad}(a_{iji} a_{iii}), \quad a_{ijj} = f_{ij}(a_{jjk}) \text{ for any } i, j, k \in I,$$

and in particular

$$f_{ii} = \mathrm{Ad}(a_{iii}), \quad a_{iij} = a_{ijj}.$$

Let $\mathcal{B}_i = \mathcal{A}_i$, $h_i = \mathrm{id}_{\mathcal{A}_i}$, and $c_{ij} = a_{iji}$. Then (A.2.2) gives

$$g_{ij} = \mathrm{Ad}(a_{iji}^{-1}) f_{ij}, \quad b_{ijk} = g_{ij}(a_{jkj}^{-1} a_{jki}).$$

The family $(\mathcal{A}_i,\, g_{ij},\, b_{ijk})_{i,j,k\in I}$ describing A satisfies the relations

$$g_{ii} = \mathrm{id}_{\mathcal{A}_i}, \quad b_{iij} = b_{ijj} = 1,$$

of a normalized cocycle as in [3].

A.3. **Module cocycles.** Let A be the $\mathcal{R}$-algebroid described over the open cover $\{U_i\}_{i\in I}$ of X by the family $(\mathcal{A}_i,\, f_{ij},\, a_{ijk})_{i,j,k\in I}$. The stack of left A-modules $\mathsf{Mod}(\mathsf{A})$ is then described as in Proposition A.1.1 (i) by the family

$$(\mathsf{Mod}(\mathcal{A}_i),\, \mathsf{Mod}(f_{ji}^{+}),\, \mathsf{Mod}(a_{kji}))_{i,j,k\in I}$$

(note the inversion of indices due to the fact that $\mathsf{Mod}(\cdot)$ is contravariant). By Morita theory, the functor $\mathsf{Mod}(f_{ji}^{+})$ is isomorphic to $\mathcal{P}_{ij} \otimes_{\mathcal{A}_j} (\cdot)$ for an invertible $\mathcal{A}_i \otimes_{\mathcal{R}} \mathcal{A}_j^{\mathrm{op}}$-module $\mathcal{P}_{ij}$. We thus recover the description of twisted sheaves given in [18] (see also [10]).

Proposition A.3.1. *Let* A *be as above. An object of* $\mathsf{Mod}(\mathsf{A})$ *is described by a family* $(\mathcal{M}_i,\, \varphi_{ij})_{i,j\in I}$, *where* $\mathcal{M}_i \in \mathsf{Mod}(\mathcal{A}_i)$, *and* $\varphi_{ij} \in \mathrm{Hom}_{\mathcal{A}_i}({}_{f_{ji}}\mathcal{M}_j|_{U_{ij}}, \mathcal{M}_i|_{U_{ij}})$ *are isomorphisms, such that for any* $u_k \in \mathcal{M}_k$ *one has*

$$\varphi_{ij}({}_{f_{ji}}\varphi_{jk}(u_k)) = \varphi_{ik}(a_{kji}^{-1} u_k).$$

Proof. Let C be an $\mathcal{R}$-stack as in Proposition A.1.1 (i). The statement follows by noticing that objects of $\mathsf{C}(X)$ are described by data

$$(\alpha_i,\, a_{ij})_{i,j\in I},$$

where $\alpha_i \in \mathsf{C}_i(U_i)$, and $a_{ij}\colon \mathsf{f}_{ij}(\alpha_j) \to \alpha_i$ are isomorphisms in $\mathsf{C}_i(U_{ij})$, such that

$$a_{ij} \circ \mathsf{f}_{ij}(a_{jk}) = a_{ik} \circ \mathsf{a}_{ijk}^{-1}(\alpha_k)$$

as isomorphisms $\mathsf{f}_{ij}\mathsf{f}_{jk}(\alpha_k) \xrightarrow{\sim} \alpha_i$ in $\mathsf{C}_i(U_{ijk})$. $\square$

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