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Boltzmanngasse 9 A-1090 Wien, Austria

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F. Poupaud
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Parabolic Limit and Stability of the Vlasov-Poisson-Fokker-Planck system*

F. Poupaud† J. Soler‡

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Abstract. In this paper the stability of the Vlasov-Poisson-Fokker-Planck with respect to the variation of its constant parameters, the scaled thermal velocity and the scaled thermal mean free path, is analyzed. For the case in which the scaled thermal velocity is the inverse of the scaled thermal mean free path and the latter tends to zero, a parabolic limit equation is obtained for the mass density. Depending on the space dimension and on the hypothesis for the initial data the convergence result in L^1 is weak and global in time or strong and local in time.

Key words. Vlasov-Poisson-Fokker-Planck, Kinetic Equations, Asymptotic Behavior, Diffusion Approximation.

AMS subject classifications. 35B40, 35Q99, 76X05, 82D10, 85A05.

1. Introduction

The aim of this paper is to study the limit behaviour of the Vlasov-Poisson-Fokker-Planck (VPFP) system in terms of the parameter ϵ representing the scaled thermal mean free path and where we have assumed that the scaled thermal velocity is the inverse of the scaled thermal mean free path, see the appendix for a discussion about the VPFP system and the physical constants involved.

The VPFP$_\epsilon$ system can be written in this context in terms of the scalar distribution of particles $f_\epsilon(t,x,v) \geq 0$, the mass density $\rho_\epsilon(t,x)$ and the potential

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†Laboratoire J. A. Dieudonné, U.M.R. 6621 C.N.R.S., Université de Nice, Parc Valrose, F-06108 Nice Cedex 2, France. E-mail: poupaud@unice.fr

‡Departamento de Matemática Aplicada, Facultad de Ciencias, Universidad de Granada, 18071 Granada, SPAIN. E-mail: jsoler@ugr.es

$\Phi_\epsilon(t,x)$, with $(t,x,v) \in \mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}^N$, $N = 2$ or 3, as follows

$$\epsilon^2 \frac{\partial f_\epsilon}{\partial t} \;+\; \epsilon\left((v\cdot\nabla_x) f_\epsilon \;-\; (\nabla_x \Phi_\epsilon \cdot \nabla_v) f_\epsilon\right) \;=\; L(f_\epsilon), \tag{1.1}$$

$$L(f_\epsilon) \stackrel{def}{=} \Delta_v f_\epsilon \;+\; \mathrm{div}_v(v f_\epsilon) \;=\; \mathrm{div}_v\left(e^{-\frac{v^2}{2}} \nabla_v \left(e^{\frac{v^2}{2}} f_\epsilon\right)\right), \tag{1.2}$$

$$-\nabla_x \Phi_\epsilon \;=\; \theta K_N * \rho_\epsilon, \quad \rho_\epsilon \;=\; \int_{\mathbb{R}^N} f_\epsilon \, dv, \tag{1.3}$$

$$f_\epsilon(0,x,v) \;=\; f_{0,\epsilon}(x,v), \tag{1.4}$$

where L defined as in (1.2) is the Fokker-Planck operator and K_N is the gradient of the fundamental solution of the Laplacian in dimension N. The parameter $\theta = +1$ in the electrostatic case and $\theta = -1$ in the gravitational case. The potential is given by

$$\Phi_\epsilon \;=\begin{cases} -\theta \frac{1}{4\pi}\frac{1}{|x|} * \rho_\epsilon, & \text{N=3} \\ \theta \frac{1}{2\pi}\log|x| * \rho_\epsilon, & \text{N=2.} \end{cases} \tag{1.5}$$

We will assume that the initial data satisfies

$$\int_{\mathbb{R}^{2N}} f_{0,\epsilon}\left(1 + |x|^2 + |v|^2 + |\ln f_{0,\epsilon}|\right) \, d(x,v) \;+\; \int_{\mathbb{R}^N} |\nabla_x \Phi_{0,\epsilon}|^2 \, dx \;<\; +\infty. \tag{1.6}$$

Depending on the dimension we will prove two kind of results. In dimension $N = 2$ or 3 we prove that there exists $T^* > 0$ depending only on the initial data $f_{0,\epsilon}$ such that for all $t \in [0, T^*]$ the solution f_ϵ tends strongly in L^1, as $\epsilon \to 0$, to $(2\pi)^{-\frac{N}{2}} \rho(t,x) e^{-\frac{|v|^2}{2}}$, where ρ solves the following drift-diffusion equation

$$\frac{\partial \rho}{\partial t} - \mathrm{div}_x(\nabla_x \rho - \rho \nabla_x \Phi) = 0 \tag{1.7}$$

$$\Phi = \begin{cases} -\theta \frac{1}{4\pi}\frac{1}{|x|} * \rho, & \text{N=3} \\ \theta \frac{1}{2\pi}\log|x| * \rho, & \text{N=2,} \end{cases} \tag{1.8}$$

$$\rho(0,x) = \rho_0(x), \tag{1.9}$$

ρ_0 being an accumulation point of $\rho_{0,\epsilon} = \int f_{0,\epsilon} \, dv$.

In dimension $N = 2$ and for $\theta = 1$, we can obtain a weak convergence result which is global in time. Thus, in this context we will prove that

$$\rho_\epsilon \;\rightharpoonup\; \rho \;\; \text{in} \;\; L^\infty(0,T; L^1(\mathbb{R}^2)) \;\; \text{weak*} \quad \forall T > 0,$$

where ρ solves (1.7)-(1.9). The same result holds for $\theta = -1$ under the additional hypothesis

$$\mathrm{limsup}_{\epsilon\to 0,\, t\to\infty} \int_{\mathbb{R}^2} |\nabla_x \Phi_\epsilon(t,x)|^2 \, dx \;<\infty. \tag{1.10}$$

Let us remark that the system of equations (1.7), (1.8) may blow up in finite time in the case $\theta = -1$. Therefore (1.10) cannot be satisfied in general. This assumption prevents in some sense gravitationnal collapse.

The study of the stability of solutions of the VPFP system with respect to the variation of its constant parameters, the scaled thermal velocity and the scaled thermal mean free path, and in particular the case analyzed in this paper in which the scaled thermal velocity is the inverse of the scaled thermal mean free path and the latter tends to zero, is interesting from different point of views. Further than its intrinsic mathematical interest and its applications in numerical simulations, the asymptotic limits here studied allow to establish some links between models in different frameworks: Vlasov, drift-diffusion, Euler, ... This kind of results are also related to diffusion approximation techniques which have been used widely in various contexts: transport equation of neutronics [2], radiative transfer [3], semiconductors physics [10, 12].

The techniques used in this paper are mainly based on the control of the kinetic and potential energy, the entropy of the system and also of some moments associated with the density. This implies, via the Dunford-Pettis Theorem, the weak $L^1(\mathbb{R}^{2N})$ compactness of the sequence $\{\rho_\epsilon\}_{\epsilon>0}$ for every time $t \geq 0$. In the 2-D case this allows, due to the antisymmetry property of the Poisson kernel, to pass to the limit in the nonlinear term and, hence, in the continuity equation for the density. This cancellation property for the singularity in the Poisson kernel was similarly observed in the study of the existence of solutions for the 2D Euler equations in Fluid Mechanics, see [14]. For $\theta = \pm 1$ and $N = 2$ or 3, we also obtain strong $L^1(\mathbb{R}^{2N})$ compactness for $\{f_\epsilon\}_{\epsilon>0}$ and $\{\rho_\epsilon\}_{\epsilon>0}$, but only locally in time, by means of a weighted norm for the particle distribution involving the associated Maxwellian. From this estimate we get a bound for ρ_ϵ, independent of ϵ, in some $L^p(\mathbb{R}^N)$, with $p > N/2$ and, as consequence, an uniform bound in space for the force field $\nabla_x \Phi_\epsilon$ and $\partial_t \Phi_\epsilon$.

Let us summarize the literature concerning the existence results for the VPFP problem. Classical solutions have been studied by H. D. Victory and B. P. O'Dwyer [18] who proved the existence of a locally in time smooth solution to the problem (1.1)-(1.4). G. Rein and J. Weckler [13] gave sufficient conditions to prove the global existence of classical solutions in the three dimensional case. In the more general setting of weak solutions, we can mention the works by H. D. Victory [17] and J. A. Carrillo and J. Soler [7] with initial data in L^p spaces. F. Bouchut studied in [4] and [5] the regularity of the weak solutions of this system. A. Majda and Y. Zheng [19] and G. Majda, A. Majda and Y. Zheng [11] obtained the existence of global measure solutions in the 1-D case by using the relationship with the two dimensional Euler equation with vortex sheet initial data and constructed some relevant explicit solutions, which show the phenomena of singularity formation in finite time. Recently, J. A. Carrillo and J. Soler in [8] allow for measures in Morrey spaces as initial data and prove the existence of a locally in time weak solution. Finally, J. A. Carrillo and J. Soler introduced in [9] the concept of functional solution and proved the global existence of a functional solution when the initial data are only Radon measures with bounded variation.

The rest of the paper is organized as follows: In Section 2 we obtain the *a priori* estimates on the system which imply the weak $L^1(\mathbb{R}^{2N})$ compactness for the sequences $\{f_\epsilon\}_{\epsilon>0}$ and $\{\rho_\epsilon\}_{\epsilon>0}$. In Section 3 we study the strong $L^1(\mathbb{R}^{2N})$ compactness for the above sequences and uniform in space bounds for $\nabla_x \Phi_\epsilon$ which are local in time. Section 4 is devoted to show how the $L^1(\mathbb{R}^{2N})$ compactness of $\{\rho_\epsilon\}_{\epsilon>0}$ is enough to pass to the limit for $N = 2$. In Section 5 we obtain the parabolic limit equation. Finally, in Section 6 we motivate the problem under consideration through the analysis of the VPFP and its physical constants: the scaled thermal velocity and the scaled thermal mean free path.

2. *A priori* estimates

We start by defining the concept of weak solution to the problem (1.1)-(1.4). Let $Q_T = [0,T) \times \mathbb{R}^N \times \mathbb{R}^N$. Given $f_0 \in L^1(\mathbb{R}^{2N})$ we will say that the pair $(\Phi_\epsilon, f_\epsilon)$ is a weak solution to the VPFP$_\epsilon$ problem if

1. $f_\epsilon \in L^1(\mathbb{R}^{2N})$, Φ_ϵ is given by (1.5),
2. $f_\epsilon \nabla_x \Phi_\epsilon \in L^1_{loc}(\mathbb{R}^{2N})$,
3. for any $\Psi \in C_0^\infty(Q_T)$, we have

$$\int_{Q_T} f_\epsilon \left(\epsilon^2 \frac{\partial \Psi}{\partial t} + \epsilon\left((v\cdot\nabla_x)\Psi - (\nabla_x\Phi_\epsilon \cdot \nabla_v)\Psi\right) - (v\cdot\nabla_v)\Psi + \Delta_v \Psi \right) d(t,x,v)$$
$$= -\int_{\mathbb{R}^{2N}} f_{0,\epsilon}(x,v)\Psi(0,x,v)\; d(x,v) \qquad (2.11)$$

If $(\Phi_\epsilon, f_\epsilon)$ is a weak solution to the VPFP$_\epsilon$, then the distribution function f_ϵ can be equivalently obtained as a fixed point of the nonlinear integral equation

$$f_\epsilon(t,x,v) \;=\; \frac{1}{\epsilon^N}\int_{\mathbb{R}^{2N}} G(\frac{t}{\epsilon^2},\frac{x}{\epsilon},v,\frac{\xi}{\epsilon},\nu)\; f_{0,\epsilon}(\xi,\nu)\; d(\xi,\nu) \;-$$
$$\frac{1}{\epsilon^{N+1}}\int_0^t \int_{\mathbb{R}^{2N}} \nabla_\nu G(\frac{s}{\epsilon^2},\frac{x}{\epsilon},v,\frac{\xi}{\epsilon},\nu)\; \nabla_x\Phi_\epsilon(t-s,\xi) f_\epsilon(t-s,\xi,\nu)\; d(s,\xi,\nu), \quad (2.12)$$

where G is the fundamental solution to the linear operator

$$\frac{\partial \tilde f}{\partial t} \;+\; (v\cdot\nabla_x)\tilde f \;-\; L(\tilde f).$$

The fundamental solution G can be written as follows

$$G(t,x,v,\xi,\nu) \;=\; G_0\left(t, x-\xi-\eta(t)\nu, v - e^{-t}\nu\right), \qquad (2.13)$$

with $x, v, \xi, \nu \in \mathbb{R}^N, t \geq 0$ and

$$G_0(t,x,v) \;=\; \frac{1}{(4\pi)^N \;D(t)^{\frac{N}{2}}}\; e^{-\frac{1}{4}\varphi_0(t,x,v)},$$

where

$$D(t) \;=\; \frac{\eta(2t)}{2}t - \eta(t)^2 \quad \text{and} \quad \eta(t) = 1 - e^{-t}.$$

An explicit formula for φ_0 was developed in [18]:

$$\varphi_0(t,x,v) \;=\; \frac{1}{D(t)} \int_0^t |\eta(s)v - e^{-s}x|^2 \;\; ds.$$

The properties of solutions that we will study in this Section can rigorously be obtained from (2.11) by combining the formal arguments to be exposed here with the choice of an apropriate sequence of test functions in (2.11) for every studied property. Since a similar rigorous approach that the one given in [1] and [6] can be easily adapted for the properties studied in this Section, we omit the sometimes tedious and standard regularization procedure in order to give the main ideas. We refer to [1] and [6] to complete the proofs.

The first result gives us the mass conservation property as well as an equation for the kinetic energy, the potential energy and the entropy of the system, i.e., for the free energy functional.

Lemma 2.1 *Assume that the initial distribution of particles verifies that* $(1 + |v|^2 + \ln f_{0,\epsilon}) f_{0,\epsilon} \in L^1(\mathbb{R}^{2N})$ *and* $|\nabla \Phi_{0,\epsilon}| \in L^2(\mathbb{R}^N)$. *Then, we have that*

1. *the total mass of the system, i.e. the* $L^1(\mathbb{R}^{2N})$ *norm of* f_ϵ, *is preserved;*
2. *the following equation is verified by* $(\Phi_\epsilon, f_\epsilon)$*:*

$$\epsilon^2 \frac{d}{dt}\left(\int_{\mathbb{R}^{2N}} \left(\frac{|v|^2}{2} + \ln f_\epsilon \right) f_\epsilon \; d(x,v) + \frac{\theta}{2} \int_{\mathbb{R}^N} |\nabla_x \Phi_\epsilon|^2 \; dx \right)$$
$$= \; - \int_{\mathbb{R}^{2N}} \left| v\sqrt{f_\epsilon} + 2\nabla_v \sqrt{f_\epsilon} \right|^2 \; d(x,v) \qquad (2.14)$$

Proof. The mass conservation

$$\|f_\epsilon(t,\cdot,\cdot)\|_{L^1(\mathbb{R}^{2N})} \;=\; \|f_{0,\epsilon}\|_{L^1(\mathbb{R}^{2N})} \qquad (2.15)$$

follows formally by integrating Equation (1.1) in $\mathbb{R}^N_v$, which gives the continuity equation for the mass density

$$\partial_t \rho_\epsilon + \frac{1}{\epsilon} \mathrm{div}_x j_\epsilon = 0, \qquad (2.16)$$

in the sense of distributions, where j_ϵ is the current density defined by

$$j_\epsilon \;=\; \int_{\mathbb{R}^N} v f_\epsilon \; dv,$$

and then integrating in $\mathbb{R}^N_x$. Also, this property can be obtained by integrating Equation (2.12) in $\mathbb{R}^N_x \times \mathbb{R}^N_v$.

On the other hand, the equation for the balance of energy can be deduced multiplying (1.1) by $|v|^2$, integrating the result with respect to x and v and then using the divergence theorem. Thus we obtain

$$\begin{aligned} &\epsilon^2 \frac{d}{dt} \int_{\mathbb{R}^{2N}} \frac{|v|^2}{2} f_\epsilon \; d(x,v) \;+\; \epsilon \int_{\mathbb{R}^{2N}} (v \cdot \nabla_x \Phi_\epsilon) f_\epsilon \; d(x,v) \\ &= \; N \int_{\mathbb{R}^{2N}} f_\epsilon \; d(x,v) \;-\; \int_{\mathbb{R}^{2N}} |v|^2 f_\epsilon \; d(x,v) \end{aligned} \tag{2.17}$$

The second term in the left hand side of (2.17) can be written as follows

$$\int_{\mathbb{R}^{2N}} (v \cdot \nabla_x \Phi_\epsilon) f_\epsilon \; d(x,v) \;=\; -\int_{\mathbb{R}^N} \Phi_\epsilon \; \mathrm{div}_x \left(\int_{\mathbb{R}^N} v f_\epsilon \; dv \right) \; dx.$$

Then, taking into account the continuity equation (2.16) for ρ_ϵ we find

$$\int_{\mathbb{R}^{2N}} (v \cdot \nabla_x \Phi_\epsilon) f_\epsilon \; d(x,v) \;=\; \epsilon \int_{\mathbb{R}^N} \Phi_\epsilon \frac{\partial}{\partial t} \left(\int_{\mathbb{R}^N} f_\epsilon \; dv \right) \; dx.$$

Now, thanks to (1.3) we obtain

$$\int_{\mathbb{R}^{2N}} (v \cdot \nabla_x \Phi_\epsilon) f_\epsilon \; d(x,v) \;=\; \epsilon \frac{\theta}{2} \frac{d}{dt} \int_{\mathbb{R}^N} |\nabla_x \Phi_\epsilon|^2 \; dx.$$

Using this equality, (2.17) becomes

$$\begin{aligned} &\epsilon^2 \frac{1}{2} \frac{d}{dt} \left(\int_{\mathbb{R}^{2N}} |v|^2 f_\epsilon \; d(x,v) + \theta \int_{\mathbb{R}^N} |\nabla_x \Phi_\epsilon|^2 \; dx \right) \\ &= \; N \int_{\mathbb{R}^{2N}} f_\epsilon \; d(x,v) \;-\; \int_{\mathbb{R}^{2N}} |v|^2 f_\epsilon \; d(x,v). \end{aligned} \tag{2.18}$$

Finally, the balance of entropy identity is formally obtained by multiplying (1.1) by $\ln f_\epsilon$, integrating in x and v and using again the divergence theorem which yields

$$\begin{aligned} &\epsilon^2 \frac{d}{dt} \int_{\mathbb{R}^{2N}} f_\epsilon \ln f_\epsilon \; d(x,v) \\ &= \; -\int_{\mathbb{R}^{2N}} |\nabla_v f_\epsilon|^2 \frac{1}{f_\epsilon} \; d(x,v) \;+\; N \int_{\mathbb{R}^{2N}} f_\epsilon \; d(x,v) \end{aligned} \tag{2.19}$$

Adding (2.18) and (2.19) we have

$$\epsilon^2 \frac{d}{dt}\left(\int_{\mathbb{R}^{2N}}\left(\frac{|v|^2}{2}+\ln f_\epsilon\right) f_\epsilon \; d(x,v)+\frac{\theta}{2}\int_{\mathbb{R}^N}|\nabla_x\Phi_\epsilon|^2 \; dx\right)$$

$$= \int_{\mathbb{R}^{2N}}\left(2Nf_\epsilon-|v|^2f_\epsilon-|\nabla_v f_\epsilon|^2\frac{1}{f_\epsilon}\right) \; d(x,v). \tag{2.20}$$

Then, since

$$\int_{\mathbb{R}^{2N}} Nf_\epsilon \; d(x,v) \;=\; -\int_{\mathbb{R}^{2N}} v\cdot\nabla_v f_\epsilon \; d(x,v),$$

we can write the second member of (2.20) as

$$-\int_{\mathbb{R}^{2N}}\frac{1}{f_\epsilon}\left|vf_\epsilon+\nabla_v f_\epsilon\right|^2 \; d(x,v)$$

Using now the identity $(\sqrt{f_\epsilon})^{-1}\nabla_v f_\epsilon = 2\nabla_v\sqrt{f_\epsilon}$ we find the announced result (2.14).

Previous Lemma allows to deduce, due to the negativity of the right hand side of (2.14), that the free energy functional

$$E_\epsilon(t) \stackrel{def}{=} \int_{\mathbb{R}^{2N}}\left(\frac{|v|^2}{2}+\ln f_\epsilon\right) f_\epsilon \; d(x,v)+\frac{\theta}{2}\int_{\mathbb{R}^N}|\nabla_x\Phi_\epsilon|^2 \; dx$$

is bounded. However, to conclude that the kinetic energy, the potential energy and the entropy of the VPFP$_\epsilon$ system are uniformly bounded with respect to $t\in[0,T]$, $\forall T>0$, we must prove that the above functional is also bounded from below.

Lemma 2.2 *Assume $\theta=1$ and that the initial distribution of particles verifies that $(1+|v|^2+\ln f_{0,\epsilon})f_{0,\epsilon}\in L^1(\mathbb{R}^{2N})$ and $|\nabla\Phi_{0,\epsilon}|\in L^2(\mathbb{R}^N)$. Let us also assume for the case $\theta=-1$ the hypothesis (1.10).*

1. *The functional $E_\epsilon(t)$ is bounded from below and, as consequence, the following quantities are bounded for any $t\in[0,T]$, with bounds which are independent of ϵ and t,*

$$\int_{\mathbb{R}^{2N}} f_\epsilon|\ln f_\epsilon| \; d(x,v)\;,\;\int_{\mathbb{R}^{2N}}|v|^2 f_\epsilon \; d(x,v)\;,\;\int_{\mathbb{R}^N}|\nabla\Phi_\epsilon(t,x)|^2 \; dx.$$

2. *The sequence $\{f_\epsilon\}_{\epsilon>0}$ is weakly compact in $L^1(\mathbb{R}^{2N})$ for $t\in[0,T]$, $\forall T>0$.*

Proof. From Lemma 2.1 we deduce that

$$E_\epsilon(t)\le E_\epsilon(0) \tag{2.21}$$

Define $\ln^- f_\epsilon \stackrel{def}{=} \max\{-\ln f_\epsilon, 0\}$ and $\ln^+ f_\epsilon \stackrel{def}{=} \max\{\ln f_\epsilon, 0\}$, which gives

$$\ln f_\epsilon \;=\; \ln^+ f_\epsilon \;-\; \ln^- f_\epsilon. \tag{2.22}$$

To obtain a bound from below for $E_\epsilon(t)$, we split the domain in three parts $\{f_\epsilon > e^{-1}\}$, $\{e^{-1} \geq f_\epsilon > e^{-1}e^{-\frac{1}{b}\frac{|v|^2}{2}}\}$ and $\{f_\epsilon \leq e^{-1}e^{-\frac{1}{b}\frac{|v|^2}{2}}\}$ to find

$$f_\epsilon \ln^- f_\epsilon \leq \left(f_\epsilon + e^{-1}e^{-\frac{1}{b}\frac{|v|^2}{2}} \right) \left(1 + \frac{1}{b}\frac{|v|^2}{2} \right), \tag{2.23}$$

where $b > 0$ is a constant to be precised. Then, combining (2.21), (2.22) and (2.23) we can obtain, see [6],

$$\left(1 - \frac{1}{b}\right) \int_{\mathbb{R}^{2N}} |v|^2 f_\epsilon \; d(x,v) \;+\; \theta \int_{\mathbb{R}^N} |\nabla \Phi_\epsilon|^2 \; dx \;+\; \int_{\mathbb{R}^{2N}} f_\epsilon \ln^+ f_\epsilon \; d(x,v)$$
$$\leq\; E_\epsilon(0) \;+\; c,$$

where c is a constant independent of ϵ. This estimate together with (1.10) for $\theta = -1$ gives the first assertion of the Lemma.

Finally, the estimates given in i) and the Dunford-Pettis theorem imply that the sequence $\{f_\epsilon\}_{\epsilon>0}$ is weakly compact in $L^1(\mathbb{R}^{2N})$.

As a consequence of Lemma 2.2 we have the following result

Lemma 2.3 *Let us assume that the initial distribution of particles verifies that* $(1 + |v|^2 + \ln f_{0,\epsilon}) f_{0,\epsilon} \in L^1(\mathbb{R}^{2N})$ *and* $|\nabla \Phi_{0,\epsilon}| \in L^2(\mathbb{R}^N)$ *and that* $\nabla_x \Phi_\epsilon$ *verifies (1.10) for* $\theta = -1$. *Then, we have*

1. *the function* $h_\epsilon \stackrel{def}{=} \epsilon^{-1}\left(v\sqrt{f_\epsilon} - 2\nabla_v \sqrt{f_\epsilon}\right)$ *verifies*

$$||h_\epsilon(t,\cdot,\cdot)||_{L^2(\mathbb{R}^{2N})} \leq c, \tag{2.24}$$

where c *is a constant independent of* ϵ *and* t;

2. *the current density* j_ϵ *is of order* ϵ *in* $L^\infty(0,T; L^1(\mathbb{R}^{2N}))$.

Proof. The first assertion follows from (2.14) and the uniform bounds given in Lemma 2.2 for the kinetic energy, the potential energy and the entropy of the system.

The current density can be written in terms of h_ϵ as follows

$$j_\epsilon = \epsilon \int_{\mathbb{R}^N} h_\epsilon \sqrt{f_\epsilon} \; dv$$

which implies, using the first assertion of the Lemma and the mass consevation, that

$$j_\epsilon \;=\; O(\epsilon) \text{ in } L^\infty(0,T; L^1(\mathbb{R}^{2N})),$$

where $O(\epsilon)$ means order ϵ.

A similar compactness property can be deduced for the sequence of densities.

Lemma 2.4 *Under the hypothesis* $(1+|x|^2+|v|^2+\ln f_{0,\epsilon})f_{0,\epsilon} \in L^1(\mathbb{R}^{2N})$, $|\nabla\Phi_{0,\epsilon}| \in L^2(\mathbb{R}^N)$ *and (1.10) for* $\theta = -1$*, we have that* $|x|\rho_\epsilon \in L^1(\mathbb{R}^N)$ *and, as consequence, the sequence* $\{\rho_\epsilon\}_{\epsilon>0}$ *is weakly compact in* $L^1(\mathbb{R}^N)$ *for* $t \in [0,T]$, $\forall T > 0$.

Proof. To deduce that $|x|\rho_\epsilon \in L^1(\mathbb{R}^N)$ we first multiply (1.1) by $|x|$ and integrate in $\mathbb{R}^N_x \times \mathbb{R}^N_v$ to obtain the following equation

$$\epsilon^2 \frac{d}{dt}\int_{\mathbb{R}^{2N}} |x| f_\epsilon \, d(x,v) \;=\; \epsilon \int_{\mathbb{R}^{2N}} \frac{x}{|x|}\cdot\left(\int_{\mathbb{R}^3} v f_\epsilon \, dv\right) dx.$$

Then, aplying Lemma 2.3 ii), which gives $j_\epsilon = O(\epsilon)$ in $L^\infty(0,T;L^1(\mathbb{R}^{2N}))$, and the hypothesis $|x|^2\rho_{0,\epsilon} \in L^1(\mathbb{R}^N)$, which implies, using the mass conservation, $|x|\rho_{0,\epsilon} \in L^1(\mathbb{R}^N)$, we deduce the expected result.

The weak compactness in $L^1(\mathbb{R}^N)$ is again a consequence of the first assertion andof the Dunford-Pettis theorem applied to the sequence $\{\rho_\epsilon\}_{\epsilon>0}$.

Remark 2.1. Let us note that under our hypothesis we cannot assure that the inertial momentum

$$\int_{\mathbb{R}^{2N}} |x|^2 f_\epsilon \, d(x,v)$$

remains bounded for $t > 0$. In fact, from the equation for the inertial momentum

$$\epsilon^2 \frac{d}{dt}\int_{\mathbb{R}^{2N}} |x|^2 f_\epsilon \, d(x,v) \;=\; 2\epsilon \int_{\mathbb{R}^N} x\cdot\left(\int_{\mathbb{R}^N} v f_\epsilon \, dv\right) dx \tag{2.25}$$

we cannot obtain a bound uniform with respect to ϵ.

The above weak compactness property in $L^1(\mathbb{R}^N)$ is not enough to pass to the limit on the nonlinear term of equation (1.1). Note that we cannot use L^p, $p > 1$, *a priori* estimates because of they all depend on ϵ as it can be directly calculated from (2.12), see also [7] and [17]. In the two next sections we will give some results that provide some extra compactness properties which will depend on the dimension N and on the hypothesis on the initial data.

3. Strong convergence in a bounded time interval

In this Section we will obtain bounds for ρ_ϵ in some L^p, $p > N/2$, independent of ϵ which implies $\nabla_x\Phi_\epsilon \in L^\infty(\mathbb{R}^N)$. These estimates give us strong convergence locally in time. With this in mind, we define the norm

$$|||f|||_p \stackrel{def}{=} \left(\int_{\mathbb{R}^{2N}} |f|^p e^{(p-1)\frac{|v|^2}{2}} \, d(x,v)\right)^{1/p}. \tag{3.26}$$

Lemma 3.5 *Assume that* $e^{\frac{1}{p'}\frac{|v|^2}{2}} f_{0,\epsilon} \in L^p(\mathbb{R}^{2N})$, *with* $p > N/2$. *Then, there exists a finite* $T^* > 0$ *and a constant* $c > 0$ *depending on* $f_{0,\epsilon}$ *and independent of* ϵ *such that*

1. *the distribution of particles verifies*
$$|||f_\epsilon(t,\cdot)|||_p \leq c, \ \forall t \in [0, T^*]; \tag{3.27}$$

2. *the following estimate*
$$||\rho_\epsilon(t,\cdot)||_{L^p(\mathbb{R}^N)} \leq c, \ \forall t \in [0, T^*] \tag{3.28}$$
holds for the density;

3. *the potential verifies*
$$||\partial_t \Phi_\epsilon(t,\cdot)||_{L^\infty(\mathbb{R}^N)} + ||\nabla_x \Phi_\epsilon(t,\cdot)||_{L^\infty(\mathbb{R}^N)} < \infty. \tag{3.29}$$

Proof. Let H be a convex regular function to be precised. If we multiply the right hand side of (1.1) by $H'(e^{\frac{|v|^2}{2}} f_\epsilon)$, we have

$$\int_{\mathbb{R}^N} -L(f_\epsilon) H'(e^{\frac{|v|^2}{2}} f_\epsilon) \, dv \ = \ \int_{\mathbb{R}^N} e^{-\frac{|v|^2}{2}} H''(e^{\frac{|v|^2}{2}} f_\epsilon) \left| e^{\frac{|v|^2}{2}} \nabla_v f_\epsilon \right|^2 dv.$$

We define $q_H(f_\epsilon, f_\epsilon)$ as follows

$$q_H(f_\epsilon, f_\epsilon) := \int_{\mathbb{R}^N} e^{-\frac{|v|^2}{2}} H''(e^{\frac{|v|^2}{2}} f_\epsilon) \left| \nabla_v \left(e^{\frac{|v|^2}{2}} f_\epsilon \right) \right|^2 dv. \tag{3.30}$$

Proceeding in the same way with the other terms in equation (1.1) we first find for the nonlinear one the following estimate

$$\begin{aligned} \int_{\mathbb{R}^N} -(\nabla_x \Phi_\epsilon \cdot \nabla_v) f_\epsilon \, H'(e^{\frac{|v|^2}{2}}) \, dv \ &= \ \int_{\mathbb{R}^N} \nabla_x \Phi_\epsilon \ f_\epsilon \ H''(e^{\frac{|v|^2}{2}}) \cdot \nabla_v \left(e^{\frac{|v|^2}{2}} f_\epsilon \right) dv \leq \\ & |\nabla_x \Phi_\epsilon| \int_{\mathbb{R}^N} e^{-\frac{|v|^2}{2}} H''(e^{\frac{|v|^2}{2}}) e^{\frac{|v|^2}{2}} f_\epsilon \nabla_v \left(e^{\frac{|v|^2}{2}} f_\epsilon \right) dv \leq \\ & |\nabla_x \Phi_\epsilon| \left(\int_{\mathbb{R}^N} f_\epsilon^2 H''(e^{\frac{|v|^2}{2}}) e^{\frac{|v|^2}{2}} dv \right)^{1/2} q_H(f_\epsilon, f_\epsilon)^{1/2}. \end{aligned} \tag{3.31}$$

Similarly for the others terms we have

$$\int_{\mathbb{R}^N} \frac{\partial f_\epsilon}{\partial t} H'(e^{\frac{|v|^2}{2}} f_\epsilon) \, dv \ = \ \frac{\partial}{\partial t} \int_{\mathbb{R}^N} H(e^{\frac{|v|^2}{2}} f_\epsilon) e^{-\frac{|v|^2}{2}} dv \tag{3.32}$$

and

$$\int_{\mathbb{R}^{2N}} (v \cdot \nabla_x) f_\epsilon H'(e^{\frac{|v|^2}{2}}) \; d(x,v) \; = \; 0. \tag{3.33}$$

Therefore, combining (3.30)-(3.33) we have that a solution of (1.1) satisfies

$$\frac{d}{dt}\int_{\mathbb{R}^{2N}} H(e^{\frac{|v|^2}{2}} f_\epsilon)e^{-\frac{|v|^2}{2}} \; d(x,v) \; + \; \frac{1}{\epsilon^2}\int_{\mathbb{R}^N} q_H(f_\epsilon, f_\epsilon) \; dx$$

$$\leq \frac{1}{\epsilon}\|\nabla_x \Phi_\epsilon\|_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N} \left(\int f_\epsilon^2 H''(e^{\frac{|v|^2}{2}} f_\epsilon) e^{\frac{|v|^2}{2}} dv \right)^{1/2} q_H(f_\epsilon, f_\epsilon)^{1/2} \; dx$$

$$\leq \frac{1}{2\epsilon^2}\int_{\mathbb{R}^N} q_H(f_\epsilon, f_\epsilon) dx \; + \; \frac{1}{2}\|\nabla_x \Phi_\epsilon\|^2_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^{2N}} f_\epsilon^2 H''\left(f_\epsilon e^{\frac{|v|^2}{2}}\right) e^{\frac{|v|^2}{2}} \; d(x,v).$$

Then, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^{2N}} H\left(f_\epsilon e^{\frac{|v|^2}{2}}\right) e^{-\frac{|v|^2}{2}} \; d(x,v) \; + \; \frac{1}{2\epsilon^2}\int_{\mathbb{R}^N} q_H(f_\epsilon, f_\epsilon) \; dx$$

$$\leq \frac{1}{2}\|\nabla_x \Phi_\epsilon(t,\cdot)\|^2_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^{2N}} \left(f_\epsilon e^{\frac{|v|^2}{2}}\right)^2 H''\left(f_\epsilon e^{\frac{|v|^2}{2}}\right) e^{-\frac{|v|^2}{2}} \; d(x,v). \tag{3.34}$$

We choose, for $1 < p < \infty$, $H(t) = t^p$. Then, using the norm (3.27), (3.34) becomes

$$\frac{d}{dt}|||f_\epsilon(t,\cdot)|||_p \leq \frac{1}{4}p(p-1)\|\nabla_x \Phi_\epsilon(t,\cdot)\|_{L^\infty(\mathbb{R}^N)}|||f_\epsilon(t,\cdot)|||_p, \tag{3.35}$$

where we have used the positivity of $q_H(f_\epsilon, f_\epsilon)$.

On the other hand, using Hölder's inequality it is straightforward to find that

$$\|\rho_\epsilon(t,\cdot)\|_{L^p(\mathbb{R}^N)} \leq c(p)|||f_\epsilon(t,\cdot)|||_p,$$

where

$$c(p) = \left(\int_{\mathbb{R}^N} e^{-\frac{|v|^2}{2}} \; dv\right)^{1/p'}.$$

As a consequence of the Hardy-Littlewood-Sobolev Theorem (see [16]), for $p > N/2$, we find

$$\|\nabla_x \Phi_\epsilon(t,\cdot)\|_{L^\infty(\mathbb{R}^N)} \leq c(p)\left(\|\rho_\epsilon(t,\cdot)\|_{L^1(\mathbb{R}^N)} + \|\rho_\epsilon(t,\cdot)\|_{L^p(\mathbb{R}^N)}\right).$$

Therefore, combining the above estimates in (3.35) we have

$$\frac{d}{dt}|||f_\epsilon(t,\cdot)|||_p \leq c(p)\left(\|f_{0,\epsilon}\|_{L^1(\mathbb{R}^6)} + |||f_\epsilon(t,\cdot)|||_p\right)|||f_\epsilon(t,\cdot)|||_p. \tag{3.36}$$

From (3.36) we deduce the announced result i).

These bounds ensure that ρ_ϵ is uniformly bounded in $L^\infty(0,T^\star;L^1(\mathrm{IR}^N)\cap L^p(\mathrm{IR}^N))$, with $p>N/2$, and j_ϵ in $L^\infty(0,T^\star;L^q(\mathrm{IR}^N))$, for $q\in]1,p]$. As a consequence, we have (3.29).

The estimates given in Lemma 3.5 imply the strong convergence in $L^1(\mathrm{IR}^{2N})$ of the sequence of functions $\{f_\epsilon\}_{\epsilon>0}$, for $t\in[0,T^\star]$, by applying a result of F. Bouchut and J. Dolbeault, see [6]. Also, (3.29) allows to pass to the limit in the nonlinear term.

Let us now give some consequences of the previous Lemma 3.5 which will be useful in Section 5 to deduce the limit equation and that improve the estimates in Lemma 2.3.

Lemma 3.6 *Under the hypothesis of Lemma 3.5, we have*

1. $q_H(f_\epsilon,f_\epsilon)$ *verifies*
$$q_H(f_\epsilon,f_\epsilon)=O(\epsilon^2)\ \text{in}\ L^\infty(0,T^*;L^1(\mathrm{IR}^{2N}));$$
2. *the current density* j_ϵ *is of order* ϵ *in* $L^\infty(0,T^*;L^2(\mathrm{IR}^N))$.

Proof. The first assertion can be deduced form (3.34), which also implies

$$\int_{\mathrm{IR}^{2N}} f_\epsilon^{p-2}\left|\nabla_v\left(f_\epsilon e^{\frac{|v|^2}{2}}\right)\right|^2 e^{(p-3)\frac{|v|^2}{2}}\,dt\,d(x,v) \;=\; O(\epsilon^2). \tag{3.37}$$

Then, for $p=2$ in (3.37) we obtain that

$$\nabla_v\left(f_\epsilon e^{\frac{|v|^2}{2}}\right) \;=\; O(\epsilon)\ \text{in}\ L^\infty(0,T^*;L^2(\mathrm{IR}^{2N}))$$

and, hence, we have that

$$j_\epsilon \;=\; \int_{\mathrm{IR}^N}\nabla_v\left(f_\epsilon e^{\frac{|v|^2}{2}}\right)e^{-\frac{|v|^2}{2}}\,dv$$

is $O(\epsilon)$ in $L^\infty(0,T^*;L^2(\mathrm{IR}^N))$.

4. Global in time weak convergence in N=2

¿From Lemma 2.4 we have that

$$\rho_\epsilon\,(1+|x|+|\mathrm{ln}\rho_\epsilon|)\ \text{is bounded in}\ L^\infty(0,T;L^1(\mathrm{IR}^2)) \tag{4.38}$$

with bounds independent of ϵ, which gives the weak $L^1(\mathrm{IR}^2)$ convergence of the sequence $\{\rho_\epsilon\}_{\epsilon>0}$ for $t\in[0,T]$, $\forall T>0$. This property is enough to pass to the limit in dimension $N=2$ on the continuity equation for the density. In fact, as we will see in the next Section the following nonlinear term appears in the equation for ρ_ϵ

$$\nabla_x \Phi_\epsilon \rho_\epsilon \;=\; \left(\frac{1}{2\pi}\frac{x}{|x|^2} * \rho_\epsilon\right)\rho_\epsilon,$$

or, by using (1.3), in a weak sense

$$\int_0^T \int_{\mathbb{R}^2} \nabla_x \Phi_\epsilon(t,x)\rho_\epsilon(t,x)\varphi(t,x)\; dt\; dx$$

$$= \frac{1}{2\pi}\int_0^T \int_{\mathbb{R}^4} \frac{x-y}{|x-y|^2}\rho_\epsilon(t,y)\rho_\epsilon(t,x)\varphi(t,x)\; dt\; d(y,x),$$

where $\varphi \in C_0^\infty([0,T)\times \mathbb{R}^2)$.

Note that due to the antisymmetric property of the kernel K_2 we can write

$$\int_0^T \int_{\mathbb{R}^4} \Psi(t,x,y)\; dt\; d(y,x) \;=\; -\int_0^T \int_{\mathbb{R}^4} \Psi(t,y,x)\; dt\; d(y,x),$$

being

$$\Psi(t,x,y) \;=\; \frac{1}{2\pi}\frac{x-y}{|x-y|^2}\rho_\epsilon(t,y)\rho_\epsilon(t,x)\varphi(t,x).$$

Hence, the nonlinear term can be written as follows

$$\int_0^T \int_{\mathbb{R}^4} \Psi(t,x,y)\; dt\; d(y,x) \;=\; \int_0^T \int_{\mathbb{R}^4} \frac{\Psi(t,x,y)-\Psi(t,y,x)}{2}\; dt\; d(y,x)$$

$$\leq \frac{1}{2\pi}\int_0^T \int_{\mathbb{R}^4} \rho_\epsilon(t,y)\rho_\epsilon(t,x)\frac{|\varphi(t,x)-\varphi(t,y)|}{2|x-y|}\; dt\; d(y,x).$$

Since φ is regular, in particular Lipschitz, the above expression for the nonlinear term allows to pass to the limit with the only property of the weak $L^1(\mathbb{R}^2)$ convergence of the sequence $\{\rho_\epsilon\}_{\epsilon>0}$, for $t\in[0,T]$, $\forall T>0$. Then, we have

$$\nabla_x \Phi_\epsilon \rho_\epsilon \;=\; \left(\frac{1}{2\pi}\frac{x}{|x|^2} * \rho_\epsilon\right)\rho_\epsilon \;\rightharpoonup\; \left(\frac{1}{2\pi}\frac{x}{|x|^2} * \rho\right)\rho \;=\; \nabla_x \Phi \rho \;\text{ in } \mathcal{D}'. \quad (4.39)$$

This cancellation property of the 2-D Poisson kernel was used previously in the framework of the study of weak solutions for the Euler equations, see S.Schochet [14] and the references therein. We also remark that the above proof shows that

$$\nabla_x \Phi_\epsilon \;\rightharpoonup\; \frac{1}{2\pi}\frac{x}{|x|^2} * \rho$$

5. The parabolic limit equation

¿From the results in Section 2 we have that

$$f_\epsilon \left(1+|x|+|v|^2+|\ln f_\epsilon|\right) \text{ is bounded in } L^\infty(0,T;L^1(\mathbb{R}^{2N})), \tag{5.40}$$

$$\nabla_x \Phi_\epsilon \text{ is bounded in } L^\infty(0,T;L^2(\mathbb{R}^N)) \tag{5.41}$$

and

$$h_\epsilon \;=\; \frac{1}{\epsilon}\left(2\nabla_v\sqrt{f_\epsilon}+v\sqrt{f_\epsilon}\right) \text{ is bounded in } L^\infty(0,T;L^2(\mathbb{R}^{2N})), \tag{5.42}$$

for all $T>0$.

Using (5.40), the fact that the current density is $O(\epsilon)$ in $L^\infty(0,T;L^1(\mathbb{R}^N))$, which is given in Lemma 2.3, and the continuity conservation law for ρ_ϵ in the sense of distributions

$$\partial_t \rho_\epsilon + \frac{1}{\epsilon}\mathrm{div}_x j_\epsilon = 0,$$

we can deduce that ρ_ϵ lies in a weakly compact set of L^1 and $\partial_t\rho_\epsilon$ is bounded in $L^\infty(0,T;W^{-1,1}(\mathbb{R}^N))$, which provides the continuity in time of the sequence $\{\rho_\epsilon\}_{\epsilon>0}$.

We will try to obtain the convergence properties of j_ϵ/ϵ, as $\epsilon\to 0$, to obtain the parabolic limit from the continuity equation for ρ_ϵ. Then, multiplying equation (1.1) by v/ϵ, we find that

$$\epsilon\partial_t j_\epsilon \;+\; \mathrm{div}_x \int_{\mathbb{R}^N} v\otimes v f_\epsilon dv \;+\; N\nabla_x\Phi_\epsilon\rho_\epsilon \;=\; \frac{N}{\epsilon} j_\epsilon \tag{5.43}$$

is satisfied in the sense of distributions.

Taking into account that

$$\int_{\mathbb{R}^N} v\otimes\nabla_v f_\epsilon \; dv \;=\; -N\rho_\epsilon I_N,$$

where I_N is the identity matrix of $\mathbb{R}^N$, we have

$$\int_{\mathbb{R}^N} v\otimes v f_\epsilon \; dv - N\rho_\epsilon I_N$$

$$=\; \epsilon\int_{\mathbb{R}^N} h_\epsilon\otimes v\sqrt{f_\epsilon}\; dv \;=\; O(\epsilon) \text{ in } L^\infty(0,T;L^1(\mathbb{R}^N)). \tag{5.44}$$

In the same way, using now Lemma 3.6 we have

$$\int_{\mathbb{R}^N} v\otimes v f_\epsilon \; dv - N\rho_\epsilon I_N \;=\; O(\epsilon) \text{ in } L^\infty(0,T^*;L^2(\mathbb{R}^N)). \tag{5.45}$$

Also, for the strong convergence result we have that the current density j_ϵ is of order ϵ in $L^\infty(0,T^*;L^2(\mathbb{R}^N))$, see Lemma 3.6 and the nonlinear term converges in $L^1(\mathbb{R}^{2N})$ for $t \in [0,T^*]$. We pass to the limit in a similar way in both cases of weak convergence ($N=2$ and $t \in [0,\infty)$) and of strong convergence ($N = 2,3$ and $t \in [0,T^*]$), with more properties in the strong convergence case. Therefore we omit the continuous reference to every case from now on.

Thus, we pass to the limit in (5.43). For the nonlinear term we find

$$\nabla_x \Phi_\epsilon \rho_\epsilon \;=\; (\theta K_N * \rho_\epsilon)\,\rho_\epsilon \;\rightharpoonup\; (\theta K_N * \rho)\,\rho \;=\; \nabla_x \Phi \rho. \tag{5.46}$$

On the other hand, (5.44) or (5.45) imply

$$\int_{\mathbb{R}^N} v \otimes v f_\epsilon dv \;\rightharpoonup\; N\rho\ Id \tag{5.47}$$

Therefore, we conclude

$$\frac{1}{\epsilon} j_\epsilon \;\rightharpoonup\; -\nabla_x \rho - \nabla_x \Phi \rho. \tag{5.48}$$

Taking into account this relation in the continuity equation for ρ_ϵ, we obtain that ρ verifies

$$\partial_t \rho - \mathrm{div}_x(\nabla_x \rho + \nabla_x \Phi \rho) = 0 \tag{5.49}$$

in the sense of distributions.

Since ρ_ϵ lies in a compact set of $C^0(0,T;W^{-1,1}(\Omega))$ for every compact $\Omega \subset \mathbb{R}^N$, we have also that

$$\rho_{0,\epsilon}(x) \;=\; \int_{\mathbb{R}^N} f_{0,\epsilon}(x,v)\ dv \;\rightharpoonup\; \rho(0,x) \tag{5.50}$$

which gives the Cauchy data. We also get :

$$\nabla_x \phi_\epsilon \rightharpoonup K_N * \rho \tag{5.51}$$

Then, we have proved the following result

Theorem 5.7 *Assume that*

$$\int_{\mathbb{R}^{2N}} f_{0,\epsilon}\left(1 + |x|^2 + |v|^2 + |\ln f_{0,\epsilon}|\right)\ d(x,v) \;+\; \int_{\mathbb{R}^N} |\nabla_x \Phi_{0,\epsilon}|^2\ dx \;<\; +\infty. \tag{5.52}$$

Then, we have that

1. *for $\theta = 1$ and $N = 2$ the sequence $\{\rho_\epsilon\}_{\epsilon>0}$ converges in $L^\infty((0,\infty);L^1(\mathbb{R}^2)$ weak)weak$*$ towards a solution ρ of (5.49), (5.51) with initial data given by (5.50); the same result is still valid assuming for $\theta = -1$ the following hypothesis*

$$\mathit{limsup}_{\epsilon\to 0,\ t\to\infty} \int_{\mathbb{R}^2} |\nabla_x \Phi_\epsilon(t,x)|^2\ dx \;<\infty; \tag{5.53}$$

2. *if the initial data also satisfies* $e^{\frac{1}{p'}\frac{|v|^2}{2}} f_{0,\epsilon} \in L^p(\mathbb{R}^{2N})$, *with* $N = 2$ *or* 3 *and* $p > N/2$, *then there exists a finite* $T^* > 0$ *such that the sequence* $\{f_\epsilon\}_{\epsilon>0}$ *converges strongly in* $L^1(\mathbb{R}^{2N})$ *towards* $(2\pi)^{-\frac{N}{2}}\rho(t,x)e^{-\frac{|v|^2}{2}}$, *for* $t \in [0, T^*]$, *and* $\rho \in L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ *solves (5.49), (5.51) with initial data given by (5.50).*

6. Appendix: The VPFP system and the physical constants

The idea of this section is to write the VPFP system in terms of the physical constants: the scaled thermal velocity and the scaled thermal mean free path. This will allow to study the behavior of solutions with respect to these constants, see [15]. Consider the VPFP system in the case of charged particles interacting through electrostatic forces.

$$\frac{\partial f}{\partial t} + (v \cdot \nabla_x) f + \frac{q}{m}(\nabla_x \Phi \cdot \nabla_v) f \;=\; L(f) \tag{6.54}$$

$$-\epsilon_0 \Delta_x \Phi \;=\; -\theta\rho, \quad \rho(t,x) \;=\; \int_{\mathbb{R}^N} f(t,x,v)\, dv \tag{6.55}$$

$$f(0,x,v) \;=\; f_0(x,v) \tag{6.56}$$

where

$$L(f) \;=\; \frac{\mu}{\tau}\nabla_v \big(e^{\frac{-|v|^2}{2\mu}} \nabla_v (e^{\frac{|v|^2}{2\mu}} f)\big),$$

$\theta = 1$, m is the particle mass, ϵ_0 the permitivity of vacuum, τ the relaxation time and where $\sqrt{\mu}$ is the thermal velocity.

There is a microscopic variation of v which is $\sqrt{\mu}$ and a macroscopic mean velocity associated with the distribution of particles f given by

$$u_0 = \frac{\int_{\mathbb{R}^N} vf\, dv}{\int_{\mathbb{R}^N} f\, dv}.$$

Hence, we choose a scaling such that

$$v \longrightarrow \sqrt{\mu} v$$

$$x \longrightarrow Rx$$

$$t \longrightarrow Tt$$

with $\frac{R}{T} = u_0$. To adimensionalize the Poisson equation, we introduce a characteristic value of concentration M and a characteristic variation of the potential Φ_0 over a typical length R. We perform the change of unknowns

$$f \longrightarrow \frac{M}{\sqrt{\mu^3}} f,$$

$$\Phi \longrightarrow \Phi_0 \Phi,$$

choosing $\Phi_0 = \frac{1}{\epsilon_0} M R^2$, to obtain

$$\Delta_x \Phi \;=\; \theta \rho$$

We remark that we control only two constants (the rest are physical constants) M, which depends on the size of the initial data and R (or u_0). We are now ready to adimensionalize the Fokker-Planck equation by using the re-scaling

$$\alpha \;=\; \frac{\sqrt{\mu}}{u_0}$$

for the scaled thermal velocity, and

$$\beta \;=\; \frac{\tau\sqrt{\mu}}{R}$$

for the scaled thermal mean free path. Then, our system reads

$$\frac{\partial f}{\partial t} + \alpha(v \cdot \nabla_x) f + \frac{1}{\beta}(\nabla_x \Phi \cdot \nabla_v) f \;=\; \frac{\alpha}{\beta} L(f), \tag{6.57}$$

$$\Delta_x \Phi \;=\; \theta \rho, \tag{6.58}$$

$$f(0, x, v) \;=\; f_0(x, v), \tag{6.59}$$

where

$$L(f) \;=\; \nabla_v (e^{\frac{-|v|^2}{2}} \nabla_v (e^{\frac{|v|^2}{2}} f)).$$

The same result holds (with different physical constants) for massive particles interacting through gravitationnal forces. In this case $\theta = -1$.

Now, the idea is to study the stability of solutions to the VPFP system with respect to α and β.

As soon as we choose $\alpha = 1/\beta$ equation (6.57) becomes equation (1.1), which has been studied in this paper. The analysis of solutions to the system (1.1)-(1.5) as β goes to zero leads to the parabolic limit of the VPFP system.

The hyperbolic limit consists in assuming that $\alpha = 1$ and $\beta \to 0$. We conjecture that in this case we will find the following limit behaviour

$$f \longrightarrow (2\pi)^{\frac{N}{2}} \rho(t, x) e^{-\frac{|v - \nabla_x \Phi|^2}{2}}$$

where $\rho(t,x)$ satisfies the following continuity equation with non-bounded energy:

$$\frac{\partial \rho}{\partial t} + \nabla_x(\rho \nabla_x \Phi) = 0,$$

which will be studied in a forthcoming publication.

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