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ON THE DOUBLE COMMUTATION METHOD

F. GESZTESY AND G. TESCHL

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ABSTRACT. We provide a complete spectral characterization of the double commutation method for general Sturm-Liouville operators which inserts any finite number of prescribed eigenvalues into spectral gaps of a given background operator. Moreover, we explicitly determine the transformation operator which links the background operator to its doubly commuted version (resulting in extensions and considerably simplified proofs of spectral results even for the special case of Schrödinger-type operators).

1. INTRODUCTION

Methods of inserting (and removing) eigenvalues in spectral gaps of a given one-dimensional Schrödinger operator H associated with differential expressions of the type $-\frac{d^2}{dx^2}+q$ on (a,∞), $a \geq -\infty$ have recently attracted an enormous amount of attention. This is due to their prominent role in diverse fields such as the inverse scattering approach introduced by Deift and Trubowitz [7], supersymmetric quantum mechanics (see, e.g., [23] and the references therein), level comparison theorems (cf. [2] and the literature cited therein), as a tool to construct soliton solutions of the Korteweg-de Vries (KdV) hierarchy relative to (general) KdV background solutions (see, e.g., [3], [6], [7], [8], Ch. 4, [12], [15], [18], [23], [25]-[27], [30], Sect. 6.6, [33]–[36]), and in connection with Bäcklund transformations for the KdV hierarchy (cf., e.g., [10], [11], [13], [18], [20], [21], [23], [28], [29], [38]).

Historically, these methods of inserting eigenvalues go back to Jacobi [24] and Darboux [5] with decisive later contributions by Crum [4], Schmincke [34], and, especially, Deift [6]. Two particular methods, shortly to be discussed in an informal manner in (1.1)–(1.6) below, turned out to be of special importance: The single commutation method, also called the Crum-Darboux method [4], [5] (actually going back at least to Jacobi [24]) and the double commutation method, to be found, e.g., in the seminal work of Gel'fand and Levitan [16].

The single commutation method, although very simply implemented, has the distinct disadvantage of relying on positivity properties of certain solutions of $H\phi = \lambda\phi$ which confines its applicability to the insertion of eigenvalues below the spectrum of H (assuming H to be bounded from below). A complete spectral characterization of this method has been provided by Deift [6] (see also [34]) on the basis of unitary

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equivalence of $A^*A|_{Ker(A)^\perp}$ and $AA^*|_{Ker(A^*)^\perp}$ for a densely defined closed linear operator A in a (complex, separable) Hilbert space.

The double commutation method on the other hand, allows one to insert eigenvalues into *any* spectral gap of H. Although relatively simply implemented also, a complete spectral characterization of the double commutation method for Schrödinger-type operators was only very recently achieved in [17] on the basis of Weyl-Titchmarsh m-function techniques. In this note we follow up on the double commutation method and provide, in particular,

- a complete spectral analysis of the double commutation method for general Sturm-Liouville operators on arbitrary intervals (a,b), $-\infty \le a < b \le \infty$
- a functional analytic approach which not only avoids the Weyl-Titchmarsh theory employed in [17], but in addition, considerably simplifies and streamlines the corresponding proofs in [17].

We emphasize that our formulation of the double commutation method for general Sturm-Liouville (SL) operators appears to be without precedent.

Before starting our analysis in Section 2, we informally describe the single and double commutation method for general Sturm-Liouville differential expressions. Consider the differential expression $(p, p', k, k' \in AC_{loc}((a,b)),\ p, k > 0,$ $q \in L^1_{loc}((a,b)),\ -\infty \le a < b \le \infty)$

$$\tau = \frac{1}{k}\Big(-\frac{d}{dx}p\frac{d}{dx} + q\Big), \tag{1.1}$$

and introduce

$$a = \frac{1}{k}\Big(\sqrt{kp}\frac{d}{dx} + \phi\Big), \quad a^* = \frac{1}{k}\Big(-\frac{d}{dx}\sqrt{kp} + \phi\Big), \tag{1.2}$$

where $\phi = (\sqrt{kp}\psi')/\psi$ and $\psi > 0$ satisfies $\tau\psi = \lambda_1\psi$. A straightforward calculation reveals

$$\tau = a^*a + \lambda_1, \quad \hat\tau = aa^* + \lambda_1 = \frac{1}{k}\Big(-\frac{d}{dx}p\frac{d}{dx} + \hat q\Big), \tag{1.3}$$

where

$$\hat q = q - \frac{p''}{2} + \frac{(p')^2}{4p} + \frac{3(k')^2p}{4k^2} - \frac{k''p}{2k} + \Big(\frac{(kp)'}{k} - 2\frac{d}{dx}p\Big)\frac{d}{dx}\ln\psi. \tag{1.4}$$

Thus (taking proper domain considerations into account) we can define two operators $H, \hat H$ on $L^2((a,b);kdx)$ associated with $\tau, \hat\tau$ which are unitarily equivalent when restricted to the orthogonal complement of the eigenspaces corresponding to λ_1, [6] (cf. also [34]). Moreover, $H - \lambda_1, \hat H - \lambda_1 \ge 0$ which is equivalent to the existence of the positive solution ψ [22]. Formulas (1.1)–(1.4) constitute the single commutation method for general SL differential expressions as discussed, e.g., by Schmincke [34]. Next we assume that ψ is square integrable near a and consider two more expressions $a_{\gamma_1}, a^*_{\gamma_1}$ as above with $\psi_{\gamma_1} = \psi/(1 + \gamma_1\int_a^x k(t)\psi(t)^2dt)$. This implies

$$aa^* = a^*_{\gamma_1}a_{\gamma_1}, \quad \tau_{\gamma_1} = a_{\gamma_1}a^*_{\gamma_1} + \lambda_1 = \frac{1}{k}\Big(-\frac{d}{dx}p\frac{d}{dx} + q_{\gamma_1}\Big), \tag{1.5}$$

where

$$q_{\gamma_1} = q + \Big(\frac{(kp)'}{k} - 2\frac{d}{dx}p\Big)\frac{d}{dx}\ln\Big(1 + \gamma_1\int_a^x k(t)\psi(t)^2dt\Big). \tag{1.6}$$

Now observe that q_{γ_1} is well defined even if ψ has zeros, and hence we expect H, H_{γ_1} (a SL operator associated with τ_{γ_1}) to be closely related even in the case where all intermediate operators are ill-defined. This was first shown in [17] for the special case of Schrödinger-type operators where $k = p = 1$. We shall prove this fact for general SL operators by explicitly computing a transformation operator for the pair τ, τ_{γ_1} which, when restricted to the orthogonal complement of the eigenspace of λ_1, turns out to be unitary. Formulas (1.1)-(1.6) sum up the double commutation method for general SL differential expressions.

Our results are relevant in connection with inverse scattering theory in nonhomogeneous media (see, e.g., [1] and the references cited therein) and yield a direct construction of N-soliton solutions relative to arbitrary background solutions of the (generalized) KdV hierarchy along the methods of [18] (see also [20]).

2. Construction of a transformation operator

Let $k \in L^1_{loc}((a,b))$ with $k > 0$. We pick $\mathfrak{H} = L^2((a,b); kdx)$ to be the underlying Hilbert space ($-\infty \le a < b \le \infty$) and define for $c \in (a,b)$,

$$L^2_{loc}([a,b); kdx) = \{f \in L^2_{loc}((a,b); kdx) | f \in L^2((a,c); kdx)\}. \tag{2.1}$$

Next, choose a positive number $\gamma > 0$, consider a fixed element $\psi \in L^2_{loc}([a,b); kdx)$, and define the following (linear) transformation

$$\begin{array}{llll} U_\gamma : & L^2_{loc}([a,b); kdx) & \to & L^2_{loc}([a,b); kdx) \\ & f(x) & \mapsto & f_\gamma(x) = f(x) - \gamma \psi_\gamma(x) \int_a^x k(t)\overline{\psi(t)} f(t) dt, \end{array} \tag{2.2}$$

where ψ_γ is defined by

$$\psi_\gamma(x) = \frac{\psi(x)}{1 + \gamma \int_a^x k(t)|\psi(t)|^2 dt}. \tag{2.3}$$

By inspection, the inverse transformation is given by

$$\begin{array}{llll} U_\gamma^{-1} : & L^2_{loc}([a,b); kdx) & \to & L^2_{loc}([a,b); kdx) \\ & g(x) & \mapsto & g(x) + \gamma \psi(x) \int_a^x k(t)\overline{\psi_\gamma(t)} g(t) dt. \end{array} \tag{2.4}$$

The restriction of U_γ to $\mathfrak{H}$ will be denoted by U_γ as well. Note that we have

$$1 - \gamma \int_a^x k(t)|\psi_\gamma(t)|^2 dt = (1 + \gamma \int_a^x k(t)|\psi(t)|^2 dt)^{-1}. \tag{2.5}$$

Lemma 2.1. *The element ψ_γ fulfills*

$$\psi_\gamma \in \mathfrak{H}, \qquad \|\psi_\gamma\|^2 = \frac{1}{\gamma}\Big(1 - \lim_{x \to b} (1 + \gamma \int_a^x k(t)|\psi(t)|^2 dt)^{-1}\Big). \tag{2.6}$$

Denote by P, P_γ the orthogonal projections onto the one dimensional subspaces of $\mathfrak{H}$ spanned by ψ, ψ_γ (set $P = 0$ if $\psi \notin \mathfrak{H}$), then the operator U_γ is unitary from $(1-P)\mathfrak{H}$ onto $(1-P_\gamma)\mathfrak{H}$.

Proof. The claims concerning ψ_γ are straightforward. Next we note that for $x \in (a,b)$ and f, f_γ as in (2.2),

$$\int_a^x k(t)\overline{\psi_\gamma(t)} f_\gamma(t) dt = \frac{\int_a^x k(t)\overline{\psi(t)} f(t) dt}{1 + \gamma \int_a^x k(t)|\psi(t)|^2 dt}. \tag{2.7}$$

A direct calculation then shows

$$\int_a^x k(t)|f_\gamma(t)|^2 dt = \int_a^x k(t)|f(t)|^2 dt - \gamma \frac{|\int_a^x k(t)\overline{\psi(t)} f(t)dt|^2}{1+\gamma\int_a^x k(t)|\psi(t)|^2 dt}, \tag{2.8}$$

which proves the lemma if $\psi \in \mathfrak{H}$. Otherwise, consider U_γ, U_γ^{-1} on the dense subspace of square integrable elements with compact support in (a,b) and take closures (cf., e.g., [39], Theorem 6.13). □

Using, e.g., the polarization identity, we obtain in addition

$$\begin{aligned} \int_a^x k(t)\overline{g_\gamma(t)} f_\gamma(t)dt \quad &= \quad \int_a^x k(t)\overline{g(t)} f(t)dt \\ &\quad - \gamma \frac{\int_a^x k(t)\overline{\psi(t)} f(t)dt \int_a^x k(t)\psi(t)\overline{g(t)}dt}{1+\gamma\int_a^x k(t)|\psi(t)|^2 dt}. \end{aligned} \tag{2.9}$$

3. Inserting a single eigenvalue

Now we turn to the SL differential expression

$$\tau = \frac{1}{k}\Big(-\frac{d}{dx}p\frac{d}{dx} + q\Big), \tag{3.1}$$

where the coefficients p, q, k are real-valued satisfying

$$p^{-1}, q, k \in L^1_{loc}((a,b)), \quad k>0, \quad kp \in AC_{loc}((a,b)). \tag{3.2}$$

We are interested in self-adjoint operators H associated with τ and separated boundary conditions.

Hypothesis (H.3.1). Let $\lambda \in \mathbb{R}$ and suppose ψ satisfies the following conditions.

(i). $\psi, p\psi' \in AC_{loc}((a,b))$ and ψ is a real-valued solution of $\tau\phi = \lambda\phi$.
(ii). ψ is square integrable near a and fulfills the boundary condition (of H) at a and b if any (i.e., if τ is limit circle (*l.c.*) at a respectively b).

Sufficient conditions for the existence of a function ψ satisfying (H.3.1) are

(a). $\lambda_1 \in \sigma_p(H)$ ($\sigma_p(.)$ the point spectrum, i.e., the set of eigenvalues), or
(b). τ is *l.c.* at a but not at b, or
(c). $\sigma(H) \neq \mathbb{R}$ (and $\lambda \in \mathbb{R}\backslash\sigma(H)$), or
(d). $\sigma(H_-^c) \neq \mathbb{R}$ (and $\lambda \in \mathbb{R}\backslash\sigma(H_-^c)$) where H_-^c is a restriction of H to $L^2((a,c))$ with $c \in (a,b)$ (finite) and, e.g., a Dirichlet boundary condition at c.

If $\lambda_1 \in \mathbb{R}$ and $\psi(\lambda_1, .)$ obeys (H.3.1), it follows that H is explicitly given by

$$Hf = \tau f, \qquad \mathfrak{D}(H) = \begin{array}{l} \{f \in \mathfrak{H} \,|\, f, pf' \in AC_{loc}((a,b)); \tau f \in \mathfrak{H}; \\ W_a(\psi(\lambda_1), f) = 0 \text{ if } \tau \text{ is } l.c. \text{ at } a \\ W_b(\psi(\lambda_1), f) = 0 \text{ if } \tau \text{ is } l.c. \text{ at } b\} \end{array} \tag{3.3}$$

with $W_x(u,v) = p(x)(u(x)v(x)' - u(x)'v(x))$ the (modified) Wronskian of $u, v \in AC_{loc}((a,b))$.

We now use Lemma 2.1 with $\psi = \psi(\lambda_1)$, $\gamma = \gamma_1$ to prove

Theorem 3.2. *Let $\lambda_1 \in \mathbb{R}$ and $\psi(\lambda_1, .)$ be a solution satisfying (H.3.1). Define U_{γ_1}, $P(\lambda_1)$, $P_{\gamma_1}(\lambda_1)$ as in (2.2) and Lemma 2.1 in terms of $\psi(\lambda_1, .)$ and set $\psi_{\gamma_1}(\lambda_1, .) = U_{\gamma_1}\psi(\lambda_1, .)$. Then the operator H_{γ_1} defined by*

$$H_{\gamma_1} f = \tau_{\gamma_1} f, \quad \mathfrak{D}(H_{\gamma_1}) = \begin{array}{l}\{f \in \mathfrak{H} \,|\, f, pf' \in AC_{loc}((a,b)); \tau_{\gamma_1} f \in \mathfrak{H};\\ W_a(\psi_{\gamma_1}(\lambda_1), f) = W_b(\psi_{\gamma_1}(\lambda_1), f) = 0\},\end{array} \tag{3.4}$$

with

$$q_{\gamma_1} = q + \Big(\frac{(kp)'}{k} - 2\frac{d}{dx}p\Big)\frac{d}{dx}\ln\Big(1 + \gamma_1 \int_a^x k(t)\psi(\lambda_1, t)^2 dt\Big) \tag{3.5}$$

is self-adjoint. Moreover, H_{γ_1} has the eigenfunction $\psi_{\gamma_1}(\lambda_1) = U_{\gamma_1}\psi(\lambda_1)$ associated with the eigenvalue λ_1. If $\psi(\lambda_1) \notin \mathfrak{H}$ (and hence τ is limit point (l.p.) at b) we have

$$H_{\gamma_1}(1 - P_{\gamma_1}(\lambda_1)) = U_{\gamma_1} H U_{\gamma_1}^{-1}(1 - P_{\gamma_1}(\lambda_1)) \tag{3.6}$$

and thus

$$\begin{array}{rclcrcl} \sigma(H_{\gamma_1}) & = & \sigma(H) \cup \{\lambda_1\}, & \quad & \sigma_{ac}(H_{\gamma_1}) & = & \sigma_{ac}(H),\\ \sigma_p(H_{\gamma_1}) & = & \sigma_p(H) \cup \{\lambda_1\}, & & \sigma_{sc}(H_{\gamma_1}) & = & \sigma_{sc}(H). \end{array} \tag{3.7}$$

(Here $\sigma_{ac}(.), \sigma_{sc}(.)$ denotes the absolutely and singularly continuous spectrum, respectively.) If $\psi(\lambda_1) \in \mathfrak{H}$ there is a unitary operator $\tilde{U}_{\gamma_1} = U_{\gamma_1} \oplus \sqrt{1 + \gamma_1\|\psi(\lambda_1)\|^2} U_{\gamma_1}$ on $(1 - P(\lambda_1))\mathfrak{H} \oplus P(\lambda_1)\mathfrak{H}$ such that

$$H_{\gamma_1} = \tilde{U}_{\gamma_1} H \tilde{U}_{\gamma_1}^{-1}. \tag{3.8}$$

Proof. It suffices to prove $H_{\gamma_1}(1 - P_{\gamma_1}(\lambda_1)) = U_{\gamma_1} H U_{\gamma_1}^{-1}(1 - P_{\gamma_1}(\lambda_1))$. Let f be square integrable near a (with $f, pf' \in AC_{loc}((a,b))$) such that τf is also square integrable near a and f fulfills the boundary condition at a (if any). Then a straightforward calculation shows

$$\tau_{\gamma_1}(U_{\gamma_1} f) = U_{\gamma_1}(\tau f) \tag{3.9}$$

and we only have to check the boundary conditions at a and b. Equation (2.8) shows that τ_{γ_1} is *l.c.* at a, respectively b, if τ is. The formula

$$W_x(\psi_{\gamma_1}(\lambda_1), U_{\gamma_1} f) = \frac{W_x(\psi(\lambda_1), f)}{1 + \gamma_1 \int_a^x k(t)\psi(\lambda_1, t)^2 dt} \tag{3.10}$$

reveals $W_a(\psi_{\gamma_1}(\lambda_1), U_{\gamma_1} f) = 0$ for $f \in \mathfrak{D}(H)$. Furthermore, we claim that

$$W_b(\psi_{\gamma_1}(\lambda_1), U_{\gamma_1} f) = 0, \qquad f \in \mathfrak{D}(H). \tag{3.11}$$

This is clear if $\psi(\lambda_1) \in \mathfrak{H}$, otherwise, i.e., if $\psi(\lambda_1) \notin \mathfrak{H}$ we use

$$|W_x(\psi_{\gamma_1}(\lambda_1), U_{\gamma_1} f)|^2 = \frac{|\int_a^x k(t)\psi(\lambda_1, t)(\lambda_1 - \tau)f(t)dt|^2}{(1 + \gamma_1 \int_a^x k(t)\psi(\lambda_1, t)^2 dt)^2}. \tag{3.12}$$

The right hand side of (3.12) tends to zero for $f \in \mathfrak{D}(H)$ as can be seen from (2.8) and the fact that U_{γ_1} is unitary. Combining the last results yields

$$(1 - P_{\gamma_1}(\lambda_1)) U_{\gamma_1} \mathfrak{D}(H) \subseteq (1 - P_{\gamma_1}(\lambda_1)) \mathfrak{D}(H_{\gamma_1}). \tag{3.13}$$

But $(1 - P_{\gamma_1}(\lambda_1)) U_{\gamma_1} \mathfrak{D}(H)$ cannot be properly contained in $(1 - P_{\gamma_1}(\lambda_1)) \mathfrak{D}(H_{\gamma_1})$ by the property of self-adjoint operators being maximal. $\square$

Remark 3.3. (i) If H has an eigenfunction $\psi(\lambda_1)$ we can remove this eigenfunction from the spectrum upon choosing $\gamma_1 = -\|\psi(\lambda_1)\|^{-2}$. Then the corresponding element $\psi_{\gamma_1}(\lambda_1)$ is not in $\mathfrak{H}$, implying that τ_{γ_1} is *l.p.* at b.
(ii) The double commutation method has the pleasant feature of leaving p invariant. This is in sharp contrast to the double commutation method for Jacobi operators [19] (the finite difference analog for (3.1)).
(iii) Let u (with $u, pu' \in AC_{loc}((a,b))$) fulfill $\tau u = zu$ (with $z \in \mathbb{C}\backslash\{\lambda_1\}$) and let

$$v(z,x) = u(z,x) + \frac{\gamma_1}{z-\lambda_1}\psi_{\gamma_1}(\lambda_1,x)W_x(\psi(\lambda_1),u(z)). \tag{3.14}$$

Then $v, pv' \in AC_{loc}((a,b))$ and v fulfills $\tau_{\gamma_1} v = zv$. If u is square integrable near a and fulfills the boundary condition at a (if any) we have $v = U_{\gamma_1} u$. We also note

$$|v(z,x)|^2 = |u(z,x)|^2 - \frac{\gamma_1 k(x)^{-1}}{|z-\lambda_1|^2}\frac{d}{dx}\left(\frac{|W_x(\psi(\lambda_1),u(z))|^2}{1+\gamma_1\int_a^x k(t)\psi(\lambda_1,t)^2 dt}\right), \tag{3.15}$$

and if $\hat{u}$, $\hat{v}$ are constructed analogously then

$$\begin{aligned} W_x(v(z),\hat{v}(\hat{z})) &= W_x(u(z),\hat{u}(\hat{z})) - \frac{\gamma_1}{1+\gamma_1\int_a^x k(t)\psi(\lambda_1,t)^2 dt} \times \\ &\quad \frac{z-\hat{z}}{(z-\lambda_1)(\hat{z}-\lambda_1)} W_x(\psi(\lambda_1),u(z))W_x(\psi(\lambda_1),\hat{u}(\hat{z})). \end{aligned} \tag{3.16}$$

(iv) Writing $U_{\gamma_1} f$ as

$$(U_{\gamma_1} f)(x) = f(x) - \int_a^x \frac{\gamma_1 k(t)\psi(\lambda_1,x)\psi(\lambda_1,t)}{1+\gamma_1\int_a^x k(s)\psi(\lambda_1,s)^2 ds} f(t)dt, \tag{3.17}$$

we see that U_{γ_1} is the transformation operator for H, H_{γ_1} in the terminology of [30] and [31].
(v) The limiting case $\gamma = \infty$ can be handled analogously producing an unitarily equivalent operator.

Finally we discuss conditions for τ_{γ_1} to be *l.p.* at a, b. Let $c \in (a,b)$ and let H_+^c denote a self-adjoint operator associated with τ on (c,b) and the boundary condition induced by $\psi(\lambda_1)$ at c (i.e., $W_c(f,\psi(\lambda_1)) = 0$, $f \in \mathfrak{D}(H_+^c)$).

Hypothesis (H.3.4). Suppose one of the following spectral conditions (i)–(iii) holds.

(i). $\sigma_{ess}(H_+^c) \neq \emptyset$.
(ii). $\sigma(H_+^c) = \sigma_d(H_+^c) = \{\lambda_n\}_{n\in\mathbb{Z}}$ with $\sum_{n\in\mathbb{Z}}(1+\lambda_n^2)^{-1} = \infty$.
(iii). H_+^c is bounded from below and $\int^b |k(x)/p(x)|^{1/2}dx = \infty$.

All conditions (i)–(iii) imply that τ is *l.p.* at b. This is clear for (i),(ii) since if H_+^c were *l.c.* at b its resolvent would be a Hilbert-Schmidt operator contradicting (i),(ii). For (iii) this follows from [17], Lemma C.1.

Theorem 3.5. *(i). τ_{γ_1} is l.p. at a if and only if τ is and l.c. at b if τ is.*
(ii). Assume (H.3.4), then τ_{γ_1} is l.p. at b.

Proof. (i) follows from (3.15). For (ii) consider the doubly commuted operator $H_{+,\gamma_1^c}^c$ of H_+^c, where $\gamma_1^c = \gamma_1/(1+\gamma_1\int_a^c \psi(\lambda_1,t)^2 dt)$. Then $\tau_{\gamma_1}|_{(c,b)} = \tau_{\gamma_1^c}^c$ and $H_{+,\gamma_1^c}^c$ also satisfies (H.3.4). Hence τ_{γ_1} is *l.p.* at b as claimed. □

Remark 3.6. (i) Removing an eigenvalue (cf. Remark 3.3 (i)) from an operator which is *l.c.* at b yields an operator which is *l.p.*. Thus τ_{γ_1} is not necessarily *l.p.* if τ is. Moreover, this shows that one cannot insert additional eigenvalues into an operator which is *l.c.* at b (remove this eigenvalue again to obtain a contradiction).
(ii) Clearly we can interchange the role of a and b. One only has to substitute $a \to b$ in the text and $\int_a \to \int^b$ in the formulas.
(iii) As long as τ is *l.c.* at a our method can be used to insert additional eigenvalues into the spectrum of H. In the special case of Schrödinger operators where $k = p = 1$ this has first been observed by Gel'fand and Levitan [16] in connection with Wigner-von Neumann examples. More recent studies can be found in [9], [32], [37] (see also [8], Section 4.4).
(iv) Theorems 3.2 and 3.5 have first been derived in [17] for the special case $p = k = 1$ and under somewhat more restrictive spectral conditions (such as $\lambda_1 \in \mathbb{R}\backslash\sigma(H)$).

Theorem 3.7. *Assume (H.3.1) and let* $m_\pm(z,\alpha)$, $m_{\pm,\gamma_1}(z,\alpha)$ *denote the Weyl m-functions of* H, H_{γ_1}, *respectively associated with the boundary condition* $\sin(\alpha)f(c)+\cos(\alpha)(pf')(c)=0$, $c\in(a,b)$. *Then we have*

$$m_{\pm,\gamma_1}(z,\tilde{\alpha}) = \frac{1+\tilde{\beta}^2}{1+\beta^2}\Big(m_\pm(z,\alpha) - \frac{\tilde{\gamma}_1}{z-\lambda_1} + \beta\Big) - \tilde{\beta}, \tag{3.18}$$

where $\beta = \cot(\alpha)$,

$$\tilde{\gamma}_1 = \frac{\gamma_1}{1+\gamma\int_a^c k(t)\psi(\lambda_1,t)^2 dt}, \quad \tilde{\beta} = \cot(\tilde{\alpha}) = \cot(\alpha) + \tilde{\gamma}\sin^2(\alpha). \tag{3.19}$$

Proof. Consider the sequences

$$\phi_{\alpha,\gamma_1}(z,x), \quad \theta_{\alpha,\gamma_1}(z,x) + \tilde{\gamma}_1\Big(\frac{1}{z-\lambda_1} + \sin^2(\alpha)\frac{1-\beta\tilde{\beta}}{1+\tilde{\beta}^2}\Big)\phi_{\alpha,\gamma_1}(z,x) \tag{3.20}$$

constructed from the fundamental system $\theta_\alpha(z,x)$, $\phi_\alpha(z,x)$ for τ, that is,

$$\theta_\alpha(z,c) = p(c)\phi'_\alpha(z,c) = \cos(\alpha), \quad p(c)\theta'_\alpha(z,c) = -\phi_\alpha(z,c) = \sin(\alpha). \tag{3.21}$$

They form a fundamental system for τ_{γ_1} corresponding to the initial conditions associated with $\tilde{\alpha}$ up to constant multiples. Now use (3.10) to evaluate

$$\begin{aligned} m_{\pm,\gamma_1}(z,\tilde{\alpha}) &= -\lim_{x\to {}^b_a} \frac{W_x(\psi_{\gamma_1}(\lambda_1),\theta_{\gamma_1,\tilde{\alpha}}(z))}{W_x(\psi_{\gamma_1}(\lambda_1),\phi_{\gamma_1,\tilde{\alpha}}(z))} \\ &= -\frac{1+\tilde{\beta}^2}{1+\beta^2}\frac{W_x(\psi_{\gamma_1}(\lambda_1),\theta_{\alpha,\gamma_1}(z)+\tilde{\gamma}_1(\dots)\phi_{\alpha,\gamma_1}(z))}{W_x(\psi_{\gamma_1}(\lambda_1),\phi_{\alpha,\gamma_1}(z))}, \end{aligned} \tag{3.22}$$

where $\theta_{\gamma_1,\tilde{\alpha}}(z,x)$, $\phi_{\gamma_1,\tilde{\alpha}}(z,x)$ is the fundamental system for τ_{γ_1} corresponding to $\tilde{\alpha}$ at c. □

4. Inserting finitely many eigenvalues

Finally we demonstrate how to iterate this method. We choose a given background operator H (with coefficients k,p,q) and pick $\gamma_1>0$, $\lambda_1\in\mathbb{R}$. Now choose $\psi(\lambda_1)$, as in Section 3 to define the transformation U_1 and the operator H_{γ_1}. Next we choose $\gamma_2>0$, $\lambda_2\in\mathbb{R}$ and another function $\psi(\lambda_2)$ to define $\psi_{\gamma_1}(\lambda_2)=U_{\gamma_1}\psi(\lambda_2)$ and corresponding operators U_{γ_1,γ_2} and H_{γ_1,γ_2}. Applying this procedure N times leads to

Theorem 4.1. *Let H be the background operator (3.3) and let $\gamma_j > 0$, $\lambda_j \in \mathbb{R}$, $1 \le j \le N$, be such that there exist corresponding solutions $\psi(\lambda_j, x)$ of $\tau\phi = \lambda_j\phi$ satisfying (H.3.1). We set $\psi_{\gamma_1,\dots,\gamma_\ell}(\lambda_j) = U_{\gamma_1,\dots,\gamma_\ell}\cdots U_{\gamma_1}\psi(\lambda_j)$ and define the following matrices ($1 \le \ell \le N$)*

$$C_\ell(x) = \left\{ \delta_{r,s} + \sqrt{\gamma_r\gamma_s}\int_a^x k(t)\psi(\lambda_r,t)\psi(\lambda_s,t)dt \right\}_{1\le r,s\le\ell}, \tag{4.1}$$

$$C_{\ell;i,j}(x) = \left\{ \begin{array}{cc} C_{\ell-1}(x)_{r,s}, & r,s\le\ell-1 \\ \sqrt{\gamma_s}\int_a^x k(t)\psi(\lambda_i,t)\psi(\lambda_s,t)dt, & s\le\ell-1, r=\ell \\ \sqrt{\gamma_r}\int_a^x k(t)\psi(\lambda_r,t)\psi(\lambda_j,t)dt, & r\le\ell-1, s=\ell \\ \int_a^x k(t)\psi(\lambda_i,t)\psi(\lambda_j,t)dt, & r=s=\ell \end{array} \right\}_{1\le r,s\le\ell}, \tag{4.2}$$

$$\Psi_\ell(\lambda_j,x) = \left\{ \begin{array}{cc} C_\ell(x)_{r,s}, & r,s\le\ell \\ \sqrt{\gamma_s}\int_a^x k(t)\psi(\lambda_j,t)\psi(\lambda_s,t)dt, & s\le\ell, r=\ell+1 \\ \sqrt{\gamma_r}\psi(\lambda_r,x), & r\le\ell, s=\ell+1 \\ \psi(\lambda_j,x), & r=s=\ell+1 \end{array} \right\}_{1\le r,s\le\ell+1}. \tag{4.3}$$

Then we have (set $C_0(x) = 1$)

$$1 + \gamma_\ell \int_a^x k(t)\psi_{\gamma_1,\dots,\gamma_{\ell-1}}(\lambda_\ell,t)^2 dt = \frac{\det C_\ell(x)}{\det C_{\ell-1}(x)} \tag{4.4}$$

and

$$q_{\gamma_1,\dots,\gamma_N} = q + \Big(\frac{(kp)'}{k} - 2\frac{d}{dx}p\Big)\frac{d}{dx}\ln\det C_N. \tag{4.5}$$

Furthermore, we obtain

$$\int_a^x k(t)\psi_{\gamma_1,\dots,\gamma_{\ell-1}}(\lambda_i,t)\psi_{\gamma_1,\dots,\gamma_{\ell-1}}(\lambda_j,t)dt = \frac{\det C_{\ell;i,j}(x)}{\det C_{\ell-1}(x)} \tag{4.6}$$

and

$$\psi_{\gamma_1,\dots,\gamma_\ell}(\lambda_j,x) = \frac{\det\Psi_\ell(\lambda_j,x)}{\det C_\ell(x)}. \tag{4.7}$$

The spectrum of $H_{\gamma_1,\dots,\gamma_N}$ is given by

$$\begin{array}{rclrcl} \sigma(H_{\gamma_1,\dots,\gamma_N}) & = & \sigma(H)\cup\{\lambda_j\}_{j=1}^N, & \sigma_{ac}(H_{\gamma_1,\dots,\gamma_N}) & = & \sigma_{ac}(H), \\ \sigma_p(H_{\gamma_1,\dots,\gamma_N}) & = & \sigma_p(H)\cup\{\lambda_j\}_{j=1}^N, & \sigma_{sc}(H_{\gamma_1,\dots,\gamma_N}) & = & \sigma_{sc}(H). \end{array} \tag{4.8}$$

Moreover,

$$\begin{aligned} & H_{\gamma_1,\dots,\gamma_N}(1 - \sum_{j=1}^N P_{\gamma_1,\dots,\gamma_N}(\lambda_j)) \\ & = (U_{\gamma_1,\dots,\gamma_N}\cdots U_{\gamma_1})H(U_{\gamma_1}^{-1}\cdots U_{\gamma_1,\dots,\gamma_N}^{-1})(1 - \sum_{j=1}^N P_{\gamma_1,\dots,\gamma_N}(\lambda_j)), \end{aligned} \tag{4.9}$$

where $P_{\gamma_1,\dots,\gamma_N}(\lambda_j)$ denotes the projection onto the one-dimensional subspace spanned by $\psi_{\gamma_1,\dots,\gamma_N}(\lambda_j)$.

Proof. It suffices to prove (4.6), (4.7) which requires a straightforward induction argument using Sylvester's determinant identity ([14], Sect. II.3). The resulting identity

$$\det C_\ell(x) \det C_{\ell;i,j}(x) - \gamma_\ell \det C_{\ell;\ell,j}(x) \det C_{\ell;i,\ell}(x) \\ = \det C_{\ell-1}(x) \det C_{\ell+1;i,j}(x), \tag{4.10}$$

together with (2.9) then proves (4.6). Similarly,

$$\det C_\ell(x) \det \Psi_{\ell-1}(\lambda_j, x) - \gamma_\ell \det \Psi_{\ell-1}(\lambda_\ell, x) \det C_{\ell;\ell,j}(x) \\ = \det C_{\ell-1}(x) \det \Psi_\ell(\lambda_j, x), \tag{4.11}$$

and (2.2) prove (4.7). The rest then follows from these two equations and Theorem 3.2. □

Remark 4.2. (i) For any element f which is square integrable near $-\infty$, $f_{\gamma_1,\dots,\gamma_\ell} = U_{\gamma_1,\dots,\gamma_\ell} \cdots U_{\gamma_1} f$ is given by substituting $\psi(\lambda_j) \to f$ in (4.7). Likewise we get the scalar product of $f_{\gamma_1,\dots,\gamma_\ell}$ and $g_{\gamma_1,\dots,\gamma_\ell}$ from (4.6) by substituting $\psi(\lambda_i) \to f$ and $\psi(\lambda_j) \to g$ in (4.2).
(ii) Equation (4.7) can be rephrased as

$$(\sqrt{\gamma_1}\psi_{\gamma_1,\dots,\gamma_N}(\lambda_1, x), \dots, \sqrt{\gamma_N}\psi_{\gamma_1,\dots,\gamma_N}(\lambda_N, x)) \\ = (C_N(x))^{-1}(\sqrt{\gamma_1}\psi(\lambda_1, x), \dots, \sqrt{\gamma_N}\psi(\lambda_N, x)), \tag{4.12}$$

where $(C_N(x))^{-1}$ denotes the inverse of $C_N(x)$.

Clearly Theorem 3.5 extends (by induction) to this more general situation.

Theorem 4.3. *(i). $\tau_{\gamma_1,\dots,\gamma_N}$ is l.p. at a if and only if τ is and l.c. at b if τ is.*
(ii). Assume (H.3.4), then $\tau_{\gamma_1,\dots,\gamma_N}$ is l.p. at b.

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Department of Mathematics, University of Missouri, Columbia, MO 65211, USA
E-mail address: mathfg@mizzou1.missouri.edu

Department of Mathematics, University of Missouri, Columbia, MO 65211, USA and Department of Theoretical Physics, Technical University of Graz, Graz, 8010, Austria
Current address: Institut für Mathematik, Strudlhofgasse 4, 1090 Wien, Austria
E-mail address: Gerald.Teschl@univie.ac.at
URL: http://www.mat.univie.ac.at/~gerald/