SCATTERING PROPERTIES AND DISPERSION ESTIMATES FOR A ONE-DIMENSIONAL DISCRETE DIRAC EQUATION

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Abstract. We derive dispersion estimates for solutions of a one-dimensional discrete Dirac equations with a potential. In particular, we improve our previous result, weakening the conditions on the potential. To this end we also provide new results concerning scattering for the corresponding perturbed Dirac operators which are of independent interest. Most notably, we show that the reflection and transmission coefficients belong to the Wiener algebra.

1. Introduction

We are concerned with the one-dimensional discrete Dirac equation

$$i\dot{\mathbf{w}}(t) := \mathcal{D}\mathbf{w}(t) = (\mathcal{D}_0 + Q)\mathbf{w}(t), \quad \mathbf{w}_n = (u_n, v_n) \in \mathbb{C}^2, \quad n \in \mathbb{Z}. \tag{1.1}$$

Here the discrete free Dirac operator $\mathcal{D}_0$ is defined by

$$\mathcal{D}_0 = \begin{pmatrix} m & d \\ d^* & -m \end{pmatrix}, \quad m > 0,$$

where $(du)_n = u_{n+1} - u_n$. For the real potential Q we assume that

$$Q_n = \begin{pmatrix} 0 & q_n \\ q_n & 0 \end{pmatrix}, \text{ where } q_n \neq 1, \quad n \in \mathbb{Z}, \tag{1.2}$$

is bounded, such that $\mathcal{D}$ gives rise to a bounded self-adjoint operator in $\mathbf{l}^2(\mathbb{Z}) = \ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$.

In the first part of our article we show that the scattering matrix of the operator $\mathcal{D}$ is in the Wiener algebra (i.e. its Fourier coefficients are summable) if the first moment of the potential is summable. We use this result to establish dispersive decay estimates for equation (1.1) under weaker assumption than in our previous results [5].

Let us introduce the weighted spaces $\ell^p_\sigma = \ell^p_\sigma(\mathbb{Z})$, $\sigma \in \mathbb{R}$, associated with the norm

$$\|u\|_{\ell^p_\sigma} = \begin{cases} \left(\sum_{n\in\mathbb{Z}}(1+|n|)^{p\sigma}|u_n|^p\right)^{1/p}, & p \in [1,\infty), \\ \sup_{n\in\mathbb{Z}}(1+|n|)^{\sigma}|u_n|, & p = \infty, \end{cases}$$

and the case $\sigma = 0$ corresponds to the standard spaces $\ell^p_0 = \ell^p$ without weight. Denote $\mathbf{l}^p_\sigma = \ell^p_\sigma \oplus \ell^p_\sigma$ and $\mathbf{l}^p = \ell^p \oplus \ell^p$.

We recall that under the condition $q \in \ell^1_1$, the spectrum of $\mathcal{D}$ consists of a purely absolutely continuous part $\Gamma = (-\sqrt{4+m^2}, -m) \cup (m, \sqrt{4+m^2})$, plus a finite number of eigenvalues located in $\mathbb{R} \setminus \overline{\Gamma}$. In addition, there could be resonances at the edges $\omega = \pm m, \pm\sqrt{4+m^2}$ of the continuous spectrum (see [5]).

As our first main result, we prove the following $\mathbf{l}^1 \to \mathbf{l}^\infty$ decay

$$\|\mathrm{e}^{-\mathrm{i}t\mathcal{D}} P_c\|_{\mathbf{l}^1 \to \mathbf{l}^\infty} = \mathcal{O}(t^{-1/3}), \quad t \to \infty, \tag{1.3}$$

under the assumptions $q \in \ell^1_1$. Here P_c is the orthogonal projection in $\mathbf{l}^2$ onto the continuous spectrum of $\mathcal{D}$.

Secondly, we establish decay in $\mathbf{l}^2_\sigma \to \mathbf{l}^2_{-\sigma}$ with $\sigma > 1/2$:

$$\|\mathrm{e}^{-\mathrm{i}t\mathcal{D}} P_c\|_{\mathbf{l}^2_\sigma \to \mathbf{l}^2_{-\sigma}} = \mathcal{O}(t^{-1/2}), \quad t \to \infty. \tag{1.4}$$

Let us emphasize that we do not require additional decay of q for (1.3)–(1.4) in the case when the edges of the continuous spectrum are resonances.

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In the remaining results we restrict ourselves to the non-resonance case. In this case, for $q \in \ell^1_2$, we show that

$$\|e^{-it\mathcal{D}} P_c\|_{\mathbf{l}^1_1 \to \mathbf{l}^\infty_{-1}} = \mathcal{O}(t^{-4/3}), \quad t \to \infty, \tag{1.5}$$

and

$$\|e^{-it\mathcal{D}} P_c\|_{\mathbf{l}^2_\sigma \to \mathbf{l}^2_{-\sigma}} = \mathcal{O}(t^{-3/2}), \quad t \to \infty, \quad \sigma > 3/2. \tag{1.6}$$

The dispersion estimates (1.3)–(1.4) have been established in our previous paper [5] under the assumption $q \in \ell^1_2$ in the non-resonance case, and under the more restrictive condition $q \in \ell^1_3$ in the resonance case. Moreover, in [5], we required $q \in \ell^1_3$ for the asymptotics (1.5)–(1.6) to hold in the non-resonance case.

To show that the extra decay of q is not necessary, we extend the approach of [2, 3], introduced in the context of discrete and continuous Schrödinger equations, which relies on a refined version of an old result of Guseinov [4]. Namely, we prove that the transmission and reflection coefficients $T(\theta)$ and $R^\pm(\theta)$ belong to the Wiener algebra $\mathcal{A}$. Let us note that in the half-line case the analogous result for the scattering data is well known (cf. Problem 3.2.1 in [6]) and was used by Weder [8] to prove a corresponding result in the half-line case.

Our approach can be summarized as follows: To prove that $T(\theta), R^\pm(\theta) \in \mathcal{A}$, we first compute the Fourier coefficients of the Jost solutions $\mathbf{h}^\pm(\theta) = (h^\pm_1(\theta), h^\pm_1(\theta))$. The main difficulty here is the presence of the factors $\lambda \pm m$, where $\lambda = \sqrt{m^2 + 2 - e^{i\theta} - e^{-i\theta}}$, in the Green function (formula (3.1) below). This implies that the Fourier series for $\mathbf{h}^\pm(\theta)$ contain all powers of $e^{i\theta}$ contrary to the Schrödinger case, where corresponding Fourier series contain nonnegative powers only. Nevertheless, we obtain the Fourier series only with nonnegative powers of $e^{i\theta}$ for $(h^\pm_1(\theta), (m+\lambda)h^\pm_1(\theta))$ in the case $\lambda > 0$ (and for $((m-\lambda)h^\pm_1(\theta), h^\pm_1(\theta))$ in the case $\lambda < 0$), see formulas (3.3) and (6.2) below.

Using these Fourier series, we then derive the Gelfand–Levitan–Marchenko equations (4.9)–(4.10) for the Fourier coefficients $\mathcal{F}^\pm_n$ of $R^\pm(\theta)$. The extra factors $\lambda \pm m$ cancel and do not appear in these equations. Moreover, these equations have a standard form and provide estimates for $\mathcal{F}^\pm_n$ similar to the estimates of [7, §10], (see also §3.5 in [6]).

To prove the decay estimates (1.3)–(1.6) we apply the spectral Fourier–Laplace representation

$$e^{-it\mathcal{D}} P_c = \frac{1}{2\pi i} \int_\Gamma e^{-it\lambda} (\mathcal{R}(\lambda + i0) - \mathcal{R}(\lambda - i0))\, d\lambda.$$

Expressing the kernels of the resolvents $\mathcal{R}(\lambda \pm i0)$ in terms of Jost solutions and using the scattering relation (4.6), we get oscillatory integrals with amplitudes from the Wiener algebra $\mathcal{A}$. This integral representation implies (1.3)–(1.6) by a suitable version of the van der Corput lemma.

We remark that the derivation of the Gelfand–Levitan–Marchenko equations for arbitrary self-adjoint perturbations Q remains an open problem.

2. Jost solutions

Here we recall some spectral properties of equation (1.1) which we obtain in [5] using the Jost solutions. Set $\Gamma_+ = (m, \sqrt{4+m^2})$, and let $\Xi_+ = \{\lambda \in \mathbb{C} \setminus \overline{\Gamma}_+,\ \operatorname{Re}\lambda \ge 0\}$. For any $\lambda \in \overline{\Xi}_+$, we consider the Jost solutions $\mathbf{w} = (u, v)$ of

$$\mathcal{D}\mathbf{w} = \lambda \mathbf{w} \tag{2.1}$$

defined by the boundary conditions

$$\mathbf{w}^\pm_n(\theta) = \begin{pmatrix} u^\pm_n(\theta) \\ v^\pm_n(\theta) \end{pmatrix} \to \begin{pmatrix} 1 \\ \alpha_\mp(\theta) \end{pmatrix} e^{\pm i\theta n}, \quad n \to \pm\infty, \qquad \alpha_\pm(\theta) := \frac{e^{\pm i\theta} - 1}{m + \lambda}. \tag{2.2}$$

Here $\theta = \theta(\lambda) \in \overline{\Sigma} := \{-\pi \le \operatorname{Re}\theta \le \pi,\ \operatorname{Im}\theta \ge 0\}$ is the solution to

$$2 - 2\cos\theta = \lambda^2 - m^2.$$

These boundary condition arise naturally in (2.1) with $Q \equiv 0$. For nonzero Q with $q \in \ell^1_1$, the Jost solutions exists everywhere in $\overline{\Xi}_+$, but for $q \in \ell^1$ they only exist away from the edges of the continuous spectrum. Introduce

$$\mathbf{h}^\pm_n(\theta) = e^{\mp in\theta} \mathbf{w}^\pm_n(\theta) \tag{2.3}$$

and set

$$\overline{\Sigma}_M := \{\theta \in \overline{\Sigma} : \operatorname{Im}\theta \le M\}, \qquad \overline{\Sigma}_{M,\delta} := \{\theta \in \overline{\Sigma}_M :\ |e^{i\theta} \pm 1| > \delta\}, \quad M \ge 1, \quad 0 < \delta < \sqrt{2}.$$

Lemma 2.1. *(see* [5, Proposition 3.1]*)*
(i) Let $q \in \ell^1_s$ with $s = 0, 1, 2$. Then the functions $\mathbf{h}^\pm_n(\theta)$ can be differentiated s times on $\overline{\Sigma}_{M,\delta}$, and the following estimates hold:

$$|\frac{d^p}{d\theta^p}\mathbf{h}^\pm_n(\theta)| \le C(M,\delta)\max((\mp n)|n|^{p-1}, 1), \quad n \in \mathbb{Z}, \quad 0 \le p \le s, \quad \theta \in \overline{\Sigma}_{M,\delta}. \tag{2.4}$$

(ii) If additionally $q \in \ell^1_{s+1}$, then $\mathbf{h}^\pm_n(\theta)$ can be differentiated s times on $\overline{\Sigma}_M$, and the following estimates hold:

$$|\frac{d^p}{d\theta^p}\mathbf{h}^\pm_n(\theta)| \le C(M)\max((\mp n)|n|^{p}, 1), \quad n \in \mathbb{Z}, \quad 0 \le p \le s, \quad \theta \in \overline{\Sigma}_{M}. \tag{2.5}$$

In the case $q \in \ell^1$ Proposition 2.1 (i) implies, in particular, that for any $\theta \in \overline{\Sigma} \setminus \{0; \pm\pi\}$ we have the estimate $|\mathbf{h}^\pm_n(\theta)| \le C(\theta)$ for all $n \in \mathbb{Z}$, where $C(\theta)$ can be chosen uniformly in compact subsets of $\overline{\Sigma}$ avoiding the band edges. Together with (2.3) this implies

$$|\mathbf{w}^\pm_n(\theta)| \le C(\theta)\mathrm{e}^{\mp \operatorname{Im}(\theta)n}, \quad \theta \in \overline{\Sigma} \setminus \{0; \pm\pi\}, \quad n \in \mathbb{Z}. \tag{2.6}$$

Denote by $W(\mathbf{w}^1, \mathbf{w}^2)$ the Wronskian determinant of any two solutions $\mathbf{w}^1$ and $\mathbf{w}^2$ to (2.1):

$$W(\mathbf{w}^1, \mathbf{w}^2) := \begin{vmatrix} u^1_n & u^2_n \\ v^1_{n+1} & v^2_{n+1} \end{vmatrix}. \tag{2.7}$$

It is easy to check that $W(\mathbf{w}^1, \mathbf{w}^2)$ is independent of $n \in \mathbb{Z}$ for arbitrary solutions $\mathbf{w}^1$ and $\mathbf{w}^2$ of (2.1). Denote

$$W(\theta) = W(\mathbf{w}^+(\theta), \mathbf{w}^-(\theta)).$$

Definition 2.2. *For $\lambda \in \{m, \sqrt{4+m^2}\}$ any nonzero solution $\mathbf{w} \in \mathbf{l}^\infty$ of the equation $\mathcal{D}\mathbf{w} = \lambda\mathbf{w}$ is called a resonance function, and in this case λ is called a resonance.*

Lemma 2.3. *(see* [5, Lemmas 4.1 and 4.4]*)*
i) Let $q \in \ell^1$. Then $W(\theta) \ne 0$ for $\theta \in (-\pi, 0) \cup (0, \pi)$.
ii) Let $q \in \ell^1_1$. Then $\lambda = m$ (or $\lambda = \sqrt{4+m^2}$) is a resonance if and only if $W(0) = 0$ (or $W(\pi) = 0$).

Given the Jost solutions, we can express the resolvent $\mathcal{R}(\lambda) := (\mathcal{D} - \lambda)^{-1}$. The method of variation of parameters gives:

Lemma 2.4. *Let $q \in \ell^1$. Then for any $\lambda \in \Xi_+$, the operators $\mathcal{R}(\lambda) : \mathbf{l}^2 \to \mathbf{l}^2$ can be represented by the matrix elements as follows*

$$[\mathcal{R}(\lambda)]_{n,k} = \frac{1}{W(\theta(\lambda))}\begin{cases} \mathbf{w}^+_n(\theta(\lambda)) \otimes \mathbf{w}^-_k(\theta(\lambda)), & k \le n, \\ \mathbf{w}^-_n(\theta(\lambda)) \otimes \mathbf{w}^+_k(\theta(\lambda)), & k \ge n, \end{cases} \tag{2.8}$$

where

$$\mathbf{w}^1_k \otimes \mathbf{w}^2_n = \begin{pmatrix} u^1_k u^2_n & v^1_{k+1}u^2_n \\ u^1_k v^2_n & v^1_{k+1}v^2_n \end{pmatrix}, \qquad \mathcal{R}(\lambda)\mathbf{w}[n] = \sum_{k=-\infty}^{\infty} [\mathcal{R}(\lambda)]_{k,n}\begin{pmatrix} u_k \\ v_{k+1} \end{pmatrix}.$$

The representations (2.8), the fact that $W(\theta)$ does not vanish for $\lambda \in \Gamma_+$, and the bound (2.6) imply the limiting absorption principle for the perturbed Dirac equation.

Lemma 2.5. *(see* [5, Lemma 5.2]*) Let $q \in \ell^1$. Then the convergence*

$$\mathcal{R}(\lambda \pm \mathrm{i}\varepsilon) \to \mathcal{R}(\lambda \pm \mathrm{i}0), \quad \varepsilon \to 0+, \quad \lambda \in \Gamma_+ \tag{2.9}$$

holds in $\mathcal{L}(\mathbf{l}^2_\sigma, \mathbf{l}^2_{-\sigma})$ with $\sigma > 1/2$. Here

$$[\mathcal{R}(\lambda \pm \mathrm{i}0)]_{n,k} = \frac{1}{W(\theta_\pm)}\begin{cases} \mathbf{w}^+_n(\theta_\pm) \otimes \mathbf{w}^-_k(\theta_\pm) & \text{for } k \le n, \\ \mathbf{w}^+_k(\theta_\pm) \otimes \mathbf{w}^-_n(\theta_\pm) & \text{for } k \ge n, \end{cases} \tag{2.10}$$

with

$$\theta_+ := \theta(\lambda^2 - m^2 + \mathrm{i}0) \in [0, \pi], \qquad \theta_- := \theta(\lambda^2 - m^2 - \mathrm{i}0) \in [-\pi, 0].$$

3. Fourier properties of $\mathbf{h}_n^\pm(\theta)$

Green's functions $G^\pm(n,\theta)$ of equation (2.1) read:

$$G^\pm(n,\theta)=\begin{cases}\frac{(m+\lambda)}{2\mathrm{i}\sin\theta}\begin{pmatrix}\mathrm{e}^{\pm\mathrm{i}\theta n}-\mathrm{e}^{\mp\mathrm{i}\theta n} & \alpha_\pm\mathrm{e}^{\pm\mathrm{i}\theta n}-\alpha_\mp\mathrm{e}^{\mp\mathrm{i}\theta n}\\ \alpha_\mp\mathrm{e}^{\pm\mathrm{i}\theta n}-\alpha_\pm\mathrm{e}^{\mp\mathrm{i}\theta n} & (\mathrm{e}^{\pm\mathrm{i}\theta n}-\mathrm{e}^{\mp\mathrm{i}\theta n})\frac{\lambda-m}{m+\lambda}\end{pmatrix}, & \mp n\geq 1,\\ 0, \quad \mp n\leq -1,\end{cases} \tag{3.1}$$

and

$$G^+(0,\theta)=\begin{pmatrix}0&0\\-1&0\end{pmatrix},\qquad G^-(0,\theta)=\begin{pmatrix}0&-1\\0&0\end{pmatrix},$$

so that

$$\begin{pmatrix}m-\lambda & d\\ d^* & -(m+\lambda)\end{pmatrix}G^\pm(\cdot,\theta)[n]=\begin{pmatrix}1&0\\0&1\end{pmatrix}\delta_{n0},\quad n\in\mathbb{Z}.$$

Applying Green's function representation, we obtain

$$\mathbf{w}_n^\pm(\theta)=\begin{pmatrix}1\\ \alpha_\mp(\theta)\end{pmatrix}\mathrm{e}^{\pm\mathrm{i}\theta n}-G^\pm(0,\theta)Q_n\mathbf{w}_n^\pm(\theta)-\sum_{k=n\pm1}^{\pm\infty}G^\pm(n-k,\theta)Q_k\mathbf{w}_k^\pm(\theta).$$

Substituting $\mathbf{w}_n^\pm(\theta)=\mathbf{h}_n^\pm(\theta)\mathrm{e}^{\pm\mathrm{i}\theta n}$, we get

$$A_n^\pm\mathbf{h}_n^\pm(\theta)=\begin{pmatrix}1\\ \alpha_\mp(\theta)\end{pmatrix}+\sum_{k=n\pm1}^{\pm\infty}\tilde{G}^\pm(k-n,\theta)Q_k\mathbf{h}_k^\pm(\theta), \tag{3.2}$$

where

$$\tilde{G}^\pm(l,\theta)=\frac{(m+\lambda)}{2\mathrm{i}\sin\theta}\begin{pmatrix}\mathrm{e}^{\pm2\mathrm{i}\theta l}-1 & \alpha_\mp\mathrm{e}^{\pm2\mathrm{i}\theta l}-\alpha_\pm\\ \alpha_\pm\mathrm{e}^{\pm2\mathrm{i}\theta l}-\alpha_\mp & (\mathrm{e}^{\pm2\mathrm{i}\theta l}-1)\frac{\lambda-m}{m+\lambda}\end{pmatrix},\quad \pm l\geq 1,$$

$$A_n^+=\begin{pmatrix}1&0\\0&1-q_n\end{pmatrix},\quad A_n^-=\begin{pmatrix}1-q_n&0\\0&1\end{pmatrix}.$$

Representation (3.2) implies

Proposition 3.1. *Let $q\in\ell_1^1$. Then the Jost solutions $\mathbf{h}^\pm$ are given by*

$$A_n^\pm\mathbf{h}_n^\pm(\theta)=\begin{pmatrix}1\\ \alpha_\mp(\theta)\end{pmatrix}+\sum_{k=\mp1}^{\pm\infty}\begin{pmatrix}a_{n,k}^\pm\\ \frac{b_{n,k}^\pm}{\lambda+m}\end{pmatrix}\mathrm{e}^{\pm\mathrm{i}k\theta}, \tag{3.3}$$

where

$$|a_{n,k}^\pm|,|b_{n,k}^\pm|\leq C_n^\pm\sum_{l=n\pm1+[k/2]}^{\pm\infty}(|q_l|+\frac{|q_l|}{|1-q_l|}). \tag{3.4}$$

Moreover,

$$C_n^\pm\leq C^\pm,\quad \text{if } \pm n\geq 0. \tag{3.5}$$

Proof. Substituting (3.3) into (3.2) and setting $z=\mathrm{e}^{\mathrm{i}\theta}$, we obtain, formally,

$$\sum_{k=\mp1}^{\pm\infty}\begin{pmatrix}a_{n,k}^\pm\\ \frac{b_{n,k}^\pm}{\lambda+m}\end{pmatrix}z^{\pm k}=\sum_{p=n\pm1}^{\pm\infty}\tilde{G}^\pm(p-n,\theta)Q_p(A_p^\pm)^{-1}\Big[\begin{pmatrix}1\\ \alpha_\mp(\theta)\end{pmatrix}+\sum_{r=\mp1}^{\pm\infty}\begin{pmatrix}a_{p,r}^\pm\\ \frac{b_{p,r}^\pm}{\lambda+m}\end{pmatrix}z^{\pm r}\Big], \tag{3.6}$$

where

$$Q_p(A_p^+)^{-1}=\begin{pmatrix}0&\tilde{q}_p\\ q_p&0\end{pmatrix},\quad Q_p(A_p^-)^{-1}=\begin{pmatrix}0&q_p\\ \tilde{q}_p&0\end{pmatrix},\quad \tilde{q}_p:=\frac{q_p}{1-q_p}. \tag{3.7}$$

Step i) First we consider the "+" case and represent $\tilde{G}^+(n,\theta)$, $n\geq1$, as the sum:

$$\tilde{G}^+(n,\theta)=\sum_{j=0}^{2n}(-1)^j\begin{pmatrix}0&0\\1&0\end{pmatrix}z^j+\sum_{j=1}^{2n-1}(-1)^j\begin{pmatrix}0&1\\0&0\end{pmatrix}z^j+\sum_{j=1}^{n}\begin{pmatrix}\lambda+m&0\\0&\lambda-m\end{pmatrix}z^{2j-1}.$$

Substituting this expression into (3.6) and omitting the "+" sign, we obtain

$$\sum_{k=\mp1}^{\infty}\begin{pmatrix}a_{n,k}\\ \frac{b_{n,k}}{\lambda+m}\end{pmatrix}z^k=\sum_{p=n+1}^{\infty}\Big[\sum_{j=0}^{2(p-n)}(-1)^jz^j\begin{pmatrix}0&0\\0&\tilde{q}_p\end{pmatrix}+\sum_{j=1}^{2(p-n)-1}(-1)^jz^j\begin{pmatrix}q_p&0\\0&0\end{pmatrix}$$
$$+\sum_{j=1}^{p-n}z^{2j-1}\begin{pmatrix}0&(\lambda+m)\tilde{q}_p\\(\lambda-m)q_p&0\end{pmatrix}\Big]\Big[\begin{pmatrix}1\\ \frac{z^{-1}-1}{\lambda+m}\end{pmatrix}+\sum_{r=\mp}^{\infty}\begin{pmatrix}a_{p,r}\\ \frac{b_{p,r}}{\lambda+m}\end{pmatrix}z^r\Big]. \tag{3.8}$$

Using $(\lambda - m)(\lambda + m) = 2 - z - z^{-1}$, we rewrite (3.8) for the first and second line separately:

$$\sum_{k=-1}^{\infty} a_{n,k} z^k = \sum_{p=n+1}^{\infty} q_p \Big[1 + \sum_{r=-1}^{\infty} a_{p,r} z^r\Big] \sum_{j=1}^{2(p-n)-1} (-1)^j z^j + \sum_{p=n+1}^{\infty} \tilde{q}_p \Big[\frac{1}{z} - 1 + \sum_{r=-1}^{\infty} b_{p,r} z^r\Big] \sum_{j=1}^{p-n} z^{2j-1},$$

$$\sum_{k=-1}^{\infty} b_{n,k} z^k = (2 - \frac{1}{z} - z) \sum_{p=n+1}^{\infty} q_p \Big[1 + \sum_{r=-1}^{\infty} a_{p,r} z^r\Big] \sum_{j=1}^{p-n} z^{2j-1} + \sum_{p=n+1}^{\infty} \tilde{q}_p \Big[\frac{1}{z} - 1 + \sum_{r=-1}^{\infty} b_{p,r} z^r\Big] \sum_{j=0}^{2(p-n)} (-1)^j z^j.$$

Equating the coefficients of equal powers of z, we obtain

$$a_{n,-1} = 0, \quad a_{n,0} = \sum_{p=n+1}^{\infty} \tilde{q}_p (1 + b_{p,-1}), \tag{3.9}$$

$$b_{n,-1} = \sum_{p=n+1}^{\infty} \tilde{q}_p (1 + b_{p,-1}), \quad b_{n,0} = -\sum_{p=n+1}^{\infty} (q_p[1 + a_{p,0}] + \tilde{q}_p[2 + b_{p,-1} - b_{p,0}]), \tag{3.10}$$

and for $k \geq 1$,

$$a_{n,k} = (-1)^k \sum_{p=n+1+[\frac{k}{2}]}^{\infty} (q_p + \tilde{q}_p) + \sum_{r=0}^{k-1} (-1)^{k+r} \sum_{p=n+1+[\frac{k-r}{2}]}^{\infty} q_p a_{p,r} + \sum_{r=-1}^{[\frac{k-1}{2}]} \sum_{p=n-r+[\frac{k+1}{2}]}^{\infty} \tilde{q}_p b_{p,f_k(r)}, \tag{3.11}$$

$$\begin{aligned} b_{n,k} &= (-1)^{k+1} \sum_{p=n+[\frac{k+1}{2}]}^{\infty} 2(q_p + \tilde{q}_p) + \sigma_k q_{n+\frac{k}{2}} + \sum_{r=0}^{k-1} (-1)^{k+r+1} \sum_{p=n+[\frac{k-r+1}{2}]}^{\infty} 2 q_p a_{p,r} \\ &- \sum_{r=0}^{k-1} \sigma_{k+r} q_{\frac{2n+k-r}{2}} a_{\frac{2n+k-r}{2}, r} + \sum_{r=-1}^{k-1} (-1)^{r+k} \sum_{p=n+[\frac{k-r+1}{2}]}^{\infty} \tilde{q}_p b_{p,r}, \end{aligned} \tag{3.12}$$

where

$$\sigma_k = \begin{cases} 0 & \text{for odd } k, \\ 1 & \text{for even } k, \end{cases} \qquad f_k(r) = \begin{cases} 2r & \text{for odd } k, \\ 2r+1 & \text{for even } k. \end{cases}$$

These equations are solved by adapting the iteration of [1]:

$$a_{n,k} = \sum_{j=0}^{\infty} a_{j,n,k}, \qquad b_{n,k} = \sum_{j=0}^{\infty} b_{j,n,k},$$

where

$$a_{0,n,0} = -b_{0,n,-1} = \sum_{p=n+1}^{\infty} \tilde{q}_p, \qquad b_{0,n,0} = -\sum_{p=n+1}^{\infty} (q_p + 2\tilde{q}_p),$$

$$a_{0,n,k} = (-1)^k \sum_{p=n+1+[\frac{k}{2}]}^{\infty} (q_p + \tilde{q}_p), \quad b_{0,n,k} = (-1)^{k+1} \sum_{p=n+[\frac{k+1}{2}]}^{\infty} 2(q_p + \tilde{q}_p) + \sigma_k q_{n+\frac{k}{2}}, \quad k \geq 1,$$

and for $j \geq 0$,

$$a_{j+1,n,1} = -b_{j+1,n,0} = \sum_{p=n+1}^{\infty} \tilde{q}_p b_{j,p,0}, \quad b_{j+1,n,1} = -\sum_{p=n+1}^{\infty} (q_p a_{j,p,1} + \tilde{q}_p[b_{j,p,0} - b_{j,p,1}]),$$

$$a_{j+1,n,k} = \sum_{r=1}^{k-1} (-1)^{k+r} \sum_{p=n+1+[\frac{k-r}{2}]}^{\infty} q_p a_{j,p,r} + \sum_{r=0}^{[\frac{k-1}{2}]} \sum_{p=n-r+[\frac{k+1}{2}]}^{\infty} \tilde{q}_p b_{j,p,f_k(r)}, \quad k \geq 1,$$

$$b_{j+1,n,k} = \sum_{r=1}^{k-1} (-1)^{k+r+1} \sum_{p=n+[\frac{k-r+1}{2}]}^{\infty} 2 q_p a_{j,p,r} + \sum_{r=0}^{k-1} (-1)^{r+k} \sum_{p=n+[\frac{k-r+1}{2}]}^{\infty} \tilde{q}_p b_{j,p,r}, \quad k \geq 1.$$

Now we define the functions

$$\eta(n)=\max\{\sum_{k=n}^{\infty}|q_k|,\sum_{k=n}^{\infty}|\tilde q_k|\},\quad \gamma(n)=\max\{\sum_{k=n}^{\infty}(k-n)|q_k|,\sum_{k=n}^{\infty}(k-n)|\tilde q_k|\}.$$

It is obvious that $|a_{0,n,k}|+|b_{0,n,k}|\le 2\eta(n+1+[k/2])$. Moreover, one can show as in [1, Lemma 3] that

$$|a_{j,n,k}|+|b_{j,n,k}|\le \frac{(2\gamma(n))^j}{j!}\eta(n+1+[k/2]).$$

Hence, (3.4) with $C_n^+=\mathrm{e}^{2\gamma(n)}$ follows.

Step ii) It is easy to check that in the "−" case, we obtain similarly to (3.8),

$$\sum_{k=0}^{-\infty}\begin{pmatrix} a^-_{n,k}\\ \frac{b^-_{n,k}}{\lambda+m}\end{pmatrix} z^{-k}=\sum_{p=n-1}^{-\infty}\Big[\sum_{j=-1}^{2(p-n)+1}(-1)^j z^{-j}\begin{pmatrix}0&0\\0&q_p\end{pmatrix}+\sum_{j=0}^{2(p-n)}(-1)^j z^{-j}\begin{pmatrix}\tilde q_p&0\\0&0\end{pmatrix}$$
$$+\sum_{j=-1}^{p-n} z^{-2j-1}\begin{pmatrix}0&(\lambda+m)q_p\\(\lambda-m)\tilde q_p&0\end{pmatrix}\Big]\Big[\begin{pmatrix}1\\ \frac{z-1}{\lambda+m}\end{pmatrix}+\sum_{r=0}^{-\infty}\begin{pmatrix}a^-_{p,r}\\ \frac{b^-_{p,r}}{\lambda+m}\end{pmatrix}z^{-r}\Big].$$

This is equivalent to the system

$$\sum_{k=0}^{-\infty}a^-_{n,k}z^{-k}=\sum_{p=n-1}^{-\infty}\tilde q_p\Big[1+\sum_{r=0}^{-\infty}a^-_{p,r}z^{-r}\Big]\sum_{j=0}^{2(p-n)}(-1)^j z^{-j}$$
$$+\sum_{p=n-1}^{-\infty}q_p\Big[z-1+\sum_{r=0}^{-\infty}b^-_{p,r}z^{-r}\Big]\sum_{j=-1}^{p-n}z^{-2j-1},$$

$$\sum_{k=0}^{-\infty}b^-_{n,k}z^{-k}=\sum_{p=n-1}^{-\infty}\tilde q_p\Big[1+\sum_{r=0}^{-\infty}a^-_{p,r}z^{-r}\Big]\sum_{j=-1}^{p-n}(2-z^{-1}-z)z^{-2j-1}$$
$$+\sum_{p=n-1}^{-\infty}q_p\Big[z-1+\sum_{r=0}^{-\infty}b^-_{p,r}z^{-r}\Big]\sum_{j=-1}^{2(p-n)+1}(-1)^j z^{-j}.$$

Equating the coefficients of equal powers of z, we obtain equations for $a^-_{n,k}$ and $b^-_{n,k}$ similar to the equations (3.9)–(3.11). In particular, we get

$$a^-_{n,0}=-b^-_{n,0}=\sum_{p=n-1}^{-\infty}\tilde q_p[1+a^-_{p,0}]. \tag{3.13}$$

□

4. The Gelfand–Levitan–Marchenko equations

The following formula is obtained by means of simple calculations:

Lemma 4.1. *For any* $\mathbf{w}^1=(u^1,v^1)$, $\mathbf{w}^2=(u^2,v^2)$,

$$\sum_{j=m}^{n}\Big(\mathbf{w}^1_j\cdot(\mathcal{D}\mathbf{w}^2)_j-(\mathcal{D}\mathbf{w}^1)_j\cdot\mathbf{w}^2_j\Big)=-W_n(\mathbf{w}^1,\mathbf{w}^2)+W_{m-1}(\mathbf{w}^1,\mathbf{w}^2), \tag{4.1}$$

where $\mathbf{w}^1_j\cdot\mathbf{w}^2_j=u^1_ju^2_j+v^1_jv^2_j$, *and* $W_j(\mathbf{w}^1,\mathbf{w}^2)=u^1_jv^2_{j+1}-u^2_jv^1_{j+1}$.

Let now $\mathbf{w}^1$ and $\mathbf{w}^2$ be solutions to (2.1). Then

$$\frac{d}{d\lambda}(\mathcal{D}-\lambda)\mathbf{w}^k=(\mathcal{D}-\lambda)\frac{d}{d\lambda}\mathbf{w}^k-\mathbf{w}^k=0,\quad k=1,2,$$

and (4.1) implies

$$-W_n(\mathbf{w}^1,\frac{d}{d\lambda}\mathbf{w}^2)+W_{m-1}(\mathbf{w}^1,\frac{d}{d\lambda}\mathbf{w}^2)=\sum_{j=m}^{n}\Big(\mathbf{w}^1_j\cdot(\mathcal{D}\frac{d}{d\lambda}\mathbf{w}^2)_j-(\mathcal{D}\mathbf{w}^1)_j\cdot\frac{d}{d\lambda}\mathbf{w}^2_j\Big)=\sum_{j=m}^{n}\mathbf{w}^1_j\cdot\mathbf{w}^2_j.$$

Using this formula, we obtain

Lemma 4.2. *(cf. [7, Lemma 2.4]) Let $\mathbf{w}^{\pm}(\lambda)$ be solutions to (2.1), square summable near $\pm\infty$. Then*

$$W_n(\mathbf{w}^+(\lambda), \frac{d}{d\lambda}\mathbf{w}^+(\lambda)) = \sum_{j=n+1}^{\infty} \mathbf{w}_j^+(\lambda)\cdot\mathbf{w}_j^+(\lambda),$$
$$W_n(\mathbf{w}^-(\lambda), \frac{d}{d\lambda}\mathbf{w}^-(\lambda)) = -\sum_{j=-\infty}^{n} \mathbf{w}_j^-(\lambda)\cdot\mathbf{w}_j^-(\lambda). \tag{4.2}$$

Let now λ_l be an isolated eigenvalue of $\mathcal{D}$. In this case $W(\mathbf{w}^+(\lambda_l), \mathbf{w}^-(\lambda_l)) = 0$, and hence $\mathbf{w}^{\pm}(\lambda_l)$ differ only by a (nonzero) constant multiple $\varkappa_l$: $\mathbf{w}^-(\lambda_l) = \varkappa_l\mathbf{w}^+(\lambda_l)$. Hence,

$$\frac{d}{d\lambda}W(\mathbf{w}^+(\lambda), \mathbf{w}^-(\lambda))\Big|_{\lambda=\lambda_l} = W_n(\frac{1}{\varkappa_l}\mathbf{w}^-(\lambda_l), \frac{d}{d\lambda}\mathbf{w}^-(\lambda_l)) + W_n(\frac{d}{d\lambda}\mathbf{w}^+(\lambda_l), \varkappa_l\mathbf{w}^+(\lambda_l))$$
$$= -\sum_{j\in\mathbb{Z}} \mathbf{w}_j^+(\lambda_l)\cdot\mathbf{w}_j^-(\lambda_l) = -\varkappa_l\sum_{j\in\mathbb{Z}} \mathbf{w}_j^+(\lambda_l)\cdot\mathbf{w}_j^+(\lambda_l) \tag{4.3}$$

by (4.2). Thus the poles of the resolvent at isolated eigenvalues are simple. Denote $z = \mathrm{e}^{\mathrm{i}\theta}$. From $2 - z - z^{-1} = \lambda^2 - m^2$ we obtain $\dfrac{d\lambda}{dz} = \dfrac{1-z^2}{2z^2\lambda}$. Therefore,

$$\frac{d}{dz}W(\mathbf{w}^+(\lambda(z)), \mathbf{w}^-(\lambda(z)))\Big|_{z=z_l} = \frac{z_l^2-1}{2z_l^2\lambda_l}\sum_{j\in\mathbb{Z}} \mathbf{w}_j^+(z_l)\cdot\mathbf{w}_j^-(z_l), \quad \lambda_l = \lambda(z_l). \tag{4.4}$$

Now we consider the Jost solution $\mathbf{w}^{\pm}(\theta)$, $\theta = \theta(\lambda)$, defined in (2.2). Denote

$$W^{\pm}(\theta) = W(\mathbf{w}^{\mp}(\theta), \mathbf{w}^{\pm}(-\theta)).$$

Recall that the quantities

$$T(\theta) = \frac{2\mathrm{i}\sin\theta}{(m+\lambda)W(\theta)}, \quad R^{\pm}(\theta) = \pm\frac{W^{\pm}(\theta)}{W(\theta)}, \quad \lambda\in\Gamma_+, \tag{4.5}$$

are known as the transmission and reflection coefficients. For these coefficients the following scattering relations hold (see [5])

$$T(\theta)\mathbf{w}^{\mp}(\theta) = R^{\pm}(\theta)\mathbf{w}^{\pm}(\theta) + \mathbf{w}^{\pm}(-\theta), \quad \theta\in[-\pi,\pi]. \tag{4.6}$$

Denote $\tilde{T}(z) = T(\theta(z))$, $\tilde{R}^{\pm}(z) = R^{\pm}(\theta(z))$. Let $F_n^{\pm}$ be the Fourier coefficients of $R^{\pm}$:

$$F_n^{\pm} := \frac{1}{2\pi\mathrm{i}}\int_{|z|=1}\tilde{R}^{\pm}(z)z^{\pm n}\frac{dz}{z}. \tag{4.7}$$

Since $|\tilde{R}^{\pm}(z)| \le 1$ (see [1, 7, 5]), Parseval's identity implies $\sum\limits_{n\in\mathbb{Z}}|F_n^{\pm}|^2 = \frac{1}{2\pi\mathrm{i}}\int_{|z|=1}|\tilde{R}^{\pm}(z)|^2\frac{dz}{z} \le 1$. Hence, $F^{\pm}\in\ell^2(\mathbb{Z})$. Let $\lambda_l \ge 0$, $l = 1, ..., N$, be the poles of the resolvent and let

$$\mathcal{F}_n^{\pm} = F_n^{\pm} + \sum_{l=1}^{N}\gamma_l^{\pm}z_l^{\pm n}, \quad \text{where} \quad \gamma_l^{\pm} = \frac{2\lambda_l}{(m+\lambda_l)\sum\limits_{j\in\mathbb{Z}}\mathbf{w}_j^{\pm}(z_l)\cdot\mathbf{w}_j^{\pm}(z_l)}. \tag{4.8}$$

Now we derive the Gelfand–Levitan–Marchenko equations for $\mathcal{F}^{\pm}$.

Proposition 4.3. *(cf. [7, Equations (10.71), (10.76)]) Let $q\in\ell_1^1$. Then*
i) For any $j\ge 0$, the following equations hold:

$$\begin{cases} a_{n,j}^+ + \mathcal{F}_{2n+j}^+ + \sum\limits_{p=0}^{\infty}\mathcal{F}_{2n+p+j}^+a_{n,p}^+ = \frac{\tilde{T}(0)(1+a_{n,0}^-)}{1-q_n}\delta_{j,0}, \\ b_{n,j-1}^+ + \mathcal{F}_{2n+j}^+ - \mathcal{F}_{2n+j-1}^+ + \sum\limits_{p=-1}^{\infty}\mathcal{F}_{2n+p+j-1}^+b_{n,p}^+ \\ \quad = (1-q_n)\Big[\tilde{T}(0)(1+b_{n,0}^-)\delta_{j-1,0} + \big(\tilde{T}'(0)(b_{n,0}^- - 1) + \tilde{T}(0)(b_{n,-1}^- + 1)\big)\delta_{j-1,-1}\Big]. \end{cases} \tag{4.9}$$

ii) For any $j\le 0$, the following equations hold:

$$\begin{cases} a_{n,j}^- + \mathcal{F}_{2n+j}^- + \sum\limits_{p=0}^{-\infty}\mathcal{F}_{2n+p+j}^-a_{n,p}^- = (1-q_n)[\tilde{T}(0)(1+a_{n,0}^+)\delta_{j,0}], \\ b_{n,j}^- + \mathcal{F}_{2n+j}^- + \sum\limits_{p=0}^{-\infty}\mathcal{F}_{2n+p+j}^-b_{n,p}^- = \dfrac{\tilde{T}(0)(b_{n,0}^+ - 1)}{1-q_n}\delta_{j,0}. \end{cases} \tag{4.10}$$

iii) The following estimates hold

$$|\mathcal{F}_n^{\pm}| \le M_n^{\pm} \sum_{p=[\frac{n}{2}]}^{\pm\infty} (|q_p| + |\tilde{q}_p|), \tag{4.11}$$

where $M_n^{\pm}$ are terms of order zero as $n \to \pm\infty$.

Proof. i). Consider (4.6) with upper signs:

$$\begin{cases} \tilde{T}(z)\tilde{u}^-(z) = \tilde{u}^+(z^{-1}) + \tilde{R}^+(z)\tilde{u}^+(z), \\ \tilde{T}(z)\tilde{w}^-(z) = \tilde{w}^+(z^{-1}) + \tilde{R}^+(z)\tilde{w}^+(z), \end{cases} \tag{4.12}$$

where $\tilde{u}^{\pm}(z) = u^{\pm}(\theta(z))$, $\tilde{w}^{\pm}(z) := (m + \lambda(z))v^{\pm}(\theta(z))$. We multiply the first equation by $(2\pi i)^{-1}z^{n+j}$, $j = 0, 1...$, and integrate around the unit circle. Using (3.3), we first evaluate the right hand side:

$$\frac{1}{2\pi \mathrm{i}} \int\limits_{|z|=1} \tilde{u}_n^+(z^{-1}) z^{n+j} \frac{dz}{z} = a_{n,j}^+, \qquad \frac{1}{2\pi \mathrm{i}} \int\limits_{|z|=1} \tilde{R}^+(z)\tilde{u}_n^+(z) z^{n+j} \frac{dz}{z} = \sum_{p=0}^{\infty} F_{2n+p+j}^+ a_{n,p}^+. \tag{4.13}$$

Next we evaluate the left hand side. From (4.4) and (4.5) it follows that

$$\begin{aligned} \mathrm{res}_{z_l} \tilde{T}(z)\tilde{u}_n^-(z) z^{n+j-1} &= \mathrm{res}_{z_l} \frac{(z^2-1)\tilde{u}_n^-(z) z^{n+j-2}}{(m+\lambda)W(\tilde{\mathbf{w}}^+(z), \tilde{\mathbf{w}}^-(z))} \\ &= \frac{2\lambda_l \tilde{u}_n^-(z_l) z_l^{n+j}}{(m+\lambda_l)\sum\limits_{j\in\mathbb{Z}} \mathbf{w}_j^+(z_l)\cdot \mathbf{w}_j^-(z_l)} = \gamma_l^+ \tilde{u}_n^+(z_l) z_l^{n+j}. \end{aligned}$$

Using (3.3) and the residue theorem (take a contour inside the unit disk enclosing all poles and let this contour approach the unit circle), we obtain

$$\begin{aligned} \frac{1}{2\pi \mathrm{i}} \int_{|z|=1} \tilde{T}(z)\tilde{u}_n^-(z) z^{n+j} \frac{dz}{z} &= -\sum_{l=1}^{N} \gamma_l^+ \tilde{u}_n^+(z_l) z_l^{n+j} + \tilde{T}(0)(\tilde{h}_n^-(0))_1 \delta_{j,0} \\ = -\sum_{p=0}^{\infty} a_{n,p}^+ \sum_{l=1}^{N} \gamma_l^+ z_l^{2n+p+j} &+ \frac{\tilde{T}(0)(1+a_{n,0}^-)}{1-q_n}\delta_{j,0}, \end{aligned} \tag{4.14}$$

where $\tilde{T}(0) < \infty$ (see Appendix A), and $(\tilde{h}_n^-)_1$ is the first component of the vector $\tilde{h}_n^{-1}$. Substituting (4.13) and (4.14) into the first equation of (4.12), we obtain the first equation of (4.9).

Now consider the second equation of (4.12). Similarly to (4.13)–(4.14), we obtain for $j = -1, 0, 1, ...$

$$\begin{aligned} \frac{1}{2\pi \mathrm{i}} \int_{|z|=1} \tilde{w}_n^+(z^{-1}) z^{n+j} \frac{dz}{z} &= \frac{b_{n,j}^+}{1-q_n}, \\ \frac{1}{2\pi \mathrm{i}} \int_{|z|=1} \tilde{R}^+(z)\tilde{w}_n^+(z) z^{n+j} \frac{dz}{z} &= \sum_{p=-1}^{\infty} F_{2n+p+j}^+ \frac{b_{n,p}^+}{1-q_n}, \\ \frac{1}{2\pi \mathrm{i}} \int_{|z|=1} \tilde{T}(z)\tilde{w}_n^-(z) z^{n+j} \frac{dz}{z} &= -\sum_{p=-1}^{\infty} b_{n,p}^+ \sum_{l=1}^{N} \gamma_l^+ z_l^{2n+p+j} + \tilde{T}(0)(1+b_{n,0}^-)\delta_{j,0} \\ &\quad + \big(\tilde{T}'(0)(b_{n,0}^- - 1) + \tilde{T}(0)(b_{n,-1}^- + 1)\big)\delta_{j,-1}, \end{aligned}$$

where $\tilde{T}(0), \tilde{T}'(0) < \infty$ (see Appendix A). Then the second equation of (4.9) follows.

ii) Equation (4.6) with lower signs reads

$$\begin{cases} \tilde{T}(z)\tilde{u}^+(z) = \tilde{u}^-(z^{-1}) + \tilde{R}^-(z)\tilde{u}^-(z), \\ \tilde{T}(z)\tilde{w}^+(z) = \tilde{w}^-(z^{-1}) + \tilde{R}^-(z)\tilde{w}^-(z). \end{cases}$$

Multiplying by $(2\pi \mathrm{i})^{-1}z^{n+j}$, $j = 0, -1, -2\dots$, and integrating around the unit circle, we obtain (4.10).

iii) Note that $|a_{n,p}^{\pm}| < 1$ for sufficiently large $\pm n$ by (3.4). Hence, equations (4.9)–(4.10) together with the estimates (3.4) imply

$$|\mathcal{F}^{\pm}_{2n+j}| \le |a^{\pm}_{n,j}| + \sum_{p=0}^{\pm\infty} |\mathcal{F}^{\pm}_{2n+p+j} a^{\pm}_{n,p}|$$
$$\le C^{\pm}_n \Big(\mathcal{Q}^{\pm}\big(n \pm 1 + [\tfrac{j}{2}]\big) + \sum_{p=0}^{\pm\infty} |\mathcal{F}^{+}_{2n+p+j}| \mathcal{Q}^{\pm}\big(n \pm 1 + [\tfrac{p}{2}]\big)\Big), \quad \pm j \ge 1,$$

where $\mathcal{Q}^{\pm}(n) = \sum_{l=n}^{\pm\infty}(|q_l| + |\tilde{q}_l|)$. Then (4.11) follows by arguments from [4] and [7, Section 10.3]. □

5. The Wiener algebra

Recall that the Wiener algebra is the set of all integrable functions whose Fourier coefficients are integrable:

$$\mathcal{A} = \Big\{ f(\theta) = \sum_{m\in\mathbb{Z}} \hat{f}_m e^{im\theta} \,\Big|\, \|\hat{f}\|_{\ell^1} < \infty \Big\}.$$

We set

$$\|f\|_{\mathcal{A}} = \|\hat{f}\|_{\ell^1}, \qquad \|(f_1, f_2)\|_{\mathcal{A}} = \|(\hat{f}_1, \hat{f}_2)\|_{l^1}.$$

Since $\lambda = \lambda(\theta) = \sqrt{2 - 2\cos\theta + m^2} \in C^\infty([-\pi,\pi])$ we have $\lambda + m$, $\frac{1}{\lambda+m} \in \mathcal{A}$. Hence, the representation (3.3) and the estimates (3.4) imply that

$$\mathbf{h}^{\pm}_n(\theta), \mathbf{w}^{\pm}_n(\theta) \in \mathcal{A} \quad \text{if} \quad q \in \ell^1_1. \tag{5.1}$$

Consequently, the Wronskians $W(\theta)$ and $W^{\pm}(\theta)$ also belong to $\mathcal{A}$.

Theorem 5.1. *If $q \in \ell^1_1$, then $T(\theta)$, $R^{\pm}(\theta) \in \mathcal{A}$.*

Proof. Due to Lemma 2.3, $W(\theta)$ can vanish only at the edges of continuous spectra, i.e. when $\theta = 0, \pi$, which correspond to the resonant cases. (We identify the points π and $-\pi$ and consider the Jost solutions as functions on the unit circle.) In the case $W(0)W(\pi) \ne 0$, $W(\theta)^{-1} \in \mathcal{A}$ by Wiener's lemma. Therefore, $T(\theta), R^{\pm}(\theta) \in \mathcal{A}$. It remains to consider the case $W(0)W(\pi) = 0$.

Lemma 5.2. *Let $W(0) = 0$. Then the following representations hold*

$$(m+\lambda)W(\theta) = (1 - e^{i\theta})\Phi(\theta), \quad (m+\lambda)W^{\pm}(\theta) = (1 - e^{i\theta})\Phi^{\pm}(\theta), \quad \lambda = \lambda(\theta),$$

where $\Phi(\theta), \Phi^{\pm}(\theta) \in \mathcal{A}$. Moreover, if $W(\pi) = 0$ then $\Phi(\theta) \ne 0$ for $\theta \in (-\pi,\pi)$, and if $W(\pi) \ne 0$ then $\Phi(\theta) \ne 0$ for $\theta \in [-\pi,\pi]$.

Proof. Denote $w^{\pm}_n(\theta) := (m+\lambda)v^{\pm}_n(\theta)$. Since

$$W(0) = u^{+}_0(0)\frac{w^{-}_1(0)}{2m} - \frac{w^{+}_1(0)}{2m}u^{-}_0(0) = 0, \tag{5.2}$$

we have two possible combinations (because the solutions $\mathbf{w}^{\pm}_n(0)$ cannot vanish at two consecutive points):

$$(a): \quad u^{+}_0(0)u^{-}_0(0) \ne 0 \qquad \text{and} \qquad (b): \quad w^{+}_1(0)w^{-}_1(0) \ne 0.$$

We will only consider the case (a). The case (b) is treated similarly. By (2.7) and (5.2) we get

$$(m+\lambda)W(\theta) = u^{+}_0(\theta)u^{-}_0(\theta)\left(\frac{V^{+}(\theta)}{u^{+}_0(0)u^{+}_0(\theta)} - \frac{V^{-}(\theta)}{u^{-}_0(0)u^{-}_0(\theta)}\right), \tag{5.3}$$

where $V^{\pm}(\theta) := u^{\pm}_0(\theta)w^{\pm}_1(0) - u^{\pm}_0(0)w^{\pm}_1(\theta)$.

Step i) Let us prove that

$$V^{\pm}(\theta) = (1 - e^{i\theta})\Psi^{\pm}(\theta), \quad V^{\pm}(\theta) = (1 + e^{i\theta})\tilde{\Psi}^{\pm}(\theta) \tag{5.4}$$

with

$$\Psi^{\pm}(\theta),\ \tilde{\Psi}^{\pm}(\theta) \in \mathcal{A}. \tag{5.5}$$

We consider the case "+" and the first equality in (5.4) only. Representation (3.3) implies

$$u^{+}_n(\theta) = \sum_{k=n}^{\infty} \tilde{a}^{+}_{n,k} z^k, \qquad w^{+}_n(\theta) = (m+\lambda)v^{+}_n(\theta) = \sum_{k=n-1}^{\infty} \tilde{b}^{+}_{n,k} z^k, \quad z = e^{i\theta}, \tag{5.6}$$

where

$$\tilde{a}^{+}_{n,k} = \delta_{n,k} + a^{+}_{n,k-n}, \qquad \tilde{b}^{+}_{n,k} = \frac{\delta_{k,-1} - \delta_{k,0} + b^{+}_{n,k-n}}{1 - q_n}. \tag{5.7}$$

We will use summation by parts, i.e., the following identity,

$$\sum_{k=s}^{\infty}(f(k)-f(k+1))g(k)=\sum_{k=s}^{\infty}f(k)(g(k)-g(k-1))+f(s)g(s-1), \tag{5.8}$$

which is valid for all $f\in\ell^1(\mathbb{Z}_+)$, $g\in\ell^\infty(\mathbb{Z}_+)$ or vice versa. Introduce

$$a_n(s)=\sum_{k=s}^{\infty}\tilde a^+_{n,k},\qquad b_n(s)=\sum_{k=s}^{\infty}\tilde b^+_{n,k}, \tag{5.9}$$

which are well defined due to (3.4). Note, that $a_n(n)=u_n^+(0)$ and $b_n(n-1)=w_n^+(0)$. Applying (5.8) to (5.6) and using (5.9), we obtain

$$u_n^+(\theta)=\sum_{k=n}^{\infty}(a_n(k)-a_n(k+1))z^k=\sum_{k=n}^{\infty}a_n(k)z^k(1-z^{-1})+u_n^+(0)z^{n-1},$$
$$w_n^+(\theta)=\sum_{k=n-1}^{\infty}(b_n(k)-b_n(k+1))z^k=\sum_{k=n-1}^{\infty}b_n(k)z^k(1-z^{-1})+w_n^+(0)z^{n-2}.$$

Abbreviate $\zeta(z)=(z-1)/z$, then

$$u_0^+(\theta)=\zeta(z)\sum_{k=1}^{\infty}a_0(k)z^k+u_0^+(0),\qquad w_1^+(\theta)=\zeta(z)\sum_{k=1}^{\infty}b_1(k)z^k+w_1^+(0). \tag{5.10}$$

Multiplying the first equation of (5.10) by $w_1^+(0)$ and the second equation by $u_0^+(0)$, their difference is equal to

$$V^+(\theta)=u_0^+(\theta)w_1^+(0)-w_1^+(\theta)u_0^+(0)=(1-\mathrm{e}^{\mathrm{i}\theta})\sum_{k=0}^{\infty}g(k)\mathrm{e}^{\mathrm{i}k\theta}, \tag{5.11}$$

where

$$g(k)=a_0(k)w_1^+(0)-b_1(k)u_0^+(0). \tag{5.12}$$

Note that by (3.4) and (5.9), we have $g(\cdot)\in\ell^\infty(\mathbb{Z}_+)$. It remains to show that

$$g(\cdot)\in\ell^1(\mathbb{Z}_+). \tag{5.13}$$

The Gelfand–Levitan–Marchenko equations (4.9) imply

$$\tilde a_{0,j}+\sum_{p=0}^{\infty}\mathcal{F}_{p+j}\tilde a_{0,p}=0,\qquad \tilde b_{1,j}+\sum_{p=0}^{\infty}\mathcal{F}_{p+j}\tilde b_{1,p}=0,\quad j\ge 2.$$

Summing both equalities from $s\ge 2$ to ∞ gives

$$a_0(s)+\sum_{j=s}^{+\infty}\sum_{p=0}^{+\infty}\mathcal{F}_{p+j}[a_0(p)-a_0(p+1)]=0,$$
$$b_1(s)+\sum_{j=s}^{+\infty}\sum_{p=0}^{+\infty}\mathcal{F}_{p+j}[b_1(p)-b_1(p+1)]=0.$$

Applying (5.8), we obtain

$$a_0(s)+\sum_{j=s}^{+\infty}\Big(\sum_{p=0}^{+\infty}(\mathcal{F}_{p+j}-\mathcal{F}_{p+j-1})a_0(p)+a_0(0)\mathcal{F}_{j-1}\Big)=0,$$
$$b_1(s)+\sum_{j=s}^{+\infty}\Big(\sum_{p=0}^{+\infty}(\mathcal{F}_{p+j}-\mathcal{F}_{p+j-1})b_1(p)+b_1(0)\mathcal{F}_{j-1}\Big)=0.$$

Hence,

$$a_0(s)+u_0^+(0)\sum_{j=s}^{+\infty}\mathcal{F}_{j-1}-\sum_{p=0}^{+\infty}a_0(p)\mathcal{F}_{p+s-1}=0,$$
$$b_1(s)+w_1^+(0)\sum_{j=s}^{+\infty}\mathcal{F}_{j-1}-\sum_{p=0}^{+\infty}b_1(p)\mathcal{F}_{p+s-1}=0$$

by (5.9). We multiply the first equation by $w_1^+(0)$, the second by $u_0^+(0)$, subtract the second equation from the first, and use (5.12) to arrive at

$$g(s) - \sum_{p=0}^{+\infty} g(p)\mathcal{F}_{p+s-1} = 0. \tag{5.14}$$

Any bounded solution to (5.14) with a kernel satisfying (4.11) belongs to $\ell^1(\mathbb{Z}_+)$ as proved in [6]. Hence, (5.13) follows.

Step ii) Substituting (5.4) into (5.3), we obtain

$$(m+\lambda)W(\theta) = (1-\mathrm{e}^{\mathrm{i}\theta})\left(\frac{u_0^-(\theta)}{u_0^+(0)}\Psi^+(\theta) - \frac{u_0^+(\theta)}{u_0^-(0)}\Psi^-(\theta)\right) = (1-\mathrm{e}^{\mathrm{i}\theta})\Phi(\theta),$$

where $\Phi(\theta)\in\mathcal{A}$ by (5.5) and (5.1). We observe that if $W(\pi)=0$ then $\Phi(\theta)\neq 0$ for $\theta\in(-\pi,\pi)$, and if $W(\pi)\neq 0$ then $\Phi(\theta)\neq 0$ for $\theta\in[-\pi,\pi]$.

Since $W(0)=0$ implies $W^\pm(0)=0$ then we can also get similarly $(m+\lambda)W^\pm(\theta)=(1-\mathrm{e}^{\mathrm{i}\theta})\Phi^\pm(\theta)$ with $\Phi^\pm(\theta)\in\mathcal{A}$. □

Analogously, $W(\pi)=0$ implies

$$(m+\lambda)W(\theta) = (1+\mathrm{e}^{\mathrm{i}\theta})\tilde{\Phi}(\theta), \quad (m+\lambda)W^\pm(\theta) = (1+\mathrm{e}^{\mathrm{i}\theta})\tilde{\Phi}^\pm(\theta)$$

with $\tilde{\Phi},\tilde{\Phi}^\pm\in\mathcal{A}$ and $\tilde{\Phi}(\theta)\neq 0$ for $\theta\in[-\pi,\pi]$ if $W(0)\neq 0$. Thus, if W vanishes at only one endpoint, this finishes the proof. If W vanishes at both endpoints, we can use a smooth cut-off function to combine both representations into $(m+\lambda)W(\theta) = (1-\mathrm{e}^{2\mathrm{i}\theta})\breve{\Phi}(\theta)$ (respectively, $(m+\lambda)W^\pm(\theta) = (1-\mathrm{e}^{2\mathrm{i}\theta})\breve{\Phi}^\pm(\theta)$) with $\breve{\Phi},\breve{\Phi}^\pm\in\mathcal{A}$ and $\breve{\Phi}(\theta)\neq 0$ for $\theta\in[-\pi,\pi]$. □

6. The case $\operatorname{Re}\lambda \le 0$.

In the case $\lambda\in\Xi_- = \{\lambda\in\mathbb{C}\setminus\overline{\Gamma}_-,\ \operatorname{Re}\lambda\le 0\}$, where $\Gamma_- = (-\sqrt{4+m^2}, -m)$, the Jost solutions of (2.1) are defined according the boundary conditions

$$\check{\mathbf{w}}_n^\pm(\theta) = \begin{pmatrix} \check{u}_n^\pm(\theta) \\ \check{v}_n^\pm(\theta)\end{pmatrix} \to \begin{pmatrix} \check{\alpha}_\pm(\theta) \\ 1\end{pmatrix}\mathrm{e}^{\pm\mathrm{i}\theta n}, \quad n\to\pm\infty, \quad \text{where } \check{\alpha}_\pm(\theta) = \frac{\mathrm{e}^{\pm\mathrm{i}\theta}-1}{\lambda-m}. \tag{6.1}$$

Obviously, Lemmas 2.1 and 2.3 hold also for $\check{\mathbf{h}}_n^\pm(\theta) = \check{\mathbf{w}}_n^\pm(\theta)\mathrm{e}^{\mp\mathrm{i}\theta n}$ and $\check{W}(\theta) = W(\check{w}^+(\theta),\check{w}^-(\theta))$. Furthermore, for any $\lambda\in\Xi_-$, the matrix elements of the resolvent $\mathcal{R}(\lambda) := (\mathcal{D}-\lambda)^{-1} : \mathbf{l}^2\to\mathbf{l}^2$ are given by

$$[\mathcal{R}(\lambda)]_{n,k} = \frac{1}{\check{W}(\theta(\lambda))}\begin{cases} \check{\mathbf{w}}_n^+(\theta(\lambda))\otimes\check{\mathbf{w}}_k^-(\theta(\lambda)), & k\le n, \\ \check{\mathbf{w}}_n^-(\theta(\lambda))\otimes\check{\mathbf{w}}_k^+(\theta(\lambda)), & k\ge n.\end{cases}$$

For $\lambda\in\Gamma_-$, the following convergence

$$\mathcal{R}(\lambda\pm\mathrm{i}\varepsilon)\to\mathcal{R}(\lambda\pm\mathrm{i}0), \quad \varepsilon\to 0+$$

holds in $\mathcal{L}(\mathbf{l}^2_\sigma,\mathbf{l}^2_{-\sigma})$ with $\sigma>1/2$. Here

$$[\mathcal{R}(\lambda\pm\mathrm{i}0)]_{n,k} = \frac{1}{\check{W}(\theta_\pm)}\begin{cases} \check{\mathbf{w}}_n^+(\theta_\pm)\otimes\check{\mathbf{w}}_k^-(\theta_\pm) \ \text{ for } \ n\le k, \\ \\ \check{\mathbf{w}}_k^+(\theta_\pm)\otimes\check{\mathbf{w}}_n^-(\theta_\pm) \ \text{ for } \ n\ge k.\end{cases}$$

Calculations similar to calculations in the Proposition 3.1 lead to the representations

$$A_n^\pm\check{\mathbf{h}}_n^\pm(\theta) = \begin{pmatrix}\check{\alpha}_\mp(\theta) \\ 1\end{pmatrix} + \sum_{k=\mp1}^{\pm\infty}\begin{pmatrix}\frac{\check{a}^\pm_{n,k}}{\lambda-m} \\ \check{b}^\pm_{n,k}\end{pmatrix}\mathrm{e}^{\pm\mathrm{i}k\theta}, \tag{6.2}$$

where

$$|\check{a}^\pm_{n,k}|, |\check{b}^\pm_{n,k}| \le \check{C}_n^\pm \sum_{l=n+1+[k/2]}^{\pm\infty}(|q_l| + \frac{|q_l|}{|1-q_l|}), \tag{6.3}$$

and

$$\check{C}_n^\pm \le \check{C}^\pm, \quad \text{if } \pm n\ge 0. \tag{6.4}$$

Denote

$$\check{W}^\pm(\theta) = W(\check{\mathbf{w}}^\mp(\theta),\check{\mathbf{w}}^\pm(-\theta)), \quad \check{T}(\theta) = \frac{2\mathrm{i}\sin\theta}{(\lambda-m)\check{W}(\theta)}, \quad \check{R}^\pm(\theta) = \pm\frac{\check{W}^\pm(\theta)}{\check{W}(\theta)}, \quad \lambda\in\Gamma_-.$$

Finally, $\check{T}(\theta), \check{R}^\pm(\theta)\in\mathcal{A}$ for $q\in\ell^1_1$. This follows similarly to Theorem 5.1 using the corresponding Gelfand–Levitan–Marchenko equations and estimates of type (4.11) for its coefficients.

7. Dispersive decay

We will use the following variant of the van der Corput lemma.

Lemma 7.1. *(see* [2]*) Consider the oscillatory integral*

$$I(t)=\int_a^b \mathrm{e}^{\mathrm{i}t\phi(\theta)}f(\theta)d\theta,\qquad -\pi\le a<b\le\pi, \tag{7.1}$$

where $\phi(\theta)$ is a real-valued smooth function and $f\in\mathcal{A}$. If $|\phi^{(k)}(\theta)|>0$ for some $k\ge 2$ and for any $\theta\in[a,b]$ then

$$|I(t)|\le C_k\big(t\min_{[a,b]}|\phi^{(k)}(\theta)|\big)^{-1/k}\|f\|_{\mathcal{A}},\quad t\ge 1,$$

where C_k is a universal constant.

Theorem 7.2. *Let $q\in\ell^1_1$. Then the asymptotics* (1.3) *and* (1.4) *hold.*

Proof. We split $\mathrm{e}^{-\mathrm{i}t\mathcal{D}}P_c$ as follows

$$\mathrm{e}^{-\mathrm{i}t\mathcal{D}}P_c=\mathrm{e}^{-\mathrm{i}t\mathcal{D}}P_c^+ +\mathrm{e}^{-\mathrm{i}t\mathcal{D}}P_c^-,\qquad \mathrm{e}^{-\mathrm{i}t\mathcal{D}}P_c^\pm=\frac{1}{2\pi\mathrm{i}}\int\limits_{\Gamma_\pm}\mathrm{e}^{-\mathrm{i}t\lambda}(\mathcal{R}(\lambda+\mathrm{i}0)-\mathcal{R}(\lambda-\mathrm{i}0))\,d\lambda, \tag{7.2}$$

and prove asymptotics (1.3) and (1.4) for the first summand only. Using (2.10), we express the matrix elements of $\mathrm{e}^{-\mathrm{i}t\mathcal{D}}P_c^+$ in terms of the Jost solutions, (cf. [5, Formula 6.5]):

$$\big[\mathrm{e}^{-\mathrm{i}t\mathcal{D}}P_c^+\big]_{n,k}=\frac{1}{2\pi\mathrm{i}}\int_{-\pi}^{\pi}\frac{\mathrm{e}^{-\mathrm{i}t\sqrt{2-2\cos\theta+m^2}}}{\sqrt{2-2\cos\theta+m^2}}\,\frac{\mathbf{w}_k^+(\theta)\otimes\mathbf{w}_n^-(\theta)}{W(\theta)}\sin\theta\,d\theta,\quad n\le k, \tag{7.3}$$

and by symmetry $\big[\mathrm{e}^{-\mathrm{i}t\mathcal{D}}P_c\big]_{n,k}=\big[\mathrm{e}^{-\mathrm{i}t\mathcal{D}}P_c\big]_{k,n}$ for $n\ge k$.

Step i) We first prove asymptotics of type (1.3) for $\mathrm{e}^{-\mathrm{i}t\mathcal{D}}P_c^+$. It is enough to check that

$$\sup_{n,k}\big|\,\big[\mathrm{e}^{-\mathrm{i}t\mathcal{D}}P_c^+\big]_{n,k}\big|\le Ct^{-1/3},\quad t\to\infty. \tag{7.4}$$

We suppose $n\le k$ for notational simplicity. Using (4.5), we rewrite (7.3) as

$$\big[\mathrm{e}^{-\mathrm{i}t\mathcal{D}}P_c^+\big]_{n,k}=-\frac{1}{4\pi}\int_{-\pi}^{\pi}(m+\lambda)\frac{\mathrm{e}^{-\mathrm{i}t[g(\theta)-\frac{k-n}{t}\theta]}}{g(\theta)}\,T(\theta)\mathbf{h}_k^+(\theta)\otimes\mathbf{h}_n^-(\theta)d\theta, \tag{7.5}$$

where $g(\theta):=\sqrt{2-2\cos\theta+m^2}$. We also apply the scattering relations (4.6) to get the representations

$$T(\theta)\mathbf{h}_k^+(\theta)\otimes\mathbf{h}_n^-(\theta)=\begin{cases}R^-(\theta)\mathbf{h}_n^-(\theta)\otimes\mathbf{h}_k^-(\theta)\mathrm{e}^{-2\mathrm{i}k\theta}+\mathbf{h}_n^-(\theta)\otimes\mathbf{h}_k^-(-\theta), & n\le k\le 0,\\ R^+(\theta)\mathbf{h}_k^+(\theta)\otimes\mathbf{h}_n^+(\theta)\mathrm{e}^{2\mathrm{i}n\theta}+\mathbf{h}_k^+(\theta)\otimes\mathbf{h}_n^+(-\theta), & 0\le n\le k.\end{cases} \tag{7.6}$$

Using the facts

$$k-n-2k=-(k+n)=|k+n|,\quad n\le k\le 0, \tag{7.7}$$

$$k-n+2n=k+n=|k+n|,\quad 0\le n\le k, \tag{7.8}$$

and abbreviating $v:=\frac{k-n}{t}\ge 0$, $\tilde v:=\frac{|n+k|}{t}\ge 0$, we finally rewrite (7.5) as

$$\big[\mathrm{e}^{-\mathrm{i}t\mathcal{D}}P_c^+\big]_{n,k}=-\frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{\mathrm{e}^{-\mathrm{i}t\Phi_v(\theta)}}{g(\theta)}\,Y^1_{n,k}(\theta)d\theta-\frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{\mathrm{e}^{-\mathrm{i}t\tilde\Phi_v(\theta)}}{g(\theta)}\,Y^2_{n,k}(\theta)d\theta, \tag{7.9}$$

where

$$Y^1_{n,k}(\theta)=\frac{m+\lambda}{g(\theta)}\left\{\begin{array}{l}T(\theta)\mathbf{h}_k^+(\theta)\otimes\mathbf{h}_n^-(\theta),\ n\le 0\le k,\\ \mathbf{h}_n^-(\theta)\otimes\mathbf{h}_k^-(-\theta),\ n\le k\le 0,\\ \mathbf{h}_k^+(\theta)\otimes\mathbf{h}_n^+(-\theta),\ 0\le n\le k,\end{array}\right.$$

$$Y^2_{n,k}(\theta)=\frac{m+\lambda}{g(\theta)}\left\{\begin{array}{l}0,\ n\le 0\le k,\\ R^-(\theta)\mathbf{h}_n^-(\theta)\otimes\mathbf{h}_k^-(\theta),\ n\le k\le 0,\\ R^+(\theta)\mathbf{h}_n^+(\theta)\otimes\mathbf{h}_k^+(\theta),\ 0\le n\le k,\end{array}\right.$$

and

$$\Phi_v(\theta)=g(\theta)-v\theta,\quad \tilde\Phi_v(\theta)=g(\theta)-\tilde v\theta. \tag{7.10}$$

We observe that the matrix functions $Y^j_{n,k}(\theta)$ belong to $\mathcal{A}$, and the ℓ_1 -norm of its Fourier coefficients can be estimated by a value, which does not depend on n and k. Indeed, (3.4)–(3.5) imply that

$$\sup_{\pm n>0}\sum_{k=0}^{\pm\infty}|a^{\pm}_{n,k}|+\sup_{\pm n>0}\sum_{k=0}^{\pm\infty}|b^{\pm}_{n,k}|\le C<\infty.$$

Hence,

$$\|\hat{\mathbf{h}}^{\pm}_n\|_{\mathbf{l}^1}\le C,\quad \text{for}\quad \pm n\ge 0, \tag{7.11}$$

and Theorem 5.1 implies that

$$\|Y^j_{n,k}(\cdot)\|_{\mathcal{A}}\le C. \tag{7.12}$$

Denote $\varkappa:=(2+m^2-\sqrt{4m^2+m^4})/2$, $0<\varkappa<1$. It is easy to check that if $v\ne\sqrt{\varkappa}$ then the phase function $\Phi_v(\theta)$ has at most two non-degenerate stationary points. In the case $v=\sqrt{\varkappa}$ there exists a unique degenerate stationary point $\theta_0=\arccos\varkappa$, $0<\theta_0<\pi/2$, such that $\Phi'''(\theta_0)=\sqrt{\varkappa}\ne 0$. The function $\tilde\Phi_v(\theta)$ has the same properties.

We split the domain of integration in (7.9) into regions where either the second or third derivative of the phases is nonzero and apply Lemma 7.1 together with (7.12) to obtain the asymptotics (7.4).

Step ii) Now we prove the asymptotics of type (1.4) for $\mathrm{e}^{-\mathrm{i}t\mathcal{D}}P^+_c$. Denote $G=\max\limits_{\theta\in[-\pi,\pi]}|g'''(\theta)|$ and set

$$\mathbf{J}_{\pm}=\{\theta:\ |\theta\mp\theta_0|\le\nu|\theta_0|\},\quad \mathbf{J}=[-\pi,\pi]\setminus(\mathbf{J}_+\cup\mathbf{J}_-), \tag{7.13}$$

where $\nu=\min\{\frac12,\sqrt{\frac{2\sqrt{\varkappa}}{3G\theta_0^2}}\}$. We represent $\mathrm{e}^{-\mathrm{i}t\mathcal{D}}P^+_c$ as the sum

$$\mathrm{e}^{-\mathrm{i}t\mathcal{D}}P^+_c=\mathcal{K}^{\pm}(t)+\mathcal{K}(t), \tag{7.14}$$

where

$$[\mathcal{K}^{\pm}(t)]_{n,k}=-\frac{1}{4\pi}\int_{\mathbf{J}_\pm}\Big[\mathrm{e}^{-\mathrm{i}t\Phi_v(\theta)}Y^1_{n,k}(\theta)+\mathrm{e}^{-\mathrm{i}\tilde\Phi_v(\theta)}Y^2_{n,k}(\theta)\Big]\frac{d\theta}{g(\theta)},$$
$$[\mathcal{K}(t)]_{n,k}=-\frac{1}{4\pi}\int_{\mathbf{J}}\Big[\mathrm{e}^{-\mathrm{i}t\Phi_v(\theta)}Y^1_{n,k}(\theta)+\mathrm{e}^{-\mathrm{i}\tilde\Phi_v(\theta)}Y^2_{n,k}(\theta)\Big]\frac{d\theta}{g(\theta)}.$$

The van der Corput Lemma 7.1 with $k=2$ together with (7.12) imply

$$\sup_{n,k\in\mathbb{Z}}|[\mathcal{K}(t)]_{n,k}|\le Ct^{-1/2},\quad t\ge 1.$$

Hence, the asymptotics of type (1.4) for $\mathcal{K}(t)$ follow. It remains to prove the asymptotics for $\mathcal{K}^{\pm}(t)$. Since $W(\theta)\ne 0$ for $\theta\in\mathbf{J}_\pm$, then

$$|\frac{d}{d\theta}T(\theta))|,\ |\frac{d}{d\theta}R^{\pm}(\theta)|\le C,\quad \theta\in\mathbf{J}_\pm \tag{7.15}$$

by Proposition 2.1 (i). Hence, (2.4) and (7.15) imply

$$|Y_{n,k}|+|\frac{d}{d\theta}Y_{n,k}|\le C,\ \theta\in\mathbf{J}_\pm,\ j=1,2. \tag{7.16}$$

Moreover,

$$|\Phi'_v(-\theta_0\pm\theta)|=|\frac{-\sin(\theta_0\mp\theta)}{\sqrt{2-2\cos(\theta_0\pm\theta)+m^2}}-v|\ge\frac{\sin(\theta_0\mp\theta)}{\sqrt{4+m^2}}\ge\frac{\sin(\theta_0/2)}{\sqrt{4+m^2}}>C>0,\quad \theta\in J_-.$$

Therefore, applying integration by parts, we obtain

$$\sup_{n,k\in\mathbb{Z}}|[\mathcal{K}^-(t)]_{n,k}|\le Ct^{-1},\quad t\ge 1,$$

which implies asymptotics of type (1.4) for $\mathcal{K}^-(t)$. Finally, the asymptotics for $\mathcal{K}^+(t)$ follow by Lemma 6.3 from [5] (with $p=0$) due to (7.16). □

8. The non-resonant case

Theorem 8.1. *Let $q \in \ell^1_2$. Then in the non-resonant case the asymptotics* (1.5) *hold.*

Proof. It suffices to show that

$$| \left[\mathrm{e}^{-\mathrm{i}t\mathcal{D}} P_c \right]_{n,k} | \le C(1+|n|)(1+|k|)t^{-4/3}, \quad t \ge 1. \tag{8.1}$$

For $n \le k$ and $\omega \in \Gamma_+$ we represent the jump of the resolvent as (cf. [2, p.13])

$$\mathcal{R}(\lambda + \mathrm{i}0) - \mathcal{R}(\lambda - \mathrm{i}0) = \frac{(m+\lambda)|T(\theta)|^2}{-2\mathrm{i}\sin\theta}[\mathbf{w}^+_k(\theta) \otimes \mathbf{w}^+_n(-\theta) + \mathbf{w}^-_k(\theta) \otimes \mathbf{w}^-_n(-\theta)].$$

Inserting this into (7.2) and integrating by parts, we get

$$\left[\mathrm{e}^{-\mathrm{i}t\mathcal{D}} P_c^+ \right]_{n,k} = \left[\mathcal{P}^+(t) \right]_{n,k} + \left[\mathcal{P}_-(t) \right]_{n,k},$$

where

$$\begin{aligned} \left[\mathcal{P}^\pm(t)\right]_{n,k} &:= \frac{\mathrm{i}}{4\pi t}\int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}tg(\theta)} \frac{d}{d\theta}\Big[\frac{(m+\lambda(\theta))|T(\theta)|^2}{\sin\theta} \mathrm{e}^{\pm\mathrm{i}\theta(k-n)} \mathbf{h}^\pm_k(\theta) \otimes \mathbf{h}^\pm_n(-\theta)\Big] d\theta \\ &= \frac{1}{4\pi t}\int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}t(g(\theta)\mp\frac{k-n}{t})} \big(\mp (k-n) + \mathrm{i}\frac{d}{d\theta}\big) \frac{(m+\lambda(\theta))|T(\theta)|^2}{\sin\theta} \mathbf{h}^\pm_k(\theta) \otimes \mathbf{h}^\pm_n(-\theta). \end{aligned} \tag{8.2}$$

First, note that $T(\theta)\mathbf{h}^\pm_p(\theta) \in \mathcal{A}$ if $q \in \ell^1_1$, and

$$\|T(\cdot)\mathbf{h}^\pm_p(\cdot)\|_{\mathcal{A}} \le C, \quad \forall p \in \mathbb{Z}. \tag{8.3}$$

Indeed, for $\pm p \ge 0$ it follows from (7.11) and Theorem 5.1, and for $\pm p < 0$ from the scattering relation

$$T(\theta)\mathbf{h}^\pm_p(\theta) = R^\mp(\theta) h^\mp_p(\theta) \mathrm{e}^{\mp 2\mathrm{i}p\theta} + h^\mp_p(-\theta) \tag{8.4}$$

(see (4.6)). Further, representation (3.3) and the bounds (3.4)–(3.5) imply

$$\frac{d}{d\theta}\mathbf{h}^\pm_p(\theta) \in \mathcal{A} \quad \text{if} \quad q \in \ell^1_2. \tag{8.5}$$

Therefore, $\frac{d}{d\theta}W(\theta) := W'(\theta) \in \mathcal{A}$. Since in the non-resonant case $W^{-1}(\theta) \in \mathcal{A}$, we also infer

$$T'(\theta), \qquad (R^\pm(\theta))' \in \mathcal{A} \tag{8.6}$$

by Wiener's lemma. For the derivatives of $\mathbf{h}^\pm_p$ bounds of the type (7.11) hold. Namely,

$$\|\frac{d}{d\theta}\mathbf{h}^\pm_p(\cdot)\|_{\mathcal{A}} \le C \text{ for } \pm p \ge 0. \tag{8.7}$$

Denote

$$\mathbf{a}^\pm_p(j) := \sum_{s=j}^{\pm\infty} |a^\pm_{p,s}|, \qquad \mathbf{b}^\pm_p(j) := \sum_{s=j}^{\pm\infty} |b^\pm_{p,s}|.$$

From (3.4) it follows that if $q \in \ell^1_2$, then $a^\pm_{p,s},\ b^\pm_{p,s} \in \ell^1_1(\mathbb{Z}_\pm)$ for any fixed p. Hence,

$$\mathbf{a}^\pm_p(\cdot),\ \mathbf{b}^\pm_p(\cdot) \in \ell^1(\mathbb{Z}_\pm). \tag{8.8}$$

Based on this observation we prove the following

Lemma 8.2. *Let $q \in \ell^1_2$ and $W(0)W(\pi) \neq 0$. Then*

$$\Big\| \frac{T(\theta)\mathbf{h}^\pm_p(\theta)}{\sin\theta} \Big\|_{\mathcal{A}} \le C(1+|p|), \quad p \in \mathbb{Z}. \tag{8.9}$$

Proof. Note that $\dfrac{T(\theta)}{\sin\theta} = \dfrac{2\mathrm{i}}{(m+\lambda)W(\theta)}$ by (4.5). Hence, for $p \in \mathbb{Z}_\pm \cup \{0\}$ the bound (8.9) follows from (7.11) and Theorem 5.1. Consider the case $p \in \mathbb{Z}_\mp \cup \{0\}$. The scattering relations (4.6) imply

$$T(\theta)\mathbf{h}^\pm_p(\theta) = (R^\mp(\theta)+1)\mathbf{h}^\mp_p(\theta)\mathrm{e}^{\mp 2\mathrm{i}p\theta} - (\mathbf{h}^\mp_p(\theta) - \mathbf{h}^\mp_p(-\theta))\mathrm{e}^{\mp 2\mathrm{i}p\theta} + \mathbf{h}^\mp_p(-\theta)(1 - \mathrm{e}^{\mp 2\mathrm{i}p\theta}). \tag{8.10}$$

Using (3.3), (3.9), (3.10), and (3.13), we obtain

$$\frac{\mathbf{h}^{\mp}_{p,1}(\theta)-\mathbf{h}^{\mp}_{p,1}(-\theta)}{\sin\theta}=\varkappa_p^{\mp}\sum_{s=\mp1}^{\mp\infty}a^{\mp}_{p,s}\frac{\mathrm{e}^{\mp\mathrm{i}s\theta}-\mathrm{e}^{\pm\mathrm{i}s\theta}}{\sin\theta}$$

$$=\mp2\mathrm{i}\varkappa_p^{\mp}\sum_{s=\mp1}^{\mp\infty}a^{\mp}_{p,s}\times\begin{cases}(\mathrm{e}^{-\mathrm{i}(s-1)\theta}+...+\mathrm{e}^{-2\mathrm{i}\theta}+1+\mathrm{e}^{2\mathrm{i}\theta}+...+\mathrm{e}^{\mathrm{i}(s-1)\theta})\text{ for odd }s\\(\mathrm{e}^{-\mathrm{i}(s-1)\theta}+...+\mathrm{e}^{-\mathrm{i}\theta}+\mathrm{e}^{\mathrm{i}\theta}+...+\mathrm{e}^{\mathrm{i}(s-1)\theta})\text{ for even }s\end{cases}$$

$$=\mp2\mathrm{i}\varkappa_p^{\mp}\sum_{j=-\infty}^{\infty}\Big(a^{\mp}_{p,\mp|j|\mp1}+a^{\mp}_{p,\mp|j|\mp3}+a^{\mp}_{p,\mp|j|\mp5}+....\Big)\mathrm{e}^{\mathrm{i}j\theta},$$

where $\varkappa_p^{+}=(A_p^{+})^{-1}_{11}=1$, $\varkappa_p^{-}=(A_p^{-})^{-1}_{11}=1/(1-q_p)$. Similarly,

$$\frac{\mathbf{h}^{\mp}_{p,2}(\theta)-\mathbf{h}^{\mp}_{p,2}(-\theta)}{\sin\theta}=\frac{\varkappa_p^{\mp}}{m+\lambda}\Big[\mp2\mathrm{i}\pm2\mathrm{i}b^{\mp}_{p,\pm1}+\sum_{s=0}^{\mp\infty}b^{\mp}_{p,s}\frac{\mathrm{e}^{\mp\mathrm{i}s\theta}-\mathrm{e}^{\pm\mathrm{i}s\theta}}{\sin\theta}\Big]$$

$$=\mp\frac{2\mathrm{i}\varkappa_p^{\mp}}{m+\lambda}\Big[1-b^{\mp}_{p,\pm1}+\sum_{j=-\infty}^{\infty}\Big(b^{\mp}_{p,\mp|j|\mp1}+b^{\mp}_{p,\mp|j|\mp3}+b^{\mp}_{p,\mp|j|\mp5}+....\Big)\mathrm{e}^{\mathrm{i}j\theta}\Big]$$

Property (8.8) then implies

$$\Big\|\frac{\mathbf{h}^{\mp}_{p}(\theta)-\mathbf{h}^{\mp}_{p}(-\theta)}{\sin\theta}\Big\|_{\mathcal{A}}\le C,\quad p\in\mathbb{Z}_{\mp}.\tag{8.11}$$

In particular,

$$\frac{u_0^{\mp}(\theta)-u_0^{\mp}(-\theta)}{\sin\theta}=\frac{\mathbf{h}^{\mp}_{0,1}(\theta)-\mathbf{h}^{\mp}_{0,1}(-\theta)}{\sin\theta},\qquad\frac{v_1^{\mp}(\theta)-v_1^{\mp}(-\theta)}{\sin\theta}=\frac{\mathrm{e}^{\mp\mathrm{i}\theta}\mathbf{h}^{\mp}_{1,2}(\theta)-\mathrm{e}^{\pm\mathrm{i}\theta}\mathbf{h}^{\mp}_{1,2}(-\theta)}{\sin\theta}\in\mathcal{A},$$

which implies that

$$\frac{R^{\mp}(\theta)+1}{\sin\theta}=\frac{1}{W(\theta)}\frac{W(\theta)+W^{+}(\theta)}{\sin\theta}\in\mathcal{A}.\tag{8.12}$$

Finally,

$$\Big\|\frac{1-\mathrm{e}^{\pm2\mathrm{i}p\theta}}{\sin\theta}\Big\|_{\mathcal{A}}\le2|p|.\tag{8.13}$$

Substituting (8.11), (8.12), and (8.13) into (8.10) we get (8.9). □

Now we return to the representation (8.2). Let $|k|\le|n|$. In this case $|k-n|\le2|n|$, and applying (8.3) and (8.9) to the factors $T(-\theta)\mathbf{h}^{\pm}_n(-\theta)$ and $T(\theta)\mathbf{h}^{\pm}_k(\theta)/\sin\theta$, respectively, we obtain

$$\Big\|(k-n)\frac{|T(\theta)|^2\mathbf{h}^{\pm}_k(\theta)\otimes\mathbf{h}^{\pm}_n(-\theta)}{\sin\theta}\Big\|_{\mathcal{A}}\le C(1+|n|)(1+|k|).\tag{8.14}$$

The case $|n|\le|k|$ is handled similarly. Furthermore, applying (8.9) to both $T(-\theta)\mathbf{h}^{\pm}_n(-\theta)/\sin\theta$ and $T(\theta)\mathbf{h}^{\pm}_k(\theta)/\sin\theta$ we obtain

$$\Big\|\frac{|T(\theta)|^2\mathbf{h}^{\pm}_k(\theta)\otimes\mathbf{h}^{\pm}_n(-\theta)}{\sin^2\theta}\Big\|_{\mathcal{A}}\le C(1+|n|)(1+|k|).\tag{8.15}$$

To complete the proof we need one more property.

Lemma 8.3. *Let $q\in\ell^1_2$ and $W(0)W(\pi)\ne0$. Then*

$$\Big\|\frac{d}{d\theta}(T(\theta)\mathbf{h}^{\pm}_p(\theta))\Big\|_{\mathcal{A}}\le C(1+|p|),\quad p\in\mathbb{Z}.\tag{8.16}$$

Proof. Since $T'(\theta)$ are elements of $\mathcal{A}$ for $q\in\ell^1_2$ by (8.6), then for $p\in\mathbb{Z}_{\pm}\cup\{0\}$ the statement of the Lemma is evident in view of (8.7). To get it for $p\in\mathbb{Z}_{\mp}$ we use (8.6), (8.7), and the formula

$$\frac{d}{d\theta}(T(\theta)\mathbf{h}^{\pm}_p(\theta))=\frac{d}{d\theta}\big(R^{\mp}(\theta)\mathbf{h}^{\mp}_p(\theta)\big)\,\mathrm{e}^{\mp2\mathrm{i}p\theta}\mp2\mathrm{i}p\,\mathrm{e}^{\pm2\mathrm{i}p\theta}R^{\mp}(\theta)\mathbf{h}^{\mp}_p(\theta)+\frac{d}{d\theta}\mathbf{h}^{\mp}_p(-\theta).$$

□

Now (8.9), (8.14), (8.15) and (8.16) imply

$$\|\big(\mp(k-n)+\mathrm{i}\frac{d}{d\theta}\big)\frac{|T(\theta)|^2}{\sin\theta}\mathbf{h}^{\pm}_k(\theta)\otimes\mathbf{h}^{\pm}_n(-\theta)\|_{\mathcal{A}}\le C(1+|n|)(1+|k|).\tag{8.17}$$

Finally, we split the domain of integration in (8.2) into regions where either the second or third derivative of the phase is nonzero. Then Lemma 7.1 together with (8.17) imply (8.1).

Theorem 8.4. *Let $q\in\ell^1_2$. Then in the non-resonant case the asymptotics* (1.6) *hold.*

We consider the case $n\le k$ and obtain the asymptotics of type (1.6) for $\mathcal{P}^+(t)$ defined in (8.2). Namely, we should prove that

$$\|\mathcal{P}^+(t)\|_{\mathbf{l}^2_\sigma\to\mathbf{l}^2_{-\sigma}}\le C(t^{-3/2}),\quad t\to\infty,\quad \sigma>3/2. \tag{8.18}$$

As in the proof of Theorem 7.2 (ii) we consider the integrals over $\mathbf{J}_\pm$ and over $\mathbf{J}$ separately. Namely, taking into account the scattering relation (8.4), we split $\mathcal{P}^+(t)$ according to

$$\mathcal{P}^+(t)=\mathcal{M}(t)+\sum_\pm\Big[\mathcal{M}_1^\pm(t)+\mathcal{M}_2^\pm(t)+\mathcal{M}_3^\pm(t)+\mathcal{M}_4^\pm(t)+\mathcal{M}_5^\pm(t)\Big], \tag{8.19}$$

where

$$[\mathcal{M}(t)]_{n,k}=\frac{1}{4\pi t}\int_{\mathbf{J}}\mathrm{e}^{-\mathrm{i}t\Phi_v(\theta)}\big(n-k+\mathrm{i}\frac{d}{d\theta}\big)\frac{(m+\lambda)|T(\theta)|^2}{\sin\theta}\mathbf{h}_k^+(\theta)\otimes\mathbf{h}_n^+(-\theta),$$

and

$$[\mathcal{M}_j^\pm(t)]_{n,k}=\frac{1}{4\pi t}\int_{\mathbf{J}_\pm}\mathrm{e}^{-\mathrm{i}t\Phi_{v_j}(\theta)}Z_{n,k}^j(\theta)d\theta,\quad j=1,...,5. \tag{8.20}$$

Here we set $\Phi_{v_j}(\theta)=g(\theta)-v_j\theta$ with

$$v_1=v=\frac{k-n}{t},\quad v_2=v_3=\frac{k+n}{t},\quad v_4=-\frac{k+n}{t},\quad v_5=\frac{n-k}{t},$$

and

$$Z_{n,k}^1(\theta)=\begin{cases}\big(n-k+\mathrm{i}\frac{d}{d\theta}\big)\frac{(m+\lambda)|T(\theta)|^2}{\sin\theta}\mathbf{h}_k^+(\theta)\otimes\mathbf{h}_n^+(-\theta), & 0\le n\le k,\\[2mm] \big(n-k+\mathrm{i}\frac{d}{d\theta}\big)\frac{(m+\lambda)T(\theta)}{\sin\theta}\mathbf{h}_k^+(\theta)\otimes\mathbf{h}_n^-(\theta), & n<0\le k,\\[2mm] \big(n-k+\mathrm{i}\frac{d}{d\theta}\big)\frac{m+\lambda}{\sin\theta}\mathbf{h}_k^-(-\theta)\otimes\mathbf{h}_n^-(\theta), & n\le k<0,\end{cases}$$

$$Z_{n,k}^2(\theta)=\begin{cases}0,\quad 0\le n\le k,\\[2mm] \big(-n-k+\mathrm{i}\frac{d}{d\theta}\big)\frac{(m+\lambda)T(\theta)}{\sin\theta}R^-(-\theta)\mathbf{h}_k^+(\theta)\otimes\mathbf{h}_n^-(-\theta), & n<0\le k,\ |n|>k,\\[2mm] \big(-n-k+\mathrm{i}\frac{d}{d\theta}\big)\frac{(m+\lambda)}{\sin\theta}R^-(-\theta)\mathbf{h}_k^-(-\theta)\otimes\mathbf{h}_n^-(-\theta), & n\le k<0,\end{cases}$$

$$Z_{n,k}^3(\theta)=\begin{cases}0,\quad 0\le n\le k\quad\text{and}\quad n\le k<0,\\[2mm] \big(-n-k+\mathrm{i}\frac{d}{d\theta}\big)\frac{(m+\lambda)T(\theta)}{\sin\theta}R^-(-\theta)\mathbf{h}_k^+(\theta)\otimes\mathbf{h}_n^-(-\theta), & n<0\le k,\ |n|\le k,\end{cases}$$

$$Z_{n,k}^4(\theta)=\begin{cases}0,\quad 0\le n\le k\quad\text{and}\quad n<0\le k,\\[2mm] \big(k+n+\mathrm{i}\frac{d}{d\theta}\big)\frac{(m+\lambda)R^-(\theta)}{\sin\theta}\mathbf{h}_k^-(\theta)\otimes\mathbf{h}_n^-(\theta), & n\le k\le 0,\end{cases}$$

$$Z_{n,k}^5(\theta)=\begin{cases}0,\quad 0\le n\le k\quad\text{and}\quad n<0\le k,\\[2mm] \big(k-n+\mathrm{i}\frac{d}{d\theta}\big)\frac{(m+\lambda)|R^-(\theta)|^2}{\sin\theta}\mathbf{h}_k^-(\theta)\otimes\mathbf{h}_n^-(-\theta), & n\le k<0.\end{cases}$$

Note, that the sign of each $v_j=v_j(n,k)$ in the representation (8.20) for $[\mathcal{M}_j^\pm(t)]_{n,k}$ does not depend on n,k. Namely, for $t>0$ one has

$$\begin{cases}v_j\ge 0 & \text{for}\quad j=1,3,4,\\ v_j\le 0 & \text{for}\quad j=2,5.\end{cases}$$

Lemma 7.1 with $s=2$ and (8.17) imply

$$|[\mathcal{M}(t)]_{n,k}|\le Ct^{-3/2}(1+|n|)(1+|k|),\quad n,k\in\mathbb{Z},\quad t\ge 1.$$

Hence, the asymptotics of type (8.18) for $\mathcal{M}(t)$ follow. Further, Proposition 2.1 (i) implies

$$|\frac{d^p}{d\theta^p}T(\theta))|,\ |\frac{d^p}{d\theta^p}R^\pm(\theta)|\le C,\quad 0\le p\le 2,\quad \theta\in\mathbf{J}_\pm.$$

Respectively,

$$|Z_{n,k}^j(\theta)|+|\frac{d}{d\theta}Z_{n,k}^j(\theta)|\le C(1+\max\{|n|,|k|\}),\quad n,k\in\mathbb{Z},\quad \theta\in\mathbf{J}_\pm,\quad j=1,\dots 4. \tag{8.21}$$

The operators $\mathcal{M}_j^{\pm}(t)$ with $j=1,3,4$ and the operators $\mathcal{M}_j^{\mp}(t)$ with $j=2,5$ are estimated in the same way as the operators $\mathcal{K}^{\pm}(t)$ in the proof of Theorem 7.2. Namely, applying integration by parts, we obtain

$$|[\mathcal{M}_j^-(t)]_{n,k}|\le Ct^{-2}(1+|n|)(1+|k|),\quad n,k\in\mathbb{Z},\quad j=1,3,4,\quad t\ge 1.$$
$$|[\mathcal{M}_j^+(t)]_{n,k}|\le Ct^{-2}(1+|n|)(1+|k|),\quad n,k\in\mathbb{Z},\quad j=2,5,\quad t\ge 1.$$

Hence, asymptotics of type (8.18) for $\mathcal{M}_j^-(t)$ with $j=1,3,4$ and for $\mathcal{M}_j^+(t)$ with $j=2,5$ follow. Further, applying [5, Lemma 6.3] with $p=1$, we obtain the asymptotics for $\mathcal{M}_j^+(t)$ with $j=1,3,4$. The asymptotics for $\mathcal{M}_j^-(t)$ with $j=2,5$ follow by the same lemma with $\mathbf{J}_+$ replaced by $\mathbf{J}_-$. □

Appendix A. The calculation of $\tilde{T}(0)$

Representation (3.3) implies

$$\begin{aligned}(m+\lambda)W(\theta)&=\tilde{u}_n^+(z)\tilde{w}_{n+1}^-(z)-\tilde{u}_n^-(z)\tilde{w}_{n+1}^+(z)=\frac{1}{z}\big[1+\sum_{k=0}^{\infty}a_{n,k}^+z^k\big]\big[z-1+\sum_{k=0}^{\infty}b_{n+1,-k}^-z^k\big]\\&\quad-\frac{z}{(1-q_n)(1-q_{n+1})}\big[1+\sum_{k=0}^{\infty}a_{n,-k}^-z^k\big]\big[\frac{1}{z}-1+\sum_{k=-1}^{\infty}b_{n+1,k}^+z^k\big]\\&=\frac{A_{-1}}{z}+A_0+A_1z+...,\quad z\to 0,\end{aligned}$$

where

$$A_{-1}=(1+a_{n,0}^+)(1+b_{n+1,0}^-)=(1+a_{0,0}^+)(b_{1,0}^--1)$$

does not depend on n. Assume that $b_{1,0}^-=1$. Then (3.13) implies that $b_{0,0}^--\tilde{q}_0(b_{0,0}^--1)=1$. Hence, $b_{0,0}^-=1$. Repeating this, we obtain that $b_{n,0}^-=1$ for all $n\le 1$, which contradicts (3.4)–(3.5). Similarly, if $a_{0,0}^+=-1$ then $a_{n,0}^+=-1$ for all $n\ge 0$ by (3.9), which again contradicts (3.4)–(3.5). Therefore, $A_{-1}\neq 0$. Moreover,

$$\tilde{T}(z)=\frac{2\mathrm{i}\sin\theta}{(m+\lambda)W(\theta)}\sim\frac{1-z^2}{A_{-1}+A_0z+A_1z^2+...},\quad z\to 0.$$

Hence,

$$\tilde{T}(0)=\frac{1}{A_{-1}}<\infty,\quad \tilde{T}'(0)=\frac{-A_0}{A_{-1}^2}<\infty.$$

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