

DISPERSION ESTIMATES FOR SPHERICAL SCHRÖDINGER EQUATIONS: THE EFFECT OF BOUNDARY CONDITIONS

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Dedicated with great pleasure to Petru A. Cojuhari on the occasion of his 65th birthday

ABSTRACT. We investigate the dependence of the $L^1 \rightarrow L^\infty$ dispersive estimates for one-dimensional radial Schrödinger operators on boundary conditions at 0. In contrast to the case of additive perturbations, we show that the change of a boundary condition at zero results in the change of the dispersive decay estimates if the angular momentum is positive, $l \in (0, 1/2)$. However, for nonpositive angular momenta, $l \in (-1/2, 0]$, the standard $O(|t|^{-1/2})$ decay remains true for all self-adjoint realizations.

1. INTRODUCTION

We are concerned with the one-dimensional Schrödinger equation

$$i\dot{\psi}(t, x) = H_\alpha \psi(t, x), \quad H_\alpha := -\frac{d^2}{dx^2} + \frac{l(l+1)}{x^2}, \quad (t, x) \in \mathbb{R} \times \mathbb{R}_+, \quad (1.1)$$

with the angular momentum $|l| < \frac{1}{2}$ and self-adjoint boundary conditions at $x = 0$ parameterized by a parameter $\alpha \in [0, \pi)$ (the definition is given in Section 2, see (2.1)–(2.2) — for recent discussion of this family of operators see [1, 4]). More precisely, we are interested in the dependence of the $L^1 \rightarrow L^\infty$ dispersive estimates associated to the evolution group e^{-itH_α} on the parameters $\alpha \in [0, \pi)$ and $l \in (-1/2, 1/2)$.

On the whole line such results have a long tradition and we refer to Weder [22], Goldberg and Schlag [9], Egorova, Kopylova, Marchenko and Teschl [5], as well as the reviews [10, 18]. On the half line, the case $l = 0$ with a Dirichlet boundary condition was treated by Weder [23]. The case of general l and the Friedrichs boundary condition at 0 ($\alpha = 0$ in our notation)

$$\lim_{x \rightarrow 0} x^l ((l+1)f(x) - xf'(x)) = 0, \quad l \in \left(-\frac{1}{2}, \frac{1}{2}\right), \quad (1.2)$$

was recently considered in Kovařík and Truc [14] and they proved (see Theorem 2.4 in [14]) that

$$\|e^{-itH_0}\|_{L^1(\mathbb{R}_+) \rightarrow L^\infty(\mathbb{R}_+)} = \mathcal{O}(|t|^{-1/2}), \quad t \rightarrow \infty. \quad (1.3)$$

It was proved in [13] that this estimate remains true under additive perturbations. More precisely (see [13, Theorem 1.1]), let $H = H_0 + q$, where the potential q is a

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real integrable on \mathbb{R}_+ function. If in addition

$$\int_0^1 |q(x)| dx < \infty \quad \text{and} \quad \int_1^\infty x^{\max(2, l+1)} |q(x)| dx < \infty, \quad (1.4)$$

and there is neither a resonance nor an eigenvalue at 0, then

$$\|e^{-itH} P_c(H)\|_{L^1(\mathbb{R}_+) \rightarrow L^\infty(\mathbb{R}_+)} = \mathcal{O}(|t|^{-1/2}), \quad t \rightarrow \infty. \quad (1.5)$$

Here $P_c(H)$ is the orthogonal projection in $L^2(\mathbb{R}_+)$ onto the continuous spectrum of H .

The main result of the present paper shows that the decay estimates (1.3) and (1.5) are no longer true for $\alpha \in (0, \pi)$ if $l \in (0, 1/2)$. In other words, this means that singular rank one perturbations destroy these decay estimates if $l \in (0, 1/2)$ (since the change of a boundary condition can be considered as a rank one perturbation in the resolvent sense). Namely, consider first the operator $H_{\pi/2}$, which is associated with the following boundary condition at $x = 0$:

$$\lim_{x \rightarrow 0} x^{-l-1} (lf(x) + xf'(x)) = 0, \quad l \in \left(-\frac{1}{2}, \frac{1}{2}\right). \quad (1.6)$$

Theorem 1.1. *Let $|l| < 1/2$. Then*

$$\|e^{-itH_{\pi/2}}\|_{L^1(\mathbb{R}_+) \rightarrow L^\infty(\mathbb{R}_+)} = \mathcal{O}(|t|^{-1/2}), \quad t \rightarrow \infty, \quad (1.7)$$

for all $l \in (-1/2, 0]$, and

$$\|e^{-itH_{\pi/2}}\|_{L^1(\mathbb{R}_+, \max(x^{-l}, 1)) \rightarrow L^\infty(\mathbb{R}_+, \min(x^l, 1))} = \mathcal{O}(|t|^{-1/2+l}), \quad t \rightarrow \infty, \quad (1.8)$$

whenever $l \in (0, 1/2)$. The last estimate is sharp.

In the remaining case $\alpha \in (0, \pi/2) \cup (\pi/2, \pi)$, the decay estimate is given by the the next theorem.

Theorem 1.2. *Let $|l| < 1/2$ and $\alpha \in (0, \pi/2) \cup (\pi/2, \pi)$. Then*

$$\|e^{-itH_\alpha} P_c(H_\alpha)\|_{L^1(\mathbb{R}_+) \rightarrow L^\infty(\mathbb{R}_+)} = \mathcal{O}(|t|^{-1/2}), \quad t \rightarrow \infty, \quad (1.9)$$

for all $l \in (-1/2, 0]$, and

$$\|e^{-itH_\alpha} P_c(H_\alpha)\|_{L^1(\mathbb{R}_+, \max(x^{-l}, 1)) \rightarrow L^\infty(\mathbb{R}_+, \min(x^l, 1))} = \mathcal{O}(|t|^{-1/2}), \quad t \rightarrow \infty, \quad (1.10)$$

whenever $l \in (0, 1/2)$.

Notice that in the case $l \in (0, 1/2)$ we need to consider weighted L^1 and L^∞ spaces since functions contained in the domain of H_α might be unbounded near 0.

Finally, let us briefly outline the content of the paper. In the next section we define the operator H_α and collect its basic spectral properties. Section 3 contains the proof of Theorem 1.1. In particular, we compute explicitly the kernel of the evolution group $e^{-itH_{\pi/2}}$ and this enables us to prove (1.7) and (1.8) by using the estimates for Bessel functions J_ν (all necessary facts on Bessel functions are contained in Appendix A). Theorem 1.2 is proved in Section 4. Its proof is based on the use of a version of the van der Corput lemma, which is given in Appendix B. Also Appendix B contains necessary facts about the Wiener algebras $\mathcal{W}_0(\mathbb{R})$ and $\mathcal{W}(\mathbb{R})$. In the final section we formulate some sufficient conditions for a function $f(H)$ of a 1-D Schrödinger operator H to be an integral operator.

2. SELF-ADJOINT REALIZATIONS AND THEIR SPECTRAL PROPERTIES

Let $l \in (-1/2, 1/2)$ and denote by H_{\max} the maximal operator associated with

$$\tau = -\frac{d^2}{dx^2} + \frac{l(l+1)}{x^2}$$

in $L^2(\mathbb{R}_+)$. Note that τ is limit point at infinity and limit circle at $x = 0$ since $|l| < 1/2$. Therefore, self-adjoint restrictions of H_{\max} (or in other words, self-adjoint realizations of τ in $L^2(\mathbb{R}_+)$) form a 1-parameter family. More precisely (see, e.g., [7] and also [1]), the following limits

$$\Gamma_0 f := \lim_{x \rightarrow 0} W_x(f, x^{l+1}), \quad \Gamma_1 f := \frac{-1}{2l+1} \lim_{x \rightarrow 0} W_x(f, x^{-l}) \quad (2.1)$$

exist and are finite for all $f \in \text{dom}(H_{\max})$. Self-adjoint restrictions H_α of H_{\max} are parameterized by the following boundary conditions at $x = 0$:

$$\text{dom}(H_\alpha) = \{f \in \text{dom}(H_{\max}) : \sin(\alpha) \Gamma_1 f = \cos(\alpha) \Gamma_0 f\}, \quad \alpha \in [0, \pi). \quad (2.2)$$

Note that the case $\alpha = 0$ corresponds to the Friedrichs extension of $H_{\min} = H_{\max}^*$.

Let $\phi(z, x)$ and $\theta(z, x)$ be the fundamental system of solutions of $\tau u = zu$ given by

$$\begin{aligned} \phi(z, x) &= C_l^{-1} \sqrt{\frac{\pi x}{2}} z^{-\frac{2l+1}{4}} J_{l+\frac{1}{2}}(\sqrt{zx}), \\ \theta(z, x) &= C_l \sqrt{\frac{\pi x}{2}} \frac{z^{\frac{2l+1}{4}}}{\sin((l+\frac{1}{2})\pi)} J_{-l-\frac{1}{2}}(\sqrt{zx}), \end{aligned} \quad (2.3)$$

where J_ν is the Bessel function of order ν (see Appendix A) and

$$C_l = \frac{\sqrt{\pi}}{\Gamma(l+\frac{3}{2})2^{l+1}}. \quad (2.4)$$

The Weyl solution normalized by $\Gamma_0 \psi = 1$ is given by

$$\psi(z, x) = \theta(z, x) + m(z)\phi(z, x) = C_l i z^{\frac{2l+1}{4}} \sqrt{\frac{\pi x}{2}} H_{l+1/2}^{(1)}(\sqrt{zx}) \in L^2(0, \infty), \quad (2.5)$$

where $H_\nu^{(1)}$ is the Hankel function of the first kind [17, Chapter X.2], and

$$m(z) = -C_l^2 \frac{(-z)^{l+1/2}}{\sin((l+\frac{1}{2})\pi)}, \quad z \in \mathbb{C} \setminus \mathbb{R}_+, \quad (2.6)$$

is the Weyl function associated with H_0 . Here the branch cut of the root is taken along the negative real axis. Notice that

$$d\rho(\lambda) = \frac{C_l^2}{\pi} \mathbb{1}_{[0, \infty)}(\lambda) \lambda^{l+\frac{1}{2}} d\lambda \quad (2.7)$$

is the corresponding spectral measure. It follows from (A.1) that

$$\phi(z, x) = x^{l+1}(1 + o(1)), \quad \theta(z, x) = \frac{x^{-l}}{2l+1}(1 + o(1)),$$

as $x \rightarrow 0$ and, moreover,

$$\Gamma_0 \theta = \Gamma_1 \phi = 1, \quad \Gamma_1 \theta = \Gamma_0 \phi = 0.$$

Set

$$\begin{aligned}\phi_\alpha(z, x) &:= \cos(\alpha)\phi(z, x) + \sin(\alpha)\theta(z, x), \\ \theta_\alpha(z, x) &:= \cos(\alpha)\theta(z, x) - \sin(\alpha)\phi(z, x),\end{aligned}\tag{2.8}$$

for all $z \in \mathbb{C}$. Therefore, $W(\theta_\alpha, \phi_\alpha) = 1$ and

$$\psi_\alpha(z, x) := \theta_\alpha(z, x) + m_\alpha(z)\phi_\alpha(z, x), \quad m_\alpha(z) = \frac{m(z)\cos(\alpha) + \sin(\alpha)}{\cos(\alpha) - m(z)\sin(\alpha)},\tag{2.9}$$

is a Weyl solution normalized by $W(\psi_\alpha, \phi_\alpha) = 1$. Hence

$$G_\alpha(z; x, y) = \begin{cases} \phi_\alpha(z, x)\psi_\alpha(z, y), & x \leq y, \\ \phi_\alpha(z, x)\psi_\alpha(z, y), & x \geq y, \end{cases}\tag{2.10}$$

is the Green's function of H_α . The absolutely continuous spectrum remains unchanged, $\sigma_{\text{ac}}(H_\alpha) = [0, \infty)$, but there is one additional eigenvalue

$$E_\alpha = - \left(\frac{\cot(\alpha) \cos(l\pi)}{C_l^2} \right)^{\frac{2}{2l+1}}\tag{2.11}$$

if $\frac{\pi}{2} < \alpha < \pi$. Finally, since

$$\text{Im } m_\alpha(z) = \frac{\text{Im } m(z)}{|\cos(\alpha) - m(z)\sin(\alpha)|^2},\tag{2.12}$$

we get the absolutely continuous part of the corresponding spectral measure of the operator H_α :

$$\begin{aligned}\rho'_\alpha(\lambda)d\lambda &= \frac{1}{\pi} \text{Im } m_\alpha(\lambda + i0)d\lambda \\ &= \frac{1}{\pi} \frac{C_l^2 \lambda^{l+1/2} \mathbb{1}_{[0, \infty)}(\lambda)}{(\cos(\alpha) - C_l^2 \sin(\alpha) \tan(\pi l) \lambda^{l+1/2})^2 + C_l^4 \sin^2(\alpha) \lambda^{2l+1}} d\lambda.\end{aligned}\tag{2.13}$$

3. PROOF OF THEOREM 1.1

Similar to the case $\alpha = 0$ (see [14]), the kernel of the evolution group $e^{-itH_{\pi/2}}$ can be computed explicitly.

Lemma 3.1. *Let $|l| < 1/2$. Then the evolution group $e^{-itH_{\pi/2}}$ is an integral operator for all $t \neq 0$ and its kernel is given by*

$$[e^{-itH_{\pi/2}}](x, y) = \frac{i^{l-1/2}}{2t} e^{i\frac{x^2+y^2}{4t}} \sqrt{xy} J_{-l-1/2} \left(\frac{xy}{2t} \right),\tag{3.1}$$

for all $x, y > 0$ and $t \neq 0$.

Proof. First, notice that

$$\phi_{\pi/2}(z, x) = \theta(z, x), \quad m_{\pi/2}(z) = -1/m(z),$$

and then define the spectral transformation $U: L^2(\mathbb{R}_+) \rightarrow L^2(\mathbb{R}_+; \rho_{\pi/2})$ by

$$U: f \mapsto \hat{f}, \quad \hat{f}(\lambda) := \int_{\mathbb{R}_+} \theta(\lambda, x) f(x) dx,$$

for every $f \in L^2_c(\mathbb{R}_+)$. Notice that U extends to an isometry on $L^2(\mathbb{R}_+)$ and its inverse $U^{-1}: L^2(\mathbb{R}_+; \rho_{\pi/2}) \rightarrow L^2(\mathbb{R}_+)$ is given by

$$U^{-1}: g \mapsto \check{g}, \quad \check{g}(x) := \int_{\mathbb{R}_+} \theta(\lambda, x) g(\lambda) d\rho_{\pi/2}(\lambda),$$

for all $g \in L_c^2(\mathbb{R}_+; \rho_{\pi/2})$. Therefore, we get by using (2.3) and (2.13)

$$\begin{aligned} (e^{-(it+\varepsilon)H_{\pi/2}} f)(x) &= (U^{-1}e^{-(it+\varepsilon)\lambda}Uf)(x) = (U^{-1}e^{-(it+\varepsilon)\lambda}\check{f})(x) \\ &= \int_{\mathbb{R}_+} \theta(\lambda, x)e^{-(it+\varepsilon)\lambda} \int_{\mathbb{R}_+} \theta(\lambda, y)f(y) dy d\rho_{\pi/2}(\lambda) \\ &= \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} e^{-(it+\varepsilon)\lambda} \frac{\sqrt{xy}}{2} J_{-l-\frac{1}{2}}(\sqrt{\lambda}x)J_{-l-\frac{1}{2}}(\sqrt{\lambda}y)f(y) dy d\lambda. \end{aligned}$$

Since $|l| < 1/2$, (A.1) implies that

$$|J_{-l-1/2}(k)| \leq \frac{2^{l+1/2}}{\Gamma(1/2-l)k^{l+1/2}}(1 + \mathcal{O}(k)) \quad (3.2)$$

as $k \rightarrow 0$. Noting that $f \in L_c^2(\mathbb{R}_+)$ and using (3.2), Fubini's theorem implies

$$(e^{-(it+\varepsilon)H_{\pi/2}} f)(x) = \int_{\mathbb{R}_+} f(y) \int_{\mathbb{R}_+} e^{-(it+\varepsilon)\lambda} \frac{\sqrt{xy}}{2} J_{-l-\frac{1}{2}}(\sqrt{\lambda}x)J_{-l-\frac{1}{2}}(\sqrt{\lambda}y)d\lambda dy. \quad (3.3)$$

The integral

$$[e^{-(it+\varepsilon)H_{\pi/2}}](x, y) := \frac{\sqrt{xy}}{2} \int_0^\infty e^{-it\lambda} J_{-l-\frac{1}{2}}(\sqrt{\lambda}x)J_{-l-\frac{1}{2}}(\sqrt{\lambda}y)d\lambda \quad (3.4)$$

is known as Weber's second exponential integral [21, §13.31] (cf. also [6, (4.14.39)]) and hence

$$(e^{-(it+\varepsilon)H_{\pi/2}} f)(x) = \frac{1}{\varepsilon + it} \int_0^\infty e^{-\frac{x^2+y^2}{4(\varepsilon+it)}} \frac{\sqrt{xy}}{2} I_{-l-\frac{1}{2}}\left(\frac{xy}{2(\varepsilon+it)}\right) f(y) dy,$$

where I_ν is the modified Bessel function (see [17, Chapter X] and in particular formula (10.27.6) there)

$$I_\nu(z) = \sum_{n=0}^\infty \frac{(z/2)^{\nu+2n}}{n!\Gamma(\nu+m+1)} = e^{\mp i\nu\pi/2} J_\nu(\pm iz), \quad -\pi \leq \arg(z) \leq \pi/2. \quad (3.5)$$

The estimate (A.2) implies

$$|J_{-l-1/2}(k)| \leq k^{-1/2}(1 + \mathcal{O}(k^{-1})) \quad (3.6)$$

as $k \rightarrow \infty$. Therefore, there is $C > 0$ which depends only on l and such that

$$|\sqrt{k}J_{-l-1/2}(k)| \leq C \left(\frac{1+k}{k}\right)^l, \quad k > 0. \quad (3.7)$$

By (3.7) we deduce

$$\frac{\sqrt{xy}}{2|\varepsilon+it|} \left| e^{-\frac{x^2+y^2}{4(\varepsilon+it)}} I_{-l-\frac{1}{2}}\left(\frac{xy}{2(\varepsilon+it)}\right) \right| \leq C \sqrt{\frac{1}{|\varepsilon+it|}} \left| 1 + \frac{2(\varepsilon+it)}{xy} \right|^l,$$

which is uniformly (wrt. ε) bounded on compact sets $K \subset \subset \mathbb{R}_+ \times \mathbb{R}_+$. Thus we can apply dominated convergence and hence the claim follows. \square

In particular, we immediately arrive at the following estimate.

Corollary 3.2. *Let $|l| < 1/2$. Then there is a constant $C > 0$ which depends only on l and such that the inequality*

$$|[e^{-itH_{\pi/2}}](x, y)| \leq \frac{C}{\sqrt{2t}} \left(\frac{2t + xy}{xy} \right)^l \quad (3.8)$$

holds for all $x, y > 0$ and $t > 0$.

Proof. Applying (3.7) to (3.1), we arrive at (3.8). \square

Remark 3.3. *For any fixed x and $y \in \mathbb{R}_+$, we get from (A.1)*

$$\left| e^{-itH_{\pi/2}}(x, y) \right| \sim \frac{\sqrt{xy}}{2t} \left(\frac{xy}{4t} \right)^{-l-1/2} = \frac{1}{t^{1/2-l}} \left(\frac{xy}{2} \right)^{-l} \quad (3.9)$$

Moreover, in view of (A.1) one can see that

$$\left| e^{-itH_{\pi/2}}(x, y) \right| \geq c_l t^{l-1/2} \left(\frac{xy}{2} \right)^{-l}, \quad (3.10)$$

whenever $xy < t$ with some constant $c_l > 0$, which depends only on l .

Now we are ready to prove our first main result.

Proof of Theorem 1.1. If $l \in (-1/2, 0]$, then

$$\left(\frac{2t + xy}{xy} \right)^l \leq 1$$

for all $x, y > 0$ and $t \geq 0$. This immediately implies (1.7).

Assume now that $l \in (0, 1/2)$. Clearly,

$$\frac{2t + xy}{xy} = 1 + 2\frac{t}{xy} \leq 3t \max(x^{-1}, 1) \max(y^{-1}, 1)$$

for all $t \geq 1$ and $x, y > 0$. Indeed, the latter follows from the weaker estimate

$$\frac{t}{xy} \leq t \max(x^{-1}, 1) \max(y^{-1}, 1), \quad t \geq 1, \quad x, y > 0,$$

which is equivalent to $1 \leq \max(x, 1) \max(y, 1)$ for all $x, y > 0$. Therefore,

$$\left(\frac{2t + xy}{xy} \right)^l \leq 3^l t^l \max(x^{-l}, 1) \max(y^{-l}, 1), \quad t \geq 1, \quad x, y > 0,$$

which proves (1.8). Remark 3.3 shows that (1.8) is sharp. \square

4. PROOF OF THEOREM 1.2

Let us consider the following improper integrals:

$$I_1(t; x, y) := \sqrt{xy} \int_{\mathbb{R}_+} e^{-itk^2} J_{l+\frac{1}{2}}(kx) J_{l+\frac{1}{2}}(ky) \operatorname{Im} m_\alpha(k^2) k^{-2l} dk, \quad (4.1)$$

$$I_2(t; x, y) := \sqrt{xy} \int_{\mathbb{R}_+} e^{-itk^2} J_{l+\frac{1}{2}}(kx) J_{-l-\frac{1}{2}}(ky) \operatorname{Im} m_\alpha(k^2) k dk, \quad (4.2)$$

$$I_3(t; x, y) := \sqrt{xy} \int_{\mathbb{R}_+} e^{-itk^2} J_{-l-\frac{1}{2}}(kx) J_{-l-\frac{1}{2}}(ky) \operatorname{Im} m_\alpha(k^2) k^{2l+2} dk, \quad (4.3)$$

where $x, y > 0$ and $t \neq 0$. Moreover, here and below we shall use the convention $\operatorname{Im} m_\alpha(k^2) := \operatorname{Im} m_\alpha(k^2 + i0) = \lim_{\varepsilon \downarrow 0} \operatorname{Im} m_\alpha(k^2 + i\varepsilon)$ for all $k \in \mathbb{R}$. Denote the corresponding integrand by A_j , that is, $I_j(t) = \int_{\mathbb{R}_+} e^{-itk^2} A_j(k; x, y) dk$. Our aim is

to use Lemma B.2 (plus the remarks after this lemma) and hence we need to show that each A_j belongs to the Wiener algebra $\mathcal{W}(\mathbb{R})$, that is, coincide with a function which is the Fourier transform of a finite measure.

We also need the following estimates, which follow from (2.13)

$$\operatorname{Im} m_\alpha(k^2) = \begin{cases} C_l^2 |k|^{2l+1}, & \alpha = 0, \\ \frac{\cos^2(\pi l)}{C_l^2 \sin^2(\alpha)} |k|^{-2l-1} + \mathcal{O}(|k|^{-4l-2}), & \alpha \neq 0, \end{cases} \quad k \rightarrow \infty, \quad (4.4)$$

and

$$\operatorname{Im} m_\alpha(k^2) = \begin{cases} \frac{C_l^2}{\cos(\alpha)^2} |k|^{2l+1} + \mathcal{O}(|k|^{4l+2}), & \alpha \neq \pi/2, \\ C_l^{-2} \cos^2(\pi l) |k|^{-2l-1}, & \alpha = \pi/2, \end{cases} \quad k \rightarrow 0. \quad (4.5)$$

4.1. The integral I_1 . Consider the function

$$J(r) := \sqrt{r} J_{l+\frac{1}{2}}(r) = \frac{r^{l+1}}{2^{l+1/2}} \sum_{n=0}^{\infty} \frac{(-r^2/4)^n}{n! \Gamma(\nu + n + 1)}, \quad r \geq 0.$$

Note that $J(r) \sim r^{l+1}$ as $r \rightarrow 0$ and $J(r) = \sqrt{\frac{2}{\pi}} \sin(r - \frac{l\pi}{2}) + O(r^{-1})$ as $r \rightarrow +\infty$ (see (A.2)). Moreover, $J'(r) \sim r^l$ as $r \rightarrow 0$ and $J'(r) = \sqrt{\frac{2}{\pi}} \cos(r - \frac{l\pi}{2}) + O(r^{-1})$ as $r \rightarrow +\infty$ (see (A.4)). In particular, $\tilde{J}(r) := J(r) - \sqrt{\frac{2}{\pi}} \sin(r - \frac{l\pi}{2})$ is in $H^1(\mathbb{R}_+)$. Moreover, we can define $J(r)$ for $r < 0$ such that it is locally in H^1 and $J(r) = \sqrt{\frac{2}{\pi}} \sin(r - \frac{l\pi}{2})$ for $r < -1$. By construction we then have $\tilde{J} \in H^1(\mathbb{R})$ and thus \tilde{J} is the Fourier transform of an integrable function (see Lemma B.3). Moreover, $\sin(r - \frac{l\pi}{2})$ is the Fourier transform of the sum of two Dirac delta measures and so J is the Fourier transform of a finite measure. By scaling, the total variation of the measures corresponding to $J(kx)$ is independent of x .

Next consider the function

$$F(k) := \frac{\operatorname{Im} m_\alpha(k^2)}{|k|^{2l+1}} = \frac{C_l^2}{(\cos(\alpha) - C_l^2 \sin(\alpha) \tan(\pi l) |k|^{2l+1})^2 + C_l^4 \sin^2(\alpha) |k|^{4l+2}}.$$

By Corollary B.6, F is in the Wiener algebra $\mathcal{W}_0(\mathbb{R})$.

Now it remains to note that

$$I_1(t) = \int_{\mathbb{R}_+} e^{-itk^2} A_1(k^2; x, y) dk = \int_{\mathbb{R}_+} e^{-itk^2} J(kx) J(ky) F(k) dk, \quad (4.6)$$

and applying Lemma B.2 we end up with the estimate

$$|I_1(t; x, y)| \leq Ct^{-1/2}, \quad t > 0, \quad (4.7)$$

with a positive constant $C > 0$ independent of $x, y > 0$.

4.2. The integral I_2 . Assume first that $l \in (0, 1/2)$ and write

$$A_2(k^2; x, y) = J(kx) Y(ky) \frac{\chi_l(k)}{\chi_l(ky)} \frac{\operatorname{Im} m_\alpha(k^2)}{\chi_l(k)},$$

where

$$J(r) = \sqrt{r} J_{l+\frac{1}{2}}(r), \quad Y(r) = \chi_l(r) \sqrt{r} J_{-l-\frac{1}{2}}(r), \quad \chi_l(r) = \frac{|r|^l}{1 + |r|^l}.$$

The asymptotic behavior (4.4) and (4.5) of $\operatorname{Im} m_\alpha$ shows that

$$M(k) = \frac{\operatorname{Im} m_\alpha(k^2)}{\chi_l(k)} = \begin{cases} |k|^{1+l}, & k \rightarrow 0, \\ |k|^{-2l-1}, & |k| \rightarrow \infty, \end{cases}$$

and hence $M \in H^1(\mathbb{R})$, which implies that M is in the Wiener algebra $\mathcal{W}_0(\mathbb{R})$.

We continue $J(r)$, $Y(r)$ to the region $r < 0$ such that they are continuously differentiable and satisfy

$$J(r) = \sqrt{\frac{2}{\pi}} \sin\left(r - \frac{\pi l}{2}\right), \quad Y(r) = \sqrt{\frac{2}{\pi}} \cos\left(r + \frac{\pi l}{2}\right),$$

for $r < -1$. Then $\tilde{J}(r) := J(r) - \sqrt{\frac{2}{\pi}} \sin(r - \frac{\pi l}{2})$ and $\tilde{Y}(r) := Y(r) - \sqrt{\frac{2}{\pi}} \cos(r + \frac{\pi l}{2})$ are in $H^1(\mathbb{R})$. In fact, they are continuously differentiable and hence it suffices to look at their asymptotic behavior. For $r < -1$ they are zero and for $r > 1$ they are $O(r^{-1})$ and their derivative is $O(r^{-1})$ as can be seen from the asymptotic behavior of Bessel functions (see Appendix A). Hence both J and Y are Fourier transforms of finite measures. By scaling the total variation of the measures corresponding to $J(kx)$ and $Y(ky)$ are independent of x and y , respectively.

It remains to consider the function $\chi_l(k)/\chi_l(ky)$. Observe that

$$h_{y,l}(k) := 1 - \frac{\chi_l(k)}{\chi_l(ky)} = 1 - \frac{1 + |ky|^l}{y^l + |ky|^l} = \frac{1 - y^{-l}}{1 + |k|^l} = (1 - y^{-l})(1 - \chi_l(k)).$$

By Corollary B.6, $1 - \chi_l \in \mathcal{W}_0(\mathbb{R})$. Therefore, applying Lemma B.2, we obtain the following estimate

$$|I_2(t; x, y)| \leq Ct^{-1/2} \max(1, y^{-l}), \quad t > 0, \quad (4.8)$$

whenever $l \in (0, 1/2)$.

Consider now the remaining case $l \in (-1/2, 0]$. Write

$$A_2(k^2; x, y) = J(kx)Y(ky) \operatorname{Im} m_\alpha(k^2),$$

where

$$J(r) = \sqrt{r} J_{l+\frac{1}{2}}(r), \quad Y(r) = \sqrt{r} J_{-l-\frac{1}{2}}(r).$$

Noting that $Y(r) \sim r^{-l}$ as $r \rightarrow 0$ and using Lemma B.3, we can continue J and Y to the region $r < 0$ such that both J and Y are Fourier transforms of finite measures.

It remains to consider $\operatorname{Im} m_\alpha(k^2)$ given by (2.13). However, by Corollary B.6, this function is in the Wiener algebra $\mathcal{W}_0(\mathbb{R})$ and hence applying Lemma B.2, we end up with the estimate

$$|I_2(t; x, y)| \leq Ct^{-1/2}, \quad t > 0, \quad (4.9)$$

whenever $l \in (-1/2, 0]$.

4.3. The integral I_3 . Again let us consider two cases. Assume first that $l \in (-1/2, 0]$ and then write

$$A_3(k^2; x, y) = Y(kx)Y(ky) \operatorname{Im} m_\alpha(k^2) k^{2l+1},$$

where

$$Y(r) = \sqrt{r} J_{-l-\frac{1}{2}}(r), \quad r > 0.$$

Notice that

$$|k|^{2l+1} \operatorname{Im} m_\alpha(k^2) = \frac{C_l^2 k^{4l+2}}{(\cos(\alpha) - C_l^2 \sin(\alpha) \tan(\pi l) k^{2l+1})^2 + C_l^4 \sin^2(\alpha) k^{4l+2}},$$

which is the sum of a constant and a function of the form (B.5), and hence it belongs to the Wiener algebra $\mathcal{W}(\mathbb{R})$ by Corollary B.6. Arguing as in the previous subsection and applying Lemma B.2, we arrive at the following estimate

$$|I_3(t; x, y)| \leq Ct^{-1/2}, \quad t > 0, \quad (4.10)$$

whenever $l \in (-1/2, 0]$.

If $l \in (0, 1/2)$, write

$$A_3(k^2; x, y) = Y(kx)Y(ky) \frac{\chi_l(k)}{\chi_l(kx)} \frac{\chi_l(k)}{\chi_l(ky)} \frac{\operatorname{Im} m_\alpha(k^2)}{\chi_l^2(k)},$$

where

$$Y(r) = \chi_l(r) \sqrt{r} J_{-l-\frac{1}{2}}(r), \quad \chi_l(r) = \frac{|r|^l}{1+|r|^l}.$$

Notice that

$$\begin{aligned} M(k) &:= \frac{\operatorname{Im} m_\alpha(k^2) |k|^{2l+1}}{\chi_l^2(k)} \\ &= \frac{C_l^2 |k|^{2l+2} (1+k^l)^2}{(\cos(\alpha) - C_l^2 \sin(\alpha) \tan(\pi l) |k|^{2l+1})^2 + C_l^4 \sin^2(\alpha) |k|^{4l+2}} \end{aligned}$$

Clearly, by Corollary B.6, $M \in \mathcal{W}(\mathbb{R})$. Therefore, similar to the previous subsection, we end up with the estimate

$$|I_3(t; x, y)| \leq Ct^{-1/2} \max(1, x^{-l}) \max(1, y^{-l}), \quad t > 0, \quad (4.11)$$

whenever $l \in (0, 1/2)$.

4.4. Proof of Theorem 1.2. We begin with the representation of the integral kernel of the evolution group.

Lemma 4.1. *Let $|l| < 1/2$ and $\alpha \in [0, \pi)$. Then the evolution group $e^{-itH_\alpha} P_c(H_\alpha)$ is an integral operator and its kernel is given by*

$$[e^{-itH_\alpha} P_c(H_\alpha)](x, y) = \frac{2}{\pi} \int_{\mathbb{R}_+} e^{-itk^2} \phi_\alpha(k^2, x) \phi_\alpha(k^2, y) \operatorname{Im} m_\alpha(k^2) k \, dk, \quad (4.12)$$

where the integral is to be understood as an improper integral.

Proof. By (2.3) and (2.8),

$$\begin{aligned} \phi_\alpha(k^2, x) &= \cos(\alpha) \phi(k^2, x) + \sin(\alpha) \theta(k^2, x) \\ &= \sqrt{\frac{\pi x}{2}} \left(C_l^{-1} \cos(\alpha) k^{-l-1/2} J_{l+\frac{1}{2}}(kx) + C_l k^{l+1/2} \frac{\sin(\alpha)}{\cos(\pi l)} J_{-l-\frac{1}{2}}(kx) \right), \end{aligned}$$

and hence

$$\phi_\alpha(k^2, x) \phi_\alpha(k^2, y) = \frac{\pi}{2} \sqrt{xy} \left(\frac{\cos^2(\alpha)}{C_l^2} k^{-2l-1} J_{l+\frac{1}{2}}(kx) J_{l+\frac{1}{2}}(ky) \right. \quad (4.13)$$

$$\left. + \frac{\sin(2\alpha)}{2 \cos(\pi l)} (J_{l+\frac{1}{2}}(kx) J_{-l-\frac{1}{2}}(ky) + J_{-l-\frac{1}{2}}(kx) J_{l+\frac{1}{2}}(ky)) \right) \quad (4.14)$$

$$\left. + C_l^2 k^{2l+1} \frac{\sin^2(\alpha)}{\cos^2(\pi l)} J_{-l-\frac{1}{2}}(kx) J_{-l-\frac{1}{2}}(ky) \right). \quad (4.15)$$

By our considerations in the previous subsections, we have

$$\phi_\alpha(k^2, x) \phi_\alpha(k^2, y) \operatorname{Im} m_\alpha(k^2) k \in \mathcal{W}(\mathbb{R})$$

with norm uniformly bounded for x, y restricted to any compact subset of $(0, \infty)$. Moreover, we have $e^{-i(t-i\varepsilon)H_\alpha} P_c(H_\alpha) \rightarrow e^{-itH_\alpha} P_c(H_\alpha)$ as $\varepsilon \downarrow 0$ in the strong operator topology. By Lemma C.1, $e^{-i(t-i\varepsilon)H_\alpha} P_c(H_\alpha)$ is an integral operator for all $\varepsilon > 0$ and, moreover, the kernel converges uniformly on compact sets by Lemma C.2. Hence $e^{-itH_\alpha} P_c(H_\alpha)$ is an integral operator whose kernel is given by the limits of the kernels of the approximating operators, that is, by (4.12). \square

Proof of Theorem 1.2. Combining (4.7), (4.8), (4.9), (4.10) and (4.11), we arrive at the following decay estimate for the kernel of the evolution group

$$\left| [e^{-itH_\alpha} P_c(H_\alpha)](x, y) \right| \leq Ct^{-1/2} \times \begin{cases} 1, & l \in (-1/2, 0], \\ \max(1, x^{-l}) \max(1, y^{-l}), & l \in (0, 1/2). \end{cases} \quad (4.16)$$

This completes the proof of Theorem 1.2. \square

APPENDIX A. BESSEL FUNCTIONS

Here we collect basic formulas and information on Bessel functions (see, e.g., [17, 21]). We start with the definition:

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(-z^2/4)^n}{n! \Gamma(\nu + n + 1)}. \quad (A.1)$$

The asymptotic behavior as $|z| \rightarrow \infty$ is given by

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}} \left(\cos(z - \nu\pi/2 - \pi/4) + e^{|\operatorname{Im} z|} \mathcal{O}(|z|^{-1}) \right), \quad |\arg z| < \pi. \quad (A.2)$$

Noting that

$$J'_\nu(z) = -J_{\nu+1}(z) + \frac{\nu}{z} J_\nu(z) = J_{\nu-1}(z) - \frac{\nu}{z} J_\nu(z), \quad (A.3)$$

one can show that the derivative of the remainder satisfies

$$\left(\sqrt{\frac{\pi z}{2}} J_\nu(z) - \cos\left(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right) \right)' = e^{|\operatorname{Im} z|} \mathcal{O}(|z|^{-1}), \quad |z| \rightarrow \infty. \quad (A.4)$$

APPENDIX B. THE VAN DER CORPUT LEMMA AND THE WIENER ALGEBRA

We will need the classical van der Corput lemma (see, e.g., [19, page 334]):

Lemma B.1. *Consider the oscillatory integral*

$$I(t) = \int_a^b e^{itk^2 + ick} A(k) dk.$$

If $A \in \text{AC}(a, b)$, then

$$|I(t)| \leq C_2 |t|^{-1/2} (\|A\|_\infty + \|A'\|_1), \quad |t| \geq 1,$$

where $C_2 \leq 2^{8/3}$ is a universal constant.

Note that we can apply the above result with $(a, b) = (-\infty, \infty)$ by considering the limit $(-a, a) \rightarrow (-\infty, \infty)$.

Our proof will be based on the following variant of the van der Corput lemma (see, e.g., [13, Lemma A.2]).

Lemma B.2. *Let $(a, b) \subseteq \mathbb{R}$ and consider the oscillatory integral*

$$I(t) = \int_a^b e^{itk^2} A(k) dk.$$

If $A \in \mathcal{W}(\mathbb{R})$, i.e., A is the Fourier transform of a signed measure

$$A(k) = \int_{\mathbb{R}} e^{ikp} d\alpha(p),$$

then the above integral exists as an improper integral and satisfies

$$|I(t)| \leq C_2 |t|^{-1/2} \|A\|_{\mathcal{W}}, \quad |t| > 0.$$

where $\|A\|_{\mathcal{W}} := \|\alpha\| = |\alpha|(\mathbb{R})$ denotes the total variation of α and C_2 is the constant from the van der Corput lemma.

In this respect we note that if A_1 and A_2 are two such functions, then (cf. p. 208 in [2])

$$(A_1 A_2)(k) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}} e^{ikp} d(\alpha_1 * \alpha_2)(p)$$

is associated with the convolution

$$\alpha_1 * \alpha_2(\Omega) = \iint \mathbb{1}_{\Omega}(x+y) d\alpha_1(x) d\alpha_2(y),$$

where $\mathbb{1}_{\Omega}$ is the indicator function of a set Ω . Note that

$$\|\alpha_1 * \alpha_2\| \leq \|\alpha_1\| \|\alpha_2\|.$$

Let $\mathcal{W}_0(\mathbb{R})$ be the Wiener algebra of functions $C(\mathbb{R})$ which are Fourier transforms of L^1 functions,

$$\mathcal{W}_0(\mathbb{R}) = \left\{ f \in C(\mathbb{R}) : f(k) = \int_{\mathbb{R}} e^{ikx} g(x) dx, \quad g \in L^1(\mathbb{R}) \right\}.$$

Clearly, $\mathcal{W}_0(\mathbb{R}) \subset \mathcal{W}(\mathbb{R})$. Moreover, by the Riemann–Lebesgue lemma, $f \in C_0(\mathbb{R})$, that is, $f(k) \rightarrow 0$ as $k \rightarrow \infty$ if $f \in \mathcal{W}_0(\mathbb{R})$. A comprehensive survey of necessary and sufficient conditions for $f \in C(\mathbb{R})$ to be in the Wiener algebras $\mathcal{W}_0(\mathbb{R})$ and $\mathcal{W}(\mathbb{R})$ can be found in [15], [16]. We need the following statements.

Lemma B.3. *If $f \in L^2(\mathbb{R})$ is locally absolutely continuous and $f' \in L^p(\mathbb{R})$ with $p \in (1, 2]$, then f is in the Wiener algebra $\mathcal{W}_0(\mathbb{R})$ and*

$$\|f\|_{\mathcal{W}} \leq C_p (\|f\|_{L^2(\mathbb{R})} + \|f'\|_{L^p(\mathbb{R})}), \quad (\text{B.1})$$

where $C_p > 0$ is a positive constant, which depends only on p .

Proof. Since the Fourier transform is unitary on $L^2(\mathbb{R})$, it suffices to show that $\hat{f} \in L^1(\mathbb{R})$. First of all, the Cauchy–Schwarz inequality implies $\hat{f} \in L^1_{\text{loc}}(\mathbb{R})$ and, in particular,

$$\int_{-1}^1 |\hat{f}(\lambda)| d\lambda \leq \sqrt{2} \left(\int_{-1}^1 |\hat{f}(\lambda)|^2 d\lambda \right)^{1/2} \leq \sqrt{2} \|f\|_{L^2(\mathbb{R})}. \quad (\text{B.2})$$

On the other hand, $f' \in L^p(\mathbb{R})$ and hence the Hausdorff–Young inequality implies $\lambda \hat{f}(\lambda) \in L^q(\mathbb{R})$ with $1/p + 1/q = 1$. Applying the Hölder inequality and then the

Hausdorff–Young inequality once again, we get

$$\begin{aligned} \int_{|\lambda|>1} |\hat{f}(\lambda)| d\lambda &\leq 2 \int_{|\lambda|>1} \frac{1}{1+|\lambda|} |\lambda \hat{f}(\lambda)| d\lambda \\ &\leq 2 \left(\int_{\mathbb{R}} \frac{1}{(1+|\lambda|)^p} d\lambda \right)^{1/p} \left(\int_{\mathbb{R}} |\lambda \hat{f}(\lambda)|^q d\lambda \right)^{1/q} \leq C'_p \|f'\|_{L^p(\mathbb{R})}, \end{aligned}$$

which completes the proof. \square

Remark B.4. *The case $p = 2$ is due to Beurling [15, Theorem 5.3]. A similar result was obtained by S. G. Samko. Namely, if $f \in L^1(\mathbb{R}) \cap AC_{\text{loc}}(\mathbb{R})$ is such that $f, f' \in L^p(\mathbb{R})$ with some $p \in (1, 2]$, then $f \in \mathcal{W}_0(\mathbb{R})$ (see Theorem 6.8 in [15]).*

The next result is also due to Beurling (see, e.g., Theorem 5.4 in [15]).

Theorem B.5 (Beurling). *Let $f \in C_0(\mathbb{R})$ be even and $f, f' \in AC_{\text{loc}}(\mathbb{R})$. If*

$$C := \int_{\mathbb{R}_+} k |f''(k)| dk < \infty, \quad (\text{B.3})$$

then $f \in \mathcal{W}_0(\mathbb{R})$ and $\|f\|_{\mathcal{W}} \leq C$.

Consider the following functions, which appear in Section 4:

$$\chi_l(k) = \frac{|k|^l}{1+|k|^l}, \quad l > 0, \quad (\text{B.4})$$

$$f_{l,p}(k) = \frac{|k|^p}{a+b|k|^l+|k|^{2l}}, \quad 2l > p \geq 0, \quad (\text{B.5})$$

where $a, b \in \mathbb{R}$ are such that $a+b|k|^p+|k|^{2p} > 0$ for all $k \in \mathbb{R}$. As an immediate corollary of Beurling's result we get

Corollary B.6. $\chi_l \in \mathcal{W}(\mathbb{R})$, $1 - \chi_l \in \mathcal{W}_0(\mathbb{R})$, and $f_{l,p} \in \mathcal{W}_0(\mathbb{R})$.

APPENDIX C. INTEGRAL KERNELS

There are various criteria for operators in L^p spaces to be integral operators (see, e.g., [3]). Below we present a simple sufficient condition on a function K for $K(H)$ to be an integral operator, where H is a one-dimensional Schrödinger operator. More precisely, let H be a singular Schrödinger operator on $L^2(a, b)$ as in [11] or [12] with corresponding entire system of solutions $\theta(z, x)$ and $\phi(z, x)$. Recall

$$(H - z)^{-1} f(x) = \int_a^b G(z, x, y) f(y) dy, \quad (\text{C.1})$$

where

$$G(z, x, y) = \begin{cases} \phi(z, x)\psi(z, y), & y \geq x, \\ \phi(z, y)\psi(z, x), & y \leq x, \end{cases} \quad (\text{C.2})$$

is the Green function of H and $\psi(z, x)$ is the Weyl solution normalized by $W(\theta, \psi) = 1$ (cf. [20, Lem. 9.7]). We start with a simple lemma ensuring that a function $K(H)$ is an integral operator. To this end recall that $K(H)$ is defined as $U^{-1}KU$ with K the multiplication operator in $L^2(\mathbb{R}, d\rho)$, ρ the associated spectral measure, and $U : L^2(a, b) \rightarrow L^2(\mathbb{R}, d\rho)$ the spectral transformation

$$(Uf)(\lambda) = \int_a^b \phi(\lambda, x) f(x) dx. \quad (\text{C.3})$$

Lemma C.1. *Suppose H is bounded from below and $|K(\lambda)| \leq C(1 + |\lambda|)^{-1}$ or otherwise $|K(\lambda)| \leq C(1 + |\lambda|)^{-2}$. Then $K(H)$ is an integral operator*

$$(K(H)f)(x) = \int_a^b K(x, y)f(y)dy, \quad (\text{C.4})$$

with kernel

$$K(x, y) = \int_{\mathbb{R}} K(\lambda)\phi(\lambda, x)\phi(\lambda, y)d\rho(\lambda). \quad (\text{C.5})$$

In particular, $(1 + |\cdot|)^{-1/2}\phi(\cdot, x) \in L^2(\mathbb{R}, d\rho)$ and $K(x, \cdot) \in L^2(a, b)$ for every $x \in (a, b)$.

Proof. Note that (cf. [11, Lemma 3.6])

$$(UG(z; x, \cdot))(\lambda) = \frac{\phi(\lambda, x)}{z - \lambda}.$$

If H is bounded from below then $G(z; x, \cdot)$ is in the form domain of H for fixed x and every $z \in \mathbb{C} \setminus \sigma(H)$ (cf. [8, (A.6)]) and we obtain from [11, Lemma 3.6] that $(1 + |\lambda|)^{-1/2}\phi(\lambda, x) \in L^2(\mathbb{R}, d\rho)$. In the general case we at least have $G(z; x, \cdot) \in L^2(a, b)$ and thus $(1 + |\lambda|)^{-1}\phi(\lambda, x) \in L^2(\mathbb{R}, d\rho)$. Hence we can use Fubini's theorem to evaluate

$$\begin{aligned} K(H)f(x) &= U^{-1}KUf(x) = \int_{\mathbb{R}} \phi(x, \lambda)K(\lambda) \left(\int_a^b \phi(\lambda, y)f(y)dy \right) d\rho(\lambda) \\ &= \int_a^b K(x, y)f(y)dy. \quad \square \end{aligned}$$

As a consequence we obtain that (4.12) holds at least for $\text{Im}(t) < 0$. To take the limit $\text{Im}(t) \rightarrow 0$ we need the following result which follows from [5, Lemma 3.1].

Lemma C.2. *Consider the improper integral*

$$F(\varepsilon) = \int_{-\infty}^{\infty} e^{-i(t+i\varepsilon)k^2} f(k)dk, \quad \varepsilon \leq 0,$$

where

$$f(k) = \int_{\mathbb{R}} e^{ikp} d\alpha(p), \quad |\alpha|(\mathbb{R}) < \infty.$$

Then

$$F(\varepsilon) = \frac{1}{\sqrt{4\pi i(t + i\varepsilon)}} \int_{\mathbb{R}} e^{-\frac{p^2}{4(t+i\varepsilon)}} d\alpha(p).$$

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