ON THE CAUCHY PROBLEM FOR THE KORTEWEG–DE VRIES EQUATION WITH STEPLIKE FINITE-GAP INITIAL DATA II. PERTURBATIONS WITH FINITE MOMENTS

IRYNA EGOROVA AND GERALD TESCHL

Abstract. We solve the Cauchy problem for the Korteweg–de Vries equation with steplike quasi-periodic, finite-gap initial conditions under the assumption that the perturbations have a given number of derivatives with finite moments.

1. Introduction

The purpose of this paper paper is to investigate the Cauchy problem for the Korteweg–de Vries (KdV) equation

$$q_t(x,t) = -q_{xxx}(x,t) + 6q(x,t)q_x(x,t), \qquad q(x,0) = q(x), \tag{1.1}$$

(where subscripts denote partial derivatives as usual) for the case of real-valued steplike initial conditions $q(x)$. More precisely, we will assume that $q(x)$ is asymptotically close to (in general) different quasi-periodic, finite-gap potentials $p_\pm(x)$ in the sense that

$$\pm\int_0^{\pm\infty} \left| \frac{d^n}{dx^n}\big(q(x) - p_\pm(x)\big)\right| (1+|x|^{m_0})dx < \infty, \quad 0 \le n \le n_0, \tag{1.2}$$

for some positive integers m_0, n_0. Here by quasi-periodic, finite-gap potentials we mean algebro-geometric, quasi-periodic, finite-gap potentials which arise naturally as the stationary solutions of the KdV hierarchy as discussed in [30] (further details will be given in Section 2). If (1.2) holds for all m_0, n_0 we call q a Schwartz-type perturbation.

Ever since the seminal work of Gardner et al. [28] in 1967 the inverse scattering transform has become one of the main tools used for solving this Cauchy problem. Numerous articles have been devoted to this subject since the publication of the GGKM paper. In particular, the case that the initial condition is asymptotically close to $p_\pm(x) = 0$ is well understood. We refer to the monographs by Eckhaus and Van Harten [17], Marchenko [48], Novikov, Manakov, Pitaevskii, and Zakharov [50], or Faddeev and Takhtajan [23].

There are two natural cases which have been considered in the past when extending this classical situation. The first case is that of equal quasi-periodic, finite-gap potentials $p_-(x) = p_+(x)$ and the second is the case of steplike constant asymptotics $p_\pm(x) = c_\pm$ (with $c_- \ne c_+$). The aim of our present paper is to combine

2000 *Mathematics Subject Classification.* Primary 35Q53, 37K15; Secondary 37K20, 81U40.
Key words and phrases. KdV, inverse scattering, finite-gap background, steplike.
Research supported by the Austrian Science Fund (FWF) under Grant No. Y330.
J. d'Analyse Math. **115:1**, 71–101 (2011).

both cases and to solve some open problems in these special cases (to be discussed in detail below) along the way.

The underlying scattering theory in the case of asymptotically periodic solutions was first investigated by Firsova [24]–[26]. The first ones to consider the Cauchy problem with a periodic background seem to be Kuznetsov and Mikhaĭlov [47], who informally treated the Korteweg–de Vries equation with the Weierstraß elliptic function as background solution. It turns out, due to the poles of the Baker–Akhiezer functions, which reflect the fact that the underlying hyperelliptic Riemann surface is no longer simply connected, that the periodic case is much more complicated. The only known results concerning the existence of the solution seem to be by Ermakova [21], [22] and Firsova [27] (where the evolution of the scattering data for periodic background was given). However, both works are incomplete from the point of view of a rigorous application of the inverse scattering method. Surprisingly, much more is know about the asymptotical behavior (assuming existence) of such solutions; see for example [2], [4]–[6], [34], [41]–[45], [51]. A complete and rigorous treatment of the inverse scattering transform for the KdV equation in the case of initial conditions which are Schwartz-type perturbations of finite-gap solutions was given only recently by Grunert and the present authors [19].

Let us now turn to the case of steplike constant potentials, $p_\pm(x) = c_\pm$. The foundations for scattering theory are completely understood and were given in Buslaev and Fomin [11], Davies and Simon [15], Cohen and Kappeler [13], Gesztesy [29], and Aktosun [1].

The corresponding Cauchy problem for the KdV equation was first investigated by Khruslov [41], who derives the time evolution of the scattering data and analyzes the long-time asymptotics. Later, Cohen [12] solved the case that $q(x)$ is the Heaviside step function. Kappeler [38], based on some advances in scattering theory of Cohen and Kappeler [14], showed how to handle general initial conditions with only a fixed number of moments finite. The time evolution of the scattering data for the entire KdV hierarchy was computed recently by Khasanov and Urazboev [40]. However, while Kappeler's result is impressive from a technical point of view, it still does not give a satisfactory answer, since it only determines the decay properties of the solution near one side, whereas only very mild information is given concerning the decay properties at the other side. In particular, even if one starts with a Schwartz-type initial condition, the results in [38] do not guarantee that the solution stays within this class. The reason for this is that [38] (as well as [12]) does not use the full inverse scattering machinery but only a half-sided approach. For further results, where the initial condition is supported on a half-line, see Rybkin [52] and the references therein. The case of power-like asymptotic behavior (including some unbounded initial conditions) was investigated by Bondareva and Shubin [7], [8];see also [39] for the case of the mKdV equation. Finally, we mention that in the discrete steplike finite-gap case (Toda lattice), the same problem was completely solved in [20]. For analysis of the corresponding long-time asymptotic behavior, see [9], [16], [35], [36], [37], [46] and [53].

To state our main result we denote the spectra of the one-dimensional finite-gap Schrödinger operators $L_\pm = -\partial_x^2 + p_\pm$ associated with the potentials $p_\pm(x)$ by

$$\sigma_\pm = [E_0^\pm, E_1^\pm] \cup \cdots \cup [E_{2j-2}^\pm, E_{2j-1}^\pm] \cup \cdots \cup [E_{2r_\pm}^\pm, \infty). \tag{1.3}$$

The various possible locations of the two spectra are illustrated in the following example.

Example. Let L_+ be the two-band operator with spectrum $\sigma_+ = [E_1, E_2] \cup [E_4, +\infty)$ and L_- the three band operator with spectrum $\sigma_- = [E_1, E_2] \cup [E_3, E_4] \cup [E_5, +\infty)$, where $E_1 < E_2 < \cdots < E_5$.

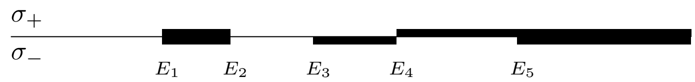

FIGURE 1. Typical locations of σ_- and σ_+.

To shed some additional light on this we recall that the Marchenko kernel $F_\pm(x, y, t)$ (cf. (3.1)) consists of three summands $F_\pm(x, y, t) = F_{\pm,D}(x, y, t) + F_{\pm,H}(x, y, t) + F_{\pm,R}(x, y, t)$, the first summand being a sum over all eigenvalues, the second one an integral over $\sigma_\mp \backslash \sigma_\pm$, and the last one an integral over $\sigma_\pm$. The crucial part is to show decay properties of $F_\pm(x, y, t)$ (and its partial derivatives). The first term $F_{\pm,D}(x, y, t)$ is as nice as one can wish for and can thus be ignored. To get the necessary decay for the remaining two terms one needs to use integration by parts. In the classical case (and more general at points E_1 and E_2 in our example), the corresponding boundary terms arising during integration by parts will vanish and the required decay follows. However in a steplike situation (at points E_4 and also E_5 for the $-$ case), this is no longer true. Moreover, at a point like E_5 the integrand of $F_{+,R}$ has a non-differentiable singularity which prevents an immediate integration by parts. By just working with the other kernel one can evade this obstacle at the price of loosing the information about decay of the solution at this side. Clearly these problems evaporate if points like E_4 and E_5 in our example are absent. This was the case analyzed in [19]. (In [19] are also found some necessary technical ingredients, which we use freely here). Note that while this restriction (which says that the respective spectral bands either coincide or are disjoint) excludes the steplike constant case, it includes the case of short-range perturbations of arbitrary quasi-periodic, finite-gap solutions.

It is the aim of the present paper to overcome these problems. To this end, rather than looking at the terms $F_{\pm,H}(x, y, t)$ and $F_{\pm,R}(x, y, t)$ individually, we will in fact show that the boundary terms mutually cancel. While this sounds like a pretty straightforward strategy, this cancellation is by no means obvious and is nothing short of a small miracle. Interestingly enough, points like E_4 and E_5 (the first one being absent in the steplike constant case) turn out to require somewhat different miracles, the first point being more involved.

Our result settles the aforementioned open problem of steplike constant Schwartz-type perturbations as a special case. Moreover, based on the recent advances in inverse scattering theory with steplike quasi-periodic, finite-gap backgrounds in [10] (cf. also [31]) and our preparations in [19], we are able to handle not only steplike constant but also arbitrary steplike quasi-periodic, finite-gap backgrounds. For more on the history of this problem and additional literature, see [19].

Next, let us state our main result. Denote by $C^n(\mathbb{R})$ the set of functions $x \in \mathbb{R} \mapsto q(x) \in \mathbb{R}$ which have n continuous derivatives with respect to x [1] and by $C^n_k(\mathbb{R}^2)$ the set of functions $(x, t) \in \mathbb{R}^2 \mapsto q(x, t) \in \mathbb{R}$ which have n continuous derivatives with respect to x and k continuous derivatives with respect to t.

[1] here $C^0(\mathbb{R}) = C(\mathbb{R})$

Theorem 1.1. *Let $p_\pm(x,t)$ be two real-valued, quasi-periodic, finite-gap solutions of the KdV equation corresponding to arbitrary quasi-periodic, finite-gap initial data $p_\pm(x) = p_\pm(x,0)$. Let $m_0 \ge 8$ and $n_0 \ge m_0 + 5$ be fixed natural numbers. Suppose that $q(x) \in C^{n_0}(\mathbb{R})$ is a real-valued function such that* (1.2) *holds. Then there exists a unique classical solution $q(x,t) \in C_1^{n_0-m_0-2}(\mathbb{R}^2)$ of the initial-value problem for the KdV equation* (1.1) *satisfying*

$$\pm\int_0^{\pm\infty} \left|\frac{\partial^n}{\partial x^n}\big(q(x,t) - p_\pm(x,t)\big)\right| (1+|x|^{\lfloor \frac{m_0}{2}\rfloor - 2})dx < \infty, \quad n \le n_0 - m_0 - 2, \tag{1.4}$$

and

$$\pm\int_0^{\pm\infty} \left|\frac{\partial}{\partial t}\big(q(x,t) - p_\pm(x,t)\big)\right| (1+|x|^{\lfloor \frac{m_0}{2}\rfloor - 2})dx < \infty, \tag{1.5}$$

for all $t \in \mathbb{R}$.

In particular, this theorem shows that the KdV equation has a solution in the class of steplike Schwartz-type perturbations of finite-gap potentials:

Corollary 1.2. *Let $p_\pm(x,t)$ be two real-valued, quasi-periodic, finite-gap solutions of the KdV equation corresponding to arbitrary quasi-periodic, finite-gap initial data $p_\pm(x) = p_\pm(x,0)$. In addition, suppose, that $q(x)$ is a steplike Schwartz-type perturbation of $p_\pm(x)$. Then the solution $q(x,t)$ of the initial-value problem for the KdV equation* (1.1) *is a steplike Schwartz-type perturbation of $p_\pm(x,t)$ for all $t \in \mathbb{R}$.*

The above results can also be used to solve analogous Cauchy problems for the modified KdV equation [18]. Furthermore, it might also be of independent interest that for uniqueness the following weaker requirement is sufficient.

Theorem 1.3. *Let $p_\pm(x,t)$ be two real-valued, quasi-periodic, finite-gap solutions of the KdV equation corresponding to arbitrary quasi-periodic, finite-gap initial data $p_\pm(x) = p_\pm(x,0)$. Suppose $q(x,t)$ is a solution of the KdV Cauchy problem satisfying*

$$\pm\int_0^{\pm\infty} \left(|q(x,t) - p_\pm(x,t)| + \left|\frac{\partial}{\partial t}\big(q(x,t) - p_\pm(x,t)\big)\right|\right)(1+x^2)dx < \infty, \tag{1.6}$$

then $q(x,t)$ is unique in this class of solutions.

Proof. The assumption are sufficient to prove the time evolution of the scattering data [19, Lemma 5.3]. Moreover, by [10, Corollary 4.4] the scattering data uniquely determine $q(x,t)$ and the claim follows. □

2. The inverse scattering transform for the KdV equation with steplike finite-gap initial data

In [19], we established the inverse scattering transform for the KdV equation in the case of Schwartz-type perturbations. In this section, we review the necessary steps and identify the changes required for the present, more general, situation. These changes are implemented in the next section. For further information and for the history of finite-gap solutions see, for example, [30], [32], [48], or [50]. For further information on the underlying scattering theory and its history see [10].

To set the stage let

$$L_\pm(t) = -\frac{d^2}{dx^2} + p_\pm(x,t) \tag{2.1}$$

be two one-dimensional Schrödinger operators, corresponding to two real-valued, quasi-periodic, finite-gap solutions $p_\pm(x,t)$ of the KdV equation that are associated with the spectra

$$\sigma_\pm = [E_0^\pm, E_1^\pm] \cup \dots \cup [E_{2j-2}^\pm, E_{2j-1}^\pm] \cup \dots \cup [E_{2r_\pm}^\pm, \infty) \tag{2.2}$$

and the Dirichlet divisors

$$\Big\{ \big(\mu_1^\pm(t), \sigma_1^\pm(t)\big), \dots, \big(\mu_{r_\pm}^\pm(t), \sigma_{r_\pm}^\pm(t)\big) \Big\}, \tag{2.3}$$

respectively. Here we assume without loss of generality that all gaps are open, that is, $E_{2j-1}^\pm < E_{2j}^\pm$ for $j = 1, 2, ..., r_\pm$. We will abbreviate $\mu_j^\pm(0) = \mu_j^\pm$, $\sigma_j^\pm(0) = \sigma_j^\pm$.

Let us cut the complex plane along the spectrum $\sigma_\pm$ and denote the upper and lower sides of the cuts by $\sigma_\pm^{\mathrm{u}}$ and $\sigma_\pm^{\mathrm{l}}$. Denote the corresponding points of the cuts by λ^{u} and λ^{l}, respectively. In particular, this means

$$f(\lambda^{\mathrm{u}}) := \lim_{\varepsilon\downarrow 0} f(\lambda + \mathrm{i}\varepsilon), \qquad f(\lambda^{\mathrm{l}}) := \lim_{\varepsilon\downarrow 0} f(\lambda - \mathrm{i}\varepsilon), \qquad \lambda \in \sigma_\pm. \tag{2.4}$$

Set

$$Y_\pm(\lambda) = -\prod_{j=0}^{2r_\pm} (\lambda - E_j^\pm), \tag{2.5}$$

and introduce the functions

$$g_\pm(\lambda, t) = -\frac{\prod_{j=1}^{r_\pm}(\lambda - \mu_j^\pm(t))}{2Y_\pm^{1/2}(\lambda)}, \tag{2.6}$$

where the branch of the square root is chosen such that

$$\frac{1}{\mathrm{i}} g_\pm(\lambda^{\mathrm{u}}, t) = \mathrm{Im}(g_\pm(\lambda^{\mathrm{u}}, t)) > 0 \quad \text{for} \quad \lambda \in \sigma_\pm, \quad t \in \mathbb{R}_+. \tag{2.7}$$

Denote by

$$\psi_\pm(\lambda, x, t) = c_\pm(\lambda, x, t) + m_\pm(\lambda, t) s_\pm(\lambda, x, t) \tag{2.8}$$

the Weyl solutions of the equations

$$L_\pm(t) y = \lambda y, \tag{2.9}$$

normalized according to $\psi_\pm(\lambda, 0, t) = 1$ and such that $\psi_\pm(\lambda, \cdot, t) \in L^2(\mathbb{R}_\pm)$ for $\lambda \in \mathbb{C} \setminus \sigma_\pm$. Here, $m_\pm(t)$ are the Weyl functions and $c_\pm(\lambda, x, t)$ and $s_\pm(\lambda, x, t)$ are solutions of (2.9), that satisfy the initial conditions

$$c_\pm(\lambda, 0, t) = s'_\pm(\lambda, 0, t) = 1, \quad s_\pm(\lambda, 0, t) = c'_\pm(\lambda, 0, t) = 0. \tag{2.10}$$

The functions $\psi_\pm$ admit the well-known representation

$$\psi_\pm(\lambda, x, t) = u_\pm(\lambda, x, t) \mathrm{e}^{\pm \mathrm{i}\theta_\pm(\lambda) x}, \quad \lambda \in \mathbb{C} \setminus \sigma_\pm, \tag{2.11}$$

where $\theta_\pm(\lambda)$ are the quasimomenta and the functions $u_\pm(\lambda, x, t)$ are quasiperiodic with respect to x with the same basic frequencies as the potentials $p_\pm(x, t)$. The quasimomenta are holomorphic for $\lambda \in \mathbb{C} \setminus \sigma_\pm$ and normalized according to

$$\frac{d\theta_\pm}{d\lambda} > 0 \quad \text{for} \quad \lambda \in \sigma_\pm^{\mathrm{u}}, \qquad \theta_\pm(E_0^\pm) = 0. \tag{2.12}$$

This normalization implies

$$\frac{d\theta_\pm}{d\lambda} = \frac{\mathrm{i} \prod_{j=1}^{r_\pm}(\lambda - \zeta_j^\pm)}{Y_\pm^{1/2}(\lambda)}, \qquad \zeta_j^\pm \in (E_{2j-1}^\pm, E_{2j}^\pm), \tag{2.13}$$

and therefore, the quasimomenta are real-valued on $\sigma_\pm^{\mathrm{u},1}$. Note, that in the case where $p_\pm(x,t)\equiv 0$ we have $\theta_\pm(\lambda)=\sqrt{\lambda}$, $u_\pm(\lambda,x,t)\equiv 1$ and $m_\pm(\lambda,t)=\pm\mathrm{i}\sqrt{\lambda}$. In the general finite-gap cases the two Weyl m-functions associated with $L_\pm$ are given by ([30, eq. (1.165)])

$$m_\pm(\lambda,t)=\frac{H_\pm(\lambda,t)\pm Y_\pm^{1/2}(\lambda)}{\prod_{j=1}^{r_\pm}(\lambda-\mu_j^\pm(t))},\quad \breve{m}_\pm(\lambda,t)=\frac{H_\pm(\lambda,t)\mp Y_\pm^{1/2}(\lambda)}{\prod_{j=1}^{r_\pm}(\lambda-\mu_j^\pm(t))}. \tag{2.14}$$

Here $H_\pm(\lambda,t)$ are polynomials in λ of $\deg(H_\pm)\le r_\pm-1$ with real-valued coefficients which are smooth with respect to t. Moreover,

$$H_\pm(\mu_j^\pm(t),t)=0 \quad \text{for} \quad \mu_j^\pm(t)\in\partial\sigma_\pm. \tag{2.15}$$

Associated with the second Weyl m-function $\breve{m}_\pm(\lambda,t)$ is the second Weyl solution

$$\begin{aligned}\breve{\psi}_\pm(\lambda,x,t)&=c_\pm(\lambda,x,t)+\breve{m}_\pm(\lambda,t)s_\pm(\lambda,x,t)\\ &=\breve{u}_\pm(\lambda,x,t)\mathrm{e}^{\mp\mathrm{i}\theta_\pm(\lambda)x},\quad \lambda\in\mathbb{C}\setminus\sigma_\pm,\end{aligned} \tag{2.16}$$

that satisfies $\breve{\psi}_\pm(\lambda,\cdot,t)\in L^2(\mathbb{R}_\mp)$ for $\lambda\in\mathbb{C}\setminus\sigma_\pm$. The Wronski determinant, $\mathsf{W}(f,g)(x)=f(x)g'(x)-f'(x)g(x)$, of the functions $\psi_\pm$ and $\breve{\psi}_\pm$ is given by

$$\mathsf{W}(\psi_\pm(\lambda,.,t),\breve{\psi}_\pm(\lambda,.,t))=\pm g_\pm(\lambda,t)^{-1}. \tag{2.17}$$

Introduce the Lax operators corresponding to the finite-gap solutions $p_\pm(x,t)$,

$$L_\pm(t)=-\partial_x^2+p_\pm(x,t), \tag{2.18}$$

$$P_\pm(t)=-4\partial_x^3+6p_\pm(x,t)\partial_x+3\partial_x p_\pm(x,t). \tag{2.19}$$

Then the following result is valid ([3], [30])

Lemma 2.1. *The functions*

$$\hat{\psi}_\pm(\lambda,x,t)=\mathrm{e}^{\alpha_\pm(\lambda,t)}\psi_\pm(\lambda,x,t), \tag{2.20}$$

where

$$\alpha_\pm(\lambda,t):=\int_0^t\left(2(p_\pm(0,s)+2\lambda)m_\pm(\lambda,s)-\frac{\partial p_\pm(0,s)}{\partial x}\right)ds, \tag{2.21}$$

satisfy the system of equations

$$L_\pm(t)\hat{\psi}_\pm=\lambda\hat{\psi}_\pm, \tag{2.22}$$

$$\frac{\partial\hat{\psi}_\pm}{\partial t}=P_\pm(t)\hat{\psi}_\pm. \tag{2.23}$$

Set

$$\begin{aligned}M_\pm(t)&=\{\mu_j^\pm(t)\mid \mu_j^\pm(t)\in(E_{2j-1}^\pm,E_{2j}^\pm)\text{ and } m_\pm(\lambda,t)\text{ has a simple pole}\},\\ \hat{M}_\pm(t)&=\{\mu_j^\pm(t)\mid\mu_j^\pm(t)\in\{E_{2j-1}^\pm,E_{2j}^\pm\}\},\end{aligned} \tag{2.24}$$

and introduce the functions

$$\begin{aligned}\delta_\pm(\lambda,t)&:=\prod_{\mu_j^\pm(t)\in M_\pm(t)}(\lambda-\mu_j^\pm(t)),\\ \hat{\delta}_\pm(\lambda,t)&:=\prod_{\mu_j^\pm(t)\in M_\pm(t)}(\lambda-\mu_j^\pm(t))\prod_{\mu_j^\pm(t)\in\hat{M}_\pm(t)}\sqrt{\lambda-\mu_j^\pm(t)},\end{aligned} \tag{2.25}$$

where $\prod = 1$ if the index set is empty. These functions allow us to remove the singularities of the Weyl solutions $\psi_\pm(\lambda, x, t)$ whenever necessary.

Next, we collect now some facts from scattering theory for Schrödinger operators with smooth steplike finite-gap potentials (cf. [10], [19]). To shorten notations throughout this discussion, we omit the dependence on t.

Let $n_1 \geq 0$ and $m_1 \geq 2$ be given natural numbers and let $q(x) \in C^{n_1}(\mathbb{R})$ be a real-valued function such that

$$\pm \int_0^{\pm\infty} \left| \frac{d^n}{dx^n} \big(q(x) - p_\pm(x)\big) \right| (1 + |x|^{m_1}) dx < \infty, \quad \forall\, 0 \leq n \leq n_1. \tag{2.26}$$

Consider the *perturbed* operator

$$L := -\frac{d^2}{dx^2} + q(x) \tag{2.27}$$

with a potential $q(x)$ that satisfies (2.26). The spectrum of L consists of a purely absolutely continuous part $\sigma := \sigma_+ \cup \sigma_-$, plus a finite number of discrete eigenvalues $\sigma_d = \{\lambda_1, \dots, \lambda_p\}$ situated in the gaps $\sigma_d \subset \mathbb{R} \setminus \sigma$. The set $\sigma^{(2)} := \sigma_+ \cap \sigma_-$ is the spectrum of multiplicity two for the operator L, and the set $\sigma_+^{(1)} \cup \sigma_-^{(1)}$ with $\sigma_\pm^{(1)} = \operatorname{clos}(\sigma_\pm \setminus \sigma_\mp)$ is the spectrum of multiplicity one.

The Jost solutions of the spectral equation

$$\left(-\frac{d^2}{dx^2} + q(x)\right) \phi(x) = \lambda \phi(x), \quad \lambda \in \mathbb{C}, \tag{2.28}$$

are defined by the requirement that they asymptotically look like the Weyl solutions of the background operators as $x \to \pm\infty$.

Lemma 2.2. *Assume $q(x)$ satisfies* (2.26)*. Then there exist solutions $\phi_\pm(\lambda, x)$, $\lambda \in \mathbb{C}$, of* (2.28) *satisfying*

$$\phi_\pm(\lambda, x) = \psi_\pm(\lambda, x)(1 + o(1)), \qquad x \to \pm\infty. \tag{2.29}$$

The Jost solutions $\phi_\pm(\lambda, .)$ are meromorphic with respect to $\lambda \in \mathbb{C} \backslash \sigma_\pm$ and have the same poles as $\psi_\pm(\lambda, .)$. The functions $\hat\delta(\lambda)\phi_\pm(\lambda, \cdot)$ are continuous up to the boundary $\sigma_\pm^{\mathrm{u}} \cup \sigma_\pm^{\mathrm{l}}$. Moreover, $\hat\delta(\lambda)\phi_\pm(\lambda, \cdot)$ are m_1 times differentiable with respect to $\lambda \in \operatorname{int}(\sigma_\pm^{\mathrm{u}} \cup \sigma_\pm^{\mathrm{l}})$ and $m_1 - 1$ times continuously differentiable with respect to the local variable $\sqrt{\lambda - E}$ near $E \in \partial\sigma_\pm$.

Proof. Set

$$J_\pm(\lambda, x, y) = \frac{\psi_\pm(\lambda, y)\breve\psi_\pm(\lambda, x) - \psi_\pm(\lambda, x)\breve\psi_\pm(\lambda, y)}{\mathsf{W}(\psi_\pm(\lambda), \breve\psi_\pm(\lambda))}. \tag{2.30}$$

Then the Jost solutions of (2.28) formally satisfy the integral equation

$$\phi_\pm(\lambda, x) = \psi_\pm(\lambda, x) - \int_x^{\pm\infty} J_\pm(\lambda, x, y)(q(y) - p_\pm(y))\phi_\pm(\lambda, y) dy. \tag{2.31}$$

To remove the singularities of $\psi_\pm(\lambda, x)$ near $\lambda \in M_\pm \cup \hat M_\pm$, one can multiply the whole equation by $\hat\delta_\pm(\lambda)$.

Similarly, the x derivatives satisfy

$$\frac{\partial}{\partial x}\phi_\pm(\lambda, x) = \frac{\partial}{\partial x}\psi_\pm(\lambda, x) - \int_x^{\pm\infty} \left(\frac{\partial}{\partial x} J_\pm(\lambda, x, y)\right)(q(y) - p_\pm(y))\phi_\pm(\lambda, y) dy.$$

Hence, existence of the Jost solutions together with their derivatives follows from existence of solutions of these integral equations. Existence is proved by the method of successive iterations in the usual manner. Observe that since at points $\lambda \in \partial\sigma_\pm$ the second solution grows linearly, the above kernel can only be estimated by $C|x-y|$ near such points. □

We also need to know the asymptotic behavior of the Jost solutions as $\lambda \to \infty$. To determine it, we recall the well-know expansion (cf. the proof of Lemma 1.19 in [30])

$$\psi_\pm(\lambda, x) = \exp\left(\pm\mathrm{i}\sqrt{\lambda}x + \int_0^x \left(\sum_{j=1}^n \frac{\kappa_j^\pm(y)}{(\pm 2\mathrm{i}\sqrt{\lambda})^j} + \frac{\tilde{\kappa}_{n,\pm}(\sqrt{\lambda}, y)}{(\pm 2\mathrm{i}\sqrt{\lambda})^n}\right) dy\right), \tag{2.32}$$

up to any order n, where

$$\kappa_1^\pm(x) = p_\pm(x), \quad \kappa_{j+1}^\pm(x) = -\frac{\partial}{\partial x}\kappa_j^\pm(x) - \sum_{i=1}^{j-1} \kappa_{j-i}^\pm(x)\kappa_i^\pm(x), \tag{2.33}$$

and the error term satisfies

$$\frac{\partial^l}{\partial k^l}\tilde{\kappa}_{n,\pm}(k, x) = o(1), \quad l = 0, 1, \dots \tag{2.34}$$

for fixed x as $k \to \infty$.

Lemma 2.3. *Assume $q(x)$ satisfies* (2.26)*. Then the Jost solutions have the asymptotic expansions*

$$\phi_\pm(\lambda, x) = \psi_\pm(\lambda, x)\left(1 + \frac{\phi_{\pm,1}(x)}{\lambda^{1/2}} + \dots + \frac{\phi_{\pm,n_1+1}(x)}{\lambda^{(n_1+1)/2}} + o\big(\lambda^{-(n_1+1)/2}\big)\right) \tag{2.35}$$

which can be differentiated m_1 times with respect to $\lambda^{1/2}$. An analogous expansion holds for $\frac{\partial}{\partial x}\phi_\pm(\lambda, x)$.

Proof. To obtain the asymptotic expansion, consider $\tilde{\phi}_\pm(\lambda, x) = \frac{\phi_\pm(\lambda,x)}{\psi_\pm(\lambda,x)}$, which satisfy

$$\tilde{\phi}_\pm(\lambda, x) = 1 - \int_x^{\pm\infty} \tilde{J}_\pm(\lambda, x, y)(q(y) - p_\pm(y))\tilde{\phi}_\pm(\lambda, y)dy, \tag{2.36}$$

where $\tilde{J}_\pm(\lambda, x, y) = J_\pm(\lambda, x, y)\frac{\psi_\pm(\lambda,y)}{\psi_\pm(\lambda,x)}$. Next recall (2.11), (2.16), and (2.17) which imply

$$\tilde{J}_\pm(\lambda, x, y) = \pm g_\pm(\lambda)\left(u_\pm(\lambda, y)^2\frac{\breve{u}_\pm(\lambda, x)}{u_\pm(\lambda, x)}\mathrm{e}^{\pm 2\mathrm{i}\theta_\pm(\lambda)(x-y)} - \breve{u}_\pm(\lambda, y)u_\pm(\lambda, y)\right),$$

where $u_\pm(\lambda, x)$, $\breve{u}_\pm(\lambda, x)$ are quasi-periodic with respect to x and have convergent expansions around ∞ with respect to $\theta_\pm(\lambda)^{-1}$. Now use the fact that

$$\int_0^\infty \mathrm{e}^{2\mathrm{i}\theta(\lambda)y} f(\lambda, y)dy = \sum_{j=1}^n \frac{f_j}{\theta(\lambda)^j} + o(\theta(\lambda)^{-n})$$

provided $f(\lambda, x)$ is n times differentiable with respect to x, the first $n-1$ derivatives have an asymptotic expansion with respect to $\theta(\lambda)^{-1}$ of order n and the n'th derivative satisfies $\lim_{\lambda\to\infty} \frac{\partial^n}{\partial x^n} f(\lambda, x) = g(x)$ in $L^1(0, \infty)$. This follows from n partial integrations and the Riemann-Lebesgue Lemma (cf. also [49, Theorem 3.2]).

As in the previous lemma, the claims for the derivatives follow by considering the corresponding integral equations. □

Corollary 2.4. *Assume $q(x)$ satisfies* (2.26). *Then the Weyl m-functions $m_{q,\pm}(\lambda,x)=\frac{\phi_\pm'(\lambda,x)}{\phi_\pm(\lambda,x)}$ have the asymptotic expansion*

$$m_{q,\pm}(\lambda,x)=\pm\mathrm{i}\sqrt{\lambda}+\sum_{j=1}^{n_1}\frac{\kappa_j(x)}{(\pm 2\mathrm{i}\sqrt{\lambda})^j}+o(\lambda^{-n_1/2}), \tag{2.37}$$

which can be differentiated m_1 times with respect to $\lambda^{1/2}$. The coefficients $\kappa_j(x)$ are given by (2.33) *with $q(x)$ in place of $p_\pm(x)$.*

Proof. Existence of the expansion follows from the previous lemma, and the expansion coefficients follow by comparing coefficients in the Riccati equation

$$\frac{\partial}{\partial x}m_{q,\pm}(\lambda,x)+m_{q,\pm}(\lambda,x)^2+\lambda-q(x)=0.$$

□

The Jost solutions can be represented, with the help of the transformation operators, as

$$\phi_\pm(\lambda,x)=\psi_\pm(\lambda,x)\pm\int_x^{\pm\infty}K_\pm(x,y)\psi_\pm(\lambda,y)dy, \tag{2.38}$$

where $K_\pm(x,y)$ are real-valued functions that satisfy

$$K_\pm(x,x)=\pm\frac{1}{2}\int_x^{\pm\infty}(q(y)-p_\pm(y))dy. \tag{2.39}$$

Moreover, as a consequence of [10, (A.15)], we have the estimate

$$\left|\frac{\partial^{n+l}}{\partial x^n\partial y^l}K_\pm(x,y)\right|\le C_\pm(x)\left(Q_\pm(x+y)+\sum_{j=0}^{n+l-1}\left|\frac{\partial^j}{\partial x^j}\big(q(\frac{x+y}{2})-p_\pm(\frac{x+y}{2})\big)\right|\right), \tag{2.40}$$

for $\pm y>\pm x$, where $C_\pm(x)=C_{n,l,\pm}(x)$ are continuous positive functions decaying as $x\to\pm\infty$, and

$$Q_\pm(x):=\pm\int_{\frac{x}{2}}^{\pm\infty}\big|q(y)-p_\pm(y)\big|dy. \tag{2.41}$$

Formula (2.38) shows that the Jost solutions inherit all singularities of the background Weyl functions $m_\pm(\lambda)$ and Weyl solutions $\psi_\pm(\lambda)$. In particular, as a direct consequence of formulas (2.8), (2.13), (2.14), (2.15), (2.38), and Lemma 2.2 we have the following result.

Lemma 2.5. *Let $E\in\partial\sigma_\pm$ and let $\varepsilon>0$ be such that $[E-\varepsilon,E+\varepsilon]\cap\partial\sigma_\pm=\{E\}$ and $\varepsilon<\mathrm{dist}(\mu_j^\pm,E)$ if $\mu_j^\pm\ne E$.*
(i) *Let $\mu_j^\pm=E$. Introduce the functions*

$$\phi_{\pm,E}(\lambda,x):=\mathrm{i}\left(\theta_\pm(\lambda)-\theta_\pm(E)\right)\phi_\pm(\lambda,x),\ \ g_{\pm,E}(\lambda):=\left|\theta_\pm(\lambda)-\theta_\pm(E)\right|^{-2}g_\pm(\lambda) \tag{2.42}$$

for $\lambda[E-\varepsilon,E+\varepsilon]$. The functions $\phi_{\pm,E}(\lambda,x)$ admit the representation

$$\phi_{\pm,E}(\lambda,x)=c_{\pm,E}(\lambda,x)+\mathrm{i}\left(\theta_\pm(\lambda)-\theta_\pm(E)\right)s_{\pm,E}(\lambda,x), \tag{2.43}$$

where $c_{\pm,E}(\cdot,x), s_{\pm,E}(\cdot,x) \in C^{m_0-1}([E-\varepsilon, E+\varepsilon])$ *and* $c_{\pm,E}(\cdot,x), s_{\pm,E}(\cdot,x) \in \mathbb{R}$. *Analogous representations hold for* $\frac{\partial}{\partial x}\phi_{\pm,E}(\lambda,x)$. *Moreover,*

$$\phi_{\pm,E}(\lambda,x) \in \mathbb{R}, \qquad \lambda \in [E-\varepsilon, E+\varepsilon] \setminus \sigma_\pm, \tag{2.44}$$

and

$$g_{\pm,E}(\lambda)^{-1} = \pm \mathsf{W}(\phi_{\pm,E}, \overline{\phi_{\pm,E}}), \qquad \lambda \in (E-\varepsilon, E+\varepsilon) \cap \sigma_\pm. \tag{2.45}$$

(ii) *Let* $\mu_j^\pm \neq E$. *The functions* $\phi_\pm(\lambda,x)$ *admit the same representation* (2.43) *on the set* $[E-\varepsilon/2, E+\varepsilon/2]$.

Next, recalling (2.25), set

$$\tilde{\phi}_\pm(\lambda,x) = \delta_\pm(\lambda)\phi_\pm(\lambda,x) \tag{2.46}$$

so that the functions $\tilde{\phi}_\pm(\lambda,x)$ have no poles in the interior of the gaps of the spectrum σ. For each eigenvalue λ_k, we introduce the corresponding norming constants

$$\left(\gamma_k^\pm\right)^{-2} = \int_{\mathbb{R}} \tilde{\phi}_\pm^2(\lambda_k,x)dx. \tag{2.47}$$

Furthermore, recall the scattering relations

$$T_\mp(\lambda)\phi_\pm(\lambda,x) = \overline{\phi_\mp(\lambda,x)} + R_\mp(\lambda)\phi_\mp(\lambda,x), \quad \lambda \in \sigma_\mp^{\mathrm{u,l}}, \tag{2.48}$$

where the transmission and reflection coefficients are defined as usual, by

$$T_\pm(\lambda) := \frac{\mathsf{W}(\overline{\phi_\pm(\lambda)}, \phi_\pm(\lambda))}{\mathsf{W}(\phi_\mp(\lambda), \phi_\pm(\lambda))}, \qquad R_\pm(\lambda) := -\frac{\mathsf{W}(\phi_\mp(\lambda), \overline{\phi_\pm(\lambda)})}{\mathsf{W}(\phi_\mp(\lambda), \phi_\pm(\lambda))}, \quad \lambda \in \sigma_\pm^{\mathrm{u,l}}. \tag{2.49}$$

Lemma 2.6. *Suppose that* $q(x) \in C^{n_1}(\mathbb{R})$ *satisfies* (2.26). *Then the scattering data*

$$\begin{aligned}\mathcal{S} = \Big\{ & R_+(\lambda),\ T_+(\lambda),\ \lambda \in \sigma_+^{\mathrm{u,l}};\ R_-(\lambda),\ T_-(\lambda),\ \lambda \in \sigma_-^{\mathrm{u,l}};\\ & \lambda_1, \dots, \lambda_p \in \mathbb{R}\setminus\sigma,\ \gamma_1^\pm, \dots, \gamma_p^\pm \in \mathbb{R}_+ \Big\}\end{aligned} \tag{2.50}$$

has the following properties:

I. (a) $T_\pm(\lambda^{\mathrm{u}}) = \overline{T_\pm(\lambda^{\mathrm{l}})}$ *for* $\lambda \in \sigma_\pm$.
$R_\pm(\lambda^{\mathrm{u}}) = \overline{R_\pm(\lambda^{\mathrm{l}})}$ *for* $\lambda \in \sigma_\pm$.

(b) $\dfrac{T_\pm(\lambda)}{\overline{T_\pm(\lambda)}} = R_\pm(\lambda)$ *for* $\lambda \in \sigma_\pm^{(1)}$.

(c) $1 - |R_\pm(\lambda)|^2 = \dfrac{g_\pm(\lambda)}{g_\mp(\lambda)}|T_\pm(\lambda)|^2$ *for* $\lambda \in \sigma^{(2)}$.

(d) $\overline{R_\pm(\lambda)}T_\pm(\lambda) + R_\mp(\lambda)\overline{T_\pm(\lambda)} = 0$ *for* $\lambda \in \sigma^{(2)}$.

(e) $T_\pm(\lambda) = 1 + O\Big(\frac{1}{\sqrt{\lambda}}\Big)$ *for* $\lambda \to \infty$.

(f) $R_\pm(\lambda) = o\Big(\frac{1}{(\sqrt{\lambda})^{n_1+1}}\Big)$ *for* $\lambda \to \infty$.

II. *The functions* $T_\pm(\lambda)$ *can be extended as meromorphic functions to the domain* $\mathbb{C}\setminus\sigma$ *and satisfy*

$$\frac{1}{T_+(\lambda)g_+(\lambda)} = \frac{1}{T_-(\lambda)g_-(\lambda)} =: -W(\lambda), \tag{2.51}$$

where $W(\lambda)$ *possesses the following properties:*

(a) *The function* $\tilde{W}(\lambda) = \delta_+(\lambda)\delta_-(\lambda)W(\lambda)$ *is holomorphic in the domain* $\mathbb{C}\setminus\sigma$, *with simple zeros at the points* λ_k, *where*

$$\frac{d\tilde{W}}{d\lambda}(\lambda_k) = (\gamma_k^+\gamma_k^-)^{-1}. \tag{2.52}$$

In addition, it satisfies

$$\overline{\tilde{W}(\lambda^{\mathrm{u}})} = \tilde{W}(\lambda^{\mathrm{l}}), \quad \lambda\in\sigma \quad \text{and} \quad \tilde{W}(\lambda)\in\mathbb{R} \quad \text{for} \quad \lambda\in\mathbb{R}\setminus\sigma. \tag{2.53}$$

(b) *The function* $\hat{W}(\lambda) = \hat{\delta}_+(\lambda)\hat{\delta}_-(\lambda)W(\lambda)$ *is continuous on the set* $\mathbb{C}\setminus\sigma$ *up to the boundary* $\sigma^{\mathrm{u}}\cup\sigma^{\mathrm{l}}$. *Moreover, this function is* m_1-1 *times differentiable with respect to* λ *on the set* $\big(\sigma^{\mathrm{u}}\cup\sigma^{\mathrm{l}}\big)\setminus(\partial\sigma_-\cup\partial\sigma_+)$ *and* m_1-1 *times continuously differentiable with respect to the local variable* $\sqrt{\lambda-E}$ *for* $E\in\partial\sigma_-\cup\partial\sigma_+$. *It can have zeros on the set* $\partial\sigma_-\cap\partial\sigma_+$ *and does not vanish at the other points of the set* σ. *If* $\hat{W}(E)=0$ *for* $E\in\partial\sigma_-\cap\partial\sigma_+$, *then* $\hat{W}(\lambda)=\sqrt{\lambda-E}(C(E)+o(1))$, $C(E)\neq 0$.

III. (a) *The reflection coefficients* $R_\pm(\lambda)$ *are continuous functions on* $\sigma_\pm^{\mathrm{u}}\cup\sigma_\pm^{\mathrm{l}}$. *They are also* m_1 *times differentiable with respect to* λ *on the sets* $\sigma_\pm^{\mathrm{u,l}}\setminus\{\partial\sigma_+\cup\partial\sigma_-\}$ *and* m_1-j *times differentiable with respect to the coordinate* $\sqrt{\lambda-E}$ *with* $E\in\{\partial\sigma_+\cup\partial\sigma_-\}\cap\sigma_\pm^{\mathrm{u,l}}$, *where* $j=1$ *if* $\hat{W}(E)\neq 0$ *and* $j=2$ *if* $\hat{W}(E)\neq 0$. *The asymptotics* **I. (f)** *hold for all derivatives as well.*

(b) *If* $E\in\partial\sigma_\pm$ *and* $\hat{W}(E)\neq 0$, *then*

$$R_\pm(E) = \begin{cases} -1 & \text{for } E\notin\hat{M}_\pm, \\ 1 & \text{for } E\in\hat{M}_\pm. \end{cases} \tag{2.54}$$

Proof. Except for **I. (f)** and the corresponding statement for the derivatives in **III. (a)** everything follows as in [19, Lemma 4.1]. To prove the missing items, we need to prove that $\mathsf{W}(\phi_\pm,\overline{\phi_\mp}) = o(\lambda^{-n_1/2})$ and all its necessary derivatives with respect to $\sqrt{\lambda}$. But this follows from

$$\mathsf{W}(\phi_\pm,\overline{\phi_\mp}) = \phi_-(\lambda,x)\overline{\phi_+(\lambda,x)}(m_{q,-}(\lambda,x)-\overline{m_{q,+}(\lambda,x)})$$

since $\phi_-(\lambda,x)\overline{\phi_+(\lambda,x)} = O(1)$ by Lemma 2.3 and $m_{q,-}(\lambda,x)-\overline{m_{q,+}(\lambda,x)} = o(\lambda^{-n_1/2})$ by Corollary 2.4. □

Next, recall the associated Gelfand–Levitan–Marchenko (GLM) equations

$$K_\pm(x,y)+F_\pm(x,y)\pm\int_x^{\pm\infty} K_\pm(x,\xi)F_\pm(\xi,y)d\xi = 0, \quad \pm y>\pm x, \tag{2.55}$$

where[2]

$$\begin{aligned} F_\pm(x,y) = \frac{1}{2\pi\mathrm{i}}\oint_{\sigma_\pm} & R_\pm(\lambda)\psi_\pm(\lambda,x)\psi_\pm(\lambda,y)g_\pm(\lambda)d\lambda + \\ &+\frac{1}{2\pi\mathrm{i}}\int_{\sigma_\mp^{(1),\mathrm{u}}} |T_\mp(\lambda)|^2\psi_\pm(\lambda,x)\psi_\pm(\lambda,y)g_\mp(\lambda)d\lambda \\ &+\sum_{k=1}^p(\gamma_k^\pm)^2\tilde{\psi}_\pm(\lambda_k,x)\tilde{\psi}_\pm(\lambda_k,y). \end{aligned} \tag{2.56}$$

[2]Here we have used the notation $\oint_{\sigma_\pm} f(\lambda)d\lambda := \int_{\sigma_\pm^{\mathrm{u}}} f(\lambda)d\lambda - \int_{\sigma_\pm^{\mathrm{l}}} f(\lambda)d\lambda$.

As in [19, Lemma 4.2], we have the following result:

Lemma 2.7. *Under the same assumptions as in Lemma 2.6, the functions $F_\pm(x,y)$ satisfy*

IV. $F_\pm(x,y) \in C^{(n_1+1)}(\mathbb{R}^2)$. *There exist real-valued continuous functions $\tilde q_\pm(x)$, with $x^{m_1}\tilde q_\pm \in L^1(\mathbb{R}_\pm)$, and monotone positive continuous functions $C_\pm(x)$, $Q_\pm(x)$, which decay as $x \to \pm\infty$, with $x^{m_1-1}Q_\pm(x) \in L^1(\mathbb{R}_\pm)$, such that for $\pm x > \pm a$, $\pm y > \pm a$ and $0 \le n+l \le n_1+1$ the inequalities hold*

$$\left|\frac{\partial^{n+l}}{\partial x^n \partial y^l} F_\pm(x,y)\right| \le C_\pm(a)\left(Q_\pm(x+y) + \tilde q_\pm(x+y)(1-\delta_{n+l,0})\right). \tag{2.57}$$

Here $\delta_{n,m}$ is the Kronecker delta and $a \in \mathbb{R}$ is an arbitrary fixed number. Moreover,

$$\pm\int_0^{\pm\infty}\left|\frac{d^n}{dx^n}F_\pm(x,x)\right|(1+|x|^{m_1})dx < \infty, \qquad 1 \le n \le n_1+1. \tag{2.58}$$

Proof. The GLM equation (2.55) and (2.56) are derived in [10]. Estimate (2.57) follows directly from (2.55) and (2.40). Equation (2.58) is then immediate from (2.57). □

As demonstrated in [10] and [19], properties **I**–**IV** are necessary and sufficient for a set $\mathcal{S}$ to be the set of scattering data for operator L with a potential $q(x)$ satisfy (2.26).

Now the procedure of solving of the inverse scattering problem is as follows:

Let $L_\pm$ be two one-dimensional finite-gap Schrödinger operators associated with the potentials $p_\pm(x)$. Let $\mathcal{S}$ be given data as in (2.50) satisfying **I**–**IV**. Define corresponding kernels $F_\pm(x,y)$ via (2.56). As shown in [10], under condition **IV** the GLM equations (2.55) have unique smooth real-valued solutions $K_\pm(x,y)$, that satisfy estimates of type (2.57). In particular,

$$\pm\int_0^{\pm\infty}(1+|x|^{m_1})\left|\frac{d^n}{dx^n}K_\pm(x,x)\right|dx < \infty, \qquad 1 \le n \le n_1+1. \tag{2.59}$$

Now introduce the functions

$$q_\pm(x) = p_\pm(x) \mp 2\frac{d}{dx}K_\pm(x,x) \tag{2.60}$$

and note that (2.59) reads

$$\pm\int_0^{\pm\infty}\left|\frac{d^n}{dx^n}\big(q_\pm(x) - p_\pm(x)\big)\right|(1+|x|^{m_1})dx < \infty, \quad 0 \le n \le n_1. \tag{2.61}$$

We obtain the following result.

Theorem 2.8 ([10]). *Let the set of data $\mathcal{S}$, defined as in* (2.50)*, satisfy properties* ***I**–**IV***. *Then the functions $q_\pm(x)$ defined by* (2.60) *satisfy* (2.61) *and coincide, $q_-(x) \equiv q_+(x) =: q(x)$. Moreover, the set $\mathcal{S}$ is the set of scattering data for the Schrödinger operator* (2.27) *with potential $q(x)$ satisfying* (2.26).

Our next step is to describe a formal scheme for using the inverse scattering method to solve the initial value problem for the KdV equation with initial conditions $q(x)$ satisfying (1.2) with with some quasi-periodic, finite-gap potentials $p_\pm(x)$ and fixed $m_0 \ge 8$ and $n_0 \ge m_0+5$. Consider the corresponding scattering data $\mathcal{S} = \mathcal{S}(0)$ which obey conditions **I**–**IV** with $n_1 = n_0$ and $m_1 = m_0$. Let $p_\pm(x,t)$

be the finite-gap solution of the KdV equation with initial conditions $p_\pm(x)$ and let $m_\pm(\lambda,t)$, $\breve m_\pm(\lambda,t)$ $\psi_\pm(\lambda,x,t)$, $\alpha_\pm(\lambda,t)$ be defined by (2.14), (2.8) and (2.21) as above. Also set

$$\breve\alpha_\pm(\lambda,t)=\int_0^t\left(2(p_\pm(0,s)+2\lambda)\breve m_\pm(\lambda,s)-\frac{\partial p_\pm(0,s)}{\partial x}\right)ds. \tag{2.62}$$

Introduce the set $\mathcal{S}(t)$ by

$$\begin{aligned}\mathcal{S}(t)=\Big\{&R_+(\lambda,t),\ T_+(\lambda,t),\ \lambda\in\sigma_+^{\mathrm{u,l}};\ R_-(\lambda,t),\ T_-(\lambda,t),\ \lambda\in\sigma_-^{\mathrm{u,l}};\\ &\lambda_1,\dots,\lambda_p\in\mathbb{R}\setminus\sigma,\ \gamma_1^\pm(t),\dots,\gamma_p^\pm(t)\in\mathbb{R}_+\Big\},\end{aligned} \tag{2.63}$$

where $\lambda_k(t)$, $R_\pm(\lambda,t)$, $T_\pm(\lambda,t)$ and $\gamma_k^\pm(t)$ are defined ([19, Lemma 5.3]) by:

$$R_\pm(\lambda,t)=R_\pm(\lambda,0)\mathrm{e}^{\alpha_\pm(\lambda,t)-\breve\alpha_\pm(\lambda,t)},\quad \lambda\in\sigma_\pm, \tag{2.64}$$

$$T_\mp(\lambda,t)=T_\mp(\lambda,0)\mathrm{e}^{\alpha_\pm(\lambda,t)-\breve\alpha_\mp(\lambda,t)},\quad \lambda\in\mathbb{C}, \tag{2.65}$$

$$\big(\gamma_k^\pm(t)\big)^2=\big(\gamma_k^\pm(0)\big)^2\,\frac{\delta_\pm^2(\lambda_k,0)}{\delta_\pm^2(\lambda_k,t)}\mathrm{e}^{2\alpha_\pm(\lambda_k,t)}, \tag{2.66}$$

where $\alpha_\pm(\lambda,t)$, $\breve\alpha_\pm(\lambda,t)$, $\delta_\pm(\lambda,t)$ are defined in (2.21), (2.62), (2.25) respectively.

In [19], it is proved, that these data satisfy **I–III** with $g_\pm(\lambda,t)$, defined by (2.6) and $\delta_\pm(\lambda)$, $\hat\delta_\pm(\lambda)$ defined by (2.25).

Introduce

$$\begin{aligned}F_\pm(x,y,t)=&\frac{1}{2\pi\mathrm{i}}\oint_{\sigma_\pm}R_\pm(\lambda,t)\psi_\pm(\lambda,x,t)\psi_\pm(\lambda,y,t)g_\pm(\lambda,t)d\lambda+\\ &+\frac{1}{2\pi\mathrm{i}}\int_{\sigma_\mp^{(1),\mathrm{u}}}|T_\mp(\lambda,t)|^2\psi_\pm(\lambda,x,t)\psi_\pm(\lambda,y,t)g_\mp(\lambda,t)d\lambda\\ &+\sum_{k=1}^p(\gamma_k^\pm(t))^2\tilde\psi_\pm(\lambda_k,x,t)\tilde\psi_\pm(\lambda_k,y,t).\end{aligned} \tag{2.67}$$

Suppose that we are able to prove that $F_\pm$ satisfy

$$\left|\frac{\partial^{n+l}}{\partial x^n\partial y^l}F_\pm(x,y,t)\right|+\left|\frac{\partial^2}{\partial x\partial t}F_\pm(x,y,t)\right|\le\frac{C}{|x+y|^{m_1+2}}\quad n+l\le n_1+1, \tag{2.68}$$

as $x,y\to\pm\infty$ for some $m_1\ge 2$, $n_1\ge 3$, and $C=C(n_1,m_1,t)$. Then (2.68) implies that condition **IV** holds with $Q_\pm(x)=(1+|x|^{m_1+2})^{-1}$, $\tilde q_\pm(x)=0$, and $C_\pm(a)$, that exists due to continuity of functions $F_\pm(x,y,t)$ together with their derivatives. Thus Theorem 2.8 ensures the unique solvability of the time dependent GLM equations

$$K_\pm(x,y,t)+F_\pm(x,y,t)\pm\int_x^{\pm\infty}K_\pm(x,\xi,t)F_\pm(\xi,y,t)d\xi=0,\quad \pm y>\pm x, \tag{2.69}$$

and yields the function

$$q(x,t)=p_\pm(x,t)\mp 2\frac{d}{dx}K_\pm(x,x,t). \tag{2.70}$$

By construction q satisfies (cf. (2.61))

$$\pm\int_0^{\pm\infty}\left|\frac{\partial^n}{\partial x^n}\big(q(x,t)-p_\pm(x,t)\big)\right|(1+|x|^{m_1})dx<\infty,\quad 0\le n\le n_1, \tag{2.71}$$

and, as in [19], one concludes that (2.68) also implies differentiability with respect to t such that

$$\pm\int_0^{\pm\infty}\left|\frac{\partial}{\partial t}\big(q(x,t)-p_\pm(x,t)\big)\right|(1+|x|^{m_1})dx<\infty. \tag{2.72}$$

Moreover, by following the arguments in Section 6 of [19] verbatim (see, in particular, Lemma 6.3 and Corollary 2.3) one establishes that $q(x,t)$ solves the associated initial-value problem of the KdV equation. Thus, to prove Theorems 1.1, it is sufficient to prove the inequality (2.68) with $m_1=\lfloor\frac{m_0}{2}\rfloor-2$, $n_1=n_0-m_0-2$.

3. Proof of the main result

To obtain (2.68) we follow the approach, developed in [19]. First of all, recall that the functions $F_\pm(x,y,t)$ are given by

$$\begin{aligned}F_\pm(x,y,t)=&\frac{1}{2\pi\mathrm{i}}\oint_{\sigma_\pm}R_\pm(\lambda,0)\hat\psi_\pm(\lambda,x,t)\hat\psi_\pm(\lambda,y,t)g_\pm(\lambda,0)d\lambda+\\&+\frac{1}{2\pi\mathrm{i}}\int_{\sigma_\mp^{(1),\mathrm{u}}}|T_\mp(\lambda,0)|^2\hat\psi_\pm(\lambda,x,t)\hat\psi_\pm(\lambda,y,t)g_\mp(\lambda,0)d\lambda\\&+\sum_{k=1}^p(\gamma_k^\pm(0))^2\breve\psi_\pm(\lambda_k,x,t)\breve\psi_\pm(\lambda_k,y,t),\end{aligned} \tag{3.1}$$

where $\hat\psi_\pm(\lambda,x,t)$ are defined by (2.20) and we have set

$$\breve\psi_\pm(\lambda,x,t):=\delta_\pm(\lambda,0)\hat\psi_\pm(\lambda,x,t). \tag{3.2}$$

Furthermore, recall that the functions $\hat\psi_\pm(\lambda,x,t)$ inherit their singularities from $\psi_\pm(\lambda,x,0)$; that is, they have simple poles on $M_\pm(0)$ and square-root singularities $\hat M_\pm(0)$. Consequently, the functions (3.2) are bounded and smooth in small vicinities of the points λ_k. Moreover, all integrands in (3.1) have only integrable singularities (cf. [10, Sect. 5]) and thus all three summands in (3.1) are well defined. Our aim is to study the decay of $F_\pm(x,y,t)$ as x, y tend to $\pm\infty$, respectively.

First of all, we observe, that the third summand in (3.1) (corresponding to the discrete spectrum) together with all its derivatives decays exponentially as $x+y\to\pm\infty$. Therefore, it satisfies (2.68) for all natural m_1 and n_1. In the second summand, $\hat\psi_\pm(\lambda,x,t)\hat\psi_\pm(\lambda,y,t)$ together with all derivatives decays exponentially with respect to $(x+y)\to\pm\infty$ for $\lambda\notin\sigma_\pm$. Hence we have to estimate this summand only in small vicinities of the points $\sigma_\pm\cap\sigma_\mp^{(1)}$.

Our strategy is as follows. In both integrals of (3.1) we make a change of variables from λ to the quasimomentum variables $\theta_\pm$ and use (2.11) to represent the integrands as $\mathrm{e}^{\pm\theta_\pm(x+y)}\rho_\pm(\lambda(\theta_\pm),x,y,t)$, where $\rho_\pm$ together with their derivatives are smooth and uniformly bounded with respect to $x,y\in\mathbb{R}$. Moreover, since these functions are differentiable with respect to $\theta_\pm$ (and also bounded with respect to x and y), we will integrate by parts both integrals in (3.1) as many times as possible and then prove that the boundary terms either cancel or vanish.

To investigate the validity of integration by parts for the first summand in (3.1) we use (2.11)–(2.13) to represent the first summand as

$$\begin{aligned} F_{\pm,R}(x,y,t) &:= 2\,\mathrm{Re}\int_{\sigma_\pm^{\mathrm{u}}} R_\pm(\lambda,t)\psi_\pm(\lambda,x,t)\psi_\pm(\lambda,y,t)\frac{g_\pm(\lambda,t)}{2\pi\mathrm{i}}d\lambda \\ &= \mathrm{Re}\int_0^\infty \mathrm{e}^{\pm\mathrm{i}(x+y)\theta_\pm}\rho_\pm(\theta_\pm,x,y,t)d\theta_\pm, \end{aligned} \tag{3.3}$$

where

$$\rho_\pm(\theta_\pm,x,y,t) := \frac{1}{2\pi}R_\pm(\lambda,0)u_\pm(\lambda,x,t)u_\pm(\lambda,y,t)\mathrm{e}^{2\alpha_\pm(\lambda,t)}\prod_{j=1}^{r_\pm}\frac{\lambda-\mu_j^\pm}{\lambda-\zeta_j^\pm}, \tag{3.4}$$

with $\lambda=\lambda(\theta_\pm)$. Since the integrand in (3.3) is not continuous at $\theta_\pm(E_{2k+1}^\pm)=\theta_\pm(E_{2k+2}^\pm)$, we regard this integral as

$$F_{\pm,R}(x,y,t) = \mathrm{Re}\sum_{k=0}^{r_\pm+1}\int_{\theta_\pm(E_{2k}^\pm)}^{\theta_\pm(E_{2k+1}^\pm)} \mathrm{e}^{\pm\mathrm{i}(x+y)\theta}\rho_\pm(\theta,x,y,t)d\theta, \tag{3.5}$$

where we have set

$$E_{2r_\pm+1}^\pm = E_{2r_\pm+2}^\pm = \tilde{E} > \max\{E_{2r_+}^+, E_{2r_-}^-\},$$

and $E_{2r_\pm+3}^\pm=+\infty$ for notational convenience.

The boundary terms arising from integration by parts (except for the last one, corresponding to $+\infty$) become

$$\mathrm{Re}\lim_{\lambda\to E}\frac{\mathrm{e}^{\pm\mathrm{i}\theta_\pm(E)(x+y)}\frac{\partial^s\rho_\pm(\theta_\pm,x,y,t)}{\partial\theta_\pm^s}}{\big(\mathrm{i}(x+y)\big)^{s+1}}, \quad E\in\partial\sigma_\pm\cup\tilde{E},\ s=0,1,\dots,m. \tag{3.6}$$

The number m of possible integrations by parts is directly related to the smoothness of $R_\pm(\lambda,0)$ and thus the to the values of m_0 and n_0. To estimate the boundary terms in (3.5) we distinguish three cases:

1) $E\in\partial\sigma_\pm\cap\partial\sigma$ (points E_1, E_2 in our example and also point E_3 for $F_{-,R}(x,y,t)$);
2) $E\in\partial\sigma_\pm\cap\mathrm{int}(\sigma_\mp)$ (the point E_5 for $F_{-,R}(x,y,t)$);
3) $E\in\partial\sigma_-^{(1)}\cap\partial\sigma_+^{(1)}$ (the point E_4).

In the first case, the boundary terms (3.6) will vanish. In the second and the third cases, however, these terms do not vanish, but we will prove, that they cancel with a corresponding terms from the second summand in (3.1). Finally, the two boundary terms stemming from our artificial boundary point $\tilde{E}$ will cancel and hence do not need to be taken into account.

The following result which takes care of 1), is an immediate consequence of the proof of [19, Lemma 6.2].

Lemma 3.1. *Let $E\in\partial\sigma_\pm\cap\partial\sigma$. Then the following limits exists and assume either real or purely imaginary values:*

$$\lim_{\lambda\to E,\,\lambda\in\sigma_\pm}\mathrm{e}^{\pm\mathrm{i}\theta_\pm(E)(x+y)}\frac{\partial^s}{\partial\theta_\pm^s}\rho_\pm(\theta_\pm,x,y,t)\in\mathrm{i}^s\mathbb{R}, \tag{3.7}$$

for $s=0,\dots,m_0-1$ if $\hat{W}(E)\neq0$ and $s=0,\dots,m_0-2$ if $\hat{W}(E)=0$.

This lemma shows, that the boundary terms (3.6) vanish at the points corresponding to case 1). Before turning to the cases 2) and 3) let us first start by discussing smoothness of the integrand $\rho_\pm(\theta, x, y, t)$ in (3.5).

Since except for $R_\pm(\lambda, 0)$, all other parts of $\rho_\pm(\theta_\pm, x, y, t)$ are smooth with respect to $\lambda \in \mathrm{int}(\sigma_\pm)$, it suffices to look at $R_\pm(\lambda, 0)$. By Lemma 2.6, **III. (a)** the latter function has m_0 derivatives with respect to λ (and consequently also with respect to $\theta_\pm$) as long as we stay in the interior of $\sigma_\pm$ and away from boundary points of $\sigma_\mp$. Hence no such points pose any problems; the only problematic points are those in $\partial\sigma_\mp \cap \mathrm{int}(\sigma_\pm)$ (the point E_5 in our example for $F_{+,R}(x, y, t)$). Hence we will address this issue first.

Let $E \in \partial\sigma_\mp \cap \mathrm{int}(\sigma_\pm)$ be such a point. As already pointed out, only $R_\pm(\lambda, 0)$ matters and by Lemma 2.6, **III. (a)** we can write it locally as a smooth function of $\sqrt{\lambda - E}$. Thus we obtain

$$\frac{\partial^s \rho_\pm(\theta_\pm, x, y, t)}{\partial\theta_\pm^s} = O\left(\frac{1}{\sqrt{(\lambda - E)^{2s-1}}}\right). \tag{3.8}$$

Since this singularity is non-integrable for $s \geq 2$, integration by parts is not an option near such points. Hence we ill split off the leading behavior near such a point. The leading term near each such point can be computed explicitly and the remainder can be handled by integration by parts.

Since the last interval $(\tilde{E}, \infty)$ does not contain such points we can restrict our attention to finite intervals. Moreover, for notational convenience we will restrict ourselves to the case of $F_{+,R}$.

Abbreviate $\theta = \theta_+$ and denote by

$$E_i \in \partial\sigma_- \cap \left(E_{2j}^+, E_{2j+1}^+\right), \quad i = 1, \dots, N,$$

our *bad* points. Let $\varepsilon > 0$ and introduce the cutoff functions

$$B_i(\theta) := B(\frac{\theta - \theta(E_i)}{\varepsilon}), \quad i = 1, \dots, N, \tag{3.9}$$

where

$$B(\xi) = \begin{cases} \mathrm{e}^{-\xi^2}\left(1 - \xi^{2m_0}\right)^{m_0}, & \text{for } |\xi| \leq 1, \\ 0, & \text{for } |\xi| \geq 1. \end{cases} \tag{3.10}$$

We choose $\varepsilon > 0$ so small that the supports of the functions $B_i(\theta)$ neither intersect nor contain small vicinities of the points $\theta(E_{2j}^+)$ and $\theta(E_{2j+1}^+)$. Moreover, we have

$$\begin{aligned} \frac{d^s B_i}{d\theta^s}(\theta(E_i) \pm \varepsilon) = 0,, \ s = 0, \dots, m_0 - 1, \\ \frac{d^s B_i}{d\theta^s}(\theta(E_i)) = 0, \ s = 1, \dots, 2m_0 + 1. \end{aligned} \tag{3.11}$$

Now we can rewrite the j-th summand in (3.5) (except for the last one) as

$$\int_{\theta(E_{2j}^+)}^{\theta(E_{2j+1}^+)} \mathrm{e}^{\mathrm{i}(x+y)\theta}\rho_+(\theta,x,y,t)d\theta =$$
$$= \int_{\theta(E_{2j}^+)}^{\theta(E_{2j+1}^+)} \mathrm{e}^{\mathrm{i}(x+y)\theta}\left(1-\sum_{i=1}^N B_i(\theta)\right)\rho_+(\theta,x,y,t)d\theta +$$
$$+ \sum_{i=1}^N \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}(x+y)\theta} B_i(\theta)\rho_+(\theta,x,y,t)d\theta.$$

Because of (3.11) the first term can be integrated by parts m_0 times and thus is covered by Lemma 3.1. For the second term, we switch to the local variable $z=\sqrt{\theta-\theta(E_i)}$ and use a Taylor expansion for the integrand,

$$\rho_+(\theta,x,y,t) = \rho_0^{(i)}(x,y,t) + \rho_1^{(i)}(x,y,t)z + \cdots + \rho_{m_0-1}^{(i)}(x,y,t)z^{m_0-2} + \beta_i(\theta),$$

where $\beta_i(\theta) = O\left(z^{m_0-1}\right)$ has $\lfloor \frac{m_0}{2} \rfloor$ integrable derivatives with respect to θ in a small vicinity of the point $\theta(E_i)$. By construction

$$\frac{\partial^s (B_i\beta_i)}{\partial\theta^s}(\theta(E_i)\pm\varepsilon) = 0, \quad s=0,\dots,\lfloor \frac{m_0}{2} \rfloor,$$

and thus

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}(x+y)\theta} B_i(\theta)\beta_i(\theta)d\theta = O\left((x+y)^{-\lfloor \frac{m_0}{2}\rfloor}\right). \tag{3.12}$$

To compute the remaining terms, observe that

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}(x+y)\theta} B_i(\theta)\left(\sqrt{\theta-\theta(E_i)}\right)^{\nu} d\theta =$$
$$= (\varepsilon)^{\nu/2+1}\mathrm{e}^{\mathrm{i}(x+y)\theta(E_i)} \int_{-1}^{1} \mathrm{e}^{-\zeta^2+\mathrm{i}\varepsilon(x+y)\zeta}\left(1-\zeta^{2m_0}\right)^{m_0}\zeta^{\nu/2}d\zeta,$$

and note that we can extend the integral over the interval $(1,1)$ to the interval$(-\infty,\infty)$, since

$$\int_{\pm 1}^{\pm\infty} \mathrm{e}^{-\zeta^2+\mathrm{i}\varepsilon(x+y)\zeta}\left(1-\zeta^{2m_0}\right)^{m_0}\zeta^{\nu/2}d\zeta = O\big((x+y)^{-m_0-1}\big).$$

Now we simply expand

$$\left(1-\zeta^{2m_0}\right)^{m_0} = 1 - m_0\zeta^{2m_0} + \cdots + (-1)^{m_0}\zeta^{2m_0^2}$$

and evaluate the integral by invoking the integral representation [33, 9.241][3] for the parabolic cylinder functions $\mathcal{D}_\kappa(z)$ (cf. [33], [54]). This gives

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\varepsilon(x+y)\zeta}\mathrm{e}^{-\zeta^2}\zeta^\kappa d\zeta =$$
$$= (-\mathrm{i})^\kappa 2^{-\kappa/2}\sqrt{\pi}\exp\left(-\frac{\varepsilon^2(x+y)^2}{8}\right)\mathcal{D}_\kappa\left(\frac{\varepsilon(x+y)}{\sqrt{2}}\right), \quad \mathrm{Re}(\kappa) > -1. \tag{3.13}$$

Since the parabolic cylinder functions have the following expansion [33, 9.246 1],

$$\mathcal{D}_\kappa(z) \sim z^\kappa \mathrm{e}^{-\frac{z^2}{4}}\left(1-\frac{\kappa(\kappa-1)}{2z^2}+\cdots\right), \quad |\arg(z)| < \frac{3\pi}{4},$$

[3]It also follows from 3.462 3, but this formula contains a sign error.

for large z, the integral (3.13) decays exponentially as $(x+y)\to\infty$ for any $\kappa>0$. Combining these estimates with Lemma 3.1 we obtain the following

Lemma 3.2. *Let $E_{2j}^{\pm}, E_{2j+1}^{\pm}\in\partial\sigma_\pm\cap\partial\sigma$. Then*

$$\frac{\partial^{n+l}}{\partial x^n\partial y^l}\operatorname{Re}\int_{\theta_\pm(E_{2j}^{\pm})}^{\theta_\pm(E_{2j+1}^{\pm})}\mathrm{e}^{\pm\mathrm{i}(x+y)\theta}\rho_\pm(\theta,x,y,t)d\theta=O\Big((x+y)^{-\lfloor\frac{m_0}{2}\rfloor}\Big)\tag{3.14}$$

as $x,y\to\pm\infty$ for all fixed $n,l=0,1,\dots$.

Note, that the condition $E_{2j}^{\pm},E_{2j+1}^{\pm}\in\partial\sigma_\pm\cap\partial\sigma$ is only used to take care of the boundary terms obtained from integration by parts and can hence be replaced with any other condition which takes care of these terms.

Now we come to case 2) and study the behavior of the boundary terms at the points $E\in\partial\sigma_\pm\cap\operatorname{int}\sigma_\mp$. In this case formula (3.7) remains valid only for $s=0$, so we need to take the second summand in (3.1) into account.

For notational convenience we consider only the $+$ case and assume, without loss of generality, that $E=E_{2j}^{+}$. In this case, $\sigma^{(2)}$ is located to the right of E and $\sigma_-^{(1)}$ to the left. Moreover, without loss of generality, we assume that the other boundary terms are already covered by the previous considerations so that we do not have to worry about them.

Choose $\varepsilon>0$ so small that

$$[\lambda(\theta_+(E)+\mathrm{i}\varepsilon),E]\subset\Big((\xi_j^+,E]\cap\sigma_-^{(1)}\Big),\quad (E,\lambda(\theta_+(E+\varepsilon))]\subset\operatorname{int}\sigma^{(2)}.$$

Introduce in these two small intervals the two new (positive) variables

$$h:=\frac{\theta_+-\theta_+(E)}{\mathrm{i}},\quad k:=\theta_+-\theta_+(E).\tag{3.15}$$

We compare the boundary terms at the point E for the two integrals:

(3.16)

$$\operatorname{Re}\int_{\theta(E)}^{\theta(E+\varepsilon)}\mathrm{e}^{\mathrm{i}(x+y)\theta_+}\rho_+(\theta_+,x,y,t)d\theta_+=\operatorname{Re}\int_0^\varepsilon R(k)\Psi(\lambda(k),x,y,t)\mathrm{e}^{\mathrm{i}k(x+y)}dk$$

and

$$\int_{\lambda(\theta(E)+\mathrm{i}\varepsilon)}^{E}|T_-(\lambda,0)|^2\hat\psi_+(\lambda,x,t)\hat\psi_+(\lambda,y,t)\frac{g_-(\lambda,0)}{2\pi\mathrm{i}}d\lambda=$$

$$=\int_\varepsilon^0 P(h)\Psi(\lambda(h),x,y,t)\mathrm{e}^{-h(x+y)}dh,\tag{3.17}$$

with

$$\Psi(\lambda,x,y,t)=\frac{\mathrm{e}^{\mathrm{i}\theta(E)(x+y)}}{2\pi}\prod_{j=1}^{r_+}\frac{\lambda-\mu_j^+}{\lambda-\zeta_j^+}\mathrm{e}^{-\mathrm{i}(x+y)\theta}\hat\psi_+(\lambda,x,t)\hat\psi_+(\lambda,y,t),\tag{3.18}$$

and

$$R(k):=R_+(\lambda,0),\tag{3.19}$$

$$\begin{aligned}P(h):=&\frac{-\mathrm{i}}{2g_+(\lambda,0)g_-(\lambda,0)|W(\lambda,0)|^2}\\=&\frac{-\mathrm{i}}{2g_+(\lambda,0)g_-(\lambda,0)\,\mathsf{W}_0(\phi_-,\phi_+)\,\mathsf{W}_0(\overline{\phi_-},\phi_+)},\end{aligned}\tag{3.20}$$

where $\mathsf{W}_0(\cdot,\cdot) = \mathsf{W}(\cdot,\cdot)|_{t=0}$. Equation (3.20) was obtained by using (2.51) together with the fact that $\overline{g_-(\lambda,0)} = -g_-(\lambda,0)$ if $\lambda \in \sigma_-$.

Integrating (3.16) and (3.17) by parts with respect to k and h, respectively, gives

$$\begin{aligned}\int_\varepsilon^0 P(h)\Psi(\lambda(h))\mathrm{e}^{-h(x-y)}dh &= -\sum_{j=0}^{m-1}\frac{1}{(x-y)^{j+1}}\frac{\partial^j(P\Psi)}{\partial h^j}(0)\\ &+\frac{1}{(x-y)^m}\int_\varepsilon^0\frac{\partial^m(P\Psi)}{\partial h^m}\mathrm{e}^{-h(x-y)}dh + O(\mathrm{e}^{-\varepsilon(x-y)}),\end{aligned} \tag{3.21}$$

$$\begin{aligned}\mathrm{Re}\int_0^\varepsilon R(k)\Psi(\lambda(k))\mathrm{e}^{\mathrm{i}k(x-y)}dk &= \mathrm{Re}\sum_{j=0}^{m-1}\frac{1}{(-\mathrm{i}(x-y))^{j+1}}\frac{\partial^j(R\Psi)}{\partial k^j}(0)\\ &+\mathrm{Re}\,\frac{1}{(-\mathrm{i}(x-y))^m}\int_0^\varepsilon\frac{\partial^m(R\Psi)}{\partial k^m}\mathrm{e}^{\mathrm{i}k(x-y)}dk.\end{aligned} \tag{3.22}$$

For the boundary terms to cancel each other we need

$$\lim_{k\to 0}\mathrm{Re}\left(\mathrm{i}^{j+1}\frac{\partial^j(R\Psi)}{\partial k^j}(k)\right) = \lim_{h\to 0}\frac{\partial^j(P\Psi)}{\partial h^j}(h), \quad j=0,\dots,m_0-1, \tag{3.23}$$

where the left limit is taken from the side of the spectrum of multiplicity two and the right limit is taken from the side of the spectrum of multiplicity one. Since Ψ is smooth to any degree with respect to k and h near E,

$$\lim_{k\to 0}\left(\mathrm{i}^j\frac{\partial^j\Psi}{\partial k^j}(k)\right) = \lim_{h\to 0}\frac{\partial^j\Psi}{\partial h^j}(h), \quad j=0,\dots \tag{3.24}$$

We observe that to prove (3.23), it suffices to prove the following lemma.

Lemma 3.3. *Let $h, k, P(h), R(k)$ be defined by* (3.15), (3.19), *and* (3.20). *If $E \in \partial\sigma_\pm \cap \mathrm{int}(\sigma_\mp)$, then*

$$\lim_{k\to 0}\mathrm{Re}\left(\mathrm{i}^{j+1}\frac{d^jR(k)}{dk^j}\right) = \lim_{h\to 0}\frac{d^jP(h)}{dh^j}, \qquad j=0,\dots,m_0-1. \tag{3.25}$$

Proof. To prove this formula, recall that $\phi_-(\cdot,x)$, $\overline{\phi_-(\cdot,x)} \in C^{m_0}(E-\varepsilon,E+\varepsilon)$ (and similarly for the x derivative) since $E \in \mathrm{int}\,\sigma_-$. Therefore their derivatives with respect to $\sqrt{\lambda - E}$ are smooth in a vicinity of $k=0$. Without loss of generality, we suppose[4], that $E \neq \mu_j^+$, that is, the function $\phi_+(\lambda,x,0)$ as well as the functions $g_+(\lambda,0)$ and $g_-(\lambda,0)$ (see (2.6)) are also smooth with respect to $\sqrt{\lambda-E}$. For $\lambda > E$ introduce the function

$$\tilde P(k) := \frac{-\mathrm{i}}{2g_+(\lambda,0)g_-(\lambda,0)\,\mathsf{W}_0(\phi_-,\phi_+)\,\mathsf{W}_0(\overline{\phi_-},\phi_+)}.$$

Then

$$\lim_{k\to+0}\mathrm{i}^s\frac{d^s\tilde P(k)}{dk^s} = \lim_{h\to+0}\frac{d^sP}{dh^s}. \tag{3.26}$$

From $g_\pm(\lambda,0)^{-1} = \pm\mathsf{W}_0(\phi_\pm,\overline{\phi_\pm})$ we see that

$$\tilde P(k) = \frac{\mathrm{i}\;\mathsf{W}_0(\phi_-,\overline{\phi_-})\,\mathsf{W}_0(\phi_+,\overline{\phi_+})}{2\,\mathsf{W}_0(\phi_-,\phi_+)\,\mathsf{W}_0(\overline{\phi_-},\phi_+)}. \tag{3.27}$$

[4]Otherwise replace $\phi_+(\lambda,x,0)$ by $\phi_{+,E}(\lambda,x,0)$ and $g_+(\lambda,0)$ by $g_{+,E}(\lambda,0)$ (cf. (2.42)) in the subsequent considerations.

Substituting

$$\phi_+(\lambda,x,0) = \frac{\mathsf{W}_0(\phi_+,\overline{\phi_-})}{\mathsf{W}_0(\phi_-,\overline{\phi_-})}\phi_-(\lambda,x,0) - \frac{\mathsf{W}_0(\phi_+,\phi_-)}{\mathsf{W}_0(\phi_-,\overline{\phi_-})}\overline{\phi_-(\lambda,x,0)}$$

into the numerator of (3.27) gives

$$\tilde{P}(k) = \frac{\mathrm{i}}{2}\left(-\frac{\mathsf{W}_0(\phi_-,\overline{\phi_+})}{\mathsf{W}_0(\phi_-,\phi_+)} + \frac{\mathsf{W}_0(\overline{\phi_-},\overline{\phi_+})}{\mathsf{W}_0(\overline{\phi_-},\phi_+)}\right).$$

Introducing the abbreviations

$$W(k) := \mathsf{W}_0(\phi_-,\phi_+), \quad V(k) := \mathsf{W}_0(\phi_-,\overline{\phi_+}). \tag{3.28}$$

we thus have

$$R(k) = -\frac{V(k)}{W(k)}, \qquad \tilde{P}(k) = \frac{\mathrm{i}}{2}\left(-\frac{V(k)}{W(k)} + \frac{\overline{W(k)}}{\overline{V(k)}}\right). \tag{3.29}$$

Next, for all x and small $\varepsilon > 0$, we have

$$\phi_-(\lambda,x,0), \frac{\partial}{\partial x}\phi_-(\lambda,x,0) \in C^{m_0}(E-\varepsilon,E+\varepsilon).$$

Therefore, according to Lemma 2.5 **(ii)**, near $k=0$, for positive k, we have the representation

$$W(k) - V(k) = \mathrm{i}kf_1(k^2), \quad W(k) + V(k) = f_2(k^2),$$

where $f_{1,2}(\cdot) \in C^{m_0-1}([0,\varepsilon_1))$. Differentiating these relations gives

$$\lim_{k\to+0}\frac{\partial^s}{\partial k^s}V(k) = (-1)^s\lim_{k\to+0}\frac{\partial^s}{\partial k^s}W(k), \qquad s = 0,\dots,m_0-1, \tag{3.30}$$

and hence we see that $V(k) = W_{m_0-1}(-k) + o(k^{m_0-1})$, where $W_{m_0-1}(k)$ is the Taylor polynomial of degree m_0-1 for $W(k)$. Now recall

$$\mathsf{W}_0(\phi_-,\phi_+)(E) \neq 0,$$

which implies that $R^{-1}(k) = R_{m_0-1}(-k) + o(k^{m_0-1})$, where $R_{m_0-1}(k)$ is the Taylor polynomial of degree m_0-1 for $R(k)$. Thus, we finally obtain

$$\tilde{P}(k) = \frac{\mathrm{i}}{2}\left(R_{m_0-1}(k) - \overline{R_{m_0-1}(-k)}\right) + o(k^{m_0-1}), \tag{3.31}$$

from which (3.25) follows. □

Lemma 3.3 settles case 2). Case 3) will follow from the next lemma.

Lemma 3.4. *Let $h,k,P(h),R(k)$ be defined by* (3.15), (3.19), *and* (3.20). *Then, if $E \in \partial\sigma_-^{(1)} \cap \partial\sigma_+^{(1)}$,*

$$\lim_{k\to0}\mathrm{Re}\left(\mathrm{i}^{j+1}\frac{d^jR(k)}{dk^j}\right) = \lim_{h\to0}\frac{d^jP(h)}{dh^j}, \qquad j = 0,\dots,m_0-1. \tag{3.32}$$

Proof. Note that now we cannot proceed as in case 2) since now we no longer have spectrum of multiplicity two to the right of E. In particular, we cannot use $g_-(\lambda,0)^{-1} = -\mathsf{W}_0(\phi_-,\overline{\phi_-})$ for $\lambda > E$, we do not have the scattering relations at our disposal, and $\phi_-(\lambda) \notin C^{m_0}(E-\varepsilon,E+\varepsilon)$. Hence we need a different strategy.

Let[5] $E \notin \hat{M}_-(0) \cup \hat{M}_+(0)$. Consider $\phi_\pm(\lambda, x, 0)$ and note that for sufficiently small ε, we can write (see Lemma 2.5)

$$\phi_-(\lambda, x, 0) = \begin{cases} f_1^-(h^2, x) + \mathrm{i}h f_2^-(h^2, x) + o(h^{m_0-1}), & E - \varepsilon < \lambda \le E, \\ f_1^-(-k^2, x) + k f_2^-(-k^2, x) + o(k^{m_0-1}), & E + \varepsilon > \lambda \ge E, \end{cases} \tag{3.33}$$

where $f_1^-(z, x), f_2^-(z, x)$ are real-valued functions which are polynomials of degree $m_0 - 1$ with respect to z and differentiable with respect to x. Next, define

$$\breve{\phi}_-(\lambda, x, 0) = \begin{cases} f_1^-(h^2, x) - \mathrm{i}h f_2^-(h^2, x), & \lambda \le E, \\ f_1^-(-k^2, x) - k f_2^-(-k^2, x), & \lambda \ge E, \end{cases} \tag{3.34}$$

and note that $\overline{\phi_-(\lambda, x, 0)} = \breve{\phi}_-(\lambda, x, 0) + o(h^{m_0-1})$ for $E - \varepsilon < \lambda \le E$.

Similarly, we write

$$\phi_+(\lambda, x, 0) = \begin{cases} f_1^+(h^2, x) + h f_2^+(h^2, x) + o(h^{m_0-1}), & E - \varepsilon < \lambda \le E, \\ f_1^+(-k^2, x) - \mathrm{i}k f_2^+(-k^2, x) + o(k^{m_0-1}), & E + \varepsilon > \lambda \ge E, \end{cases} \tag{3.35}$$

and define

$$\breve{\phi}_+(\lambda, x, 0) = \begin{cases} f_1^+(h^2, x) - h f_2^+(h^2, x), & \lambda \le E, \\ f_1^+(-k^2, x) + \mathrm{i}k f_2^+(-k^2, x), & \lambda \ge E, \end{cases} \tag{3.36}$$

implying $\overline{\phi_+(\lambda, x, 0)} = \breve{\phi}_+(\lambda, x, 0) + o(k^{m_0-1})$ for $E + \varepsilon > \lambda \ge E$. In particular, note that

$$\mathrm{i}^j \frac{\partial^j}{\partial h^j} \breve{\phi}_\pm(\lambda, x, 0) = \frac{\partial^j}{\partial k^j} \breve{\phi}_\pm(\lambda, x, 0), \qquad \lambda = E, \quad 0 \le j \le m_0 - 1.$$

Moreover,

$$g_\pm(\lambda, 0) = \pm \mathsf{W}(\phi_\pm(\lambda, ., 0), \breve{\phi}_\pm(\lambda, ., 0)) + o\big((\lambda - E)^{(m_0-1)/2}\big), \quad \lambda \in (E - \varepsilon, E + \varepsilon). \tag{3.37}$$

While the above Wronskian depends on x ($\breve{\phi}_\pm$ do not solve (2.28) in general), the leading order is independent of x. Here and in all following Wronskians below, we set $x = 0$ (and of course $t = 0$). Now consider (cf. (3.20))

$$P(\lambda) = \frac{-\mathrm{i}}{2 g_+(\lambda, 0) g_-(\lambda, 0) \, \mathsf{W}_0(\phi_-, \phi_+) \, \mathsf{W}_0(\overline{\phi_-}, \phi_+)}, \quad \lambda < E,$$

and set

$$\tilde{P}(\lambda) = \frac{\mathrm{i}}{2} \frac{\mathsf{W}(\phi_-, \breve{\phi}_-) \, \mathsf{W}(\phi_+, \breve{\phi}_+)}{\mathsf{W}(\phi_-, \phi_+) \, \mathsf{W}(\breve{\phi}_-, \phi_+)}, \quad \lambda \in (E - \varepsilon, E + \varepsilon), \tag{3.38}$$

implying

$$P(\lambda) = \tilde{P}(\lambda) + o\big((\lambda - E)^{(m_0-2)/2}\big), \quad E - \varepsilon < \lambda \le E$$

In particular,[6]

$$\frac{\partial^j}{\partial h^j} P(\lambda) = \mathrm{i}^j \frac{\partial^j}{\partial k^j} \tilde{P}(\lambda), \qquad \lambda = E, \quad 0 \le j \le m_0 - 2. \tag{3.39}$$

[5] Again, otherwise replace $\phi_\pm(\lambda, x, 0)$ by $\phi_{\pm,E}(\lambda, x, 0)$ and $g_\pm(\lambda)$ by $g_{\pm,E}(\lambda)$ (cf. Lemma 2.5) in the subsequent considerations.

[6] Note that if $\mathsf{W}(\phi_+, \phi_-)(E) = 0$, we loose one derivative, in which case we have $\mathsf{W}(\phi_+, \phi_-) = Ck(1 + o(1))$ by Lemma 2.6 **II. (b)**.

Using the Plücker identity

$$\mathsf{W}(f_1,f_2)\,\mathsf{W}(f_3,f_4)+\mathsf{W}(f_1,f_3)\,\mathsf{W}(f_4,f_2)+\mathsf{W}(f_1,f_4)\,\mathsf{W}(f_2,f_3)=0$$

with $f_1=\phi_-$, $f_2=\breve{\phi}_-$, $f_3=\phi_+$, and $f_4=\breve{\phi}_+$ we can rewrite $\tilde{P}(\lambda)$ as

$$\tilde{P}(\lambda)=\frac{\mathrm{i}}{2}\left(-\frac{V(k)}{W(k)}+\frac{\breve{W}(k)}{\breve{V}(k)}\right),$$

where

$$W(k):=\mathsf{W}(\phi_-,\phi_+),\quad V(k):=\mathsf{W}(\phi_-,\breve{\phi}_+),$$

and

$$\breve{W}(k):=\mathsf{W}(\breve{\phi}_-,\breve{\phi}_+),\quad \breve{V}(k):=\mathsf{W}(\breve{\phi}_-,\phi_+).$$

Moreover, using (3.33)–(3.36), one can verify that

$$\frac{\breve{W}(k)}{\breve{V}(k)}=\overline{\left(\frac{V(-k)}{W(-k)}\right)}+o(k^{m_0-1}).$$

Now, since $V(k)=\mathsf{W}_0(\phi_-,\overline{\phi_+})+o(k^{m_0-1})$, we obtain

$$R(k)=-\frac{V(k)}{W(k)}+o(k^{m_0-2}). \tag{3.40}$$

This implies

$$\mathrm{Re}\left(\mathrm{i}^{j+1}\frac{\partial^j}{dk^j}R(0)\right)=\mathrm{i}^j\frac{\partial^j}{\partial k^j}\tilde{P}(0)=\frac{\partial^j P}{\partial h^j}(0),\qquad j=0,\dots,m_0-2, \tag{3.41}$$

and we are done. □

Finally we discuss the possibility of integrating the last (unbounded) integrand of (3.5) by parts. More precisely, we discuss the boundary terms corresponding to the point $E^+_{2r_++3}=+\infty$ (again the considerations are the same for the $+$ and $-$ cases, and we study only the $+$ case). To begin, we recall the well-known asymptotic expressions

$$m_+(\lambda)=\mathrm{i}\sqrt{\lambda}(1+o(1)),\quad \theta(\lambda)=\sqrt{\lambda}(1+o(1))$$

as $\lambda\to\infty$. Moreover, (recall $\alpha_+(\lambda,t)=4\mathrm{i}(\sqrt{\lambda})^3t(1+o(1))$) we also have

$$\frac{\partial^s}{\partial\theta^s}u_+(\lambda,x,t)=O(1),\qquad \frac{\partial^s}{\partial\theta^s}\mathrm{e}^{\alpha_+(\lambda,t)}=O(t\theta^{2s}).$$

As in the previous cases, the only interesting part is the reflection coefficient $R_+(\lambda,0)$ for which we have

$$\frac{\partial^s}{\partial\theta^s}R_+(\lambda,0)=O(\theta^{-n_0-1}),\qquad s=0,\dots,m_0, \tag{3.42}$$

as $\lambda\to\infty$ by Lemma 2.6 **III. (a)**. Hence we conclude that

$$\frac{\partial^s}{\partial\theta_+^s}\rho_+(\lambda(\theta),x,y,t)=O(\theta^{2s-n_0-1}),\quad \text{as } \lambda\to\infty,$$

uniformly with respect to $x,y\in\mathbb{R}$ and $t\in[0,T]$ for any $T>0$. As a consequence we can perform $m\le\lfloor\frac{n_0}{2}\rfloor$ partial integrations such that the boundary terms at ∞ vanish.

In summary, supposing (1.2), the maximum number of integration by parts is determined by Lemma 3.2; that is it is determined by the points $\operatorname{int}\sigma_+\cap\partial\sigma_-$, and is given by $\lfloor\frac{m_0}{2}\rfloor$. So, excluding case 3), we have an almost complete picture. However, up to this point, we have only looked at $F_{R,+}(x,y,t)$ and have not considered derivatives with respect to x,y,t. Fortunately, since $R_+(\lambda,0)$ is evidently independent of these variables and all other terms of the integrand (3.5) (except for the last summand in (3.5)) can be differentiated as often as we please, these derivatives do not affect our analysis. Moreover, for this last summand, one has only to take into account that a partial derivative with respect to x or y adds $O(\theta_+)$ (from $\mathrm{e}^{\mathrm{i}\theta_+(x+y)}$) and a partial derivative with respect to t adds $O(\theta_+^3)$ (from $\mathrm{e}^{\alpha_+(\lambda,t)}$). Thus we obtain the following result.

Lemma 3.5. *Let $q(x)$ satisfy (1.2). Then*

$$F_+(x,y,t)=\frac{1}{(x+y)^{\lfloor\frac{m_0}{2}\rfloor}}\left(H(x,y,t)+\int_{\tilde E}^{+\infty}\mathrm{e}^{\mathrm{i}\sqrt{\lambda}(x+y)+4\mathrm{i}(\sqrt{\lambda})^3t}\,H_1(\lambda,x,y,t)d\lambda\right), \tag{3.43}$$

where $\tilde E>\max\{E^+_{2r_+},E^-_{2r_-},1\}$. The function $H(x,y,t)$ is smooth on the set $\mathcal{D}:=[0,+\infty)\times[0,+\infty)\times[0,T]$. All partial derivatives with respect to x,y,t of function H are bounded on $\mathcal{D}$. The function $H_1(\lambda,x,y,t)$ is bounded on λ and smooth with respect to $x,y,t\in\mathcal{D}$. Moreover,

$$\frac{\partial^{l+s+k}}{\partial x^l\partial y^s\partial t^k}\,H_1(\lambda,x,y,t)=o\left(\left(\sqrt{\lambda}\right)^{l+s+3k-n_0-2+2\lfloor\frac{m_0}{2}\rfloor}\right)\quad \text{as } \lambda\to\infty, \tag{3.44}$$

uniformly on $\mathcal{D}$.

Lemma 3.5 shows that for the integral in (3.43) and its derivatives with respect to x,y,t to converge, it is sufficient that $l+s+3k+2\lfloor\frac{m_0}{2}\rfloor-n_0<0$. A comparison to (2.68) shows that to guarantee a classical solution (three derivatives with respect to x and one with respect to t) of the KdV equation, we need at least $l+s=4,k=0$ and $l+s=1,k=1$ to hold; that is, we need m_0 and n_0 to satisfy $4+2\lfloor\frac{m_0}{2}\rfloor-n_0<0$. Since we also need $\lfloor\frac{m_0}{2}\rfloor-2\geq 2$, this yields the conditions $m_0\geq 8$ and $n_0\geq 2\lfloor\frac{m_0}{2}\rfloor+5$.

In particular, if (1.2) holds for all $m_0,n_0\in\mathbb{N}$ (Schwartz-type perturbations), then the same is true for the solution. Thus this provides a generalization of the main result from [19] without any restriction on the background spectra.

Acknowledgments. We are very grateful to F. Gesztesy, E.Ya. Khruslov, and V.A. Marchenko for helpful discussions. I.E. gratefully acknowledges the extraordinary hospitality of the Faculty of Mathematics at the University of Vienna during extended stays 2008–2009, where parts of this paper were written. G.T. gratefully acknowledges the stimulating atmosphere at the Centre for Advanced Study at the Norwegian Academy of Science and Letters in Oslo during June 2009 where parts of this paper were written as part of the international research program on Nonlinear Partial Differential Equations.

References

[1] T. Aktosun, *On the Schrödinger equation with steplike potentials*, J. Math. Phys. **40:11**, 5289–5305 (1999)

[2] V. B. Baranetskii, V. P. Kotlyarov, *Asymptotic behavior in a back front domain of the solution of the KdV equation with a “step type” initial condition*, Teoret. Mat. Fiz., **126:2**, 214–227 (2001).

[3] E. D. Belokolos, A. I. Bobenko, V. Z. Enolskii, A. R. Its, and V. B. Matveev, *Algebro Geometric Approach to Nonlinear Integrable Equations*, Springer, Berlin, 1994.

[4] R.F. Bikbaev, *Time asymptotics of the solution of the nonlinear Schrödinger equation with boundary conditions of "step-like" type*, Teoret. Mat. Fiz.,**81:1**, 3–11 (1989).

[5] R.F. Bikbaev, *Structure of a shock wave in the theory of the Korteweg-de Vries equation*, Phys. Lett. A **141:5-6**, 289–293 (1989).

[6] R.F. Bikbaev and R. A. Sharipov, *The asymptotic behavior as $t \to \infty$ of the solution of the Cauchy problem for the Korteweg-de Vries equation in a class of potentials with finite-gap behavior as $x \to \pm\infty$*, Teoret. Mat. Fiz. **78:3**, 345–356 (1989).

[7] I. N. Bondareva, *The Korteweg–de Vries equation in classes of increasing functions with prescribed asymptotic behavior as $|x| \to \infty$*, Mat. USSR Sb. **50:1**, 125–135 (1985).

[8] I. Bondareva and M. Shubin, *Increasing asymptotic solutions of the Korteweg–de Vries equation and its higher analogues*, Sov. Math. Dokl. **26:3**, 716–719 (1982).

[9] A. Boutet de Monvel and I. Egorova, *The Toda lattice with step-like initial data. Soliton asymptotics*, Inverse Problems **16:4**, 955–977 (2000).

[10] A. Boutet de Monvel, I. Egorova, and G. Teschl, *Inverse scattering theory for one-dimensional Schrödinger operators with steplike finite-gap potentials*, J. d'Analyse Math. **106:1**, 271–316, (2008).

[11] V. S. Buslaev and V. N. Fomin, *An inverse scattering problem for the one-dimensional Schrödinger equation on the entire axis*, Vestnik Leningrad. Univ. **17:1**, 56–64 (1962).

[12] A. Cohen, *Solutions of the Korteweg-de Vries equation with steplike initial profile*, Comm. Partial Differential Equations **9:8**, 751–806 (1984).

[13] A. Cohen and T. Kappeler, *Scattering and inverse scattering for steplike potentials in the Schrödinger equation*, Indiana Univ. Math. J. **34:1**, 127–180 (1985).

[14] A. Cohen and T. Kappeler, *Solutions to the Korteweg–de Vries equation with initial profile in $L_1^1(R) \cap L_N^1(R^+)$*, SIAM J. Math. Anal. **18:4**, 991–1025 (1987).

[15] B. Davies and B. Simon, *Scattering theory for systems with different spatial asymptotics on the left and right*, Comm. Math. Phys. **63**, 277–301 (1978).

[16] P. Deift, S. Kamvissis, T. Kriecherbauer, and X. Zhou, *The Toda rarefaction problem*, Comm. Pure Appl. Math. **49**, no. 1, 35–83 (1996).

[17] W. Eckhaus and A. Van Harten, *The Inverse Scattering Transformation and Solitons: An Introduction*, Math. Studies 50, North-Holland, Amsterdam, 1984.

[18] I. Egorova and G. Teschl, *On the Cauchy problem for the modified Korteweg–de Vries equation with steplike finite-gap initial data*, in Nonlinear partial differential equations and hyperbolic wave phenomena, H. Holden and K. H. Karlsen (eds), Contemp. Math. **526**, Amer. Math. Soc., Providence, (2010), pp.151–158.

[19] I. Egorova, K. Grunert, and G. Teschl, *On the Cauchy problem for the Korteweg–de Vries equation with steplike finite-gap initial data I. Schwartz-type perturbations*, Nonlinearity **22**, 1431–1457 (2009).

[20] I. Egorova, J. Michor, and G. Teschl, *Inverse scattering transform for the Toda hierarchy with quasi-periodic background*, Proc. Amer. Math. Soc. **135**, 1817–1827 (2007).

[21] V. D. Ermakova, *The inverse scattering problem on the whole axis for the Schrödinger equation with nondecreasing potential of special form*, Vestnik Khar'kov. Univ. **230**, 50–60 (1982).

[22] V. D. Ermakova, *The asymptotics of the solution of the Cauchy problem for the Korteweg-de Vries equation with nondecreasing initial data of special type*, Dokl. Akad. Nauk Ukrain. SSR Ser. A **7**, 3–6 (1982).

[23] L. D. Faddeev and L. A. Takhtajan, *Hamiltonian Methods in the Theory of Solitons*, Springer, Berlin, 1987.

[24] N. E. Firsova, *An inverse scattering problem for the perturbed Hill operator*, Mat. Zametki **18:6**, 831–843 (1975).

[25] N.E. Firsova, *A direct and inverse scattering problem for a one-dimensional perturbed Hill operator* Matem. Sborn. (N.S.) **130(172)**, no. 3, 349–385 (1986).

[26] N.E. Firsova, *The Riemann surface of a quasimomentum, and scattering theory for a perturbed Hill operator* Mathematical questions in the theory of wave propagation, 7. Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov (LOMI) **51**, 183–196, (1975).

[27] N. E. Firsova, *Solution of the Cauchy problem for the Korteweg-de Vries equation with initial data that are the sum of a periodic and a rapidly decreasing function*, Math. USSR-Sb. **63:1**, 257–265 (1989).

[28] C. S. Gardner, J. M. Green, M. D. Kruskal, R. M. Miura, *Method for solving the Korteweg-de Vries equation*, Phys. Rev. Lett., **19**, 1095–1097 (1967).

[29] F. Gesztesy, *Scattering theory for one-dimensional systems with nontrivial spatial asymptotics*, in *Schrödinger operators*, Lecture Notes in Mathematics, Vol. 1218, Springer, Berlin, 1986, E. Balslev (ed.), p. 93–122.

[30] F. Gesztesy and H. Holden, *Soliton Equations and their Algebro-Geometric Solutions. Volume I:* $(1+1)$*-Dimensional Continuous Models*, Cambridge Studies in Advanced Mathematics, Vol. **79**, Cambridge University Press, Cambridge, 2003.

[31] F. Gesztesy, R. Nowell, and W. Pötz, *One-dimensional scattering theory for quantum systems with nontrivial spatial asymptotics*, Differential Integral Equations **10**, no. 3, 521–546 (1997).

[32] F. Gesztesy, R. Ratnaseelan, and G. Teschl, *The KdV hierarchy and associated trace formulas*, in "*Proceedings of the International Conference on Applications of Operator Theory*", (eds. I. Gohberg, P. Lancaster, and P. N. Shivakumar), Oper. Theory Adv. Appl., 87, Birkhäuser, Basel, 125–163 (1996).

[33] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, edited and with a preface by A. Jeffrey and D. Zwillinger, 7th ed., Elsevier/Academic Press, Amsterdam, 2007.

[34] A. R. Its and A. F. Ustinov, *Time asymptotics of the solution of the Cauchy problem for the nonlinear Schrödinger equation with boundary conditions of finite density type*, Dokl. Akad. Nauk SSSR **291:1**, 91–95 (1986).

[35] S. Kamvissis, *On the Toda shock problem*, Physica D **65**, 242–266 (1993).

[36] S. Kamvissis and G. Teschl, *Stability of periodic soliton equations under short range perturbations*, Phys. Lett. A, **364:6**, 480–483 (2007).

[37] S. Kamvissis and G. Teschl, *Stability of the periodic Toda lattice under short range perturbations*, arXiv:0705.0346.

[38] T. Kappeler, *Solutions of the Korteweg de Vries equation with steplike initial data*, J. Differential Equations **63:3**, 306–331 (1986).

[39] T. Kappeler, P. Perry, M. Shubin and P. Topalov, *Solutions of mKdV in classes of functions unbounded at infinity*, J. Geom. Anal. **18**, 443–477 (2008).

[40] A. B. Khasanov and G. U. Urazboev, *Solution of the general KdV equation in the class of step functions*, J. Math. Sciences **136:1**, 3625–3640 (2006).

[41] E. Ya. Khruslov, *Asymptotics of the Cauchy problem solution to the KdV equation with step-like initial data* Matem.sborn. **99**, 261–281 (1976).

[42] E. Ya. Khruslov and V. P. Kotlyarov, *Soliton asymptotics of nondecreasing solutions of nonlinear completely integrable evolution equations*, Advances in Soviet Mathematics **19**, 129–181 (1994).

[43] E. Ya. Khruslov and V. P. Kotlyarov, *Time asymptotics of the solution of the Cauchy problem for the modified Korteweg- de Vries equation with nondecreasing initial data*, Dokl. Akad. Nauk Ukrain. SSR Ser. A ,**10**, 61–64 (1986).

[44] E. Ya. Khruslov and V. P.Kotlyarov *Solitons of the nonlinear Schrödinger equation, which are generated by the continuous spectrum*, Teoret. Mat. Fiz., **68:2**, 172–186 (1986).

[45] E. Ya. Khruslov and H. Stephan, *Splitting of some nonlocalized solutions of the Korteweg-de Vries equation into solitons*, Mat. Fiz. Anal. Geom. **5:1-2**, 49–67 (1998).

[46] H. Krüger and G. Teschl, *Long-time asymptotics for the periodic Toda lattice in the soliton region*, Int. Math. Res. Not. **2009**, Art. ID rnp077, 36 pp (2009).

[47] E. A. Kuznetsov and A. V. Mikhailov, *Stability of stationary waves in nonlinear weakly dispersive media*, Soviet Phys. JETP **40:5**, 855–859 (1975).

[48] V. A. Marchenko, *Sturm–Liouville Operators and Applications*, Birkhäuser, Basel, 1986.

[49] A. Mikikits-Leitner and G. Teschl, *Trace formulas for Schrödinger operators in connection with scattering theory for finite-gap backgrounds*, in Spectral Theory and Analysis, J. Janas (ed.) et al., Oper. Theory Adv. Appl. **214**, Birkhäuser, Basel (2011), pp.107–124.

[50] S. P. Novikov, S. V. Manakov, L. P. Pitaevskii, and V. E. Zakharov, *Theory of Solitons. The Inverse Scattering Method*, Springer, Berlin, 1984.

[51] V. Yu. Novokshenov, Time asymptotics for soliton equations in problems with step initial conditions, J. Math. Sci. **125:5**, 717–749 (2005).

[52] A. Rybkin, *Meromorphic solutions to the KdV equation with non-decaying initial data supported on a left half-line*, Nonlinearity **23**, 1143–1167 (2010).

[53] S. Venakides, P. Deift, and R. Oba, *The Toda shock problem*, Comm. in Pure and Applied Math. **44**, 1171–1242 (1991).

[54] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, 4th ed., Cambridge University Press, Cambridge, 1927.

B. Verkin Institute for Low Temperature Physics, 47 Lenin Avenue, 61103 Kharkiv, Ukraine

E-mail address: iraegorova@gmail.com

Faculty of Mathematics, University of Vienna, Nordbergstrasse 15, 1090 Wien, Austria, and, International Erwin Schrödinger Institute for Mathematical Physics, Boltzmanngasse 9, 1090 Wien, Austria

E-mail address: Gerald.Teschl@univie.ac.at

URL: http://www.mat.univie.ac.at/~gerald/