EFFECTIVE PRÜFER ANGLES AND RELATIVE OSCILLATION CRITERIA

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Abstract. We present a streamlined approach to relative oscillation criteria based on effective Prüfer angles adapted to the use at the edges of the essential spectrum.

Based on this we provided a new scale of oscillation criteria for general Sturm–Liouville operators which answer the question whether a perturbation inserts a finite or an infinite number of eigenvalues into an essential spectral gap. As a special case we recover and generalize the Gesztesy–Ünal criterion (which works below the spectrum and contains classical criteria by Kneser, Hartman, Hille, and Weber) and the well-known results by Rofe-Beketov including the extensions by Schmidt.

1. Introduction

In this article we want to use relative oscillation theory and apply it to obtain criteria for when an edge of an essential spectral gap is an accumulation point of eigenvalues for Sturm–Liouville operators

$$\tau = \left(-\frac{d}{dx}p\frac{d}{dx} + q\right), \quad \text{on} \quad (a,b). \tag{1.1}$$

Without loss of generality we will assume that $a \in \mathbb{R}$ is a regular endpoint and that b is limit point. Furthermore, we always assume the usual local integrability assumptions on the coefficients (see Section 2).

We will assume that H_0 is a given background operator associated with $\tau_0 = (-\frac{d}{dx}p_0\frac{d}{dx} + q_0)$ (think e.g. of a periodic operator) and that E is a boundary point of the essential spectrum of H_0 (which is not an accumulation point of eigenvalues). Then we want to know when a perturbation $\tau_1 = (-\frac{d}{dx}p_1\frac{d}{dx} + q_1)$ gives rise to an infinite number of eigenvalues accumulating at E. By relative oscillation theory, this question reduces to the question of when a given operator $\tau_1 - E$ is relatively oscillatory with respect to $\tau_0 - E$ (cf. Section 3).

In the simplest case $\tau_0 = -\frac{d^2}{dx^2}$, $E = 0$, Kneser [11] showed that the borderline case is given by ($p_1 = p_0 = 1$)

$$q_1(x) = \frac{\mu}{x^2}, \tag{1.2}$$

where the critical constant is given by $\mu_c = -\frac{1}{4}$. That is, for $\mu < \mu_c$ the perturbation is oscillatory and for $\mu > \mu_c$ it is nonoscillatory. In fact, later on Hartman [5], Hille [7], and Weber [23] gave a whole scale of criteria addressing the case $\mu = \mu_c$.

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Recently this result was further generalized by Gesztesy and Ünal [4], who showed that for Sturm–Liouville operators (with $p_1 = p_0$) the borderline case for $\tau_0 - E$, $E = \inf\sigma(H_0)$, is given by

$$q_1(x) = q_0(x) + \frac{\mu}{p_0(x)u_0(x)^2 v_0(x)^2}, \tag{1.3}$$

where the critical constant is again $\mu_c = -\frac{1}{4}$. Here u_0 is a minimal (also principal) positive solution of $\tau_0 u = 0$ and v_0 is a second linearly independent solution with Wronskian $W(u_0, v_0) = 1$. Since for $p_0 = 1$, $q_0 = 0$ we have $u_0 = 1$ and $v_0 = x$, this result contains Kneser's result as a special case. Moreover, they also provided a scale of criteria for the case $\mu = \mu_c$.

While Kneser's result is classical, the analogous question for a periodic background q_0 (and $p_0 = 1$) was answered much later by Rofe-Beketov in a series of papers [14]–[18] in which he eventually showed that the borderline case is again given by

$$q_1(x) = q_0(x) + \frac{\mu}{x^2}, \tag{1.4}$$

where the critical constant μ_c can be expressed in terms of the Floquet discriminant. His result was recently extended by Schmidt [21] to the case $p_0 = p_1 \neq 1$ and Schmidt also provided the second term in the case $\mu = \mu_c$.

These results raised the question for us, if there is a generalization of the Gesztesy–Ünal result which holds inside any essential spectral gap (and not just the lowest). Clearly (1.3) makes no sense, since above the lowest edge of the essential spectrum, all solutions of $\tau_0 u = Eu$ have an infinite number of zeros. However, in the periodic background case, as in the constant background case, there is one solution u_0 which is bounded and a second solution v_0 which grows like x. Hence, at least formally, the Gesztesy–Ünal result explains why the borderline case is given by (1.4). However, their proof has positivity of $H_0 - E$ as the main ingredient and thus cannot be generalized to the case above the infimum of the spectrum.

In summary, there are two natural open problems which we want to address in this paper: First of all, the whole scale of oscillation criteria inside essential spectral gaps for critically perturbed periodic operators. Secondly, what is the analog of the Gesztesy–Ünal result (1.3) inside essential spectral gaps? Based on the original ideas of Rofe-Beketov and the extensions by Schmidt, we will provide a streamlined approach to the subject which will recover and at the same time extend all previously mentioned results. For example, we will derive an averaged version of the Gesztesy–Ünal result (including the whole scale) which, to the best of our knowledge, is new even in the case originally considered by Kneser.

Concerning the Gesztesy–Ünal result we show the following. If u_0, v_0 are two linearly independent solutions of $\tau_0 u = Eu$ with Wronskian $W(u_0, v_0) = 1$ such that there are functions $\alpha(x) > 0$ and $\beta(x) \lesseqgtr 0$ satisfying $u_0(x) = O(\alpha(x))$ and $v_0(x) - \beta(x)u_0(x) = O(\alpha(x))$ as $x \to \infty$, then $(p_0 = p_1)$

$$q_1(x) = q_0(x) + \frac{\mu\,\beta'(x)}{\alpha(x)^2\beta(x)^2}, \tag{1.5}$$

is relatively oscillatory if $\limsup_{x\to\infty} \frac{\mu}{\ell}\int_x^{x+\ell} u_0(t)^2\alpha(t)^{-2}dt < -\frac{1}{4}$ and relatively nonoscillatory if $\liminf_{x\to\infty} \frac{\mu}{\ell}\int_x^{x+\ell} u_0(t)^2\alpha(t)^{-2}dt > -\frac{1}{4}$. By virtue of d'Alembert's formula (cf. (2.5) below), this reduces to (1.3) for E at the bottom of the spectrum, where we can set $\alpha = u_0$ and $\beta = \frac{v_0}{u_0} = \int p_0^{-1} u_0^{-2}$.

We will also be able to include the case $p_0 \neq p_1$ with no additional effort and we will provide a full scale of criteria in all cases.

2. Main results

In this section we will summarize our main results. We will go from the simplest to the most general case rather than the other way round for two reasons: First of all, in our proofs, which will be given in Section 4, we will also advance in this direction and show how the general case follows from the special one. In particular, this approach will allow for much simpler proofs. Secondly, several of the special cases can be proven under somewhat weaker assumptions.

We will consider Sturm–Liouville operators on $L^2((a,b), r\,dx)$ with $-\infty \leq a < b \leq \infty$ of the form

$$\tau = \frac{1}{r}\Big(- \frac{d}{dx} p \frac{d}{dx} + q \Big), \tag{2.1}$$

where the coefficients p, q, r are real-valued satisfying

$$p^{-1}, q, r \in L^1_{loc}(a,b), \quad p, r > 0. \tag{2.2}$$

We will use τ to describe the formal differentiation expression and H for the operator given by τ with separated boundary conditions at a and/or b.

If a (resp. b) is finite and q, p^{-1}, r are in addition integrable near a (resp. b), we will say a (resp. b) is a *regular* endpoint.

Our objective is to compare two Sturm–Liouville operators τ_0 and τ_1 given by

$$\tau_j = \frac{1}{r}\Big(- \frac{d}{dx} p_j \frac{d}{dx} + q_j \Big), \qquad j = 0, 1. \tag{2.3}$$

Throughout this paper we will abbreviate

$$\Delta p = \frac{1}{p_0} - \frac{1}{p_1} = \frac{p_1 - p_0}{p_1 p_0}, \qquad \Delta q = q_1 - q_0. \tag{2.4}$$

Moreover, without loss of generality we will assume that for both operators $a \in \mathbb{R}$ is a regular endpoint and that b is limit point (i.e., $(\tau - z)u$ has at most one L^2 solution near b).

We begin with the case where E is the infimum of the spectrum of H_0. Suppose that $(\tau_0 - E)u = 0$ has a positive solution and let u_0 be the corresponding minimal (principal) positive solution of $(\tau_0 - E)u_0 = 0$ near b, that is,

$$\int^b \frac{dt}{p_0(t) u_0(t)^2} = \infty.$$

A second linearly independent solution v_0 satisfying $W(u_0, v_0) = 1$ is given by d'Alembert's formula (cf. [6, Sect. XI.6])

$$v_0(x) = u_0(x) \int_a^x \frac{dt}{p_0(t) u_0(t)^2}. \tag{2.5}$$

Recall that $\tau_1 - E$ is called nonoscillatory if one solutions of $(\tau_1 - E)u$ has a finite number of zeros in (a,b). By Sturm's comparison theorem, this is then the case for all (nontrivial) solutions.

Theorem 2.1. *Suppose $\tau_0 - E$ has a positive solution and let u_0 be a minimal positive solution. Define v_0 by d'Alembert's formula* (2.5) *and suppose*

$$\lim_{x\to b} p_0 v_0\, p_0 u_0' \Delta p = \lim_{x\to b} p_0 \Delta p = 0. \tag{2.6}$$

Then $\tau_1 - E$ is oscillatory if

$$\limsup_{x\to b} p_0 v_0^2 (u_0^2 \Delta q + (p_0 u_0')^2 \Delta p) < -\frac{1}{4} \tag{2.7}$$

and nonoscillatory if

$$\liminf_{x\to b} p_0 v_0^2 (u_0^2 \Delta q + (p_0 u_0')^2 \Delta p) > -\frac{1}{4}. \tag{2.8}$$

Remark 2.2. *(i). If u_0 is a positive solution which is not minimal near b, that is $\int^b p_0(t)^{-1} u_0(t)^{-2} dt < \infty$, then*

$$v_0(x) = u_0(x) \int_x^b \frac{dt}{p_0(t) u_0(t)^2}$$

is a minimal positive solution.

(ii). Clearly, the requirement that $\tau_0 - E$ has a positive solution can be weakened to $\tau_0 - E$ being nonoscillatory. In fact, after increasing a beyond the last zero of some solution, we can reduce the nonoscillatory case to the positive one.

(iii). Note that the coefficient r does not enter since we have chosen it to be the same for τ_0 and τ_1.

The special case $\Delta p = 0$ is the Gesztesy–Ünal oscillation criterion [4]. It is not hard to see (cf. Section 6), that it can be used to give a simple proof of Rofe-Beketov's result at the infimum of the essential spectrum (another simple proof for this case was given by Schmidt in [20], which also contains nice applications to the spectrum of radially periodic Schrödinger operators in the plane). Moreover, it is only the first one in a whole scale of oscillation criteria. To get the remaining ones, we start by demonstrating that Kneser's classical result together with all its generalizations follows as a special case.

To see this, we recall the iterated logarithm $\log_n(x)$ which is defined recursively via

$$\log_0(x) = x, \qquad \log_n(x) = \log(\log_{n-1}(x)).$$

Here we use the convention $\log(x) = \log|x|$ for negative values of x. Then $\log_n(x)$ will be continuous for $x > \mathrm{e}_{n-1}$ and positive for $x > \mathrm{e}_n$, where $\mathrm{e}_{-1} = -\infty$ and $\mathrm{e}_n = \mathrm{e}^{\mathrm{e}_{n-1}}$. Abbreviate further

$$L_n(x) = \frac{1}{\log_{n+1}'(x)} = \prod_{j=0}^{n} \log_j(x), \qquad Q_n(x) = -\frac{1}{4} \sum_{j=0}^{n-1} \frac{1}{L_j(x)^2}.$$

Here and in what follows the usual convention that $\sum_{j=0}^{-1} \equiv 0$ is used, that is, $Q_0(x) = 0$.

Corollary 2.3. *Fix some $n \in \mathbb{N}_0$ and $(a,b) = (\mathrm{e}_n, \infty)$. Let*

$$p_0(x) = 1, \qquad q_0(x) = Q_n(x)$$

and suppose

$$p_1(x) = 1 + o\Big(\frac{x}{L_n(x)}\Big). \tag{2.9}$$

Then τ_1 is oscillatory if

$$\limsup_{x\to\infty} L_n(x)^2 \left(\Delta q(x) + \frac{1}{4}\Big(\sum_{j=0}^{n-1} \frac{1}{L_j(x)}\Big)^2 \Delta p(x)\right) < -\frac{1}{4} \tag{2.10}$$

and nonoscillatory if

$$\liminf_{x\to\infty} L_n(x)^2 \left(\Delta q(x) + \frac{1}{4}\Big(\sum_{j=0}^{n-1} \frac{1}{L_j(x)}\Big)^2 \Delta p(x)\right) > -\frac{1}{4}, \tag{2.11}$$

where $\delta_n = 0$ for $n = 0$ and $\delta_n = 1$ for $n \ge 1$.

Proof. Observe

$$u_0(x) = \sqrt{L_{n-1}(x)}, \qquad v_0(x) = u_0(x)\log_n(x) = \sqrt{\log_n(x) L_n(x)}$$

(where we set $L_{-1}(x) = 1$) and check

$$q_0 = \frac{u_0''}{u_0} = \frac{1}{4}\left(\frac{L_{n-1}'}{L_{n-1}}\right)^2 + \frac{1}{2}\left(\frac{L_{n-1}'}{L_{n-1}}\right)' = \frac{1}{4}\left(\sum_{j=0}^{n-1}\frac{1}{L_j}\right)^2 - \frac{1}{2}\sum_{j=0}^{n-1}\frac{1}{L_j}\sum_{k=0}^{j}\frac{1}{L_k} = Q_n$$

using $L_n' = L_n \sum_{j=0}^{n} L_j^{-1}$. Since $v_0 u_0 = \sqrt{L_n(x)}$ and $v_0 u_0' = \frac{L_n(x)}{2}\sum_{j=0}^{n-1}\frac{1}{L_j(x)}$ we obtain

$$p_0 v_0^2 (u_0^2 \Delta q + (p_0 u_0')^2 \Delta p) = L_n(x)^2 \left(\Delta q(x) + \frac{1}{4}\Big(\sum_{j=0}^{n-1}\frac{1}{L_j(x)}\Big)^2 \Delta p(x)\right),$$

where $\sum_{j=0}^{n-1}\frac{1}{L_j(x)} = 0$ for $n = 0$ and $\sum_{j=0}^{n-1}\frac{1}{L_j(x)} = x^{-1} + o(x^{-1})$ for $n \ge 1$. Finally, (2.9) gives (2.6). □

The special case $n = 0$ and $\Delta p = 0$ is Kneser's classical result [11]. The extension to $n \in \mathbb{N}_0$ and $\Delta p = 0$ is due to Weber [23], p.53–62, and was later rediscovered by Hartman [5] and Hille [7].

In fact, there is an analogous scale of oscillation criteria which contains Theorem 2.1 as the first one $n = 0$:

Theorem 2.4. *Fix $n \in \mathbb{N}_0$. Suppose $\tau_0 - E$ has a positive solution and let u_0 be a minimal positive solution. Define v_0 by d'Alembert's formula (2.5) and suppose*

$$p_0 v_0\, p_0 u_0' \Delta p = o\Big(\frac{(v_0/u_0)^2}{L_n(v_0/u_0)^2}\Big), \qquad p_0 \Delta p = o\Big(\frac{(v_0/u_0)^2}{L_n(v_0/u_0)^2}\Big).$$

Then $\tau_1 - E$ is oscillatory if

$$\limsup_{x\to b} L_n\Big(\frac{v_0}{u_0}\Big)^2 \left(p_0 u_0^2 (u_0^2 \Delta q + (p_0 u_0')^2 \Delta p) - Q_n\Big(\frac{v_0}{u_0}\Big)\right) < -\frac{1}{4} \tag{2.12}$$

and nonoscillatory if

$$\liminf_{x\to b} L_n\Big(\frac{v_0}{u_0}\Big)^2 \left(p_0 u_0^2 (u_0^2 \Delta q + (p_0 u_0')^2 \Delta p) - Q_n\Big(\frac{v_0}{u_0}\Big)\right) > -\frac{1}{4}. \tag{2.13}$$

The special case $\Delta p = 0$ is again due to [4]. The special case $\tau_0 = -\frac{d^2}{dx^2}$ gives again Corollary 2.3, however, under the (for $n > 0$) somewhat stronger condition $\lim_{x\to\infty} x^{-2} L_n(x)^2 \Delta p(x) = 0$.

Moreover, there is even a version which takes averaged (rather than pointwise) deviations from the borderline case:

Theorem 2.5. *Suppose $\tau_0 - E$ has a positive solution on (a,∞) and let u_0 be a minimal positive solution. Define v_0 by d'Alembert's formula (2.5) and suppose*

$$p_0 v_0^2(u_0^2\Delta q + (p_0 u_0')^2\Delta p) = O(1), \qquad \lim_{x\to\infty} p_0 v_0\, p_0 u_0' \Delta p = \lim_{x\to\infty} p_0 \Delta p = 0,$$

and $\rho = (p_0 u_0 v_0)^{-1}$ satisfies $\rho = o(1)$ and $\frac{1}{\ell}\int_0^\ell |\rho(x+t)-\rho(x)|\,dt = o(\rho(x))$.

Then $\tau_1 - E$ is oscillatory if

$$\limsup_{x\to\infty} \frac{1}{\ell}\int_x^{x+\ell} p_0(t)v_0^2(t)\left(u_0(t)^2\Delta q(t) + (p_0(t)u_0'(t))^2\Delta p(t)\right)dt < -\frac{1}{4} \tag{2.14}$$

and nonoscillatory if

$$\liminf_{x\to\infty} \frac{1}{\ell}\int_x^{x+\ell} p_0(t)v_0^2(t)\left(u_0(t)^2\Delta q(t) + (p_0(t)u_0'(t))^2\Delta p(t)\right)dt > -\frac{1}{4}. \tag{2.15}$$

Again we have

Corollary 2.6. *Fix some $n \in \mathbb{N}_0$ and $(a,b) = (\mathrm{e}_n,\infty)$. Let*

$$p_0(x) = 1, \qquad q_0(x) = Q_n(x)$$

and suppose

$$q_1(x) = Q_n(x) + O\Big(\frac{1}{L_n(x)^2}\Big), \qquad p_1(x) = 1 + \begin{cases} o(1), & n = 0,\\ O\Big(\frac{x^2}{L_n(x)^2}\Big), & n \ge 1.\end{cases}$$

Then τ_1 is oscillatory if

$$\inf_{\ell>0}\limsup_{x\to\infty} \frac{1}{\ell}\int_x^{x+\ell} L_n(t)^2\left(\Delta q(t) + \frac{\delta_n}{4t^2}\Delta p(t)\right)dt < -\frac{1}{4} \tag{2.16}$$

and nonoscillatory if

$$\sup_{\ell>0}\liminf_{x\to\infty} \frac{1}{\ell}\int_x^{x+\ell} L_n(t)^2\left(\Delta q(t) + \frac{\delta_n}{4t^2}\Delta p(t)\right)dt > -\frac{1}{4}, \tag{2.17}$$

where $\delta_n = 0$ for $n = 0$ and $\delta_n = 1$ for $n \ge 1$.

To the best of our knowledge this result is new even in the special case $n = 0$, in which we have that τ_1 with $q_1 = O(x^{-2})$ and $p_1 = 1 + o(1)$ is oscillatory if

$$\inf_{\ell>0}\limsup_{x\to\infty} \frac{1}{\ell}\int_x^{x+\ell} t^2 q_1(t)dt < -\frac{1}{4} \tag{2.18}$$

and nonoscillatory if

$$\sup_{\ell>0}\liminf_{x\to\infty} \frac{1}{\ell}\int_x^{x+\ell} t^2 q_1(t)dt > -\frac{1}{4}. \tag{2.19}$$

There is also a scale of criteria given in Theorem 4.8 which contains Theorem 2.5 as the special case $n = 0$. Note that the criterion is similar in spirit to the Hille–Wintner criterion (see e.g., [22]) which states that τ_1, with q_1 integrable, is oscillatory if

$$\limsup_{x\to\infty} x\int_x^\infty q_1(t)dt < -\frac{1}{4} \tag{2.20}$$

and nonoscillatory if

$$\liminf_{x\to\infty} x \int_x^\infty q_1(t)dt > -\frac{1}{4}. \tag{2.21}$$

Result similar in spirit which are applicable at the bottom of the essential spectrum of periodic operators were given by Khrabustovskii [8], [9].

Our next aim is to extend these result to the case where we are not necessarily at the infimum of the spectrum of H_0. We will again assume that there is a *minimal* solution u_0 (i.e., one solution with minimal growth) such that all other solutions are of the form $v_0 = \tilde{v}_0 + \beta u_0$, where $\tilde{v}_0$ grows like u_0 and β is some positive or negative function, which measures how much faster v_0 grows on average with respect to u_0. For example, in the case of periodic operators we will have that u_0 (and hence $\tilde{v}_0$) is bounded and $\beta(x) = \pm x$ (the sign depending on whether we are at a lower or upper edge of the spectral band). Moreover, since expressions like $\liminf p_0 u_0^2 v_0^2 \Delta q$ will just be zero if u_0 (and v_0) have zeros, we will average over some interval. To avoid problems at finite end points we will choose $b = \infty$ from now on.

But first of all we will state our growth condition more precisely:

Definition 2.7. *A boundary point E of the essential spectrum of H_0 will be called admissible if there is a minimal solution u_0 of $(\tau_0 - E)u_0 = 0$ and a second linearly independent solution v_0 with $W(u_0, v_0) = 1$ such that*

$$\begin{pmatrix} u_0 \\ p_0 u_0' \end{pmatrix} = O(\alpha), \quad \begin{pmatrix} v_0 \\ p_0 v_0' \end{pmatrix} - \beta \begin{pmatrix} u_0 \\ p_0 u_0' \end{pmatrix} = o(\alpha\beta)$$

for some weight functions $\alpha > 0$, $\beta \lessgtr 0$, where β is absolutely continuous such that $\rho = \frac{\beta'}{\beta} > 0$ satisfies $\rho(x) = o(1)$ and $\frac{1}{\ell}\int_0^\ell |\rho(x+t) - \rho(x)|\, dt = o(\rho(x))$.

Clearly, two solutions as in Definition 2.7 can always be found if one chooses α to grow faster than any solution. However, such a choice will only produce nonoscillatory perturbations! Hence, in order to get finite critical coupling constants below, the *right* choice for α and β will be crucial. Roughly speaking α needs to chosen such that $\frac{1}{\ell}\int_x^{x+\ell} \frac{u_0(t)^2}{\alpha(t)^2} dt$ remains bounded from above and below by some positive constants as $x \to \infty$. Moreover, it turns out that the sign of β will depend on whether E is a lower or upper boundary of the essential spectrum (i.e., if the essential spectral gap starts below or above E). This is related to our requirement $W(u_0, v_0) = 1$.

Note that a second linearly independent solution v_0 with $W(u_0, v_0) = 1$ can be obtained by Rofe-Beketov's formula

$$\begin{aligned} v_0(x) =& u_0(x) \int^x \frac{(q_0(t) + p_0(t)^{-1} - E\, r(t))(u_0(t)^2 - (p_0(t)u_0'(t))^2)}{(u_0(t)^2 + (p_0(t)u_0'(t))^2)^2} dt \\ & - \frac{p_0(x)u_0'(x)}{u_0(x)^2 + (p_0(x)u_0'(x))^2} \end{aligned}$$

(the case $p_0 \neq 1$ is due to [21]). In fact, this formula can be used to show that these assumptions are satisfied for certain almost periodic potentials (see [19, Sect. 6.4]).

In this case we will need to look at the difference between the zeros of two solutions u_j, $j = 0, 1$, of $(\tau_j - E)u_j = 0$. We will call $\tau_1 - E$ is relatively nonoscillatory with respect to $\tau_0 - E$ if the difference between the number of zeros of u_1 and u_0 when restricted to (a, c) remains bounded as $c \to \infty$, and relatively oscillatory otherwise. Further details and the connection with the spectra will be given in Section 3.

Now, we come to our main result.

Theorem 2.8. *Suppose E is an admissible boundary point of the essential spectrum of τ_0, with u_0, v_0 and α, β as in Definition 2.7. Furthermore, suppose that we have*

$$\Delta q, \Delta p = O\big(\frac{\beta'}{\alpha^2\beta^2}\big). \tag{2.22}$$

Then $\tau_1 - E$ is relatively oscillatory with respect to $\tau_0 - E$ if

$$\inf_{\ell>0} \limsup_{x\to\infty} \frac{1}{\ell}\int_x^{x+\ell} \frac{\beta(t)^2}{\beta'(t)} \big(u_0(t)^2\Delta q(t) + (p_0(t)u_0'(t))^2\Delta p(t)\big)\, dt < -\frac{1}{4} \tag{2.23}$$

and relatively nonoscillatory with respect to $\tau_0 - E$ if

$$\sup_{\ell>0} \liminf_{x\to\infty} \frac{1}{\ell}\int_x^{x+\ell} \frac{\beta(t)^2}{\beta'(t)} \big(u_0(t)^2\Delta q(t) + (p_0(t)u_0'(t))^2\Delta p(t)\big)\, dt > -\frac{1}{4}. \tag{2.24}$$

We remark that the growth conditions from Definition 2.7 on the derivatives p_0u_0' and p_0v_0' are not needed if $\Delta p = 0$. Similarly, the growth conditions on u_0 and v_0 are not needed if $\Delta q = 0$.

In the case where Δq and Δp have precise asymptotics we have:

Corollary 2.9. *Suppose*

$$\Delta q = \mu\frac{\beta'}{\alpha^2\beta^2}(1+o(1)), \qquad \Delta p = \nu\frac{\beta'}{\alpha^2\beta^2}(1+o(1)). \tag{2.25}$$

Then $\tau_1 - E$ is relatively oscillatory with respect to $\tau_0 - E$ if

$$\inf_{\ell>0} \limsup_{x\to\infty} \frac{1}{\ell}\int_x^{x+\ell} \left(\mu\frac{u_0(t)^2}{\alpha(t)^2} + \nu\frac{(p_0(t)u_0'(t))^2}{\alpha(t)^2}\right) dt < -\frac{1}{4} \tag{2.26}$$

and relatively nonoscillatory with respect to $\tau_0 - E$ if

$$\sup_{\ell>0} \liminf_{x\to\infty} \frac{1}{\ell}\int_x^{x+\ell} \left(\mu\frac{u_0(t)^2}{\alpha(t)^2} + \nu\frac{(p_0(t)u_0'(t))^2}{\alpha(t)^2}\right) dt > -\frac{1}{4}. \tag{2.27}$$

Clearly the precise asymptotic requirement can be removed by a simple Sturm-type comparison argument (see Lemma 3.3 below).

In the special case where p_0, q_0, and r are periodic functions, one has $\alpha(x) = 1$, $\beta(x) = \pm x$ (with the plus sign if E is a lower band edge and the minus sign if E is an upper band edge) and can take ℓ to be the period.

Then

$$C_q = \frac{1}{\ell}\int_x^{x+\ell} u_0(t)^2 dt, \qquad C_p = \frac{1}{\ell}\int_x^{x+\ell} (p_0(t)u_0'(t))^2 dt,$$

are constants and (2.26) respectively (2.27) just read

$$\mu C_q + \nu C_p \lessgtr -\frac{1}{4}.$$

In the special case $p_0 = p_1 = 1$ we recover Rofe-Beketov's well-known result [16]–[18] since one can show (see Section 6)

$$C_q = \frac{|D'(E)|}{\ell^2}$$

for $r(x) = 1$, where D is the Floquet discriminant. In the special case $\Delta p = 0$ we recover the recent extension by Schmidt [21].

If p_0, q_0 are almost periodic and there exists an almost periodic solution at the band edge E, then E is an admissible band edge ($\alpha(x) = 1$, $\beta(x) = \pm x$) after Lemma 6.5 in [19]. By taking $\ell \to \infty$ in our formulas we recover the oscillation criteria by Rofe-Beketov ([19, Thm. 6.12]). In [19], it is furthermore shown that if the spectrum of the operator H_0 has a band-structure, obeying some growth condition, then there exist almost periodic solutions at the band edge and a formula for the critical coupling constant in terms of the band edges is provided.

Clearly, as before we can get a whole scale of criteria:

Theorem 2.10. *Fix $n \in \mathbb{N}_0$. Suppose E is an admissible boundary point of the essential spectrum of τ_0, with u_0, v_0 and α, β as in Definition 2.7. Furthermore, suppose that we have $\lim_{x\to\infty} \beta(x) = \infty$ and*

$$\Delta q, \Delta p = O\big(\frac{\beta'}{\alpha^2\beta^2}\big). \tag{2.28}$$

Abbreviate

$$Q = \frac{1}{\beta'}\left(u_0^2\Delta q + (p_0 u_0')^2\Delta p\right). \tag{2.29}$$

Then $\tau_1 - E$ is relatively oscillatory with respect to $\tau_0 - E$ at b if

$$\inf_{\ell>0}\limsup_{x\to\infty}\frac{L_n(\beta(x))^2}{\beta(x)^2}\left(\frac{1}{\ell}\int_x^{x+\ell}\beta(t)^2 Q(t)dt - \beta(x)^2 Q_n(\beta(x))\right) < -\frac{1}{4} \tag{2.30}$$

and relatively nonoscillatory with respect to $\tau_0 - E$ at b if

$$\sup_{\ell>0}\liminf_{x\to\infty}\frac{L_n(\beta(x))^2}{\beta(x)^2}\left(\frac{1}{\ell}\int_x^{x+\ell}\beta(t)^2 Q(t)dt - \beta(x)^2 Q_n(\beta(x))\right) > -\frac{1}{4}. \tag{2.31}$$

As a consequence we get:

Corollary 2.11. *Let τ_0 be periodic on (a,∞) with $r(x) = 1$ and let $n \in \mathbb{N}_0$. Define*

$$\mu_c = -\frac{\ell^2}{|D|'(E)},$$

and suppose

$$q_1 = q_0 + \mu_c\Big(Q_n + \frac{\mu}{L_n^2}\Big) + o\Big(\frac{1}{L_n^2}\Big), \qquad p_1 = p_0 + o\Big(\frac{1}{L_n^2}\Big). \tag{2.32}$$

Then $\tau_1 - E$ is relatively oscillatory with respect to $\tau_0 - E$ if

$$\mu < -\frac{1}{4} \tag{2.33}$$

and relatively nonoscillatory with respect to $\tau_0 - E$ if

$$\mu > -\frac{1}{4}. \tag{2.34}$$

Again the special case $n = 1$ and $\Delta p = 0$ is due to [21]. The assumption $r(x) = 1$ can be dropped, but then μ_c can no longer be expressed in terms of the derivative of the Floquet discriminant (alternatively one could also choose $\alpha(x) = r(x)^{-1/2}$). A nonoscillation result similar in spirit to the Hille-Wintner result mentioned earlier was given by Khrabustovskii [10].

3. Relative Oscillation Theory in a Nutshell

The purpose of this section is to provide some further details on relative oscillation theory and to show how the question of relative (non)oscillation is related to finiteness of the number of eigenvalues in essential spectral gaps. We refer to [12] and [13] for further results, proofs, and historical remarks.

Our main object will be the (modified) Wronskian

$$W_x(u_0, u_1) = u_0(x)\, p_1(x) u_1'(x) - p_0(x) u_0'(x)\, u_1(x) \tag{3.1}$$

of two functions u_0, u_1 and its zeros. Here we think of u_0 and u_1 as two solutions of two different Sturm–Liouville equations $\tau_j u_j = E u_j$ of the type (2.3).

Under these assumptions $W_x(u_0, u_1)$ is absolutely continuous and satisfies

$$W_x'(u_0, u_1) = (q_1 - q_0) u_0 u_1 + \left(\frac{1}{p_0} - \frac{1}{p_1}\right) p_0 u_0' p_1 u_1'. \tag{3.2}$$

Next we recall the definition of Prüfer variables ρ_u, θ_u of an absolutely continuous function u:

$$u(x) = \rho_u(x) \sin(\theta_u(x)), \qquad p(x) u'(x) = \rho_u(x) \cos(\theta_u(x)). \tag{3.3}$$

If $(u(x), p(x)u'(x))$ is never $(0,0)$ and u, pu' are absolutely continuous, then ρ_u is positive and θ_u is uniquely determined once a value of $\theta_u(x_0)$ is chosen by requiring continuity of θ_u.

Notice that

$$W_x(u, v) = -\rho_u(x) \rho_v(x) \sin(\Delta_{v,u}(x)), \qquad \Delta_{v,u}(x) = \theta_v(x) - \theta_u(x). \tag{3.4}$$

Hence the Wronskian vanishes if and only if the two Prüfer angles differ by a multiple of π. We take two solutions u_j, $j = 0, 1$, of $\tau_j u_j = \lambda_j u_j$ and associated Prüfer variables ρ_j, θ_j. We will call the total difference

$$\#_{(c,d)}(u_0, u_1) = \lceil \Delta_{1,0}(d)/\pi \rceil - \lfloor \Delta_{1,0}(c)/\pi \rfloor - 1 \tag{3.5}$$

the number of weighted sign flips in (c, d), where we have written $\Delta_{1,0}(x) = \Delta_{u_1,u_0}$ for brevity.

One can interpret $\#_{(c,d)}(u_0, u_1)$ as the weighted sign flips of the Wronskian $W_x(u_0, u_1)$, where a sign flip is counted as $+1$ if $q_0 - q_1$ and $p_0 - p_1$ are positive in a neighborhood of the sign flip, it is counted as -1 if $q_0 - q_1$ and $p_0 - p_1$ are negative in a neighborhood of the sign flip. In the case where the differences vanish or are of opposite sign are more subtle [12, 13].

After these preparations we are now ready for

Definition 3.1. *For τ_0, τ_1 possibly singular Sturm–Liouville operators as in* (2.3) *on (a, b), we define*

$$\underline{\#}(u_0, u_1) = \liminf_{d \uparrow b,\, c \downarrow a} \#_{(c,d)}(u_0, u_1) \quad \text{and} \quad \overline{\#}(u_0, u_1) = \limsup_{d \uparrow b,\, c \downarrow a} \#_{(c,d)}(u_0, u_1), \tag{3.6}$$

where $\tau_j u_j = \lambda_j u_j$, $j = 0, 1$.

We say that $\#(u_0, u_1)$ exists, if $\overline{\#}(u_0, u_1) = \underline{\#}(u_0, u_1)$, and write

$$\#(u_0, u_1) = \overline{\#}(u_0, u_1) = \underline{\#}(u_0, u_1). \tag{3.7}$$

in this case.

One can show that $\#(u_0, u_1)$ exists if $p_0 - p_1$ and $q_0 - \lambda_0 r - q_1 + \lambda_1 r$ have the same definite sign near the endpoints a and b.

We recall that in classical oscillation theory τ is called oscillatory if a solution of $\tau u = 0$ has infinitely many zeros.

Definition 3.2. *We call τ_1 relatively nonoscillatory with respect to τ_0, if the quantities $\underline{\#}(u_0, u_1)$ and $\overline{\#}(u_0, u_1)$ are finite for all solutions $\tau_j u_j = 0$, $j = 0, 1$. We call τ_1 relatively oscillatory with respect to τ_0, if one of the quantities $\underline{\#}(u_0, u_1)$ or $\overline{\#}(u_0, u_1)$ is infinite for some solutions $\tau_j u_j = 0$, $j = 0, 1$.*

It turns out that this definition is in fact independent of the solutions chosen. Moreover, since a Sturm-type comparison theorem holds for relative oscillation theory, we have

Lemma 3.3. *If τ_1 is relatively oscillatory with respect to τ_0 for $p_1 \le p_0$, $q_1 \le q_0$ then the same is true for any τ_2 with $p_2 \le p_1$, $q_2 \le q_1$. Similarly, if τ_1 is relatively nonoscillatory with respect to τ_0 for $p_1 \le p_0$, $q_1 \le q_0$ then the same is true for any τ_2 with $p_1 \le p_2 \le p_0$, $q_1 \le q_2 \le q_0$.*

The connection between this definition and the spectrum is given by:

Theorem 3.4. *Let H_j be self-adjoint operators associated with τ_j, $j = 0, 1$. Then*

(i) *$\tau_0 - \lambda_0$ is relatively nonoscillatory with respect to $\tau_0 - \lambda_1$ if and only if* $\dim \operatorname{Ran} P_{(\lambda_0,\lambda_1)}(H_0) < \infty$.

(ii) *Suppose* $\dim \operatorname{Ran} P_{(\lambda_0,\lambda_1)}(H_0) < \infty$ *and $\tau_1 - \lambda$ is relatively nonoscillatory with respect to $\tau_0 - \lambda$ for one $\lambda \in [\lambda_0, \lambda_1]$. Then it is relatively nonoscillatory for all $\lambda \in [\lambda_0, \lambda_1]$ if and only if* $\dim \operatorname{Ran} P_{(\lambda_0,\lambda_1)}(H_1) < \infty$.

For a practical application of this theorem one needs criteria when $\tau_1 - \lambda$ is relatively nonoscillatory with respect to $\tau_0 - \lambda$ for λ inside an essential spectral gap.

Lemma 3.5. *Let H_0 be bounded from below. Suppose a is regular (b singular) and*

(i) $\lim_{x\to b} r(x)^{-1}(q_0(x) - q_1(x)) = 0$, *$\frac{q_0}{r}$ is bounded near b, and*

(ii) $\lim_{x\to b} p_1(x) p_0(x)^{-1} = 1$.

Then $\sigma_{ess}(H_0) = \sigma_{ess}(H_1)$ and $\tau_1 - \lambda$ is relatively nonoscillatory with respect to $\tau_0 - \lambda$ for every $\lambda \in \mathbb{R} \backslash \sigma_{ess}(H_0)$.

The analogous result holds for a singular and b regular.

4. Effective Prüfer angles and relative oscillation criteria

As in the previous section, we will consider two Sturm–Liouville operators τ_j, $j = 0, 1$, and corresponding self-adjoint operators H_j, $j = 0, 1$. Now we want to answer the question, when a boundary point E of the essential spectrum of H_0 is an accumulation point of eigenvalues of H_1. By Theorem 3.4 we need to investigate if $\tau_1 - E$ is relatively oscillatory with respect to $\tau_0 - E$ or not, that is, if the difference of Prüfer angles $\Delta_{1,0} = \theta_1 - \theta_0$ is bounded or not.

Hence the first step is to derive an ordinary differential equation for $\Delta_{1,0}$. While this can easily be done, the result turns out to be not very effective for our purpose. However, since the number of weighted sign flips $\#_{(c,d)}(u_0, u_1)$ is all we are eventually interested in, any *other* Prüfer angle which gives the same result will be as good:

Definition 4.1. *We will call a continuous function ψ a Prüfer angle for the Wronskian $W(u_0, u_1)$, if $\#_{(c,d)}(u_0, u_1) = \lceil \psi(d)/\pi \rceil - \lfloor \psi(c)/\pi \rfloor - 1$ for any $c, d \in (a, b)$.*

Hence we will try to find a more effective Prüfer angle ψ than $\Delta_{1,0}$ for the Wronskian of two solutions. The right choice was found by Rofe-Beketov [15]–[18] (see also the recent monograph [19]):

Let u_0, v_0 be two linearly independent solutions of $(\tau_0 - \lambda)u = 0$ with $W(u_0, v_0) = 1$ and let u_1 be a solution of $(\tau_1 - \lambda)u = 0$. Define ψ via

$$W(u_0, u_1) = -R\sin(\psi), \qquad W(v_0, u_1) = -R\cos(\psi). \tag{4.1}$$

Since $W(u_0, u_1)$ and $W(v_0, u_1)$ cannot vanish simultaneously, ψ is a well-defined absolutely continuous function, once one value at some point x_0 is fixed.

Lemma 4.2. *The function ψ defined in* (4.1) *is a Prüfer angle for the Wronskian $W(u_0, u_1)$.*

Proof. Since $W(u_0, u_1) = -R\sin(\psi) = -\rho_{u_0}\rho_{u_1}\sin(\Delta_{1,0})$ it suffices to show that $\psi = \Delta_{1,0} \mod 2\pi$ at each zero of the Wronskian. Since we can assume $\theta_{v_0} - \theta_{u_0} \in (0, \pi)$ (by $W(u_0, v_0) = 1$), this follows by comparing signs of $R\cos(\psi) = \rho_{v_0}\rho_{u_1}\sin(\theta_{u_1} - \theta_{v_0})$. □

Lemma 4.3. *Let u_0, v_0 be two linearly independent solutions of $(\tau_0 - \lambda)u = 0$ with $W(u_0, v_0) = 1$ and let u_1 be a solution of $(\tau_1 - \lambda)u = 0$.*

Then the Prüfer angle ψ for the Wronskian $W(u_0, u_1)$ defined in (4.1) *obeys the differential equation*

$$\psi' = -\Delta q\big(u_0\cos(\psi) - v_0\sin(\psi)\big)^2 - \Delta p\big(p_0 u_0'\cos(\psi) - p_0 v_0'\sin(\psi)\big)^2, \tag{4.2}$$

where

$$\Delta p = \frac{1}{p_0} - \frac{1}{p_1}, \qquad \Delta q = q_1 - q_0.$$

Proof. Observe $R\psi' = -W(u_0, u_1)'\cos(\psi) + W(v_0, u_1)'\sin(\psi)$ and use (3.2), (4.1) to evaluate the right-hand side. □

Remark 4.4. *Special cases of the* phase equation (4.2) *have been used in the physics literature before (*[1], [2]*). Moreover, ψ was originally not interpreted as Prüfer angle for Wronskians, but defined via*

$$\begin{pmatrix} u_1 \\ p_1 u_1' \end{pmatrix} = \begin{pmatrix} v_0 & u_0 \\ p_0 v_0' & p_0 u_0' \end{pmatrix} \begin{pmatrix} -R\sin(\psi) \\ R\cos(\psi) \end{pmatrix}. \tag{4.3}$$

Augmenting the definition

$$\begin{pmatrix} u_0 & u_1 \\ p_0 u_0' & p_1 u_1' \end{pmatrix} = \begin{pmatrix} v_0 & u_0 \\ p_0 v_0' & p_0 u_0' \end{pmatrix} \begin{pmatrix} 0 & -R\sin(\psi) \\ 1 & R\cos(\psi) \end{pmatrix},$$

and taking determinants shows $W(u_0, u_1) = -R\sin(\psi)$. Similarly we obtain $W(v_0, u_1) = -R\cos(\psi)$ and hence this definition is equivalent to (4.1).

In the case $p_0 = p_1$ equation (4.2) can be interpreted as the Prüfer equation of an associated Sturm–Liouville equation with coefficients given rather implicitly by means of a Liouville-type transformation of the independent variable. Hence a standard oscillation criterion of Hille and Wintner [22, Thm 2.12] can be used. This is the original strategy by Rofe-Beketov (see [19, Sect. 6.3]).

In fact, using the transformation $\eta = \tan(\psi)$ it is straightforward to check that ψ satisfies (4.2) if η satisfies the Riccati equation

$$\eta' = -\Delta q\big(u_0 - v_0\eta\big)^2 - \Delta p\big(p_0 u_0' - p_0 v_0'\eta\big)^2. \tag{4.4}$$

Hence we obtain

Lemma 4.5. *Suppose $\Delta p = 0$ and $\Delta q > 0$. Then τ_1 is relatively (non)oscillatory with respect to τ_0 if and only if the Sturm–Liouville equation associated with*

$$p^{-1} = \Delta q\, v_0^2 \exp(2\int \Delta q\, u_0 v_0) > 0, \qquad q = -\Delta q\, u_0^2 \exp(-2\int \Delta q\, u_0 v_0) < 0$$

is (non)oscillatory.

Proof. Making another transformation $\phi = \exp(-2\int \Delta q\, u_0 v_0)\eta$ we can eliminate the linear term to obtain the Riccati equation

$$\phi' = q - \frac{1}{p}\phi^2$$

for the logarithmic derivative $\phi = \frac{pu'}{u}$ of solutions of the above Sturm–Liouville equation. □

Clearly, an analogous result holds for the case where $\Delta q = 0$ and $\Delta p > 0$.

Since most oscillation criteria are for the case $p = 1$, a Liouville-type transformation is required before they can be applied. Nevertheless, in order to handle the general case $\Delta q \neq 0$ and $\Delta p \neq 0$ we will use a more direct approach.

Even though equation (4.2) is rather compact, it is still not well suited for a direct analysis, since in general u_0 and v_0 will have different growth behavior (e.g., for $\tau_0 = -\frac{d^2}{dx^2}$ we have $u_0(x) = 1$ and $v_0(x) = x$ at the boundary of the spectrum). In order to fix this problem Schmidt [21] proposed to use yet another Prüfer angle φ given by the Kepler transformation

$$\cot(\psi) = \beta_1 \cot(\varphi) + \beta_2, \tag{4.5}$$

where $\beta_1 \lesseqgtr 0$ and β_2 are arbitrary absolutely continuous functions. It is straightforward to check that there is a unique choice for φ such that it is again absolutely continuous and satisfies $\lfloor \frac{\psi}{\pi} \rfloor = \lfloor \frac{\varphi}{\pi} \rfloor$:

$$\varphi = \begin{cases} \mathrm{sgn}(\beta_1) n\pi, & \psi = n\pi, \\ \mathrm{sgn}(\beta_1) n\pi + \mathrm{arccot}(\beta_1^{-1}(\cot(\psi) - \beta_2)), & \psi \in (n\pi, (n+1)\pi), \end{cases} \qquad n \in \mathbb{Z}, \tag{4.6}$$

where the branch of arccot is chosen to have values in $(0, \pi)$. The differential equation for φ reads as follows:

Lemma 4.6. *Let u_0, v_0 be two linearly independent solutions of $(\tau_0 - \lambda)u = 0$ with $W(u_0, v_0) = 1$ and let u_1 be a solution of $(\tau_1 - \lambda)u = 0$. Moreover, let $\beta_1 \lesseqgtr 0$ and β_2 be arbitrary absolutely continuous functions.*

Then $\mathrm{sgn}(\beta_1)\varphi$*, with* φ *defined in* (4.6)*, is a Prüfer angle* φ *for the Wronskian* $W(u_0, u_1)$ *and obeys the differential equation*

$$\begin{aligned}\varphi' =&\frac{\beta_1'}{\beta_1}\sin(\varphi)\cos(\varphi) + \frac{\beta_2'}{\beta_1}\sin^2(\varphi)\\ &-\frac{\Delta q}{\beta_1}\big(\beta_1 u_0\cos(\varphi) - (v_0 - \beta_2 u_0)\sin(\varphi)\big)^2\\ &-\frac{\Delta p}{\beta_1}\big(\beta_1 p_0 u_0'\cos(\varphi) - (p_0 v_0' - \beta_2 p_0 u_0')\sin(\varphi)\big)^2.\end{aligned} \tag{4.7}$$

Proof. Rewrite (4.2) as

$$\frac{\psi'}{\sin(\psi)^2} = -\Delta q\big(u_0\cot(\psi) - v_0\big)^2 - \Delta p\big(p_0 u_0'\cot(\psi) - p_0 v_0'\big)^2.$$

On the other hand one computes

$$\frac{\psi'}{\sin(\psi)^2} = -(\cot(\psi))' = -\left(\beta_1\cot(\varphi) + \beta_2\right)' = \beta_1\frac{\varphi'}{\sin(\varphi)^2} - \beta_1'\cot(\varphi) - \beta_2'$$

and solving for φ' gives (4.7). □

We will mainly be interested in the special case $\beta_1 = \beta_2 \equiv \beta$, where

$$\begin{aligned}\varphi' =&\frac{\beta'}{\beta}\left(\sin^2(\varphi) + \sin(\varphi)\cos(\varphi)\right)\\ &-\beta\,\Delta q\Big(u_0\cos(\varphi) - \frac{1}{\beta}(v_0 - \beta\, u_0)\sin(\varphi)\Big)^2\\ &-\beta\,\Delta p\Big(p_0 u_0'\cos(\varphi) - \frac{1}{\beta}(p_0 v_0' - \beta\, p_0 u_0')\sin(\varphi)\Big)^2.\end{aligned} \tag{4.8}$$

Note that if $\beta < 0$ then not φ, but $-\varphi$ is a Prüfer angle. However, this choice will avoid case distinctions later on.

Now we turn to applications of this result. As a warm up we will treat the case where E is the infimum of the spectrum of H_0 and prove Theorem 2.1.

Proof of Theorem 2.1. Since $\tau_0 - E$ is nonoscillatory, $\tau_1 - E$ is relatively (non)oscillatory with respect to $\tau_0 - E$ if and only if $\tau_1 - E$ is (non)oscillatory.

Set $\beta = \frac{v_0}{u_0} = \int p_0^{-1} u_0^{-2} dt$ and $\rho = \frac{\beta'}{\beta} = \frac{1}{p_0 u_0 v_0}$. Now observe that (4.8) reads

$$\begin{aligned}\varphi' =&\rho\big(\sin^2(\varphi) + \sin(\varphi)\cos(\varphi) - p_0 v_0^2 u_0^2 \Delta q\cos^2(\varphi)\\ &\quad - p_0 v_0^2\Delta p(p_0 u_0'\cos(\varphi) - \frac{1}{v_0}\sin(\varphi))^2\big)\\ =&\rho\left(\sin^2(\varphi) + \sin(\varphi)\cos(\varphi) - p_0 v_0^2(u_0^2\Delta q + (p_0 u_0')^2\Delta p)\cos^2(\varphi)\right) + o(\rho),\end{aligned}$$

where we have used (2.6) in the second step. Now use Corollary 5.2 which is applicable since $\rho > 0$ and $\int^b \rho(x)dx = \int^b \frac{\beta'(x)dx}{\beta(x)} = \lim_{x\to b}\log(\beta(x)) = \infty$. □

Now note that Corollary 2.3 in turn gives us a criterion when the differential equation for our Prüfer angle has bounded solutions:

Lemma 4.7. *Fix some* $n \in \mathbb{N}_0$*, let* Q *be a locally integrable on* (a, b) *and suppose* $\beta \lesseqgtr 0$ *is absolutely continuous with* $\rho = \frac{\beta'}{\beta} > 0$ *locally bounded and* $\lim_{x\to b}|\beta(x)| =$

∞. Then all solutions of the differential equation

$$\varphi' = \rho \left(\sin^2(\varphi) + \sin(\varphi)\cos(\varphi) - \beta^2 Q \cos^2(\varphi)\right) + o\Big(\frac{\rho\beta^2}{L_n(\beta)^2}\Big) \tag{4.9}$$

tend to ∞ if

$$\limsup_{x\to b} L_n(\beta(x))^2 \left(Q(x) - Q_n(\beta(x))\right) < -\frac{1}{4}$$

and are bounded above if

$$\liminf_{x\to b} L_n(\beta(x))^2 \left(Q(x) - Q_n(\beta(x))\right) > -\frac{1}{4}.$$

In the last case all solutions are bounded under the additional assumption $Q = Q_n(\beta) + O(L_n(\beta)^{-2})$.

Proof. The case $n = 0$ is Lemma 5.1 and hence we can assume $n \geq 1$. By a change of coordinates $y = \beta(x)$ we can reduce the claim to the case $\beta(x) = x$ (and $b = \infty$).

Now we start by showing that

$$\varphi' = \frac{1}{x}\left(\Big(1 - \frac{Ax^2}{L_n(x)}\Big)\sin^2(\varphi) + \sin(\varphi)\cos(\varphi) - x^2\Big(Q_n + \frac{B}{4L_n(x)^2}\Big)\cos^2(\varphi)\right) + o\Big(\frac{x}{L_n(x)^2}\Big)$$

has only bounded solutions if $A + B > -1$ and only unbounded solutions (tending to ∞) if $A + B < -1$. Since the error term $o(xL_n(x)^{-2})$ can be bounded by $\varepsilon x L_n(x)^{-2}(\sin^2(\varphi) + \cos^2(\varphi))$ it suffices to show this for one equation in this class by an easy sub/super-solution argument: If $A + B < -1$, then any solution of one equation with slightly smaller A and B is a sub-solution and hence forces the solution to go to ∞. Similarly, If $A + B > -1$, then any solution of one equation with slightly smaller A and B is a sub-solution and any solution of one equation with slightly larger A and B is a super-solution, which together bound the solutions.

To see the claim for one equation in this class note that unboundedness (boundedness) of solutions is equivalent to $\tau_1 = -d^2/dx^2 + Q$ being relatively (non)oscillatory with respect to $\tau_0 = -d^2/dx^2$. Hence it suffices to choose $\beta_1 = x(1 + Ax^2L_n^{-2})$, $\beta_2 = x$ and $Q = Q_n + (A + B)/(4L_n^2)$ in (4.7) and invoke Corollary 2.3.

Finally, the claim from the lemma follows from this result together with another sub/super-solution argument. □

The special cases $n = 0, 1$ are essentially due to Schmidt ([21, Prop. 3 and 4]).

With this result, we can now prove Theorem 2.4:

Proof of Theorem 2.4. Set $\beta = \frac{v_0}{u_0} = \int p_0^{-1} u_0^{-2} dt$ and $Q = p_0 u_0^2 (u_0^2 \Delta q + (p_0 u_0')^2 \Delta p)$. As in the proof of Theorem 2.1, (4.8) reads

$$\varphi' = \rho \left(\sin^2(\varphi) + \sin(\varphi)\cos(\varphi) - \beta^2 Q \cos^2(\varphi)\right) + o\Big(\frac{\rho\beta^2}{L_n(\beta)^2}\Big)$$

and invoking Lemma 4.7 finishes the proof (note that ψ and hence also φ is always bounded from below, since τ_0 is nonoscillatory). □

One might expect that this theorem remains valid if the conditions are not satisfied pointwise but in some average sense. This is indeed true and can be shown by taking averages in the differential equation for the Prüfer angle. Such an averaging procedure was first used by Schmidt [20] and further extended in [21].

Theorem 4.8. *Suppose $\tau_0 - E$ has a positive solution and let u_0 be a minimal positive solution. Define v_0 by d'Alembert's formula (2.5) and abbreviate*

$$Q(x) = p_0(x)u_0^2(x)\left(u_0(x)^2\Delta q(x) + (p_0(x)u_0'(x))^2\Delta p(x)\right), \qquad \beta(x) = \frac{v_0(x)}{u_0(x)}. \tag{4.10}$$

Suppose

$$\beta^2 Q = O(1), \qquad p_0 v_0\, p_0 u_0' \Delta p = o\Big(\frac{\beta^2}{L_n(\beta)}\Big), \quad p_0 \Delta p = o\Big(\frac{\beta^2}{L_n(\beta)}\Big),$$

and $\rho = (p_0 u_0 v_0)^{-1}$ satisfies $\rho = o(1)$ and (5.7).

Then $\tau_1 - E$ is oscillatory if

$$\inf_{\ell>0} \limsup_{x\to\infty} \frac{L_n(\beta(x))^2}{\beta(x)^2}\left(\frac{1}{\ell}\int_x^{x+\ell} \beta(t)^2 Q(t) dt - \beta(x)^2 Q_n(\beta(x))\right) < -\frac{1}{4} \tag{4.11}$$

and nonoscillatory if

$$\sup_{\ell>0} \liminf_{x\to\infty} \frac{L_n(\beta(x))^2}{\beta(x)^2}\left(\frac{1}{\ell}\int_x^{x+\ell} \beta(t)^2 Q(t) dt - \beta(x)^2 Q_n(\beta(x))\right) > -\frac{1}{4}. \tag{4.12}$$

Proof. Derive the differential equation for φ as in the proof of Theorem 2.1 and then take averages using Corollary 5.4. Observe that the error term is preserved by monotonicity of $\frac{\beta^2}{L_n(\beta)^2}$ and (5.7). □

Now we turn to the case above the infimum of the essential spectrum.

Proof of Theorem 2.10. Observe that (4.8) reads

$$\varphi' = \frac{\beta'}{\beta}\left(\sin^2(\varphi) + \sin(\varphi)\cos(\varphi) - \beta^2 Q \cos^2(\varphi)\right) + o\Big(\frac{\rho\beta^2}{L_n(\beta)^2}\Big).$$

Average over a length ℓ using Corollary 5.4 and observe that the error term is preserved by monotonicity of $\frac{\beta^2}{L_n(\beta)^2}$ and (5.7). Now apply Lemma 4.7. □

Corollary 4.9. *Suppose*

$$\rho = o\Big(\frac{\beta^2}{L_n(\beta)^2}\Big), \quad \text{and} \quad \frac{1}{\ell}\int_x^{x+\ell} \frac{u_0(t)^2}{\alpha(t)^2} dt = C_q + o\Big(\frac{\beta^2}{L_n(\beta)^2}\Big) \tag{4.13}$$

for some $\ell > 0$. Furthermore, assume

$$\Delta q = \frac{\beta'}{\alpha^2 C_q}\Big(Q_n(\beta) + \frac{\mu}{L_n(\beta)^2}\Big) + o\Big(\frac{\beta'}{\alpha^2 L_n(\beta)^2)}\Big), \qquad \Delta p = o\Big(\frac{\beta'}{\alpha^2 L_n(\beta)^2}\Big). \tag{4.14}$$

Then $\tau_1 - E$ is relatively oscillatory with respect to $\tau_0 - E$ if

$$\mu < -\frac{1}{4} \tag{4.15}$$

and relatively nonoscillatory with respect to $\tau_0 - E$ if

$$\mu > -\frac{1}{4}. \tag{4.16}$$

Proof. It is sufficient to show that

$$\frac{1}{\ell}\int_x^{x+\ell}\left(\frac{\beta(t)^2}{L_j(\beta(t))^2}-\frac{\beta(x)^2}{L_j(\beta(x))^2}\right)\frac{u_0(t)^2}{\alpha(t)^2}dt = o\Big(\frac{\beta^2(x)}{L_n(\beta(x))^2}\Big)$$

for $j = 0,\dots,n$. Since $u_0\alpha^{-1}$ is bounded, this follows since by the mean value theorem and monotonicity of β we have

$$\sup_{t\in[x,x+\ell]}\left|\frac{\beta(t)^2}{L_j(\beta(t))^2}-\frac{\beta(x)^2}{L_j(\beta(x))^2}\right| \le 2\frac{\beta(x)^2}{L_j(\beta(x))^2}\sum_{k=1}^{j}\frac{\beta(x)}{L_k(\beta(x))}\sup_{t\in[x,x+\ell]}\rho(t),$$

finishing the proof (note that $\beta/L_0(\beta)=1$ and $\lim_{\beta\to\infty}\beta/L_k(\beta)=0$ for $k\ge 1$). □

Note that the assumptions hold for periodic operators by choosing ℓ to be the period. Furthermore, inspection of the proof shows that if $|\beta|\to\infty$, then $\rho = o(\beta^2 L_n(\beta)^{-2})$ can be replaced by $\rho = O(\beta^2 L_n(\beta)^{-2})$.

5. Appendix: Averaging ordinary differential equations

In Section 4 we have reduced everything to the question if certain ordinary differential equation have bounded solutions or not. In this appendix we collect the required results for these ordinary differential equations. The results are mainly straightforward generalizations of the corresponding results from [21]. All proofs are elementary and we give them for the sake of completeness.

Lemma 5.1. *Suppose $A, B\in\mathbb{R}$ and $\rho(x)>0$ (or $\rho(x)<0$) is not integrable near b. Then the equation*

$$\varphi'(x) = \rho(x)\Big(A\sin^2\varphi(x)+\cos\varphi(x)\sin\varphi(x)+B\cos^2\varphi(x)\Big)+o(\rho(x)) \tag{5.1}$$

has only unbounded solutions if $4AB>1$ and only bounded solutions if $4AB<1$. In the unbounded case we have

$$\varphi(x) = \left(\frac{\mathrm{sgn}(A)}{2}\sqrt{4AB-1}+o(1)\right)\int^x \rho(t)dt. \tag{5.2}$$

Proof. By a straightforward computation we have

$$A\sin^2(\varphi)+\sin(\varphi)\cos(\varphi)+B\cos^2(\varphi) = \frac{A+B}{2}+\frac{\sqrt{1+(A-B)^2}}{2}\cos(2(\varphi-\varphi_0)).$$

for some constant $\varphi_0=\varphi_0(A,B)$. Hence $\psi(x)=\varphi(x)-\varphi_0$ satisfies

$$\psi'(x) = \rho(x)\left(\frac{A+B}{2}+\frac{\sqrt{1+(A-B)^2}}{2}\cos(2\psi(x))\right)+o(\rho(x)) \tag{5.3}$$

If $4AB<1$, we have $|A+B|<\sqrt{1+(A-B)^2}$ from which it follows that the right-hand side of our differential equation is strictly negative for $\psi(x) \pmod\pi$ close to $\pi/2$ and strictly positive if $\psi(x) \pmod\pi$ close to 0. Hence any solution remains in such a strip.

If $4AB>1$, we have $|A+B|>\sqrt{1+(A-B)^2}$ and thus the right-hand side is always positive, $\psi'(x)\ge C\rho(x)$, if $A,B>0$ and always negative, $\psi'(x)\le -C\rho(x)$, if $A,B<0$. Since ρ is not integrable by assumption, ψ is unbounded.

In order to derive the asymptotics, rewrite (5.3) as

$$\psi'(x) = \rho(x)\left(\frac{C+D}{2}\cos^2(\psi(x))+\frac{C-D}{2}\sin^2(\psi(x))\right)+o(\rho(x)),$$

where $C = A + B$ and $D = \sqrt{1 + (A - B)^2}$. Now, introduce

$$\tilde{\psi}(x) = \arctan\left(\sqrt{\frac{C - D}{C + D}} \tan(\psi(x))\right)$$

and observe $|\psi - \tilde{\psi}| < \pi$. Moreover,

$$\tilde{\psi}'(x) = \frac{\rho(x)}{2} \mathrm{sgn}(C + D)\sqrt{C^2 - D^2} + o(\rho(x)).$$

Hence the claim follows since by assumption $4AB > 1$, which implies $\mathrm{sgn}(C+D) = \mathrm{sgn}(A)$. □

We will also need the case where $A = 1$ and B depends on x but not necessarily converge to a limit as $x \to b$. However, by a simple sub/super-solution argument we obtain from our lemma

Corollary 5.2. *Suppose $\rho(x) > 0$ is not integrable near b. Then all solutions of the equation*

$$\varphi' = \rho\Big(\sin^2(\varphi) + \sin(\varphi)\cos(\varphi) - B\cos^2(\varphi)\Big) + o(\rho) \tag{5.4}$$

tend to ∞ as $x \to b$ if $B(x) \le B_0$ for some B_0 with $B_0 < -\frac{1}{4}$ and are bounded above if $B(x) \ge B_0$ for some B_0 with $B_0 > -\frac{1}{4}$.

In addition, we also need to look at averages: Let $\ell > 0$, and denote by

$$\overline{g}(x) = \frac{1}{\ell}\int_x^{x+\ell} g(t)dt. \tag{5.5}$$

the average of g over an interval of length ℓ.

Lemma 5.3. *Let φ obey the equation*

$$\varphi'(x) = \rho(x)f(x) + o(\rho(x)), \qquad x \in (a, \infty), \tag{5.6}$$

where $f(x)$ is bounded. If

$$\frac{1}{\ell}\int_0^\ell |\rho(x+t) - \rho(x)|\, dt = o(\rho(x)) \tag{5.7}$$

then

$$\overline{\varphi}'(x) = \rho(x)\overline{f}(x) + o(\rho(x)) \tag{5.8}$$

Moreover, suppose $\rho(x) = o(1)$. If $f(x) = A(x)g(\varphi(x))$, where $A(x)$ is bounded and $g(x)$ is bounded and Lipschitz continuous, then

$$\overline{f}(x) = \overline{A}(x)g(\overline{\varphi}(x)) + o(1). \tag{5.9}$$

Proof. To show the first statement observe

$$\begin{aligned}\overline{\varphi}'(x) = \frac{\varphi(x+\ell) - \varphi(x)}{\ell} &= \frac{1}{\ell}\int_x^{x+\ell} \rho(t)f(t)dt + o(\rho(x))\\ &= \rho(x)\overline{f}(x) + \frac{1}{\ell}\int_x^{x+\ell} (\rho(t) - \rho(x))f(t)dt + o(\rho(x)).\end{aligned}$$

Now the first claim follows from (5.7) since f is bounded. Note that (5.7) implies that the $o(\rho)$ property is preserved under averaging.

To see the second, we use

$$\begin{aligned}\overline{f}(x) &= \frac{1}{\ell}\int_x^{x+\ell} A(t)g(\varphi(t))dt\\ &= \overline{A}(x)g(\overline{\varphi}(x)) + \frac{1}{\ell}\int_x^{x+\ell} A(t)(g(\varphi(t)) - g(\overline{\varphi}(x)))dt.\end{aligned}$$

Since g is Lipschitz we can use the mean value theorem together with

$$|\varphi(x+t)) - \overline{\varphi}(x)| \le C \sup_{0\le s\le \ell} \rho(x+s)$$

to finish the proof. □

Condition (5.7) is a strong version of saying that $\overline{\rho}(x) = \rho(x)(1+o(1))$ (it is equivalent to the latter if ρ is monotone). It will be typically fulfilled if ρ decreases (or increases) polynomially (but not exponentially). For example, the condition holds if $\sup_{t\in[0,1]} \frac{\rho'(x+t)}{\rho(x)} \to 0$.

We have the next result

Corollary 5.4. *Let φ obey the equation*

$$\varphi' = \rho\Big(A\sin^2(\varphi) + \sin(\varphi)\cos(\varphi) + B\cos^2(\varphi)\Big) + o(\rho) \tag{5.10}$$

with A, B bounded functions and assume that $\rho = o(1)$ satisfies (5.7)*. Then the averaged function $\overline{\varphi}$ obeys the equation*

$$\overline{\varphi}' = \rho\Big(\overline{A}\sin^2(\overline{\varphi}) + \sin(\overline{\varphi})\cos(\overline{\varphi}) + \overline{B}\cos^2(\overline{\varphi})\Big) + o(\rho). \tag{5.11}$$

Note that in this case φ is bounded (above/below) if and only if $\overline{\varphi}$ is bounded (above/below). Furthermore, note that if $A(x)$ has a limit, $A(x) = A_0 + o(1)$, then $\overline{A}(x)$ can be replaced by the limit A_0.

6. Appendix: Periodic operators

We will now suppose that $r(x)$, $p(x)$, and $q(x)$ are ℓ-periodic functions. The purpose of this appendix is to recall some basic facts from Floquet theory in order to compute the critical coupling constant for periodic operators in terms of the derivative of the Floquet discriminant. A classical reference with further details is [3].

Denote by $c(z,x)$, $s(z,x)$ a fundamental system of solutions of $\tau u = zu$ corresponding to the initial conditions $c(z,0) = p(0)s'(z,0) = 1$, $s(z,0) = p(0)c'(z,0) = 0$. One then calls

$$M(z) = \begin{pmatrix} c(z,\ell) & s(z,\ell)\\ p(\ell)c'(z,\ell) & p(\ell)s'(z,\ell)\end{pmatrix} \tag{6.1}$$

the monodromy matrix. Constancy of the Wronskian, $W(c(z), s(z)) = 1$, implies $\det M(z) = 1$ and defining the Floquet discriminant by

$$D(z) = \mathrm{tr}(M(z)) = c(z,\ell) + p(\ell)s'(z,\ell),$$

the eigenvalues $\rho_\pm$ of M are called Floquet multipliers,

$$\rho_\pm(z) = \frac{D(z) \pm \sqrt{D(z)^2 - 4}}{2}, \qquad \rho_+(z)\rho_-(z) = 1, \tag{6.2}$$

where the branch of the square root is chosen such that $|\rho_+(z)| \le 1$. In particular, there are two solutions

$$u_\pm(z,x) = c(z,x) + m_\pm(z) s(z,x), \tag{6.3}$$

the Floquet solutions, satisfying

$$\begin{pmatrix} u_\pm(z,\ell) \\ p(\ell)u_\pm'(z,\ell) \end{pmatrix} = \rho_\pm(z) \begin{pmatrix} u_\pm(z,0) \\ p(0)u_\pm'(z,0) \end{pmatrix} = \rho_\pm(z) \begin{pmatrix} 1 \\ m_\pm(z) \end{pmatrix}. \tag{6.4}$$

Here

$$m_\pm(z) = \frac{\rho_\pm(z) - c(z,\ell)}{s(z,\ell)} \tag{6.5}$$

are called Weyl m-functions. The Wronskian of u_+ and u_- is given by

$$W(u_-(z), u_+(z)) = m_+(z) - m_-(z) = \frac{\sqrt{D(z)^2 - 4}}{s(z,\ell)}. \tag{6.6}$$

The functions $u_\pm(z,x)$ are exponentially decaying as $x \to \pm\infty$ if $|\rho_+(z)| < 1$, that is, $|D(z)| > 2$, and are bounded if $|\rho_+(z)| = 1$, that is, $|D(z)| \le 2$. Note that $u_+(z)$ and $u_-(z)$ are linearly independent for $|D(z)| \ne 2$. The spectrum of H_0 is purely absolutely continuous and given by

$$\sigma(H_0) = \{\lambda \in \mathbb{R} \,|\, |D(\lambda)| \le 2\} = \bigcup_{n=0}^{\infty} [E_{2n}, E_{2n+1}]. \tag{6.7}$$

It should be noted that $m_\pm(z)$ (and hence also $u_\pm(z,x)$) are meromorphic in $\mathbb{C}\backslash\sigma(H_0)$ with precisely one of them having a simple pole at the zeros of $s(z,\ell)$ if the zero is in $\mathbb{R}\backslash\sigma(H_0)$. If the zero is at a band edge E_n of the spectrum, both $m_\pm(z)$ will have a square root type singularity.

Lemma 6.1. *For any $z \in \mathbb{C}$ we have*

$$\dot{D}(z) = -s(z,\ell) \int_0^\ell u_+(z,t) u_-(z,t) r(t) dt, \tag{6.8}$$

where the dot denotes a derivative with respect to z.

Proof. Let $u(z,x)$, $v(z,x)$ be two solutions of $\tau u = zu$, which are differentiable with respect to z, then integrating (3.2) with $u_0 = u(z)$ and $u_1 = v(z_1)$, dividing by $z_1 - z$ and taking $z_1 \to z$ gives

$$W_\ell(\dot{v}(z), u(z)) - W_0(\dot{v}(z), u(z)) = \int_0^\ell u(z,t) v(z,t) r(t) dt.$$

Now choose $u(z) = u_-(z)$ and $v(z) = u_+(z)$ and evaluate the Wronskians

$$\begin{aligned} W_\ell(\dot{u}_+(z), u_-(z)) - W_0(\dot{u}_+(z), u_-(z)) &= \dot{\rho}_+(z)\rho_-(z) W(u_+(z), u_-(z)) \\ &= -\frac{\dot{D}(z)}{\sqrt{D(z)^2 - 4}} W(u_-(z), u_+(z)) \end{aligned}$$

to obtain the formula. □

By (6.6) u_+ and u_- are linearly independent away from the band edges E_n. At a band edge E_n we have $u_-(E_n,x) = u_+(E_n,x) \equiv u(E_n,x)$ and a second linearly independent solution is given by

$$s(E_n,x), \qquad W(u(E_n), s(E_n)) = 1.$$

Here we assume without loss of generality that $s(E_n, \ell) \neq 0$ (since we are only interested in open gaps, this can always be achieved by shifting the base point $x_0 = 0$ if necessary). It is easy to check that $s(E_n, x+\ell) = \sigma_n s(E_n, x) + s(E_n, \ell) u(E_n, x)$, where $\sigma_n = \rho_\pm(E_n) = \operatorname{sgn}(D(E_n))$. In particular, $s(E_n, x)$ is of the form

$$s(E_n, x) = \tilde{s}(E_n, x) + \frac{\sigma_n s(E_n, \ell)}{\ell} x\, u(E_n, x), \qquad \tilde{s}(E_n, x+\ell) = \sigma_n \tilde{s}(E_n, x)$$

and thus $u(E_n, x)$, $s(E_n, x)$ satisfy the requirements of Definition 2.7 with $\alpha(x) = 1$ and $\beta(x) = \operatorname{sgn}(D(E_n)) s(E_n, \ell)\ell^{-1} x$. Observe that $\beta(x) > 0$ for an upper band edge E_{2m} and $\beta(x) < 0$ for a lower band edge E_{2m+1}. Moreover, note that at the bottom of the spectrum $s(E_0, x)$ is just the second solution computed from $u(E_0, x)$ by virtue of d'Alembert's formula (2.5). Setting

$$u_0(x) = \sqrt{\frac{|s(E_n, \ell)|}{\ell}} u(E_n, x), \qquad v_0(x) = \sqrt{\frac{\ell}{|s(E_n, \ell)|}} s(E_n, x)$$

we have $\beta(x) = \operatorname{sgn}(D(E_n) s(E_n, \ell)) x$ and $\ell^{-1} \int_0^\ell u_0(t)^2 r(t) dt = \ell^{-2} |\dot{D}(E_n)|$ by Lemma 6.1.

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