STURM–LIOUVILLE OPERATORS ON TIME SCALES

JONATHAN ECKHARDT AND GERALD TESCHL

ABSTRACT. We establish the connection between Sturm–Liouville equations on time scales and Sturm–Liouville equations with measure-valued coefficients. Based on this connection we generalize several results for Sturm–Liouville equations on time scales which have been obtained by various authors in the past.

1. INTRODUCTION

Time scale calculus was introduced by Hilger in 1988 as a means of unifying differential and difference calculus. Since then this approach has had an enormous impact and developed into a new field of mathematics (see e.g. [5], [6] and the references therein). In particular, Sturm–Liouville equations on time scales have attracted substantial interest (see, e.g., [1, 2, 3, 9, 10, 11, 14, 15, 16, 17, 18, 19, 20, 21, 23, 28] and the references therein) since it contains both continuous Sturm–Liouville equations as well as their discrete analog, Jacobi equations, as special cases. However, efforts to unify these two cases go at least back to the seminal work of Atkinson [4] or the book by Mingarelli [22]. The approach chosen there is at first sight somewhat different and is based on Sturm–Liouville operators with measure-valued coefficients. In this setting, derivatives have to be understood as Radon–Nikodym derivatives and we have shown only recently [12] that the Hilger derivative, the natural derivative on a time scale, can in fact be viewed as a special Radon–Nikodym derivative.

The purpose of the present paper is to show that Sturm–Liouville equations on time scales can in fact be obtained as a special case of Sturm–Liouville equations with measure-valued coefficients. Based on this we will show how some recent results for Sturm–Liouville operators with measure-valued coefficients can be used to extend several results for Sturm–Liouville operators on time scales which have been obtained by various authors in the past.

2. STURM–LIOUVILLE DIFFERENTIAL EXPRESSIONS ON TIME SCALES

To set the stage we recall a few definitions and facts from time scale calculus [5], [6]. Let $\mathbb{T}$ be a time scale, that is, a nonempty closed subset of $\mathbb{R}$ and let a, b be the left and right endpoint of $\mathbb{T}$;

$$a = \inf \mathbb{T} \quad \text{and} \quad b = \sup \mathbb{T}.$$

We define the forward and backward shifts on $\mathbb{R}$ via

(2.1) $\sigma(t) = \begin{cases} \inf\{s \in \mathbb{T} \,|\, t < s\}, & t < b, \\ b, & t \geq b, \end{cases}$ $\rho(t) = \begin{cases} \sup\{s \in \mathbb{T} \,|\, t > s\}, & t > a, \\ a, & t \leq a, \end{cases}$

in the usual way. Note that σ is nondecreasing right continuous and ρ is nondecreasing left continuous. A point $t \in \mathbb{T}$ is called right scattered if $\sigma(t) > t$ and left scattered if $\rho(t) < t$. Associated with each time scale $\mathbb{T}$ are two Borel measure which are defined via their distribution functions σ and ρ (this procedure is standard and we refer to, e.g., [26, Sect. A.1] for a brief and concise account). For notational simplicity we denote these measures by the same letters σ and ρ. Furthermore, we define

$\mathbb{T}_\kappa = \begin{cases} \mathbb{T} \backslash \{a\}, & \text{if } a \text{ is right scattered,} \\ \mathbb{T}, & \text{else,} \end{cases}$ $\mathbb{T}^\kappa = \begin{cases} \mathbb{T} \backslash \{b\}, & \text{if } b \text{ is left scattered,} \\ \mathbb{T}, & \text{else.} \end{cases}$

The forward shift on the time scale $\mathbb{T}_\kappa$ is denoted by σ_κ, i.e.

$$\sigma_\kappa(x) = \begin{cases} \sigma(x), & \text{if } x > a, \\ \sigma(a), & \text{if } x \leq a. \end{cases}$$

Again for notational simplicity the corresponding measure is also denoted by σ_κ.

Some function f on $\mathbb{T}$ is said to be Δ-differentiable at some point $t \in \mathbb{T}$ if there is a number $f^\Delta(t)$ such that for every $\varepsilon > 0$ there is a neighborhood $U \subset \mathbb{T}$ of t such that

(2.2) $|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|, \quad s \in U.$

Analogously one may define the notion of ∇-differentiability of some function using the backward shift ρ. One can show ([16, Thm. 2.1])

(2.3) $f^\Delta(t) = f^\nabla(\sigma(t)), \qquad f^\nabla(t) = f^\Delta(\rho(t))$

for continuously differentiable functions.

A function f on $\mathbb{T}$ is locally Δ-absolutely continuous if f is Δ-absolutely continuous on each bounded subinterval $[\alpha, \beta]_\mathbb{T} = [\alpha, \beta] \cap \mathbb{T}$ with $\alpha, \beta \in \mathbb{T}$. For the notion of Δ-absolute continuity on time scales, we refer to [7]. If f is locally Δ-absolutely continuous then f is continuous on $\mathbb{T}$, f^Δ exists almost everywhere on $\mathbb{T}^\kappa$ with respect to σ and $f^\Delta \in L^1_{loc}(\mathbb{T}^\kappa; \sigma)$, i.e. f^Δ is integrable over each bounded subinterval of $\mathbb{T}^\kappa$ with respect to σ. Furthermore, note that if $\mathbb{T}$ is a bounded time scale, then f is Δ-absolutely continuous on $\mathbb{T}$, i.e. $f^\Delta \in L^1(\mathbb{T}^\kappa; \sigma)$. Of course, similar statements are true for locally ∇-absolutely continuous functions on time scales.

Now let r, p, q be complex-valued functions on $\mathbb{T}_\kappa$ such that $r, q \in L^1_{loc}(\mathbb{T}_\kappa; \sigma)$ and $1/p \in L^1_{loc}(\mathbb{T}_\kappa; \rho)$. Consider the differential expression

(2.4) $\ell u = \frac{1}{r}\left(-\left(pu^\nabla\right)^\Delta + qu \right), \quad u \in \mathfrak{D}_\ell,$

where $\mathfrak{D}_\ell$ is the maximal set of functions for which this expression makes sense. It consists of all functions u on $\mathbb{T}$ which are locally ∇-absolutely continuous and for which pu^∇ is locally Δ-absolutely continuous on $\mathbb{T}_\kappa$. Consequently, we have $\ell u \in L^1_{loc}(\mathbb{T}_\kappa; |r|\sigma_\kappa)$ for $u \in \mathfrak{D}_\ell$, where $|r|\sigma_\kappa$ is the measure σ_κ weighted with the function $|r|$. Also note that for $u \in \mathfrak{D}_\ell$, u^∇ is only locally integrable with respect to ρ, whereas the function

(2.5) $u^{[1]}(x) = p(x)u^\nabla(x), \quad x \in \mathbb{T}^\kappa,$

is locally Δ-absolutely continuous. Therefore we will mainly work with this function, referred to as the first quasi-derivative of u.

We remark that in particular in the case $p = r \equiv 1$ some authors also work with

$$(2.6) \qquad \tilde{\ell}u = -u^{\Delta\Delta} + q\,u^{\sigma}, \qquad u^{\sigma} = u \circ \sigma,$$

which is equivalent to the formulation we have chosen here by virtue of (2.3). However, this formulation does not play nice with our scalar product and does not give rise to a self-adjoint operator in general as pointed out in [9, Sect. 5].

In the remaining part of this section we will show that the differential expression ℓ may be identified with a Sturm–Liouville differential expression on $\mathbb{R}$, whose coefficients are measures. Given a function f on $\mathbb{T}_{\kappa}$ we define its extension to all of $\mathbb{R}$ by

$$(2.7) \qquad \bar{f}(t) = \begin{cases} f(t), & t \in \mathbb{T}_{\kappa}, \\ f(\sigma_{\kappa}(t)), & t \notin \mathbb{T}_{\kappa}. \end{cases}$$

Now we are able to define locally finite complex Borel measures on $\mathbb{R}$ by

$$(2.8)$$
$$\varrho(B) = \int_{B \cap \mathbb{T}_{\kappa}} r(t)d\sigma_{\kappa}(t), \quad \varsigma(B) = \int_{B} \frac{1}{\overline{p}(t)}dt \quad \text{and} \quad \chi(B) = \int_{B \cap \mathbb{T}_{\kappa}} q(t)d\sigma_{\kappa}(t),$$

for each Borel set B. Let $\mathfrak{D}_{\tau}$ be the set of all locally absolutely continuous functions f on $\mathbb{R}$ such that $\overline{p}f'$ is locally absolutely continuous with respect to σ_{κ} and consider the differential expression

$$(2.9) \qquad \tau f = -\frac{d}{d\varrho}\frac{df}{d\varsigma} + \frac{q}{r}f, \qquad f \in \mathfrak{D}_{\tau}.$$

Here the derivatives have to be interpreted as Radon–Nikodym derivatives, hence the right-hand side exists almost everywhere on $\mathbb{T}_{\kappa}$ with respect to $|r|\sigma_{\kappa}$. Consequently, we have $\tau f \in L^{1}_{loc}(\mathbb{R}; |r|\sigma_{\kappa})$ for every $f \in \mathfrak{D}_{\tau}$. Next we want to show that one may identify the spaces $\mathfrak{D}_{\ell}$ and $\mathfrak{D}_{\tau}$ and the corresponding differential expressions. Therefore, for each function $u \in \mathfrak{D}_{\ell}$ let $\hat{u}$ be the extension

$$(2.10) \qquad \hat{u}(x) = \begin{cases} u(x), & \text{if } x \in \mathbb{T}, \\ u(\sigma_{\kappa}(x)) + u^{\nabla}(\sigma_{\kappa}(x))\,(x - \sigma_{\kappa}(x)), & \text{if } x \notin \mathbb{T}, \end{cases}$$

to all of $\mathbb{R}$.

Lemma 2.1. *If $u \in \mathfrak{D}_{\ell}$ then $\hat{u} \in \mathfrak{D}_{\tau}$ with*

$$(2.11) \qquad \overline{p}\hat{u}'(x) = \overline{pu^{\nabla}}(x),$$

for almost all $x \in \mathbb{R}$ with respect to Lebesgue measure and

$$(2.12) \qquad \frac{d\overline{p}\hat{u}'}{d\sigma_{\kappa}}(x) = \left(pu^{\nabla}\right)^{\Delta}(x)$$

for almost all $x \in \mathbb{R}$ with respect to σ_{κ}.

Proof. If $u \in \mathfrak{D}_{\ell}$ then [7, Corollary 3.1] shows that $\hat{u}$ is locally absolutely continuous and [7, Lemma 4.1] shows that the first equality holds. Since by assumption pu^{∇} is locally Δ-absolutely continuous on $\mathbb{T}_{\kappa}$, the results in [12, Theorem 3.3] prove that $\overline{p}\hat{u}'$ is locally absolutely continuous with respect to σ_{κ} and also that the second equality holds. $\qquad \square$

The next lemma shows that extending functions of $\mathfrak{D}_{\ell}$ yields all functions of $\mathfrak{D}_{\tau}$.

Lemma 2.2. *If $f \in \mathfrak{D}_\tau$ then the restriction $f|_\mathbb{T}$ lies in $\mathfrak{D}_\ell$ with $\widehat{f|_\mathbb{T}} = f$. In particular, the map (2.10) is a bijection between $\mathfrak{D}_\ell$ and $\mathfrak{D}_\tau$.*

Proof. Since f is continuous and linear outside of $\mathbb{T}$, we see that f really is the extension of $f|_\mathbb{T}$, as claimed. Hence because of [7, Corollary 3.1], the restriction $f|_\mathbb{T}$ is locally ∇-absolutely continuous. Furthermore [7, Lemma 4.1] shows that

$$\overline{f|_\mathbb{T}^\nabla}(x) = f'(x)$$

for almost all $x \in \mathbb{R}$. Now since $\overline{pf|_\mathbb{T}^\nabla} = \bar{p}f'$ is locally absolutely continuous with respect to σ_κ, the results in [12, Theorem 3.3] show that $pf|_\mathbb{T}^\nabla$ is locally Δ-absolutely continuous on $\mathbb{T}_\kappa$ and hence $f|_\mathbb{T} \in \mathfrak{D}_\ell$. $\square$

Now from these lemmas one sees that the time scale Sturm–Liouville differential expression ℓ is essentially equal to the Sturm–Liouville expression τ, which's coefficients are measures.

Theorem 2.3. *For each $u \in \mathfrak{D}_\ell$ and $f \in \mathfrak{D}_\tau$ we have*

$$\ell u(x) = \tau \hat{u}(x) \quad \text{and} \quad \tau f(x) = \ell f|_\mathbb{T}(x)$$

for almost all $x \in \mathbb{R}$ with respect to $|r|\sigma_\kappa$.

Proof. For $u \in \mathfrak{D}_\ell$ we get from Lemma 2.1 that

$$\frac{d}{d\varrho}\frac{d\hat{u}}{d\varsigma} = \frac{1}{r}\frac{d}{d\sigma}\bar{p}\hat{u}' = \frac{1}{r}\left(pu^\nabla\right)^\Delta,$$

almost everywhere with respect to σ_κ, which proves the first claim. The second claim immediately follows from this and Lemma 2.2. $\square$

Using the preceding theorem and the results in [13, Theorem 3.1] one readily obtains the following existence and uniqueness theorem.

Theorem 2.4. *Let $g \in L^1_{loc}(\mathbb{T}_\kappa; |r|\sigma_\kappa)$ and $z \in \mathbb{C}$. Then for each $c \in \mathbb{T}_\kappa$, d_1, $d_2 \in \mathbb{C}$ there is a unique solution $u \in \mathfrak{D}_\ell$ of*

$$(\ell - z)u = g, \quad \text{with} \quad u(c) = d_1 \quad \text{and} \quad u^{[1]}(c) = d_2.$$

Moreover, [13, Theorem 3.6] shows that the solutions from the preceding theorem depend analytically on $z \in \mathbb{C}$. Given two functions $f, g \in \mathfrak{D}_\ell$ we define the Wronski determinant $W(f, g)$ as

$$W(f, g)(x) = f(x)g^{[1]}(x) - f^{[1]}(x)g(x), \quad x \in \mathbb{T}_\kappa.$$

This function is locally Δ-absolutely continuous on $\mathbb{T}_\kappa$ with derivative

$$W(f, g)^\Delta(x) = (g(x)\ell f(x) - f(x)\ell g(x))\, r(x)$$

for almost all $x \in \mathbb{T}_\kappa$ with respect to σ_κ. In particular if $u_1, u_2 \in \mathfrak{D}_\ell$ are two solutions of $(\ell - z)u = 0$ then the Wronskian $W(u_1, u_2)$ is constant with

$$W(u_1, u_2) \neq 0 \quad \Leftrightarrow \quad u_1,\ u_2 \text{ linearly independent.}$$

In order to obtain self-adjoint operators we make the following additional assumptions on our coefficients.

Hypothesis 2.5.
   (i) *r is positive almost everywhere with respect to σ_κ.*
   (ii) *q is real valued almost everywhere with respect to σ_κ.*

(iii) *p is real-valued and nonzero almost everywhere with respect to ρ.*
(iv) $\mathbb{T}$ *consists of more than four points.*

Note that under these assumptions, the measures ϱ, ς and χ satisfy the requirements of [13, Hypothesis 3.7]. The last assumption in Hypothesis 2.5 is necessary in order to work in Hilbert spaces which are at least two-dimensional.

3. Sturm–Liouville operators on bounded time scales

In this section we consider the case when $\mathbb{T}$ is a bounded time scale. The results are special cases of the more general results in the next section. However, since this case has attracted considerable interest in the past [1, 2, 10, 15, 16, 21], we want to single out the corresponding results. The most general results seem to be the ones in [10], where the case $p \in H^1$ and $q \in L^2$ is treated.

In order to obtain an operator in $L^2(\mathbb{T}_\kappa; r\sigma_\kappa)$ we first restrict $\mathfrak{D}_\ell$ to the subspace

$$\mathfrak{D}_\ell^2 = \left\{ f \in \mathfrak{D}_\ell \mid f, \ \ell f \in L^2(\mathbb{T}_\kappa; r\sigma_\kappa) \right\}.$$

Since this subspace does not give rise to a self-adjoint operator we have to restrict it further. Moreover, since the map (2.10) acts as the identity on $\mathbb{T}$, the operators associated with ℓ and τ are identical by virtue of Theorem 2.3. Consequently, the following theorem is an immediate consequence of the results in [13, Section 7].

Theorem 3.1. *Let φ_α, $\varphi_\beta \in [0, \pi)$ and suppose that $\varphi_\alpha \neq 0$, if $\sigma(a)$ is right scattered and $p(b) \sin \varphi_\beta \neq (b - \rho(b)) \cos \varphi_\beta$, if $\rho(b)$ is right scattered. Then the linear operator S in $L^2(\mathbb{T}_\kappa; r\sigma_\kappa)$ given by*

$$(3.1) \qquad \mathfrak{D}(S) = \left\{ f \in \mathfrak{D}_\ell^2 \ \middle| \ \begin{matrix} 0 = f(\sigma(a)) \cos \varphi_\alpha - f^{[1]}(\sigma(a)) \sin \varphi_\alpha \\ 0 = f(b) \cos \varphi_\beta - f^{[1]}(b) \sin \varphi_\beta \end{matrix} \right\}$$

and $Sf = \ell f$ for $f \in \mathfrak{D}(S)$ is well-defined and self-adjoint.

Note that here well-defined means that each function in $\mathfrak{D}(S)$ has a unique representative in $\mathfrak{D}_\ell^2$ satisfying the boundary conditions. In the cases excluded by the condition on the parameters φ_α and φ_β this fails and $\mathfrak{D}(S)$ (as defined in (3.1)) will no longer be dense. Explicitly, if $\sigma(a)$ is right scattered and $\varphi_\alpha = 0$, the boundary condition reads $f(\sigma(a)) = 0$ and $\mathfrak{D}(S)$ lacks the corresponding one-dimensional subspace. Moreover, $f^{[1]}(\sigma(a)) = p(\sigma(a)) \frac{f(\sigma(a)) - f(a)}{\sigma(a) - a}$ does not enter the boundary condition and the value of $f(a)$ cannot be determined in terms of $f(\sigma(a))$. Consequently, different values of $f(a)$ for one and the same element in $L^2(\mathbb{T}_\kappa; r\sigma_\kappa)$ will give rise to different values of ℓf and S becomes a multi-valued operator, which, however, is still self-adjoint (cf. [13, Theorem 7.6]). One can obtain a single-valued operator by removing $\sigma(a)$ from the Hilbert space. In particular, this case is covered by starting with the time scale $\mathbb{T}_\kappa$ from the outset. Similarly if $\rho(b)$ is left scattered.

The self-adjoint boundary conditions given in Theorem 3.1 above are separate, i.e. they are given for each endpoint separately. As in the classical theory of Sturm–Liouville operators there are also coupled self-adjoint boundary conditions.

Theorem 3.2. *Let $\varphi \in [0, \pi)$, $R \in \mathbb{R}^{2 \times 2}$ with $\det R = 1$ and suppose that*

$$(3.2) \qquad\qquad\qquad p(b) R_{12} \neq (b - \rho(b)) R_{22},$$

if $\sigma(a)$ and $\rho(b)$ are right scattered. Then the linear operator S in $L^2(\mathbb{T}_\kappa; r\sigma_\kappa)$ given by

$$\mathfrak{D}(S) = \left\{ f \in \mathfrak{D}^2_\ell \;\middle|\; \begin{pmatrix} f(b) \\ f^{[1]}(b) \end{pmatrix} = e^{i\varphi} R \begin{pmatrix} f(\sigma(a)) \\ f^{[1]}(\sigma(a)) \end{pmatrix} \right\}$$

and $Sf = \ell f$ for $f \in \mathfrak{D}(S)$ is well-defined and self-adjoint.

As before, the cases excluded by (3.2) give rise to a multi-valued, self-adjoint linear operator in $L^2(\mathbb{T}_\kappa; r\sigma_\kappa)$.

Finally, we also mention that the results from Volkmer [27] for measure-valued Sturm–Liouville equations apply to our situation. To this end we need the Lebesgue decomposition of the measure $d\sigma$ which follows from [8, Thm. 5.2].

Lemma 3.3. *The Lebesgue decomposition of $d\sigma$ with respect to Lebesgue measure is given by*

$$(3.3) \qquad d\sigma(t) = \chi_\mathbb{T}(t)dt + \sum_{t_n \in \mathbb{T}: \mu(t_n) > 0} \mu(t_n) d\Theta(t - t_n),$$

where $d\Theta$ is the Dirac measure centered at $t = 0$, χ_I is the characteristic function of a set, and $\mu(t) = \sigma(t) - t$ is the graininess of the time scale.

Proof. It suffices to show that the measures on the left and right-hand side of (3.3) agree on every interval $I = [\alpha, \beta]$ whose endpoints are in $\mathbb{T}$. For such an interval we have $\sigma(I) = \sigma_-(\beta) - \sigma_-(\alpha) = \beta - \alpha$. So let us turn to the other side. Since $\mathbb{T}$ is closed, it can be written as $[a, b]$ minus a countable union of disjoint open intervals $(t_n, \sigma(t_n))$. Hence $\int_{[\alpha,\beta)} \chi_\mathbb{T}(t)dt$ gives $\beta - \alpha$ minus the intervals missing in $\mathbb{T} \cap [\alpha, \beta)$ and the sum over the Dirac measures just makes up for this missing part. $\square$

Now we are ready to show

Theorem 3.4. *Let S be a self-adjoint operator from Theorem 3.1 and suppose $p > 0$. Then S has purely discrete spectrum and if $L^2(\mathbb{T}_\kappa; r\sigma_\kappa)$ is infinite dimensional, the eigenvalues $E_0 < E_1 < E_2 < \cdots$ have the following asymptotics*

$$(3.4) \qquad \lim_{n \to \infty} \frac{E_n}{n^2} = \frac{\pi^2}{L^2}, \qquad L = \int_\mathbb{T} \sqrt{\frac{r(t)}{p(t)}} dt.$$

Proof. By our assumptions, S satisfies the hypothesis of [27, Thm 5.5] and the above claim holds with $L = GM(\varrho, \varsigma)$, where $GM(\varrho, \varsigma)$ is the geometric mean of the two measures ϱ and ς (see also the summary on p14). As pointed out on p11 in [27] this geometric mean is given by $GM(\varrho, \varsigma) = \sqrt{g} d\varsigma$, where $g = r\,p\,\chi_\mathbb{T}$ is the Radon–Nikodym derivative of $d\varrho$ with respect to $d\varsigma$. $\square$

The above theorem shows that the leading asymptotics comes from the continuous part of the time scale. If the time scale has Lebesgue measure zero (i.e. $L = 0$), the leading asymptotics will change and will no longer be captured by the above theorem. See [2], [3] for some results in this direction.

4. WEYL'S ALTERNATIVE

In this section we will allow our time scale $\mathbb{T}$ to be unbounded. The crucial step in order to determine the associated self-adjoint operators is the classification of the endpoints into two cases following the original ideas of Weyl. In the special

case where $r = p \equiv 1$ and continuous q this was first investigated in [28] using the original approach via Weyl circles. Further results can be found in [19].

We say ℓ is in the limit-circle (l.c.) case at the left endpoint a if for each $z \in \mathbb{C}$ every solution of $(\ell - z)u = 0$ lies in $L^2(\mathbb{T}_\kappa; r\sigma_\kappa)$ near a, i.e. is square integrable near a with respect to $r\sigma_\kappa$. Furthermore, ℓ is said to be in the limit-point (l.p.) case at a if for each $z \in \mathbb{C}$ there is some solution of $(\ell - z)u = 0$ which does not lie in $L^2(\mathbb{T}_\kappa; r\sigma_\kappa)$ near a. Similarly one defines the limit-circle and limit-point cases for the right endpoint b. For example note that each finite endpoint is in the l.c. case. Now [13, Theorem 5.2] yields Weyl's alternative.

Theorem 4.1. *At each endpoint, τ is either in the l.c. or in the l.p. case.*

As in the case of bounded time scales we consider the subspace
$$\mathfrak{D}^2_\ell = \left\{ f \in \mathfrak{D}_\ell \mid f, \ \ell f \in L^2(\mathbb{T}_\kappa; r\sigma_\kappa) \right\},$$
of $\mathfrak{D}_\ell$. Given two functions $f, g \in \mathfrak{D}^2_\ell$ the limits
$$W(f,g)(a) = \lim_{\substack{x \downarrow a \\ x \in \mathbb{T}}} W(f,g)(\sigma(x)) \quad \text{and} \quad W(f,g)(b) = \lim_{\substack{x \uparrow b \\ x \in \mathbb{T}}} W(f,g)(\sigma(x)),$$
exist and are finite. According to [13, Lemma 5.6] it is possible to characterize the l.c. and the l.p. case in terms of the Wronskian at this endpoint.

Lemma 4.2. *ℓ is in the l.p. case at a if and only if*
$$W(f,g)(a) = 0, \quad f, \ g \in \mathfrak{D}^2_\ell.$$
Furthermore, ℓ is in the l.c. case at a if and only if there is a $f \in \mathfrak{D}^2_\ell$ such that
$$W(f, f^*)(a) = 0 \quad \text{and} \quad W(f, g)(a) \neq 0 \quad \text{for some } g \in \mathfrak{D}^2_\ell.$$
Similar results hold at the right endpoint b.

Concerning self-adjointness, the case when both endpoints are in the l.p. case is the simplest, as [13, Theorem 6.2] shows.

Theorem 4.3. *If ℓ is in the l.p. case at both endpoints, then the linear operator S given by $Sf = \ell f$ for $f \in \mathfrak{D}(S) = \mathfrak{D}^2_\ell$ is well-defined and self-adjoint.*

In order to determine self-adjoint operators in the remaining cases we introduce functionals on $\mathfrak{D}^2_\ell$, given by
$$BC^1_a(f) = W(f, w_2)(a) \quad \text{and} \quad BC^2_a(f) = W(w_1, f)(a), \quad f \in \mathfrak{D}^2_\ell,$$
if ℓ is in the l.c. case at a and
$$BC^1_b(f) = W(f, w_2)(b) \quad \text{and} \quad BC^2_b(f) = W(w_1, f)(b), \quad f \in \mathfrak{D}^2_\ell,$$
if ℓ is in the l.c. case at b. Here w_1 and w_2 are some real functions in $\mathfrak{D}^2_\ell$ with
$$(4.1) \qquad\qquad W(w_1, w_2)(a) = W(w_1, w_2)(b) = 1.$$
Note that point evaluations in a finite endpoint are a special case of these functionals, as [13, Proposition 7.1] shows.

Proposition 4.4. *One may choose real w_1, $w_2 \in \mathfrak{D}^2_\ell$ satisfying (4.1) such that the corresponding functionals satisfy*
$$BC^1_a(f) = f(\sigma(a)) \quad \text{and} \quad BC^2_a(f) = f^{[1]}(\sigma(a)), \quad f \in \mathfrak{D}^2_\ell,$$

if $\mathbb{T}$ is bounded from below and

$$BC_b^1(f) = f(b) \quad and \quad BC_b^2(f) = f^{[1]}(b), \quad f \in \mathfrak{D}_\ell^2,$$

if $\mathbb{T}$ is bounded from above.

Now the self-adjoint operators in the case when one endpoint is in the l.c. case and the other is in the l.p. case may be obtained from [13, Theorem 7.3].

Theorem 4.5. *Suppose ℓ is in the l.c. case at a, in the l.p. case at b and let $\varphi_\alpha \in [0, \pi)$ such that*

$$\cos\varphi_\alpha w_2(\sigma(a)) + \sin\varphi_\alpha w_1(\sigma(a)) \neq 0,$$

if $a \in \mathbb{R}$ and $\sigma(a)$ is right scattered. Then the linear operator S in $L^2(\mathbb{T}_\kappa; r\sigma_\kappa)$, given by

$$\mathfrak{D}(S) = \left\{ f \in \mathfrak{D}_\ell^2 \mid BC_a^1(f)\cos\varphi_\alpha - BC_a^2(f)\sin\varphi_\alpha = 0 \right\},$$

and $Sf = \ell f$ for $f \in \mathfrak{D}(S)$ is well-defined and self-adjoint.

Again the excluded case gives rise to a multi-valued, self-adjoint linear operator. If ℓ is in the l.p. case at a and in the l.c. case at b, self-adjoint operators may be given similarly in terms of the functionals BC_b^1 and BC_b^2. We now turn to the case when both endpoints are in the l.c. case. As in the case of bounded time scales, the self-adjoint operators may be divided into two classes. The case of separated boundary conditions may be obtained from [13, Theorem 7.6].

Theorem 4.6. *Suppose ℓ is in the l.c. case at both endpoints and let φ_α, $\varphi_\beta \in [0, \pi)$ such that*

$$\cos\varphi_\alpha w_2(\sigma(a)) + \sin\varphi_\alpha w_1(\sigma(a)) \neq 0,$$

if $a \in \mathbb{R}$ and $\sigma(a)$ is right scattered and

$$\cos\varphi_\beta w_2(b) + \sin\varphi_\beta w_1(b) \neq 0,$$

if $b \in \mathbb{R}$ and $\rho(b)$ is right scattered. Then the linear operator S in $L^2(\mathbb{T}_\kappa; r\sigma_\kappa)$, given by

$$\mathfrak{D}(S) = \left\{ f \in \mathfrak{D}_\ell^2 \;\middle|\; \begin{matrix} 0 = BC_a^1(f)\cos\varphi_\alpha - BC_a^2(f)\sin\varphi_\alpha \\ 0 = BC_b^1(f)\cos\varphi_\beta - BC_b^2(f)\sin\varphi_\beta \end{matrix} \right\},$$

and $Sf = \ell f$ for $f \in \mathfrak{D}(S)$ is well-defined and self-adjoint.

Now, the second class of self-adjoint operators in this case is defined via coupled boundary conditions. The corresponding results may be found in [13, Theorem 7.6].

Theorem 4.7. *Suppose ℓ is in the l.c. case at both endpoints and let $\varphi \in [0, \pi)$, $R \in \mathbb{R}^{2\times2}$ with $\det R = 1$. Furthermore if $a, b \in \mathbb{R}$ and $\sigma(a)$, $\rho(b)$ are right scattered, set*

$$\tilde{R} = \begin{pmatrix} w_2^{[1]}(b) & -w_2(b) \\ -w_1^{[1]}(b) & w_1(b) \end{pmatrix}^{-1} R \begin{pmatrix} w_2^{[1]}(\sigma(a)) & -w_2(\sigma(a)) \\ -w_1^{[1]}(\sigma(a)) & w_1(\sigma(a)) \end{pmatrix},$$

and assume that $\tilde{R}_{12} \neq 0$. Then the linear operator S in $L^2(\mathbb{T}_\kappa; r\sigma_\kappa)$, given by

$$\mathfrak{D}(S) = \left\{ f \in \mathfrak{D}_\ell^2 \;\middle|\; \begin{pmatrix} BC_b^1(f) \\ BC_b^2(f) \end{pmatrix} = e^{i\varphi} R \begin{pmatrix} BC_a^1(f) \\ BC_a^2(f) \end{pmatrix} \right\},$$

and $Sf = \ell f$ for $f \in \mathfrak{D}(S)$ is well-defined and self-adjoint.

5. Spectrum and resolvent

In this section we will provide the resolvents of self-adjoint operators given in the preceding section. As the results in [13, Section 8] show, they turn out to be integral operators. We start with the case when both endpoints are in the l.c. case.

Theorem 5.1. *Suppose ℓ is in the l.c. case at both endpoints and S is a self-adjoint operator as in Theorem 4.6 or Theorem 4.7. Then for each $z \in \rho(S)$ the resolvent R_z is an integral operator*

$$R_z f(x) = \int_{\mathbb{T}_\kappa} G_z(x,y) f(y) r(y) d\sigma_\kappa(y), \quad x \in \mathbb{T}_\kappa, \ f \in L^2(\mathbb{T}_\kappa; r\sigma_\kappa),$$

with some square integrable kernel G_z on $\mathbb{T}_\kappa \times \mathbb{T}_\kappa$. If u_1, u_2 are two linearly independent solutions of $(\ell - z)u = 0$, then there are coefficients $m_{ij}^\pm(z) \in \mathbb{C}$, i, $j \in \{1,2\}$, such that the kernel is given by

$$G_z(x,y) = \begin{cases} \sum_{i,j=1}^2 m_{ij}^+(z) u_i(x) u_j(y), & \text{if } y \leq x, \\ \sum_{i,j=1}^2 m_{ij}^-(z) u_i(x) u_j(y), & \text{if } y > x. \end{cases}$$

Since the resolvents are clearly Hilbert–Schmidt operators, as in [13, Corollary 8.2] we obtain some information about the spectrum in this case.

Corollary 5.2. *Suppose ℓ is in the l.c. case at both endpoints and S is a self-adjoint operator as in Theorem 4.6 or Theorem 4.7. Then S has purely discrete spectrum, i.e. $\sigma(S) = \sigma_d(S)$. Moreover*

$$\sum_{\substack{\lambda \in \sigma(S) \\ \lambda \neq 0}} \frac{1}{\lambda^2} < \infty \quad \text{and} \quad \dim \ker (S - \lambda) \leq 2, \quad \lambda \in \sigma(S).$$

If S is a self-adjoint operator with separated boundary conditions as in Theorem 4.3, Theorem 4.5 or Theorem 4.6, then the resolvent has a simpler form, as [13, Theorem 8.3] shows.

Theorem 5.3. *Suppose S is a self-adjoint operator as in Theorem 4.3, Theorem 4.5 or Theorem 4.6. Furthermore let $z \in \rho(S)$ and u_a, u_b be non-trivial solutions of $(\ell - z)u = 0$ such that*

$$u_a \begin{cases} \text{satisfies the boundary condition at } a \text{ if } \ell \text{ is l.c. at } a, \\ \text{lies in } L^2(\mathbb{T}_\kappa; r\sigma_\kappa) \text{ near } a \text{ if } \ell \text{ is l.p. at } a, \end{cases}$$

and

$$u_b \begin{cases} \text{satisfies the boundary condition at } b \text{ if } \ell \text{ is l.c. at } b, \\ \text{lies in } L^2(\mathbb{T}_\kappa; r\sigma_\kappa) \text{ near } b \text{ if } \ell \text{ is l.p. at } b. \end{cases}$$

Then the resolvent R_z is given by

$$R_z f(x) = \int_{\mathbb{T}_\kappa} G_z(x,y) f(y) r(y) d\sigma_\kappa(y), \quad x \in \mathbb{T}_\kappa, \ f \in L^2(\mathbb{T}_\kappa; r\sigma_\kappa),$$

where

$$G_z(x,y) = \begin{cases} W(u_b, u_a)^{-1} u_a(x) u_b(y), & \text{if } x < y, \\ W(u_b, u_a)^{-1} u_a(y) u_b(x), & \text{if } x \geq y. \end{cases}$$

Furthermore if S is a self-adjoint operator as in Theorem 5.3, then [13, Corollary 8.4] shows that all eigenvalues of S are simple.

6. Weyl–Titchmarsh theory

The associated eigenfunction expansion was considered in [11], [16], [20] for the case of bounded time scales and for semi-unbounded time scales in [17], [18]. Here we will obtain it via classical Weyl–Titchmarsh theory thereby generalizing the presently best result from [18] where the case $r = 1$, p differentiable, and q continuous is treated. For further generalizations of Weyl–Titchmarsh theory to time scale systems see [25], [24] and the references therein.

In this section assume that our time scale $\mathbb{T}$ is bounded from below and let S be a self-adjoint operator as in Theorem 4.5 or Theorem 4.6. In particular there is some $\varphi_\alpha \in [0, \pi)$ such that the boundary condition at the point $\sigma(a)$ reads

$$f(\sigma(a)) \cos \varphi_\alpha - f^{[1]}(\sigma(a)) \sin \varphi_\alpha = 0.$$

Now for each $z \in \mathbb{C}$ consider the linearly independent solutions θ_z, ϕ_z of $(\ell - z)u = 0$ with the initial conditions

$$\theta_z(\sigma(a)) = \phi_z^{[1]}(\sigma(a)) = \cos \varphi_\alpha \quad \text{and} \quad -\theta_z^{[1]}(\sigma(a)) = \phi_z(\sigma(a)) = \sin \varphi_\alpha.$$

Note that the solution ϕ_z satisfies the boundary condition at $\sigma(a)$. Given these solutions one may define a function m on the resolvent set of S by requiring that the solutions

$$\psi_z(x) = \theta_z(x) + m(z)\phi_z(x), \quad x \in \mathbb{T}, \ z \in \rho(S),$$

lie in $L^2(\mathbb{T}_\kappa; r\sigma_\kappa)$ near b and satisfy the boundary condition at b if S is self-adjoint operator as in Theorem 4.6. The function m is called the Weyl–Titchmarsh m-function of S, the solutions ψ_z, $z \in \rho(S)$ are called the Weyl solutions. Now the results in [13, Section 9 and Section 10] readily yield the following properties of m.

Theorem 6.1. *The Weyl–Titchmarsh m-function is a Herglotz–Nevanlinna function. In particular, there is a unique Borel measure μ on $\mathbb{R}$ such that*

$$m(z) = \operatorname{Re}(m(\mathrm{i})) + \int_{\mathbb{R}} \frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} d\mu(\lambda), \quad z \in \mathbb{C}\backslash\mathbb{R}.$$

The measure μ is called the spectral measure of S. Indeed the next theorem, obtained from [13, Section 10] justifies this name.

Theorem 6.2. *The mapping $\mathcal{F}$ given by*

$$\mathcal{F}f(\lambda) = \lim_{\substack{\beta \uparrow b \\ \beta \in \mathbb{T}}} \int_{(a,\beta] \cap \mathbb{T}_\kappa} \phi_\lambda(x)f(x)r(x)d\sigma_\kappa(x), \quad \lambda \in \mathbb{R}, \ f \in L^2(\mathbb{T}_\kappa; r\sigma_\kappa),$$

where the limit exists in $L^2(\mathbb{R}; \mu)$, is unitary from $L^2(\mathbb{T}_\kappa; r\sigma_\kappa)$ onto $L^2(\mathbb{R}; \mu)$ and maps S onto multiplication with the identity function in $L^2(\mathbb{R}; \mu)$.

As a consequence of this theorem we may read off the spectrum of S from the boundary behavior of the Weyl–Titchmarsh m-function in the usual way.

Corollary 6.3. *The spectrum of S is given by*

$$\sigma(S) = \operatorname{supp}(\mu) = \overline{\{\lambda \in \mathbb{R} \mid 0 < \limsup_{\epsilon \downarrow 0} \operatorname{Im}(m(\lambda + \mathrm{i}\epsilon))\}}.$$

References

[1] R. P. Agarwal, M. Bohner and P. J. Y. Wong, *Sturm–Liouville eigenvalue problems on time scales*, Appl. Math. Comput. **99** (1999), no. 2-3, 153–166.

[2] P. Amster, P. De Nápoli and J. P. Pinasco, *Eigenvalue distribution of second-order dynamic equations on time scales considered as fractals*, J. Math. Anal. Appl. **343** (2008), no. 1, 573–584.

[3] P. Amster, P. De Nápoli and J. P. Pinasco, *Detailed asymptotic of eigenvalues on time scales*, J. Difference Equ. Appl. **15** (2009), no. 3, 225–231.

[4] F. Atkinson, *Discrete and Continuous Boundary Problems*, Academic Press, New York, 1964.

[5] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2001.

[6] M. Bohner and A. Peterson (Eds.), *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2003.

[7] A. Cabada and D. R. Vivero, *Criterions for absolute continuity on time scales*, J. Difference Equ. Appl. **11** (2005), no. 11, 1013–1028.

[8] A. Cabada and D. R. Vivero, *Expression of the Lebesgue Δ-integral on time scales as a usual Lebesgue integral; application to the calculus of Delta-antiderivatives*, Math. Comput. Modelling **43** (2006), no. 1-2, 194–207.

[9] F. A. Davidson and B. P. Rynne, *Global bifurcation on time scales*, J. Math. Anal. Appl. **267** (2002), no. 1, 345–360.

[10] F. A. Davidson and B. P. Rynne, *Self-adjoint boundary value problems on time scales*, Electron. J. Differential Equations **2007**, No. 175, 10pp.

[11] F. A. Davidson and B. P. Rynne, *Eigenfunction expansions in L^2 spaces for boundary value problems on time-scales*, J. Math. Anal. Appl. **335** (2007), no. 2, 1038–1051.

[12] J. Eckhardt and G. Teschl, *On the connection between the Hilger and Radon–Nikodym derivatives*, J. Math. Anal. Appl. **385** (2012), no. 2, 1184–1189.

[13] J. Eckhardt and G. Teschl, *Sturm–Liouville operators with measure-valued coefficients*, J. Anal. Math. **120**, 151–224 (2013).

[14] L. Erbe and S. Hilger, *Sturmian theory on measure chains*, Differential Equations Dynam. Systems **1** (1993), no. 3, 223–244.

[15] L. Erbe and A. Peterson, *Eigenvalue conditions and positive solutions*, J. Differ. Equations Appl. **6** (2000), no. 2, 165–191.

[16] G. S. Guseinov, *Eigenfunction expansions for a Sturm–Liouville problem on time scales*, Int. J. Difference Equ. **2** (2007), no. 1, 93–104.

[17] G. S. Guseinov, *An expansion theorem for a Sturm–Liouville operator on semi-unbounded time scales*, Adv. Dyn. Syst. Appl. **3** (2008), no. 1, 147–160.

[18] A. Huseynov, *Eigenfunction expansion associated with the one-dimensional Schrödinger equation on semi-infinite time scale intervals*, Rep. Math. Phys. **66** (2010), no. 2, 207–235.

[19] A. Huseynov, *Limit point and limit circle cases for dynamic equations on time scales*, Hacet. J. Math. Stat. **39** (2010), no. 3, 379–392.

[20] A. Huseynov and E. Bairamov, *On expansions in eigenfunctions for second order dynamic equations on time scales*, Nonlinear Dyn. Syst. Theory **9** (2009), no. 1, 77–88.

[21] Q. Kong, *Sturm–Liouville problems on time scales with separated boundary conditions*, Results Math. **52** (2008), no. 1-2, 111–121.

[22] A. B. Mingarelli, *Volterra-Stieltjes integral equations and generalized ordinary differential expressions*, Lecture Notes in Mathematics **989**, Springer, Berlin, 1983.

[23] B. P. Rynne, *L^2 spaces and boundary value problems on time-scales*, J. Math. Anal. Appl. **328** (2007), no. 2, 1217–1236.

[24] R. Šimon Hilscher and P. Zemánek, *Weyl–Titchmarsh theory for time scale symplectic systems on half line* Abstr. Appl. Anal. **2011**, Art. ID 738520, 41pp.

[25] S. Sun, M. Bohner and S. Chen, *Weyl–Titchmarsh theory for Hamiltonian dynamic systems*, Abstr. Appl. Anal. **2010**, Art. ID 514760, 18pp.

[26] G. Teschl, *Mathematical Methods in Quantum Mechanics; With Applications to Schrödinger Operators*, Graduate Studies in Mathematics **99**, Amer. Math. Soc., Providence, RI, 2009.

[27] H. Volkmer, *Eigenvalue problems of Atkinson, Feller and Krein, and their mutual relationship*, Electron. J. Differential Equations **2005**, No. 48, 15pp.

[28] C. Zhang and L. Zhang, *Classification for a class of second-order singular equations on time scales*, in Proceedings of the 8th ACIS International Conference on Software Engineering, W. Feng and F. Gao, Eds., vol. 3, pp. 73–76, IEEE, Los Alamitos, Calif, USA, 2007.

FACULTY OF MATHEMATICS, UNIVERSITY OF VIENNA, NORDBERGSTRASSE 15, 1090 WIEN, AUSTRIA

E-mail address: `jonathan.eckhardt@univie.ac.at`

FACULTY OF MATHEMATICS, UNIVERSITY OF VIENNA, NORDBERGSTRASSE 15, 1090 WIEN, AUSTRIA, AND INTERNATIONAL ERWIN SCHRÖDINGER INSTITUTE FOR MATHEMATICAL PHYSICS, BOLTZMANNGASSE 9, 1090 WIEN, AUSTRIA

E-mail address: `Gerald.Teschl@univie.ac.at`

URL: `http://www.mat.univie.ac.at/~gerald/`