SPECTRAL THEORY FOR JACOBI OPERATORS

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ABSTRACT

The present thesis discusses various aspects of spectral theory for Jacobi operators.

The first chapter reviews Weyl-Titchmarsh theory for these operators and provides the necessary background for the following chapters.

In the second chapter we provide a comprehensive treatment of oscillation theory for Jacobi operators with separated boundary conditions. Moreover, we present a reformulation of oscillation theory in terms of Wronskians of solutions, thereby extending the range of applicability for this theory. Furthermore, these results are applied to establish the finiteness of the number of eigenvalues in essential spectral gaps of perturbed periodic Jacobi operators.

In the third chapter we offer two methods of inserting eigenvalues into spectral gaps of a given background Jacobi operator: The single commutation method which introduces eigenvalues into the lowest spectral gap of a given semi-bounded background Jacobi operator and the double commutation method which inserts eigenvalues into arbitrary spectral gaps. Moreover, we prove unitary equivalence of the commuted operators, restricted to the orthogonal complement of the eigenspace corresponding to the newly inserted eigenvalues, with the original background operator. Finally, we show how to iterate the above methods. Concrete applications include an explicit realization of the isospectral torus for algebro-geometric finite-gap Jacobi operators and the N-soliton solutions of the Toda and Kac-van Moerbeke lattice equations with respect to arbitrary background solutions.

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Chapter 1

Weyl-Titchmarsh Theory for Jacobi Operators

1.1 General Background

First of all we need to fix some notation. For $I \subseteq \mathbb{Z}$ we denote by $\ell(I)$ the set of $\mathbb{C}$-valued sequences $\{f(n)\}_{n\in I}$. For $M, N \in \mathbb{Z}\cup\{\pm\infty\}$ we abbreviate $\ell(M,N) = \ell(\{n \in \mathbb{Z}|M < n < N\})$ (sometimes we will also write $\ell(N,-\infty)$ instead of $\ell(-\infty,N)$). $\ell^2(I)$ is the Hilbert space of all square summable sequences with scalar product and norm defined as

$$\langle f, g\rangle = \sum_{n\in I} \overline{f(n)} g(n), \quad \|f\| = \sqrt{\langle f, f\rangle}, \quad f, g \in \ell^2(I). \tag{1.1}$$

Furthermore, $\ell_0(I)$ denotes the set of sequences with only finitely-many values being nonzero, $\ell^1(I)$ the set of summable sequences, $\ell^\infty(I)$ the set of bounded sequences, and $\ell^2_\pm(\mathbb{Z})$ denotes the set of sequences in $\ell(\mathbb{Z})$ which are ℓ^2 near $\pm\infty$. For brevity we focus in the following on the case $I = \mathbb{Z}$.

To set the stage, we shall consider operators on $\ell^2(\mathbb{Z})$ associated with the difference expression

$$(\tau f)(n) = a(n)f(n+1) + a(n-1)f(n-1) - b(n)f(n), \tag{1.2}$$

where $a, b \in \ell(\mathbb{Z})$ satisfy the following hypothesis.

Hypothesis H. 1.1 *Suppose $a, b \in \ell(\mathbb{Z})$ satisfy*

$$a(n) \in \mathbb{R}\backslash\{0\}, \quad b(n) \in \mathbb{R}, \quad n \in \mathbb{Z}. \tag{1.3}$$

If τ is limit point (*l.p.*) at both $\pm\infty$ (cf., e.g., [3], [5]) then τ gives rise to a unique self-adjoint operator H when defined maximally. Otherwise we need to fix a boundary condition at each endpoint where τ is limit circle (*l.c.*). Throughout this thesis we denote by $u_\pm(z, .)$, $z \in \mathbb{C}$ nontrivial solutions of $\tau u = zu$ which satisfy the boundary condition at $\pm\infty$ (if any) and $u_\pm(z,.) \in \ell^2_\pm(\mathbb{Z})$, respectively. $u_\pm(z,.)$ might not exist for $z \in \mathbb{R}$ (cf. Lemma 1.3) but if it exists it is unique up to a constant multiple.

Picking $z_0 \in \mathbb{C}\backslash\mathbb{R}$ we can characterize H by

$$H: \begin{array}{ccc} \mathfrak{D}(H) & \to & \ell^2(\mathbb{Z}) \\ f & \mapsto & \tau f \end{array}, \tag{1.4}$$

where the domain of H is explicitly given by

$$\begin{aligned}\mathfrak{D}(H) = \{f \in \ell^2(\mathbb{Z}) |\ \tau f \in \ell^2(\mathbb{Z}),\ &\lim_{n\to+\infty} W_n(u_+(z_0), f) = 0, \\ &\lim_{n\to-\infty} W_n(u_-(z_0), f) = 0\}\end{aligned} \tag{1.5}$$

and

$$W_n(f,g) = a(n)\Big(f(n)g(n+1) - f(n+1)g(n)\Big) \tag{1.6}$$

denotes the (modified) Wronskian. By $\sigma(.)$, $\sigma_p(.)$, and $\sigma_{ess}(.)$ we denote the spectrum, point spectrum (i.e., the set of eigenvalues), and essential spectrum of an operator, respectively.

A simple calculation yields Green's formula for $f, g \in \ell(\mathbb{Z})$

$$\sum_{j=m}^{n} \Big(f(\tau g) - g\tau f\Big)(j) = W_n(f,g) - W_{m-1}(f,g). \tag{1.7}$$

A glance at (1.7) shows that the modified Wronskian of two solutions of

$$\tau u = zu \tag{1.8}$$

is constant and nonzero if and only if they are linearly independent. If we choose $f = u(z)$, $g = \overline{u(z)}$ in (1.7), where $u(z)$ is a solution of (1.8) with $z \in \mathbb{C}\backslash\mathbb{R}$, we obtain

$$[u(z)]_n = [u(z)]_{m-1} - \sum_{j=m}^{n} |u(z,j)|^2, \tag{1.9}$$

where $[.]_n$ denotes the Weyl bracket

$$[u(z)]_n = \frac{W_n(u(z), \overline{u(z)})}{2\mathrm{i}\mathrm{Im}(z)} = a(n)\frac{\mathrm{Im}(u(z,n)\overline{u(z,n+1)})}{\mathrm{Im}(z)}, \quad n \in \mathbb{Z}. \tag{1.10}$$

Taking limits in (1.7) shows that $W_{\pm\infty}(f,g) = \lim_{n\to\pm\infty} W_n(f,g)$ exists if $f, g, \tau f$, and τg are square summable near $\pm\infty$.

The following sections generalize some well-known facts about Sturm–Liouville operators (to be found, e.g., in [12],[31],[71],[76]) to Jacobi operators. The presented material is essentially taken from [1],[3],[5],[9].

1.2 Weyl m-functions

Let $c_\alpha(z,.), s_\alpha(z,.)$ be the fundamental system of (1.8) corresponding to the initial conditions

$$\begin{aligned} s_\alpha(z,0) &= -\sin(\alpha), & s_\alpha(z,1) &= \cos(\alpha),\\ c_\alpha(z,0) &= \frac{\cos(\alpha)}{a(0)}, & c_\alpha(z,1) &= \frac{\sin(\alpha)}{a(0)} \end{aligned} \tag{1.11}$$

such that

$$W(c_\alpha(z), s_\alpha(z)) = 1. \tag{1.12}$$

Next pick $\lambda_1 \in \mathbb{R}$ and define the following rational function with respect to z,

$$m_N(z,\alpha) = \frac{W_N(s_\alpha(\lambda_1), c_\alpha(z))}{W_N(s_\alpha(\lambda_1), s_\alpha(z))}, \qquad N \in \mathbb{Z}\backslash\{0\}, \tag{1.13}$$

which has poles at the zeros $\lambda_j(N) \in \mathbb{R}$, $\lambda_1(N) \equiv \lambda_1$ of $W_N(s_\alpha(\lambda_1), s_\alpha(.)) = 0$. The fact that one can rewrite $m_N(z,\alpha)$ with λ_1 replaced by $\lambda_j(N)$ together with

$$\lim_{z\to\lambda_j(N)} W_N(s_\alpha(\lambda_j(N)), c_\alpha(z)) \;=\; -1, \tag{1.14}$$

$$\lim_{z\to\lambda_j(N)} \frac{W_N(s_\alpha(\lambda_j(N)), s_\alpha(z))}{z-\lambda_j(N)} \;=\; W_N(s_\alpha(\lambda_j(N)), \frac{d}{dz} s_\alpha(\lambda_j(N))) \tag{1.15}$$

imply that all poles of $m_N(z,\alpha)$ are simple. Using (1.7) to evaluate (1.15) one infers that ∓ 1 times the residue at $\lambda_j(N)$ is given by

$$\gamma_j(\alpha,N) = \Big(\sum_{n={1 \atop N+1}}^{{N \atop 0}} s_\alpha(\lambda_j(N),n)^2\Big)^{-1}, \quad N \gtrless 0. \tag{1.16}$$

The $\gamma_j(\alpha,N)$ are called norming constants. Hence one gets

$$m_N(z,\alpha) = \sum_j \frac{\mp\gamma_j(\alpha,N)}{z-\lambda_j(N)} + \begin{cases} \frac{\pm\tan(\alpha)^{\pm 1}}{a(0)}, & \alpha \in {[0,\pi) \atop (0,\pi]} \\ \frac{\pm z - b({1 \atop 0})}{a(0)^2}, & \alpha = {\pi \atop 0} \end{cases}, \quad N \gtrless 0. \tag{1.17}$$

(We note that $\lambda_j(N)$ depend on α for $j>1$.) Furthermore, the function

$$u_N(z,n) = c_\alpha(z,n) - m_N(z,\alpha) s_\alpha(z,n) \tag{1.18}$$

satisfies

$$\sum_{n={1 \atop N+1}}^{{N \atop 0}} |u_N(z,n)|^2 = \pm\frac{\mathrm{Im}(m_N(z,\alpha))}{\mathrm{Im}(z)}, \quad N \gtrless 0, \tag{1.19}$$

i.e., $\pm m_N(z,\alpha)$ are Herglotz functions for $N \gtrless 0$.

Next we want to investigate the limits $N \to \pm\infty$. Fix $z \in \mathbb{C}\backslash\mathbb{R}$. Then, as in the Sturm-Liouville case, the function $m_N(z,\alpha)$ (for different values of $\lambda_1 \in \mathbb{R}$) lies on a circle given by

$$\{m \in \mathbb{C} | [c_\alpha(z) - m s_\alpha(z)]_N = 0\}. \tag{1.20}$$

Since $[.]_N$ is decreasing in N for $N > 0$, the circle corresponding to $N+1$ lies inside the circle corresponding to N. Similarly for $N < 0$. Hence these circles either tend to a limit point or a limit circle, depending on whether

$$\sum^{\pm\infty} |s_\alpha(z,n)|^2 = \infty, \quad \text{or} \quad \sum^{\pm\infty} |s_\alpha(z,n)|^2 < \infty. \tag{1.21}$$

Accordingly, one says that τ is limit point (*l.p.*) respectively limit circle (*l.c.*) at $\pm\infty$. One can show that this definition is independent of $z \in \mathbb{C}\backslash\mathbb{R}$. Thus the pointwise convergence of $m_N(z,\alpha)$ is clear in the *l.p.* case. In the *l.c.* case both Wronskians in (1.13) converge and we may set

$$\tilde{m}_\pm(z,\alpha) = \lim_{N\to\pm\infty} m_N(z,\alpha). \tag{1.22}$$

Remark 1.2 *(i). $\tilde{m}_\pm(z,0)$ are not the usual Weyl m-functions defined in the literature. For a connection with the standard Weyl m-functions $m_\pm(z)$ see (1.47), (1.48). We have chosen to introduce $\tilde{m}_\pm(z,\alpha)$ in order to simplify our notation in various places.*
(ii). This explicit construction of converging sequences, even in the l.c. case, also works for Sturm-Liouville operators and seems to be novel to the best of our knowledge. Previously one usually proved the existence of such sequences using Helly's selection theorem (cf., e.g., [12]).

Moreover, the above sequences are locally bounded in z (fix an N and take all circles corresponding to a (sufficiently small) neighborhood of any point z and note that all following circles lie inside the ones corresponding to N) and by Vitali's theorem ([72], p. 168) they converge uniformly on every compact set in $\mathbb{C}_\pm = \{z \in \mathbb{C} | \pm \mathrm{Im}(z) > 0\}$, implying that $\pm\tilde{m}_\pm(z,\alpha)$ are again Herglotz functions.

Upon setting

$$u_\pm(z,n) = c_\alpha(z,n) - \tilde{m}_\pm(z,\alpha) s_\alpha(z,n) \tag{1.23}$$

we get a function which is square summable near $\pm\infty$

$$\sum_{n={1 \atop -\infty}}^{{\infty \atop 0}} |u_\pm(z,n)|^2 = \pm\frac{\mathrm{Im}(\tilde{m}_\pm(z,\alpha))}{\mathrm{Im}(z)}, \quad N \gtrless 0. \tag{1.24}$$

In addition,

$$W_{\pm\infty}(s_\alpha(\lambda_1), u_\pm(z)) = 0, \tag{1.25}$$

if τ is *l.c.* at $\pm\infty$. We remark that (independently of the *l.c.* and *l.p.* case at $\pm\infty$)

$$\tilde{m}_\pm(z) = \tilde{m}_\pm(z, 0) = \frac{-u_\pm(z,1)}{a(0)u_\pm(z,0)} \tag{1.26}$$

and that $\tilde{m}_\pm(z, \alpha)$ can be expressed in terms of $\tilde{m}_\pm(z, \beta)$ (use that $u_\pm$ is unique up to a constant) by

$$\tilde{m}_\pm(z, \alpha) = \frac{1}{a(0)} \frac{a(0)\cos(\beta - \alpha)\tilde{m}_\pm(z, \beta) - \sin(\beta - \alpha)}{a(0)\sin(\beta - \alpha)\tilde{m}_\pm(z, \beta) + \cos(\beta - \alpha)}. \tag{1.27}$$

1.3 Weyl-Titchmarsh Theory on $\mathbb{N}$

Let H_+ be a given self-adjoint operator associated with τ on $\mathbb{N}$ and a Dirichlet boundary condition at $n = 0$. Abbreviate $s(z, n) = s_0(z, n)$ and let $u_+(z, n)$, $z \in \mathbb{C}\backslash\sigma(H_+)$ be a solution of (1.8) which is square summable near ∞ and fulfills the boundary condition at ∞ (if any). The resolvent of H_+ then reads

$$((H_+ - z)^{-1} f)(n) = \sum_{m\in\mathbb{N}} G_+(z, m, n) f(m), \quad z \in \mathbb{C}\backslash\sigma(H_+), \tag{1.28}$$

where

$$G_+(z, m, n) = \frac{1}{W(s(z), u_+(z))} \left\{ \begin{array}{ll} s(z, n)u_+(z, m), & m \geq n \\ s(z, m)u_+(z, n), & m \leq n \end{array} \right. . \tag{1.29}$$

Since $s(z, n)$ is a polynomial in z we infer by induction

$$s(H_+, n)\delta_1 = \delta_n, \qquad \delta_n(k) = \left\{ \begin{array}{ll} 1, & k = n \\ 0, & k \neq n \end{array} \right. , \tag{1.30}$$

implying that δ_1 is a cyclic vector for H_+. If $E_+(.)$ denotes the family of spectral projections corresponding to H_+ we introduce the measure

$$d\rho_+(.) = d\langle\delta_1, E_+(.)\delta_1\rangle. \tag{1.31}$$

Equation (1.30) now shows that the polynomials $s(z, n)$, $n \in \mathbb{N}$ are orthogonal with respect to this measure, i.e.,

$$\langle s(j), s(k)\rangle = \int_{-\infty}^{\infty} s(\lambda, j)s(\lambda, k)\, d\rho_+(\lambda) = \delta_j(k), \tag{1.32}$$

implying

$$a(n) = \langle s(n+1), \lambda s(n)\rangle, \quad b(n) = -\langle s(n), \lambda s(n)\rangle, \;\; n \in \mathbb{N}. \tag{1.33}$$

Now consider the following transformation U from the set $\ell_0(\mathbb{N})$ onto the set of polynomials

$$\begin{aligned} (Uf)(\lambda) &= \sum_{n=1}^{\infty} f(n) s(\lambda, n), & (1.34)\\ (U^{-1}F)(n) &= \int_{\mathbb{R}} s(\lambda, n) F(\lambda) d\rho_+(\lambda). & (1.35) \end{aligned}$$

A simple calculation for $F(\lambda) = (Uf)(\lambda)$ shows that

$$\sum_{n=1}^{\infty} |f(n)|^2 = \int_{\mathbb{R}} |F(\lambda)|^2 d\rho_+(\lambda). \tag{1.36}$$

Thus U extends to a unitary transformation

$$\tilde{U} : \ell^2(\mathbb{N}) \to L^2(\mathbb{R}, d\rho_+) \tag{1.37}$$

(since the set of polynomials is dense in $L^2(\mathbb{R}, d\rho_+)$, [5], Theorem VII.1.7) which maps the operator H_+ to the multiplication operator by λ,

$$\tilde{U} H_+ \tilde{U}^{-1} = \tilde{H}, \tag{1.38}$$

where

$$\tilde{H} F(\lambda) = \lambda F(\lambda), \quad \mathfrak{D}(\tilde{H}) = \{F \in L^2(\mathbb{R}, d\rho_+) | \lambda F(\lambda) \in L^2(\mathbb{R}, d\rho_+)\}. \tag{1.39}$$

This is easily verified for $f \in \ell_0(\mathbb{N})$. If τ is *l.p.* at ∞ note that $\ell_0(\mathbb{N})$ is a core for H_+ and if τ is *l.c.* at ∞ note that $d\rho_+$ is a pure point measure and that eigenfunctions are mapped onto eigenfunctions (all finite linear combinations of eigenfunctions form again a core).

This implies that the spectrum of H_+ can be characterized as follows. Let the Lebesgue decomposition of $d\rho_+$ be given by

$$d\rho_+ = d\rho_{+,p} + d\rho_{+,ac} + d\rho_{+,sc}, \tag{1.40}$$

then we have ($\rho_+(\lambda) = \int_{(-\infty,\lambda]} d\rho_+$)

$$\begin{aligned} \sigma(H_+) &= \{\lambda \in \mathbb{R} | \lambda \text{ is a growth point of } \rho_+\}, & (1.41)\\ \sigma_p(H_+) &= \{\lambda \in \mathbb{R} | \lambda \text{ is a growth point of } \rho_{+,p}\}, & (1.42)\\ \sigma_{ac}(H_+) &= \{\lambda \in \mathbb{R} | \lambda \text{ is a growth point of } \rho_{+,ac}\}, & (1.43)\\ \sigma_{sc}(H_+) &= \{\lambda \in \mathbb{R} | \lambda \text{ is a growth point of } \rho_{+,sc}\}. & (1.44) \end{aligned}$$

The Stieltjes transform of the spectral function ρ_+ is called the Weyl m-function

$$m_+(z) = \int_{\mathbb{R}} \frac{d\rho_+(\lambda)}{z-\lambda}, \qquad z \in \mathbb{C}\backslash\mathbb{R}. \tag{1.45}$$

Conversely, the spectral function ρ_+ can be recovered from $m_+(z)$ by the Stieltjes inversion formula

$$\rho_+(\lambda) = \frac{-1}{\pi} \lim_{\delta\downarrow 0} \lim_{\varepsilon\downarrow 0} \int_{-\infty}^{\lambda+\delta} \mathrm{Im}(m_+(\nu + \mathrm{i}\varepsilon)) d\nu. \tag{1.46}$$

We have normalized ρ_+ such that it is right continuous and satisfies $\lim_{\lambda\to-\infty} \rho_+(\lambda) = 0$. One infers

$$m_+(z) = G_+(z,1,1) = \frac{-u_+(1)}{a(0)u_+(0)} = \tilde{m}_+(z), \tag{1.47}$$

and we remark that the local compact convergence of $m_N(z,0)$ to $\tilde{m}_+(z) = m_+(z)$ implies the convergence of the associated spectral functions at every point of continuity ([2], p. 332). The second Weyl m-function is usually defined as

$$m_-(z) = G_-(z,-1,-1) = \frac{-u_-(-1)}{a(-1)u_-(0)} = -\frac{z + b(0) + a(0)^2\tilde{m}_-(z)}{a(-1)^2}. \tag{1.48}$$

$m_\pm(z)$, like $\pm\tilde{m}_\pm(z)$, are Herglotz functions.

1.4 Weyl–Titchmarsh Theory on $\mathbb{Z}$

In Section 1.3 we have dealt with the half-line $\mathbb{N}$. In this section we extend these results to all of $\mathbb{Z}$.

Let H be a given self-adjoint operator associated with τ. Let $u_\pm(z,n)$ be a solution of (1.8) which is square summable near $\pm\infty$ (provided such a solution exists) and fulfills the boundary condition at $\pm\infty$ if any. The resolvent of H then reads

$$((H-z)^{-1}f)(n) = \sum_{m\in\mathbb{Z}} G(z,m,n)f(m), \quad z \in \rho(H), \tag{1.49}$$

where

$$G(z,m,n) = \frac{1}{W(u_-(z),u_+(z))} \begin{cases} u_-(z,n)u_+(z,m), & m \geq n \\ u_-(z,m)u_+(z,n), & m \leq n \end{cases}. \tag{1.50}$$

Consider the vector-valued polynomials

$$\underline{S}(z,n) = \big(s(z,n), c(z,n)\big), \tag{1.51}$$

where $s(z,n)$, $c(z,n)$ are solutions of (1.8) satisfying the initial conditions

$$\begin{array}{ll} s(z,0)=0, & s(z,1)=1, \\ c(z,0)=1, & c(z,1)=0. \end{array} \tag{1.52}$$

The analog of (1.30) reads

$$s(H,n)\delta_1 + c(H,n)\delta_0 = \delta_n. \tag{1.53}$$

This is obvious for $n=0,1$ and the rest follows from induction upon applying H to (1.53). If $E(.)$ denotes the spectral resolution of the identity corresponding to H we introduce the measures

$$d\rho_{j,k}(.) = d\langle\delta_j, E(.)\delta_k\rangle, \tag{1.54}$$

and the (hermitian) matrix-valued measure

$$d\rho = \left(\begin{array}{cc} d\rho_{1,1} & d\rho_{1,2} \\ d\rho_{2,1} & d\rho_{2,2} \end{array}\right). \tag{1.55}$$

By (1.53) the vector-valued polynomials are orthogonal with respect to $d\rho$

$$\begin{aligned} \langle \underline{S}(m), \underline{S}(n)\rangle &= \sum_{j,k=1}^{2} \int_{\mathbb{R}} S_j(\lambda,m)\; S_k(\lambda,n) d\rho_{j,k}(\lambda) \\ &\equiv \int_{\mathbb{R}} \underline{S}(\lambda,m) d\rho(\lambda)\; \underline{S}(\lambda,n) = \delta_n(m). \end{aligned} \tag{1.56}$$

The analogous formulas to (1.33) then read

$$a(n) = \langle \underline{S}(n+1), \lambda\underline{S}(n)\rangle, \quad b(n) = \langle \underline{S}(n), \lambda\underline{S}(n)\rangle,\;\; n\in\mathbb{Z}. \tag{1.57}$$

Next we consider the following transformation U from the set $\ell_0(\mathbb{Z})$ onto the set of vector-valued polynomials

$$(Uf)(\lambda) = \sum_{n\in\mathbb{Z}} f(n)\underline{S}(\lambda,n), \tag{1.58}$$

$$(U^{-1}\underline{F})(n) = \int_{\mathbb{R}} \underline{S}(\lambda,n)d\rho(\lambda)\underline{F}(\lambda). \tag{1.59}$$

Again a simple calculation for $\underline{F}(\lambda) = (Uf)(\lambda)$ shows that

$$\sum_{n\in\mathbb{Z}} |f(n)|^2 = \int_{\mathbb{R}} \overline{\underline{F}(\lambda)} d\rho(z)\underline{F}(\lambda). \tag{1.60}$$

Thus U extends to a unitary transformation

$$\tilde{U} : \ell^2(\mathbb{Z}) \to L^2(\mathbb{R}, d\rho) \tag{1.61}$$

which maps the operator H to the multiplication operator by λ,

$$\tilde{U} H \tilde{U}^{-1} = \tilde{H}, \tag{1.62}$$

where

$$\tilde{H}\underline{F}(\lambda) = z\underline{F}(\lambda), \quad \mathfrak{D}(\tilde{H}) = \{\underline{F} \in L^2(\mathbb{R}, d\rho) | \lambda\underline{F}(\lambda) \in L^2(\mathbb{R}, d\rho)\}. \tag{1.63}$$

In order to characterize the spectrum of H one only needs to consider the trace $d\rho^t$ of $d\rho$

$$d\rho^t = d\rho_{1,1} + d\rho_{2,2}. \tag{1.64}$$

Let the Lebesgue decomposition of $d\rho^t$ be given by

$$d\rho^t = d\rho^t_p + d\rho^t_{ac} + d\rho^t_{sc}, \tag{1.65}$$

then we have ($\rho^t(\lambda) = \int_{(-\infty,\lambda]} d\rho^t$, etc.)

$$\begin{aligned}
\sigma(H) &= \{\lambda \in \mathbb{R} | \lambda \text{ is a growth point of } \rho^t\}, && (1.66)\\
\sigma_p(H) &= \{\lambda \in \mathbb{R} | \lambda \text{ is a growth point of } \rho^t_p\}, && (1.67)\\
\sigma_{ac}(H) &= \{\lambda \in \mathbb{R} | \lambda \text{ is a growth point of } \rho^t_{ac}\}, && (1.68)\\
\sigma_{sc}(H) &= \{\lambda \in \mathbb{R} | \lambda \text{ is a growth point of } \rho^t_{sc}\}. && (1.69)
\end{aligned}$$

The Weyl-matrix $M(z)$ is defined as

$$M(z) = \int_{-\infty}^{\infty} \frac{d\rho(\lambda)}{z-\lambda}, \qquad z \in \mathbb{C}\backslash\mathbb{R}. \tag{1.70}$$

Explicit evaluation yields

$$\begin{aligned}
M(z) &= \begin{pmatrix} G(z,0,0) & G(z,1,0) \\ G(z,0,1) & G(z,1,1) \end{pmatrix} \\
&= \frac{a(0)^{-2}}{\tilde{m}_-(z) - \tilde{m}_+(z)} \begin{pmatrix} 1 & -a(0)\tilde{m}_+(z) \\ -a(0)\tilde{m}_+(z) & a(0)^2\tilde{m}_+(z)\tilde{m}_-(z) \end{pmatrix}.
\end{aligned} \tag{1.71}$$

Finally, assuming ρ to be right continuous and normalizing $\rho(-\infty) = 0$ one obtains

$$\rho_{j,k}(\lambda) = \frac{-1}{\pi} \lim_{\delta\downarrow 0} \lim_{\varepsilon\downarrow 0} \int_{-\infty}^{\lambda+\delta} \mathrm{Im}(M_{j,k}(\nu + \mathrm{i}\varepsilon)) d\nu, \quad 1 \le j,k \le 2. \tag{1.72}$$

1.5 Some Useful Lemmas

This section provides some useful results needed later on. Denote by $s(z,n)$ and $c(z,n)$ the solutions of $\tau u = zu$ corresponding to the initial conditions $s(z,0) = c(z,1) = 0$, $s(z,1) = c(z,0) = 1$.

Lemma 1.3 *Let $\lambda_0 < \lambda_1$ be such that $[\lambda_0, \lambda_1] \cap \sigma_{ess}(H_+) = \emptyset$. Then there exists a solution $u_+(z,.) \in \ell^2_+(\mathbb{Z})$ of $\tau u = zu$ satisfying the boundary condition of H at $+\infty$ (if any) which is holomorphic with respect to z for $z \in \mathbb{C}\backslash((-\infty, \lambda_0] \cup [\lambda_1, \infty))$. In addition, we can assume $u_+(z,.) \not\equiv 0$ and $\overline{u_+(z,.)} = u_+(\overline{z},.)$.*

Similarly, $[\lambda_0, \lambda_1] \cap \sigma_{ess}(H_-) = \emptyset$ implies the existence of a solution $u_-(z,.) \in \ell_-(\mathbb{Z})$ fulfilling the boundary condition of H at $-\infty$ (if any) and, as a function of z, satisfies the same conditions as $u_+(z,.)$.

Proof. Explicitly, we can set

$$u_\pm(z,n) = \Big(\prod_{\mu \in \sigma(H_+) \cap [\lambda_0, \lambda_1]} (z-\mu)\Big)\Big(a(0)^{-1} c(z,n) - \tilde{m}_\pm(z) s(z,n)\Big). \qquad (1.73)$$

□

Lemma 1.4 *Suppose $a(n) < 0$ and let $\lambda < \inf \sigma(H)$. Then we can assume*

$$u_\pm(\lambda, n) > 0, \quad n \in \mathbb{Z}, \qquad (1.74)$$

$$n\, s(\lambda, n) > 0, \quad n \in \mathbb{Z}\backslash\{0\}. \qquad (1.75)$$

They solutions $u_\pm(\lambda,.)$ are called principal solutions of $(H-\lambda)u = 0$ near $\pm\infty$ in [43].

Proof. From $(H-\lambda) > 0$ one infers $(H_{+,n} - \lambda) > 0$ and hence

$$0 < \langle \delta_{n+1}, (H_{+,n} - \lambda)^{-1} \delta_{n+1} \rangle = \frac{u_+(\lambda, n+1)}{-a(n) u_+(\lambda, n)} \qquad (1.76)$$

showing that $u_+(\lambda)$ can be chosen to be positive. Furthermore, for $n > 0$ we obtain

$$0 < \langle \delta_n, (H_+ - \lambda)^{-1} \delta_n \rangle = \frac{u_+(\lambda, n) s(\lambda, n)}{-a(0) u_+(\lambda, 0)} \qquad (1.77)$$

implying $s(\lambda, n) > 0$ for $n > 0$. Similarly one proves the remaining results. □

Let $u_\pm(z,n)$ are solutions of $\tau u = zu$ as in Lemma 1.3. Then Green's formula (1.7) implies

$$W_n(u_+(z), u_+(\tilde{z})) = (z - \tilde{z}) \sum_{j=n+1}^{\infty} u_+(z,j) u_+(\tilde{z}, j) \qquad (1.78)$$

and furthermore,

$$W_n(u_+(z), \dot{u}_+(z)) = \lim_{\tilde{z}\to z} W_n(u_+(z), \frac{u_+(z) - u_+(\tilde{z})}{z - \tilde{z}}) = \sum_{j=n+1}^{\infty} u_+(z,j)^2. \qquad (1.79)$$

Here the dot denotes the derivative with respect to z. An analogous result holds for $u_-(z,n)$. Interchanging limit and summation can be justified using (cf. Remark 2.8)

$$u_+(\tilde{z}, j) = const(\tilde{z})(H^{\beta}_{+,n-1} - \tilde{z})^{-1}\delta_n(j) \quad \text{for } j \le n \qquad (1.80)$$

(with β such that $z \notin \sigma(H^{\beta}_{+,n-1})$) and the first resolvent identity. Summarizing (compare [3], Theorem 4.2.2):

Lemma 1.5 *Let $u_\pm(z,n)$ be solutions of $\tau u = zu$ as in Lemma 1.3. Then we have*

$$W_n(u_\pm(z), \dot{u}_\pm(z)) = \begin{cases} -\sum\limits_{j=n+1}^{\infty} u_+(z,j)^2 \\ \sum\limits_{j=-\infty}^{n} u_-(z,j)^2 \end{cases}. \qquad (1.81)$$

Chapter 2

Oscillation Theory

2.1 Introduction

In 1836 Sturm originated the investigations of oscillation properties of solutions of second-order differential and difference equations [70]. Since then numerous extensions have been made. Especially, around 1948, Hartman and others have shown the following in a series of papers ([44], [45], [46]). For a given Sturm–Liouville operator H on $L^2(0,\infty)$, the dimension of the spectral projection $P_{(-\infty,\lambda)}(H)$ equals the number of zeros of certain solutions of $Hu = \lambda u$. Moreover, the dimension of $P_{(\lambda_1,\lambda_2)}(H)$ can be obtained by considering the difference of the number of zeros inside a finite interval $(0, x)$ of two solutions corresponding to their respective spectral parameters λ_1 and λ_2, and performing a limit $x \to \infty$. Only recently it was shown in [39] by F. Gesztesy, B. Simon, and myself that these limits can be avoided by using a renormalized version of oscillation theory, that is, counting zeros of Wronskians of solutions instead.

This naturally raises the question whether similar results hold for second-order difference equations. Despite a variety of literature on this subject (cf., e.g., [3], [7], [18], [27], [37], [41], Sections 14 and 37, [43], [47], [48], [49], [52], [61], [62] and the references therein) only a few things concerning the connections between oscillation properties of solutions and spectra of the corresponding operators appear to be known. In particular, the analogs of the aforementioned theorems seem to be unknown. Moreover, even the analog of the well-known fact that the n-th eigenfunction of a Sturm-Liouville operator (below the essential spectrum) has $n-1$ nodes is only known in the special case of finite Jacobi operators (i.e., finite tri-diagonal matrices) [3], Theorem 4.3.5, [27]. The present thesis aims at filling these gaps and provides a complete solution to these problems.

Now, we want to give the reader an intuitive idea of how oscillation theory works. In the sequel a solution of $\tau u = \lambda u$, $\lambda \in \mathbb{R}$ will always mean a real-valued, non-zero solution. We first need to define what we mean by a node of a real-valued sequence

$u \in \ell(\mathbb{Z})$. A point $n \in \mathbb{Z}$, is called a node of u if either

$$u(n) = 0 \quad \text{or} \quad a(n)u(n)u(n+1) > 0. \tag{2.1}$$

In the special case $a(n) < 0$, $n \in \mathbb{Z}$ a node of u is precisely a sign flip of u as one would expect. In the general case, however, one has to take the sign of $a(n)$ into account.

For simplicity we shall assume $a(n) < 0$ (cf. Lemma 2.2) and a, b bounded (implying H bounded) for the remainder of this section.

By Lemma 1.3 $u_-(\lambda, .)$ can be assumed to be continuous with respect to λ as long as λ is below the essential spectrum of H. In addition, $u_-(\lambda, .)$ can be assumed positive for λ below the spectrum of H and hence has no nodes in this case. Increasing λ one needs to observe three things: (i) Nodes of $u_-(\lambda)$ move to the right (by (2.34)) without colliding; (ii) $u_-(\lambda)$ cannot pick up nodes locally (by (2.14)); (iii) $u_-(\lambda)$ cannot lose nodes at $-\infty$. By (i) and (ii) we infer that $u_-(\lambda)$ can only pick up nodes at $+\infty$. Intuitively this happens if $u_-(\lambda) \in \ell^2(\mathbb{Z})$ (or equivalently, if λ an eigenvalue of H) and hence $\lim_{n\to\infty} u_-(\lambda, n) = 0$. Summarizing, $u_-(\lambda)$ has no nodes below the spectrum of H and picks up one additional node whenever λ is an eigenvalue of H. Since no nodes get lost we are lead to (cf. Theorem 2.13)

$$\dim \operatorname{Ran} P_{(-\infty,\lambda)}(H) = \#(u_-(\lambda)), \tag{2.2}$$

where $\#(u)$ denotes the total number of nodes of u and $P_\Omega(H)$ is the spectral projection of H corresponding to the Borel set $\Omega \subseteq \mathbb{R}$. As a corollary we conclude, as already anticipated, that the n-th eigenfunction (below the essential spectrum) has $n-1$ nodes.

To obtain the number of eigenvalues between two given values λ_1 and λ_2 it seems natural to consider $\#(u_-(\lambda_2)) - \#(u_-(\lambda_1))$. This gives nothing new below the essential spectrum and otherwise we have $\#(u) = \infty$ for any solution of $\tau u = \lambda u$ with λ above the infimum of the essential spectrum. Hence, a naive use of oscillation theory in the latter case yields $\infty - \infty$. There are two ways to overcome this problem. The first, due to [45] in the case of differential operators, uses a limiting procedure which only works for half-line operators and can be found in Theorem 2.16. The second, due to [39] in the case of differential operators, uses the fact that the nodes of the Wronskian of two solutions u_1, u_2 corresponding to λ_1, λ_2, respectively, essentially counts the additional nodes of u_2 with respect to u_1 (cf. Corollary 2.18). In this sense the Wronskian comes with a built-in renormalization. Moreover, the nodes of Wronskians behave similar to the nodes of solutions and satisfy the above properties (i), (ii), and (iii) as well. Hence, similar techniques apply.

To give rigorous proofs for the indicated results we first introduce and investigate Prüfer variables in Section 2.2. They will be our main tool in Section 2.3 and Section 2.4 where our major theorems are derived. Section 2.5 uses the results of Section 2.3 and 2.4 to investigate the spectra of short-range perturbations of periodic Jacobi operators.

2.2 Prüfer Variables

For the rest of this chapter we assume for convenience

Hypothesis H. 2.1 *Suppose*

$$a, b \in \ell(\mathbb{Z}), \qquad a(n) < 0, b(n) \in \mathbb{R}. \tag{2.3}$$

We remark that the case $a(n) \neq 0$ can be reduced to the case $a(n) > 0$ or $a(n) < 0$ (cf., e.g., [21], p. 141). In fact one has

Lemma 2.2 *Assume (H.1.1) and let H be a Jacobi operator associated with the difference expression (3.25). Introduce a_ε by*

$$a_\varepsilon(n) = \varepsilon(n)a(n),\ \varepsilon(n) \in \{+1, -1\}, \quad n \in \mathbb{Z} \tag{2.4}$$

and the unitary operator U_ε by

$$U_\varepsilon = \{\tilde{\varepsilon}(n)\delta_{m,n}\}_{m,n\in\mathbb{Z}}, \quad \tilde{\varepsilon}(n) \in \{+1, -1\},\ \tilde{\varepsilon}(n)\tilde{\varepsilon}(n+1) = \varepsilon(n). \tag{2.5}$$

Then H_ε defined as

$$H_\varepsilon = U_\varepsilon^{-1} H U_\varepsilon, \tag{2.6}$$

is associated with the difference expression

$$(\tau_\varepsilon f)(n) = a_\varepsilon(n)f(n+1) + a_\varepsilon(n-1)f(n-1) - b_\varepsilon(n)f(n). \tag{2.7}$$

In particular, H_ε is unitarily equivalent to H.

In addition, by a solution of $\tau u = \lambda u$, $\lambda \in \mathbb{R}$ we will always mean a real-valued solution not vanishing identically.

Given a solution $u(\lambda, .)$ of $\tau u = \lambda u$, $\lambda \in \mathbb{R}$ we introduce Prüfer variables $\rho_u(\lambda, .)$, $\theta_u(\lambda, .)$ via

$$u(\lambda, n) = \rho_u(\lambda, n) \sin\theta_u(\lambda, n), \tag{2.8}$$
$$u(\lambda, n+1) = \rho_u(\lambda, n) \cos\theta_u(\lambda, n). \tag{2.9}$$

Notice that the Prüfer angle $\theta_u(\lambda, .n)$ is only defined up to an additive integer multiple of 2π (which depends on n).

Inserting (2.8), (2.9) into $(\tau - \lambda)u = 0$ yields

$$a(n)\cot\theta_u(\lambda, n) + a(n-1)\tan\theta_u(\lambda, n-1) = b(n) + \lambda, \tag{2.10}$$

$$\rho_u(\lambda, n)\sin\theta_u(\lambda, n) = \rho_u(\lambda, n-1)\cos\theta_u(\lambda, n-1). \tag{2.11}$$

Equation (2.10) is a discrete Riccati equation (cf. [48]) for $\cot\theta_u(n)$ and (2.11) can be solved if $\theta_u(n)$ is known provided it is replaced by

$$a(n)\rho_u(\lambda,n) = a(n-1)\rho_u(\lambda,n-1) = 0 \tag{2.12}$$

if $\sin\theta_u(\lambda,n) = \cos\theta_u(\lambda,n-1) = 0$ (use $\tau u = \lambda u$ and (2.14) below). The Wronskian of two solutions $u_{1,2}(\lambda_{1,2},n)$ reads

$$W_n(u_1(\lambda_1),u_2(\lambda_2)) = a(n)\rho_{u_1}(\lambda_1,n)\rho_{u_2}(\lambda_2,n)\sin(\theta_{u_1}(\lambda_1,n)-\theta_{u_2}(\lambda_2,n)). \tag{2.13}$$

The next lemma considers nodes of solutions and their Wronskians more closely.

Lemma 2.3 *Let $u_{1,2}$ be solutions of $\tau u_{1,2} = \lambda_{1,2}u_{1,2}$ corresponding to $\lambda_1 \neq \lambda_2$, respectively. Then*

$$u_1(n) = 0 \quad \Rightarrow \quad u_1(n-1)u_1(n+1) < 0. \tag{2.14}$$

Moreover, suppose $W_n(u_1,u_2) = 0$ but $W_{n-1}(u_1,u_2)W_{n+1}(u_1,u_2) \neq 0$, then

$$W_{n-1}(u_1,u_2)W_{n+1}(u_1,u_2) < 0. \tag{2.15}$$

Otherwise, if $W_n(u_1,u_2) = W_{n+1}(u_1,u_2) = 0$, then necessarily

$$u_1(n+1) = u_2(n+1) = 0, \quad \text{and} \quad W_{n-1}(u_1,u_2)W_{n+2}(u_1,u_2) < 0. \tag{2.16}$$

Proof. The fact $u(n) = 0$ implies $u_1(n-1)u_1(n+1) \neq 0$ (otherwise u_1 vanishes identically) and $a(n)u_1(n+1) = -a(n-1)u_1(n-1)$ (from $\tau u = \lambda u$) shows $u_1(n-1)u_1(n+1) < 0$.

Next, $W_n(u_1,u_2) = 0$ is equivalent to $u_1(n) = cu_2(n)$, $u_1(n+1) = cu_2(n+1)$ for some $c \neq 0$ and from (1.7) we infer

$$W_{n+1}(u_1,u_2) - W_n(u_1,u_2) = (\lambda_2-\lambda_1)u_1(n+1)u_2(n+1). \tag{2.17}$$

Applying the above formula gives

$$W_{n-1}(u_1,u_2)W_{n+1}(u_1,u_2) = -c^2(\lambda_2-\lambda_1)^2u_1(n)^2u_1(n+1)^2 \tag{2.18}$$

proving the first claim. If $W_n(u_1,u_2)$, $W_{n+1}(u_1,u_2)$ are both zero we must have $u_1(n+1) = u_2(n+1) = 0$ and as before $W_{n-1}(u_1,u_2)W_{n+1}(u_1,u_2) = -(\lambda_2-\lambda_1)^2u_1(n-1)u_1(n+2)u_2(n-1)u_2(n+2)$. Hence the claim follows from the first part. □

We can make the Prüfer angel $\theta_u(\lambda,.)$ unique by fixing, for instance, $\theta_u(\lambda,0)$ and requiring

$$[\![\theta_u(\lambda,n)/\pi]\!] \leq [\![\theta_u(\lambda,n+1)/\pi]\!] \leq [\![\theta_u(\lambda,n)/\pi]\!] + 1, \tag{2.19}$$

where

$$[\![x]\!] = \sup\{n \in \mathbb{Z} \,|\, n < x\}. \tag{2.20}$$

Lemma 2.4 *Let $\Omega \subseteq \mathbb{R}$ be an interval. Suppose $u(\lambda, n)$ is continuous with respect to $\lambda \in \Omega$ and (2.19) holds for one $\lambda_0 \in \Omega$. Then it holds for all $\lambda \in \Omega$ if we require $\theta_u(.,n) \in C(\Omega)$.*

Proof. Fix n and set

$$\theta_u(\lambda, n) = k\pi + \delta(\lambda), \quad \theta_u(\lambda, n+1) = k\pi + \Delta(\lambda), \quad k \in \mathbb{Z}, \tag{2.21}$$

where $\delta(\lambda) \in (0, \pi]$, $\Delta(\lambda) \in (0, 2\pi]$. If (2.19) should break down then by continuity we must have one of the following cases for some $\lambda_1 \in \Omega$. (i) $\delta(\lambda_1) = 0$ and $\Delta(\lambda_1) \in (\pi, 2\pi)$, (ii) $\delta(\lambda_1) = \pi$ and $\Delta(\lambda_1) \in (0, \pi)$, (iii) $\Delta(\lambda_1) = 0$ and $\delta(\lambda_1) \in (0, \pi)$, (iv) $\Delta(\lambda_1) = 2\pi$ and $\delta(\lambda_1) \in (0, \pi)$. Abbreviate $R = \rho(\lambda_1, n)\rho(\lambda_1, n+1)$. Case (i) implies $0 > \sin(\Delta(\lambda_1)) = \cos(k\pi)\sin(k\pi + \Delta(\lambda_1)) = R^{-1}u(\lambda_1, n+1)^2 > 0$, contradicting (i). Case (ii) is similar. Case (iii) implies $\delta(\lambda_1) = \pi/2$ and hence $1 = \sin(k\pi + \pi/2)\cos(k\pi) = R^{-1}u(\lambda_1, n)u(\lambda_1, n+2)$ contradicting (2.14). Again, case (iv) is similar. □

Let us call a point $n \in \mathbb{Z}$ a node of a solution u if either $u(n) = 0$ or $a(n)u(n)u(n+1) > 0$. Then, $[\![\theta_u(n)/\pi]\!] = [\![\theta_u(n+1)/\pi]\!]$ implies no node at n. Conversely, if $[\![\theta_u(n+1)/\pi]\!] = [\![\theta_u(n)/\pi]\!] + 1$, then n is a node by (2.14). Denote by $\#(u)$ the total number of nodes of u and by $\#_{(m,n)}(u)$ the number of nodes of u between m and n. More precisely, we shall say that a node n_0 of u lies between m and n if either $m < n_0 < n$ or if $n_0 = m$ but $u(m) \neq 0$. Hence we conclude

Lemma 2.5 *Let $m < n$. Then we have for any solution u*

$$\#_{(m,n)}(u) = [\![\theta_u(n)/\pi]\!] - \lim_{\varepsilon \downarrow 0} [\![\theta_u(m)/\pi + \varepsilon]\!] \tag{2.22}$$

and

$$\#(u) = \lim_{n \to \infty} \Big([\![\theta_u(n)/\pi]\!] - [\![\theta_u(-n)/\pi]\!] \Big). \tag{2.23}$$

Next, we recall the well-known analog of Sturm's theorem for differential equations and include a proof for the sake of completeness (cf., [3],[62]).

Lemma 2.6 *Let $u_{1,2}$ be solutions of $\tau u = \lambda u$ corresponding to $\lambda_1 \leq \lambda_2$. Suppose $m < n$ are two consecutive points which are either nodes of u_1 or zeros of $W_.(u_1, u_2)$ (the cases $m = -\infty$ or $n = +\infty$ are allowed if u_1 and u_2 are both in $\ell^2_\pm(\mathbb{Z})$ and $W_{\pm\infty}(u_1, u_2) = 0$ respectively) such that u_1 has no further node between m and n. Then u_2 has at least one node between m and $n+1$. Moreover, suppose $m_1 < \cdots < m_k$ are consecutive nodes of u_1. Then u_2 has at least $k-1$ nodes between m_1 and m_k. Hence we even have*

$$\#_{(m,n)}(u_2) \geq \#_{(m,n)}(u_1) - 1. \tag{2.24}$$

Proof. Suppose u_2 has no node between m and $n+1$. Hence we may assume (perhaps after flipping signs) that $u_1(j) > 0$ for $m < j < n$, $u_1(n) \geq 0$, and $u_2(j) > 0$ for $m \leq j \leq n$. Moreover, $u_1(m) \leq 0$, $u_1(n+1) < 0$ and $u_2(n+1) \geq 0$ provided m, n are finite. By Green's formula (1.7)

$$0 \leq (\lambda_2 - \lambda_1) \sum_{j=m+1}^{n} u_1(j) u_2(j) = W_n(u_1, u_2) - W_m(u_1, u_2). \tag{2.25}$$

Evaluating the Wronskians shows $W_n(u_1, u_2) < 0$, $W_m(u_1, u_2) > 0$ which is a contradiction.

It remains to prove the last part. We will use induction on k. The case $k = 1$ is trivial and $k = 2$ has already been proven. Denote the nodes of u_2 lower or equal than m_{k+1} by $n_k > n_{k-1} > \cdots$. If $n_k > m_k$ we are done since there are $k-1$ nodes n such that $m_1 \leq n \leq m_k$ by induction hypothesis. Otherwise we can find k_0, $0 \leq k_0 \leq k$ such that $m_j = n_j$ for $1 + k_0 \leq j \leq k$. If $k_0 = 0$ we are clearly done and we can suppose $k_0 \geq 1$. By induction hypothesis it suffices to show that there are $k - k_0$ nodes n of u_2 with $m_{k_0} \leq n \leq m_{k+1}$. By assumption $m_j = n_j$, $1 + k_0 \leq j \leq k$ are the only nodes n of u_2 such that $m_{k_0} \leq n \leq m_{k+1}$. Abbreviate $m = m_{k_0}$, $n = m_{k+1}$ and assume without restriction $u_1(m+1) > 0$, $u_2(m) > 0$. Since the nodes of u_1 and u_2 coincide we infer $0 < \sum_{j=m+1}^{n} u_1(j) u_2(j)$ and we can proceed as in the first part to obtain a contradiction. □

We call τ oscillatory if one solution of $\tau u = 0$ has an infinite number of nodes. In addition, we call τ oscillatory at $\pm\infty$ if one solution of $\tau u = 0$ has an infinite number of nodes near $\pm\infty$. We remark that if one solution of $(\tau - \lambda)u = 0$ has infinitely many nodes so has any other (corresponding to the same λ) by (2.24). Furthermore, $\tau - \lambda_1$ oscillatory implies $\tau - \lambda_2$ oscillatory for all $\lambda_2 > \lambda_1$ (again by (2.24)).

Now we turn to the special solution $s(\lambda, n)$ characterized via the initial conditions $s(\lambda, 0) = 0$, $s(\lambda, 1) = 1$. As in Lemma 1.5 we infer

$$W_n(s(\lambda), \dot{s}(\lambda)) = \sum_{j=n+1}^{0} s(\lambda, j)^2, \quad n < -1, \tag{2.26}$$

$$W_n(s(\lambda), \dot{s}(\lambda)) = \sum_{j=1}^{n} s(\lambda, j)^2, \quad n \geq 1. \tag{2.27}$$

Here the dot denotes the derivative with respect to λ. Notice also $W_{-1}(s(\lambda), \dot{s}(\lambda)) = W_0(s(\lambda), \dot{s}(\lambda)) = 0$. Evaluating the above equation using Prüfer variables shows

$$\dot{\theta}_s(\lambda, n) = \frac{\sum_{j=1}^{n} s(\lambda, j)^2}{-a(n)\rho_s(\lambda, n)^2} > 0, \quad n \geq 1, \tag{2.28}$$

$$\dot{\theta}_s(\lambda, n) = \frac{\sum_{j=n+1}^{0} s(\lambda, j)^2}{a(n)\rho_s(\lambda, n)^2} < 0, \quad n < -1. \tag{2.29}$$

Notice, again that $\dot{\theta}_s(\lambda,-1) = \dot{\theta}_s(\lambda,0) = 0$. Equation (2.28) implies that nodes of $s(\lambda,n)$ for $n \in \mathbb{N}$ move monotonically to the left without colliding (cf., [3] Theorem 4.3.4). In addition, since $s(\lambda,n)$ cannot pick up nodes locally by (2.14), all nodes must enter at ∞ and since $\dot{\theta}_s(\lambda,0) = 0$ they are trapped inside $(0,\infty)$.

We shall normalize $\theta_s(\lambda,0) = 0$ and hence $\theta_s(\lambda,-1) = -\pi/2$. Since $s(\lambda,n)$ is a polynomial in λ we easily infer $s(\lambda,n) \gtrless 0$ for fixed $n \gtrless 0$ and λ sufficiently small. This implies

$$-\pi < \theta_s(\lambda,n) < -\pi/2,\ n < -1, \quad 0 < \theta_s(\lambda,n) < \pi,\ n \geq 1, \tag{2.30}$$

for fixed n and λ sufficiently small. Moreover, dividing (2.10) by λ and letting $\lambda \to -\infty$ using (2.30) shows

$$\lim_{\lambda\to\pm\infty} \frac{\cot(\theta_s(\lambda,n))^{\pm 1}}{\lambda} = \frac{1}{a(n)}, \quad n \begin{matrix} \geq +1 \\ < -1 \end{matrix} \tag{2.31}$$

and hence

$$\theta_s(\lambda,n) = -\frac{\pi}{2} - \frac{a(n)}{\lambda} + o(\frac{1}{\lambda}),\ n < -1, \quad \theta_s(\lambda,n) = \frac{a(n)}{\lambda} + o(\frac{1}{\lambda}),\ n \geq 1, \tag{2.32}$$

as $\lambda \to -\infty$.

Analogously, let $u_\pm(\lambda,n)$ be solutions of $\tau u = \lambda u$ as in Lemma 1.3. Then Lemma 1.5 implies

$$\dot{\theta}_+(\lambda,n) = \frac{\sum_{j=n+1}^{\infty} u_+(\lambda,j)^2}{a(n)\rho_+(\lambda,n)^2} < 0, \tag{2.33}$$

$$\dot{\theta}_-(\lambda,n) = \frac{\sum_{j=-\infty}^{n} u_-(\lambda,j)^2}{-a(n)\rho_-(\lambda,n)^2} > 0, \tag{2.34}$$

where we have abbreviated $\rho_{u_\pm} = \rho_\pm$, $\theta_{u_\pm} = \theta_\pm$.

If H is bounded from below we can normalize

$$0 < \theta_\mp(\lambda,n) < \pi/2, \quad n \in \mathbb{Z}, \quad \lambda < \inf \sigma(H) \tag{2.35}$$

and we get as before

$$\theta_-(\lambda,n) = \frac{a(n)}{\lambda} + o(\frac{1}{\lambda}), \quad \theta_+(\lambda,n) = \frac{\pi}{2} - \frac{a(n)}{\lambda} + o(\frac{1}{\lambda}), \quad n \in \mathbb{Z} \tag{2.36}$$

as $\lambda \to -\infty$.

2.3 Standard Oscillation Theory

First of all we recall ([39], Lemma 5.2).

Lemma 2.7 *Let H, H_n be self-adjoint operators and $H_n \to H$ in strong resolvent sense as $n \to \infty$. Then*

$$\dim \operatorname{Ran} P_{(\lambda_1,\lambda_2)}(H) \le \liminf_{n\to\infty} \dim \operatorname{Ran} P_{(\lambda_1,\lambda_2)}(H_n). \tag{2.37}$$

Our first theorem considers half-line operators $H_\pm$ associated with a Dirichlet boundary condition at $n = 0$, that is, the following restrictions of H to the subspaces $\ell^2(\pm\mathbb{N})$,

$$\begin{array}{rccl} H_\pm : & \mathfrak{D}(H_\pm) & \to & \ell^2(\pm\mathbb{N}) \\ & f(n) & \mapsto & \left\{ \begin{array}{ll} a({}^{+1}_{-2})f(\pm 2) - b(\pm 1)f(\pm 1), & n = \pm 1 \\ (\tau f)(n), & n \begin{smallmatrix} < \\ > \end{smallmatrix} \pm 1 \end{array} \right. \end{array}, \tag{2.38}$$

with

$$\mathfrak{D}(H_\pm) = \{f \in \ell^2(\pm\mathbb{N}) | \tau f \in \ell^2(\pm\mathbb{N}), \lim_{n\to\pm\infty} W_n(u_\pm(z_0), f) = 0\}. \tag{2.39}$$

Similarly one defines finite restriction H_{n_1,n_2} to the subspaces $\ell^2(n_1, n_2)$ with Dirichlet boundary conditions at $n = n_1$ and $n = n_2$.

Remark 2.8 *We only consider the case of a Dirichlet boundary condition at $n = 0$ since the operators $H^\beta_{\pm,n_0}$ on $\ell^2(n_0, \pm\infty)$ associated with the general boundary condition*

$$f(n_0 + 1) + \beta f(n_0) = 0, \qquad \beta \in \mathbb{R} \cup \{\infty\} \tag{2.40}$$

at $n = n_0$ can be reduced to this case by a simple shift and altering the sequence b at one point. More precisely, we have

$$H^0_{+,n_0} = H_{+,n_0+1}, \quad H^\beta_{+,n_0} = H_{+,n_0} - a(n_0)\beta^{-1}\langle\delta_{n_0+1}, .\rangle\delta_{n_0+1}, \quad \beta \ne 0, \tag{2.41}$$

and

$$H^\infty_{-,n_0} = H_{-,n_0}, \quad H^\beta_{-,n_0} = H_{-,n_0+1} - a(n_0)\beta\langle\delta_{n_0}, .\rangle\delta_{n_0}, \quad \beta \ne \infty, \tag{2.42}$$

where $\delta_{n_0}(n) = 1$ if $n = n_0$ and $\delta_{n_0}(n) = 0$ otherwise. Hence all one has to do is alter the definition of $b(n_0)$ or $b(n_0 + 1)$. Analogously one defines the corresponding finite operators $H^{\beta_1,\beta_2}_{n_1,n_2}$ which will be used in the next section.

Theorem 2.9 *Let $\lambda \in \mathbb{R}$. Suppose τ is l.p. at $+\infty$ or $\lambda \in \sigma_p(H_+)$. Then*

$$\dim \operatorname{Ran} P_{(-\infty,\lambda)}(H_+) = \#_{(0,+\infty)}(s(\lambda)). \tag{2.43}$$

The same theorem holds if $+$ is replaced by $-$.

Proof. We only carry out the proof for the plus sign (the other part following from reflection). By virtue of (2.28), (2.32), and Lemma 2.5 we infer

$$\dim \operatorname{Ran} P_{(-\infty,\lambda)}(H_{0,n}) = [\![\theta_s(\lambda,n)/\pi]\!] = \#_{(0,n)}(s(\lambda)), \quad n > 1, \tag{2.44}$$

since $\lambda \in \sigma(H_{0,n})$ if and only if $\theta_s(\lambda,n) = 0 \bmod \pi$. Let $k = \#(s(\lambda))$ if $\#(s(\lambda)) < \infty$, otherwise the following argument works for arbitrary $k \in \mathbb{N}$. If we pick n so large that k nodes of $s(\lambda)$ are to the left of n we have k eigenvalues $\hat\lambda_1 < \cdots < \hat\lambda_k < \lambda$ of $H_{0,n}$. Taking an arbitrary linear combination $\eta(m) = \sum_{j=1}^k c_j s(\hat\lambda_j, m)$, $c_j \in \mathbb{C}$ for $m < n$ and $\eta(m) = 0$ for $m \ge n$ a straightforward calculation (using orthogonality of $s(\hat\lambda_j)$) yields

$$\langle \eta, H_+ \eta\rangle < \lambda \|\eta\|^2. \tag{2.45}$$

Invoking the spectral theorem shows

$$\dim \operatorname{Ran} P_{(-\infty,\lambda)}(H_\pm) \ge k. \tag{2.46}$$

For the reversed inequality we can assume $k = \#(s(\lambda)) < \infty$.

We first suppose τ is *l.p.* at $+\infty$. Consider $\tilde H_{0,n} = H_{0,n} \oplus \lambda \mathbb{I}$ on $\ell^2(0,n) \oplus \ell^2(n-1,\infty)$. Then Theorem 9.16.(i) in [76] (take $\ell_0(\mathbb{Z})$ as a core) implies strong resolvent convergence of $\tilde H_{0,n}$ to H_+ as $n \to \infty$ and by Lemma 2.7 we have

$$\dim \operatorname{Ran} P_{(-\infty,\lambda)}(H_+) \le \lim_{n\to\infty} \dim \operatorname{Ran} P_{(-\infty,\lambda)}(H_{0,n}) = k \tag{2.47}$$

completing the proof if τ is *l.p.* at $+\infty$.

Otherwise, that is, if τ is *l.c.* at $+\infty$ (implying that the spectrum of H_+ is purely discrete), λ is an eigenvalue by hypothesis. We first suppose H bounded from below. Hence it suffices to show that the n-th eigenvalue λ_n, $n \in \mathbb{N}$ has at least $n-1$ nodes. This is trivial for $n = 1$. Suppose this is true for λ_n and let m be the largest node of $s(\lambda_n)$. By $\theta_s(\lambda_{n+1}, m) > \theta_s(\lambda_n, m)$ we infer that $\theta_s(\lambda_{n+1}, m)$ has either more nodes between 0 and m or there is at least one additional node of $\theta_s(\lambda_{n+1}, m)$ larger than m by Lemma 2.6. In the case where H is not bounded from below we can label the eigenvalues λ_n, $n \in \mathbb{Z}$. The same argument as before shows that the eigenfunction corresponding to λ_m has $|m-n|$ nodes more than the one corresponding to λ_n. Letting $m \to -\infty$ shows that the eigenfunction corresponding to λ_n has infinitely many nodes. This completes the proof. □

Remark 2.10 *(i) The l.p. / $\lambda \in \sigma_p(H_+)$ assumption is crucial since we need some information about the boundary condition at $+\infty$.*
(ii) The previous remark implies the following. Let $\lambda \in \mathbb{R}$. Suppose τ is l.p. at $+\infty$ or $\lambda \in \sigma_p(H^\beta_{+,n_0})$ and $\beta \ne 0$. Then

$$\dim \operatorname{Ran} P_{(-\infty,\lambda)}(H^\beta_{+,n_0}) = \#_{(0,+\infty)}(s_\beta(\lambda, ., n_0)), \tag{2.48}$$

where $s_\beta(\lambda, ., n_0)$ is a sequence satisfying $\tau s = \lambda s$ and the boundary condition (2.40). Similar modifications apply to Theorems 2.16, 2.19, and 2.20.

As a consequence of Theorem 2.9 we infer

Corollary 2.11 *We have*

$$\dim \operatorname{Ran} P_{(-\infty,\lambda)}(H_\pm) < \infty \tag{2.49}$$

if and only if $\tau - \lambda$ *is non-oscillatory near* $\pm\infty$*, respectively, and hence*

$$\inf \sigma_{ess}(H_\pm) = \inf\{\lambda \in \mathbb{R} \,|\, (\tau - \lambda) \text{ is oscillatory at } \pm\infty\}. \tag{2.50}$$

Moreover, let $H_\pm$ *be bounded from below and* $\lambda_1 < \cdots < \lambda_k < \cdots$ *be the eigenvalues of* $H_\pm$ *below the essential spectrum of* $H_\pm$*. Then the eigenfunction corresponding to* λ_k *has precisely* $k-1$ *nodes inside* $(0, \pm\infty)$*.*

We remark that the first part of Corollary 2.11 can be found in [41], Theorem 32 (see also [47]).

Remark 2.12 *Consider the following example*

$$a(n) = -\frac{1}{2},\; n \in \mathbb{N}, \quad b(1) = 1, b(2) = b_2, b(3) = \frac{1}{2}, b(n) = 0,\; n \geq 4. \tag{2.51}$$

The essential spectrum of H_+ *is given by* $\sigma_{ess}(H_+) = [-1, 1]$ *and one might expect that* H_+ *has no eigenvalues below the essential spectrum if* $b_2 \to -\infty$*. However, since we have*

$$s(-1,0) = 0, s(-1,1) = 1, s(-1,2) = 0, s(-1,n) = -1,\; n \geq 3, \tag{2.52}$$

Theorem 2.9 shows that, independent of $b_2 \in \mathbb{R}$*, there is always precisely one eigenvalue below the essential spectrum.*

In a similar way we obtain

Theorem 2.13 *Let* $\lambda < \inf \sigma_{ess}(H)$*. Suppose* τ *is l.p. at* $-\infty$ *or* $\lambda \in \sigma_p(H)$*. Then*

$$\dim \operatorname{Ran} P_{(-\infty,\lambda)}(H) = \#(u_+(\lambda)). \tag{2.53}$$

The same theorem holds if l.p. at $-\infty$ *and* $u_+(\lambda)$ *is replaced by l.p. at* $+\infty$ *and* $u_-(\lambda)$*.*

Proof. Again it suffices to prove the minus case. If H is not bounded from below the same is true for $H_- \oplus H_+$ (which can be embedded into $\ell^2(\mathbb{Z})$ and considered as a finite rank perturbation of H). Hence H_- or H_+ (or both) is not bounded from below implying $\tau - \lambda$ oscillatory near $-\infty$ or $+\infty$ by Corollary 2.11 and we can suppose H bounded from below.

By virtue of (2.34) and (2.36) we infer

$$\dim \operatorname{Ran} P_{(-\infty,\lambda)}(H_{-,n}) = [\![\theta_-(\lambda, n)/\pi]\!], \quad n \in \mathbb{Z}. \tag{2.54}$$

We first want to show $[\![\theta_-(\lambda,n)/\pi]\!] = \#_{(-\infty,n)}(u_-(\lambda))$ or equivalently

$$\lim_{n\to\infty} [\![\theta_-(\lambda,n)/\pi]\!] = 0. \tag{2.55}$$

Suppose $\lim_{n\to\infty}[\![\theta_-(\lambda_1,n)/\pi]\!] = k \geq 1$ for some $\lambda_1 \in \mathbb{R}$ (saying that $u_-(.,n)$ loses at least one node at $-\infty$). In this case we can find n such that $\theta_-(\lambda_1,n) > k\pi$ for $m \geq n$. Now pick λ_0 such that $\theta_-(\lambda_0,n) = k\pi$. Then $u_-(\lambda_0,.)$ has a node at n but no node between $-\infty$ and n (by Lemma 2.5). Now apply Lemma 2.6 to $u_-(\lambda_0,.)$, $u_-(\lambda_1,.)$ to obtain a contradiction. The rest follows as in the proof of Theorem 2.9. □

As before we obtain

Corollary 2.14 *We have*

$$\dim \operatorname{Ran} P_{(-\infty,\lambda)}(H) < \infty \tag{2.56}$$

if and only if $\tau - \lambda$ is non-oscillatory and hence

$$\inf \sigma_{ess}(H) = \inf\{\lambda \in \mathbb{R} \,|\, (\tau-\lambda) \textit{ is oscillatory}\}. \tag{2.57}$$

Furthermore, let H be bounded from below and $\lambda_1 < \cdots < \lambda_k < \dots$ be the eigenvalues of H below the essential spectrum of H. Then the eigenfunction corresponding to λ_k has precisely $k-1$ nodes.

Remark 2.15 *Corresponding results for the projection $P_{(\lambda,\infty)}(H)$ can be obtained from $P_{(\lambda,\infty)}(H) = P_{(-\infty,-\lambda)}(-H)$. In fact, it suffices to change the definition of a node according to $u(n) = 0$ or $a(n)u(n)u(n+1) < 0$ and $P_{(-\infty,\lambda)}(H)$ to $P_{(\lambda,\infty)}(H)$ in all results of this section.*

Now we turn to the analog of [45], Theorem I.

Theorem 2.16 *Let $\lambda_1 < \lambda_2$. Suppose $\tau - \lambda_2$ is oscillatory near $+\infty$ and τ is l.p. at $+\infty$. Then*

$$\dim \operatorname{Ran} P_{(\lambda_1,\lambda_2)}(H_+) = \liminf_{n\to+\infty} \Big(\#_{(0,n)}(s(\lambda_2)) - \#_{(0,n)}(s(\lambda_1))\Big). \tag{2.58}$$

The same theorem holds if $+$ is replaced by $-$.

Proof. As before we only carry out the proof for the plus sign. Abbreviate $\Delta(n) = [\![\theta_s(\lambda_2,n)/\pi]\!] - [\![\theta_s(\lambda_1,n)/\pi]\!] = \#_{(0,n)}(s(\lambda_2)) - \#_{(0,n)}(s(\lambda_1))$. By (2.44) we infer

$$\dim \operatorname{Ran} P_{[\lambda_1,\lambda_2)}(H_{0,n}) = \Delta(n), \quad n > 2. \tag{2.59}$$

Let $k = \liminf \Delta(n)$ if $\limsup \Delta(n) < \infty$ and $k \in \mathbb{N}$ otherwise. We contend that there exists a $n \in \mathbb{N}$ such that

$$\dim \operatorname{Ran} P_{(\lambda_1,\lambda_2)}(H_{0,n}) \geq k. \tag{2.60}$$

In fact, if $k = \limsup \Delta(n) < \infty$ it follows that $\Delta(n)$ is eventually equal to k and since $\lambda_1 \notin \sigma(H_{0,m}) \cap \sigma(H_{0,m+1})$, $m \in \mathbb{N}$ we are done in this case. Otherwise we can pick n such that $\dim \operatorname{Ran} P_{[\lambda_1,\lambda_2)}(H_{0,n}) \geq k+1$. Hence $H_{0,n}$ has at least k eigenvalues $\hat{\lambda}_j$ with $\lambda_1 < \hat{\lambda}_1 < \ldots < \hat{\lambda}_k < \lambda_2$. Again let $\eta(m) = \sum_{j=1}^k c_j s(\hat{\lambda}_j, n)$, $c_j \in \mathbb{C}$ for $m < n$ and $\eta(m) = 0$ for $n \geq m$ be an arbitrary linear combination. Then

$$\|(H_+ - \frac{\lambda_2 + \lambda_1}{2})\eta\| < \frac{\lambda_2 - \lambda_1}{2}\|\eta\| \tag{2.61}$$

together with the spectral theorem implies

$$\dim \operatorname{Ran} P_{(\lambda_1,\lambda_2)}(H_+) \geq k. \tag{2.62}$$

To prove the second inequality we use that $\tilde{H}_{0,n} = H_{0,n} \oplus \lambda_2 \mathbb{I}$ converges to H_+ in strong resolvent sense as $n \to \infty$ and proceed as before

$$\dim \operatorname{Ran} P_{(\lambda_1,\lambda_2)}(H_+) \leq \liminf_{n\to\infty} P_{[\lambda_1,\lambda_2)}(\tilde{H}_{0,n}) = k \tag{2.63}$$

since $P_{[\lambda_1,\lambda_2)}(\tilde{H}_{0,n}) = P_{[\lambda_1,\lambda_2)}(H_{0,n})$. □

2.4 Renormalized Oscillation Theory

The objective of this section is to look at the nodes of the Wronskian of two solutions $u_{1,2}$ corresponding to $\lambda_{1,2}$, respectively. We call $n \in \mathbb{Z}$ a node of the Wronskian if $W_n(u_1,u_2) = 0$ and $W_{n+1}(u_1,u_2) \neq 0$ or if $W_n(u_1,u_2)W_{n+1}(u_1,u_2) < 0$. Again we shall say that a node n_0 of $W(u_1,u_2)$ lies between m and n if either $m < n_0 < n$ or if $n_0 = m$ but $W_{n_0}(u_1,u_2) \neq 0$. We abbreviate

$$\Delta_{u_1,u_2}(n) = (\theta_{u_2}(n) - \theta_{u_1}(n)) \bmod 2\pi. \tag{2.64}$$

and require

$$[\![\Delta_{u_1,u_2}(n)/\pi]\!] \leq [\![\Delta_{u_1,u_2}(n+1)/\pi]\!] \leq [\![\Delta_{u_1,u_2}(n)/\pi]\!] + 1. \tag{2.65}$$

We shall fix $\lambda_1 \in \mathbb{R}$ and a corresponding solution u_1 and choose a second solution $u(\lambda,n)$ with $\lambda \in [\lambda_1,\lambda_2]$. Now let us consider

$$W_n(u_1,u(\lambda)) = -a(n)\rho_{u_1}(n)\rho_u(\lambda,n)\sin(\Delta_{u_1,u}(\lambda,n)) \tag{2.66}$$

as a function of $\lambda \in [\lambda_1,\lambda_2]$.

Lemma 2.17 *Suppose $\Delta_{u_1,u}(\lambda_1,.)$ satisfies (2.65) then we have*

$$\Delta_{u_1,u}(\lambda,n) = \theta_u(\lambda,n) - \theta_{u_1}(n) \tag{2.67}$$

where $\theta_u(\lambda,.)$, $\theta_{u_1}(.)$ both satisfy (2.19). That is, $\Delta_{u_1,u}(.,n) \in C[\lambda_1,\lambda_2]$ and (2.65) holds for all $\Delta_{u_1,u}(\lambda,.)$ with $\lambda \in [\lambda_1,\lambda_2]$. In particular, the second inequality in (2.19) is attained if and only if n is a node of $W_.(u_1,u(\lambda))$. Moreover, denote by $\#_{(m,n)}W(u_1,u_2)$ the total number of nodes of $W_.(u_1,u_2)$ between m and n. Then

$$\#_{(m,n)}W(u_1,u_2) = [\![\Delta_{u_1,u_2}(n)/\pi]\!] - \lim_{\varepsilon\downarrow 0}[\![\Delta_{u_1,u_2}(m)/\pi+\varepsilon]\!] \tag{2.68}$$

and

$$\#W(u_1,u_2) = \#_{(-\infty,\infty)}W(u_1,u_2) = \lim_{n\to\infty}\Big([\![\Delta_{u_1,u_2}(n)/\pi]\!] - [\![\Delta_{u_1,u_2}(-n)/\pi]\!]\Big). \tag{2.69}$$

Proof. We fix n and set

$$\Delta_{u_1,u}(\lambda,n) = k\pi + \delta(\lambda), \quad \Delta_{u_1,u}(\lambda,n+1) = k\pi + \Delta(\lambda), \tag{2.70}$$

where $k \in \mathbb{Z}, \delta(\lambda_1) \in (0,\pi]$ and $\Delta(\lambda_1) \in (0,2\pi]$. Clearly (2.67) holds for $\lambda = \lambda_1$ since $W_.(u_1,u(\lambda_1))$ is constant. If (2.65) should break down we must have one of the following cases for some $\lambda_0 \ge \lambda_1$. (i) $\delta(\lambda_0) = 0$, $\Delta(\lambda_0) \in (\pi,2\pi]$, or (ii) $\delta(\lambda_0) = \pi$, $\Delta(\lambda_0) \in (0,\pi]$, or (iii) $\Delta(\lambda_0) = 2\pi$, $\delta(\lambda_0) \in (\pi,\pi]$, or (iv) $\Delta(\lambda_0) = 0$, $\delta(\lambda_0) \in (\pi,\pi]$. For notational convenience let us set $\delta = \delta(\lambda_0), \Delta = \Delta(\lambda_0)$ and $\theta_{u_1}(n) = \theta_1(n), \theta_u(\lambda_0,n) = \theta_2(n)$. Furthermore, we can assume $\theta_{1,2}(n) = k_{1,2}\pi + \delta_{1,2}$, $\theta_{1,2}(n+1) = k_{1,2}\pi + \Delta_{1,2}$ with $k_{1,2} \in \mathbb{Z}, \delta_{1,2} \in (0,\pi]$ and $\Delta_{1,2} \in (0,2\pi]$.

Suppose (i). Then

$$W_{n+1}(u_1,u(\lambda_0)) = (\lambda_0-\lambda_1)u_1(n+1)u(\lambda_0,n+1). \tag{2.71}$$

Inserting Prüfer variables shows

$$\sin(\Delta_2-\Delta_1) = \rho\cos^2(\delta_1) \ge 0 \tag{2.72}$$

for some $\rho > 0$ since $\delta = 0$ implies $\delta_1 = \delta_2$. Moreover, $k = (k_2-k_1) \bmod 2$ and $k\pi + \Delta = (k_2-k_1)\pi + \Delta_2 - \Delta_1$ implies $\Delta = (\Delta_2-\Delta_1) \bmod 2\pi$. Hence we have $\sin\Delta \ge 0$ and $\Delta \in (\pi,2\pi]$ implies $\Delta = 2\pi$. But this says $\delta_1 = \delta_2 = \pi/2$ and $\Delta_1 = \Delta_2 = \pi$. Since we have at least $\delta(\lambda_2-\varepsilon) > 0$ and hence $\delta_2(\lambda_2-\varepsilon) > \pi/2$, $\Delta_2(\lambda_2-\varepsilon) > \pi$ for $\varepsilon > 0$ sufficiently small. Thus from $\Delta(\lambda_2-\varepsilon) \in (\pi,2\pi)$ we get

$$0 > \sin\Delta(\lambda_2-\varepsilon) = \sin(\Delta_2(\lambda_2-\varepsilon)-\pi) > 0, \tag{2.73}$$

contradicting (i).

Suppose (ii). Again by (2.71) we have $\sin(\Delta_2-\Delta_1) \ge 0$ since $\delta_1 = \delta_2$. But now $(k+1) = (k_1-k_2) \bmod 2$. Furthermore, $\sin(\Delta_2-\Delta_1) = -\sin(\Delta) \ge 0$ says $\Delta = \pi$ since $\Delta \in (0,\pi]$. Again this implies $\delta_1 = \delta_2 = \pi/2$ and $\Delta_1 = \Delta_2 = \pi$. But since $\delta(\lambda)$ increases/decreases precisely if $\Delta(\lambda)$ increases/decreases for λ near λ_0 (2.65) stays valid.

Suppose (iii) or (iv). Then

$$W_n(u_1, u(\lambda_0)) = -(\lambda_0 - \lambda_1) u_1(n+1) u(\lambda_0, n+1). \tag{2.74}$$

Inserting Prüfer variables gives

$$\sin(\delta_2 - \delta_1) = -\rho \sin(\Delta_1) \sin(\Delta_2) \tag{2.75}$$

for some $\rho > 0$. We first assume $\delta_2 > \delta_1$. In this case we infer $k = (k_2 - k_1) \bmod 2$ implying $\Delta_2 - \Delta_1 = 0 \bmod 2\pi$ contradicting (2.75). Next assume $\delta_2 \leq \delta_1$. Then we obtain $(k+1) = (k_2 - k_1) \bmod 2$ implying $\Delta_2 - \Delta_1 = \pi \bmod 2\pi$ and hence $\sin(\delta_2 - \delta_1) \geq 0$ from (2.75). Thus we get $\delta_1 = \delta_2 = \pi/2$ $\Delta_1 = \Delta_2 = \pi$, and hence $\Delta_2 - \Delta_1 = 0 \bmod 2\pi$ contradicting (iii), (iv). This settles (2.67).

Furthermore, if $\Delta(\lambda) \in (0, \pi]$ we have no node at n since $\delta(\lambda) = \pi$ implies $\Delta(\lambda) = \pi$ by (ii). Conversely, if $\Delta(\lambda) \in (\pi, 2\pi]$ we have a node at n since $\Delta(\lambda) = 2\pi$ is impossible by (iii). The rest being straightforward. □

Equations (2.22), (2.67), and (2.68) imply

Corollary 2.18 *Let $\lambda_1 \leq \lambda_2$ and suppose $u_{1,2}$ satisfy $\tau u_{1,2} = \lambda_{1,2} u_{1,2}$, respectively. Then we have*

$$|\#_{(n,m)} W(u_1, u_2) - (\#_{(n,m)}(u_2) - \#_{(n,m)}(u_1))| \leq 2 \tag{2.76}$$

Now we come to a renormalized version of Theorem 2.16. We first need the result for a finite interval.

Theorem 2.19 *Fix $n_1 < n_2$ and $\lambda_1 < \lambda_2$. Then*

$$\dim \operatorname{Ran} P_{(\lambda_1, \lambda_2)}(H_{n_1, n_2}) = \#_{(n_1, n_2)} W(s(\lambda_1, ., n_1), s(\lambda_2, ., n_2)). \tag{2.77}$$

Proof. We abbreviate

$$\Delta(\lambda, n) = \Delta_{s(\lambda_1, ., n_1), s(\lambda, ., n_2)}(n) \tag{2.78}$$

and normalize (perhaps after flipping the sign of $s(\lambda_1, ., n_1)$) $\Delta(\lambda_1, n) \in (0, \pi]$. From (2.28) we infer

$$\dim \operatorname{Ran} P_{(\lambda_1, \lambda_2)}(H_{n_1, n_2}) = -\lim_{\varepsilon \downarrow 0} [\![\Delta(\lambda_2, n_1)/\pi + \varepsilon]\!] \tag{2.79}$$

since $\lambda \in \sigma(H_{n_1, n_2})$ is equivalent to $\Delta(\lambda, n_1) = 0 \bmod \pi$. Using (2.68) completes the proof. □

Theorem 2.20 *Fix $\lambda_1 < \lambda_2$ and suppose τ is in the l.p. case near $+\infty$ or $\lambda_2 \in \sigma_p(H_+)$. Then*

$$\dim \operatorname{Ran} P_{(\lambda_1, \lambda_2)}(H_+) = \#_{(0, +\infty)} W(s(\lambda_1), s(\lambda_2)). \tag{2.80}$$

The same theorem holds if $+$ is replaced by $-$.

Proof. As usual we only prove the result for H_+ and set $k = \#_{(0,\infty)}W(s(\lambda_1),s(\lambda_2))$ provided this number is finite and $k \in \mathbb{N}$ otherwise. We abbreviate

$$\Delta(\lambda, n) = \Delta_{s(\lambda_1),s(\lambda)}(n) \tag{2.81}$$

and normalize $\Delta(\lambda_1, n) = 0$ implying $\Delta(\lambda, n) > 0$ for $\lambda > \lambda_1$. Hence if we chose n so large that all k nodes are to the left of n we have

$$\Delta(\lambda, n) > k\pi. \tag{2.82}$$

Thus we can find $\lambda_1 < \hat{\lambda}_1 < \cdots < \hat{\lambda}_k < \lambda_2$ with $\Delta(\hat{\lambda}_j, n) = j\pi$. Now define

$$\eta_j(m) = \left\{ \begin{array}{ll} s(\hat{\lambda}_j, m) - \rho_j s(\lambda_1, m) & m \leq n \\ 0 & m \geq n \end{array} \right. , \tag{2.83}$$

where $\rho_j \neq 0$ is chosen such that $s(\hat{\lambda}_j, m) = \rho_j s(\lambda_1, m)$ for $m = n, n+1$. Furthermore observe that

$$\tau\eta_j(m) = \left\{ \begin{array}{ll} \hat{\lambda}_j s(\hat{\lambda}_j, m) - \lambda_1 \rho_1 s(\lambda_1, m) & m \leq n \\ 0 & m \geq n \end{array} \right. \tag{2.84}$$

and that $s(\lambda_1, m)$, $s(\hat{\lambda}_j, .)$, $1 \leq j \leq k$ are orthogonal on $1, \dots, n$. Next, let $\eta = \sum_{j=1}^k c_j \eta_j$, c_j be an arbitrary linear combination, then a short calculation verifies

$$\|(H_+ - \frac{\lambda_2 + \lambda_1}{2})\eta\| < \frac{\lambda_2 - \lambda_1}{2}\|\eta\|. \tag{2.85}$$

And invoking the spectral theorem gives

$$\dim \operatorname{Ran} P_{(\lambda_1,\lambda_2)}(H_+) \geq k. \tag{2.86}$$

To prove the reversed inequality is only necessary if $\#_{(0,\infty)}W(s(\lambda_1),s(\lambda_2)) < \infty$. In this case we look at $H_{0,n}^{\infty,\beta}$ with $\beta = s(\lambda_2, n+1)/s(\lambda_2, n)$. By Theorem 2.19 and Remark 2.10 (ii) we have

$$\dim \operatorname{Ran} P_{(\lambda_1,\lambda_2)}(\tilde{H}_{0,n}^{\infty,\beta}) = \#_{(0,n)}W(s(\lambda_1), s(\lambda_2)). \tag{2.87}$$

Now use strong resolvent convergence of $\tilde{H}_{0,n}^{\infty,\beta} = H_{0,n}^{\infty,\beta} \oplus \lambda_1 \mathbb{I}$ to H_+ (due to our *l.p.* / $\lambda_2 \in \sigma_p(H_+)$ assumption) as $n \to \infty$ to obtain

$$\dim \operatorname{Ran} P_{(\lambda_1,\lambda_2)}(H_+) \leq \liminf_{n\to\infty} \dim \operatorname{Ran} P_{(\lambda_1,\lambda_2)}(\tilde{H}_{0,n}^{\infty,\beta}) = k \tag{2.88}$$

completing the proof. □

As a consequence we infer.

Corollary 2.21 *Let $u_{1,2}$ satisfy $\tau u_{1,2} = \lambda_{1,2} u_{1,2}$. Then*

$$\#_{(0,\pm\infty)}W(u_1, u_2) < \infty \quad \Leftrightarrow \quad \dim \operatorname{Ran} P_{(\lambda_1,\lambda_2)}(H_\pm) < \infty. \tag{2.89}$$

Proof. By Corollary 2.18 the result does not depend on the choice of $u_{1,2}$. Since the proof of (2.86) does not use the *l.p.* / $\lambda_2 \in \sigma_p(H_+)$ assumption the first direction follows. Conversely, we can replace the sequence β in (2.88) by a sequence $\hat{\beta}$ such that $\tilde{H}_{0,n}^{\infty,\hat{\beta}}$ converges to H_+. Since we have $|\dim \operatorname{Ran} \tilde{H}_{0,n}^{\infty,\hat{\beta}} - \dim \operatorname{Ran} \tilde{H}_{0,n}^{\infty,\beta}| \leq 1$ the corollary is proven. □

Finally we turn to our main result for Jacobi operators H on $\mathbb{Z}$. We emphasize that to date, Theorem 2.22 appears to be the only oscillation theoretic result concerning the number of eigenvalues in essential spectral gaps of Jacobi operators on $\mathbb{Z}$.

Theorem 2.22 *Fix $\lambda_1 < \lambda_2$ and suppose $[\lambda_1, \lambda_2] \cap \sigma_{ess}(H) = \emptyset$. Then*

$$\dim \operatorname{Ran} P_{(\lambda_1,\lambda_2)}(H) = \#W(u_\mp(\lambda_1), u_\pm(\lambda_2)). \tag{2.90}$$

In addition, if τ is l.p. at $+\infty$ we even have

$$\dim \operatorname{Ran} P_{(\lambda_1,\lambda_2)}(H) = \#W(u_+(\lambda_1), u_+(\lambda_2)). \tag{2.91}$$

The same result holds if $+$ is replaced by $-$.

Proof. Since the proof is similar to the proof of Theorem 2.20 we shall only outline the first part. Let $k = \#W(u_+(\lambda_1), u_-(\lambda_2))$ if this number is finite and $k \in \mathbb{N}$ else. Pick $n > 0$ so large that all zeros of the Wronskian are between $-n$ and n. We abbreviate

$$\Delta(\lambda, n) = \Delta_{u_+(\lambda_1),u_-(\lambda)}(n) \tag{2.92}$$

and normalize $\Delta(\lambda_1, n) \in [0, \pi)$ implying $\Delta(\lambda, n) > 0$ for $\lambda > \lambda_1$. Hence if we chose $n \in \mathbb{N}$ so large that all k nodes are between $-n$ and n we can assume

$$\Delta(\lambda, n) > k\pi. \tag{2.93}$$

Thus we can find $\lambda_1 < \hat{\lambda}_1 < \cdots < \hat{\lambda}_k < \lambda_2$ with $\Delta(\hat{\lambda}_j, n) = 0 \bmod \pi$. Now define

$$\eta_j(m) = \begin{cases} u_-(\hat{\lambda}_j, m) & m \leq n \\ \rho_j u_+(\lambda_1, m) & m \geq n \end{cases}, \tag{2.94}$$

where $\rho_j \neq 0$ is chosen such that $u_-(\hat{\lambda}_j, m) = \rho_j u_+(\lambda_1, m)$ for $m = n, n+1$. Now proceed as in the previous theorems. □

Again, we infer as a consequence.

Corollary 2.23 *Let $u_{1,2}$ satisfy $\tau u_{1,2} = \lambda_{1,2} u_{1,2}$. Then*

$$\#W(u_1, u_2) < \infty \quad \Leftrightarrow \quad \dim \operatorname{Ran} P_{(\lambda_1,\lambda_2)}(H) < \infty. \tag{2.95}$$

Proof. Follows from Corollaries 2.18, 2.21, and $\dim \operatorname{Ran} P_{(\lambda_1,\lambda_2)}(H) < \infty$ if and only if $(\dim \operatorname{Ran} P_{(\lambda_1,\lambda_2)}(H_-) + \dim \operatorname{Ran} P_{(\lambda_1,\lambda_2)}(H_+)) < \infty$. □

Remark 2.24 *The most general three-term recurrence relation*

$$\tilde\tau f(n) = \tilde a(n) f(n+1) - \tilde b(n) f(n) + \tilde c(n) f(n-1), \tag{2.96}$$

with $\tilde a(n)\tilde c(n) > 0$, can be transformed to a Jacobi recurrence relation as follows. First we symmetrise $\tilde\tau$ via

$$\tilde\tau f(n) = \frac{1}{w(n)}\Big(c(n)f(n+1) + c(n-1)f(n-1) - d(n)f(n)\Big), \tag{2.97}$$

where

$$w(n) = \begin{cases} \prod\limits_{j=n_0}^{n-1} \frac{\tilde a(j)}{\tilde c(j+1)} & \text{for } n > n_0 \\ 1 & \text{for } n = n_0 \\ \prod\limits_{j=n}^{n_0-1} \frac{\tilde c(j+1)}{\tilde a(j)} & \text{for } n < n_0 \end{cases} > 0, \tag{2.98}$$

$$c(n) = w(n)\tilde a(n) = w(n+1)\tilde c(n+1), \quad d(n) = w(n)\tilde b(n). \tag{2.99}$$

The natural Hilbert space for $\tilde\tau$ is the weighted space $\ell^2(\mathbb{Z}, w)$ with scalar product

$$\langle f, g\rangle = \sum_{n\in\mathbb{Z}} w(n)\overline{f(n)}g(n), \qquad f, g \in \ell^2(\mathbb{Z}, w). \tag{2.100}$$

Let $\tilde H$ be a self-adjoint operator associated with $\tilde\tau$ in $\ell^2(\mathbb{Z}, w)$. Then the unitary operator

$$\begin{array}{rccl} U: & \ell^2(\mathbb{Z}, w) & \to & \ell^2(\mathbb{Z}) \\ & u(n) & \mapsto & \sqrt{w(n)}u(n) \end{array} \tag{2.101}$$

transforms $\tilde H$ into a Jacobi operator H in $\ell^2(\mathbb{Z})$ associated with the sequences

$$a(n) = \frac{c(n)}{\sqrt{w(n)w(n+1)}} = \mathrm{sgn}(\tilde a(n))\sqrt{\tilde a(n)\tilde c(n+1)}, \tag{2.102}$$

$$b(n) = \frac{d(n)}{w(n)} = \tilde b(n). \tag{2.103}$$

In addition we infer

$$c(n)\Big(f(n)g(n+1) - f(n+1)g(n)\Big) = a(n)\Big((Uf)(n)(Ug)(n+1) - (Uf)(n+1)(Ug)(n)\Big). \tag{2.104}$$

Hence all the results derived for Jacobi operator thus far apply to generalized Jacobi operators of the type $\tilde H$ as well.

2.5 Applications

One important class of Jacobi operators are periodic ones (cf., e.g., [8], Appendix B, [55], [60]). Instead of periodic operators themselves we are interested in short-range perturbations of these operators. In fact, we are going to prove the analog of the Theorem by Rofe-Beketov ([63], see also [33], [41], Section 67, [53]) about the finiteness of the number of eigenvalues in essential spectral gaps of the perturbed Hill operator. Since constant coefficients a, b are a special case of periodic ones our results contain results from scattering theory (cf., e.g., [10], [42]).

To set the stage, we first recall some basic facts from the theory of periodic operators. Let H_p be a Jacobi operator associated with periodic sequences $a_p < 0, b_p$, that is,

$$a_p(n+N) = a_p(n), \qquad b_p(n+N) = b_p(n), \tag{2.105}$$

for some fixed $N \in \mathbb{N}$. The spectrum of H_p is purely absolutely continuous and consists of a finite number of gaps, that is,

$$\sigma(H_p) = \bigcup_{j=0}^{g} [E_{2j}, E_{2j+1}], \qquad g \in \mathbb{N}_0, \tag{2.106}$$

with $E_0 < E_1 < \cdots < E_{2g+1}$ and $g \leq N-1$. Moreover, Floquet theory implies the existence of solutions $u_{p,\pm}(z,.)$ of $\tau_p u = zu$, $z \in \mathbb{C}$ (τ_p the difference expression corresponding to H_p) satisfying

$$u_{p,\pm}(z, n+N) = m^{\pm}(z) u_{p,\pm}(z,n), \tag{2.107}$$

where $m^{\pm}(z) \in \mathbb{C}$ are called Floquet multipliers. $m^{\pm}(z)$ satisfy $m^+(z)m^-(z) = 1$, $m^{\pm}(z)^2 = 1$ for $z \in \{E_j\}_{j=0}^{2g+1}$, $|m^{\pm}(z)| = 1$ for $z \in \sigma(H_p)$, and $|m^+(z)| < 1$ for $z \in \mathbb{C}\backslash\sigma(H_p)$. (This says in particular, that $u_{p,\pm}(z,.)$ are bounded for $z \in \sigma(H_p)$ and linearly independent for $z \in \mathbb{C}\backslash\{E_j\}_{j=0}^{2g+1}$.)

We are going to study perturbations H of H_p associated with sequences a, b satisfying $a(n) \to a_p(n)$ and $b(n) \to b_p(n)$ as $|n| \to \infty$. Clearly, H and H_p are both bounded and hence defined on the whole of $\ell^2(\mathbb{Z})$. In fact, we have

$$\sigma(H) \subseteq [\underline{c}, \overline{c}], \tag{2.108}$$

where $\underline{c} = \inf_{n\in\mathbb{Z}}(b(n) + a(n-1) + a(n))$ and $\overline{c} = \sup_{n\in\mathbb{Z}}(b(n) - a(n-1) - a(n))$. Using this notation our theorem reads:

Theorem 2.25 *Suppose a_p, b_p are given periodic sequences and H_p is the corresponding Jacobi operator. Let H be a perturbation of H_p such that*

$$\sum_{n\in\mathbb{Z}} |n(a(n) - a_p(n))| < \infty, \quad \sum_{n\in\mathbb{Z}} |n(b(n) - b_p(n))| < \infty. \tag{2.109}$$

Then we have $\sigma_{ess}(H) = \sigma(H_p)$, the point spectrum of H is finite and confined to the spectral gaps of H_p, that is, $\sigma_p(H) \subset \mathbb{R}\backslash\sigma(H_p)$. Furthermore, the essential spectrum of H_p is purely absolutely continuous.

For the proof we will need the following lemma the proof of which is elementary.

Lemma 2.26 *The Volterra sum equation*

$$f(n) = g(n) + \sum_{m=n+1}^{\infty} K(n,m) f(m), \tag{2.110}$$

with

$$|K(n,m)| \le \hat{K}(n,m), \quad \hat{K}(n+1,m) \le \hat{K}(n,m), \quad \hat{K}(n,.) \in \ell^1(0,\infty), \tag{2.111}$$

has for $g \in \ell^\infty(0,\infty)$ a unique solution $f \in \ell^\infty(0,\infty)$, fulfilling the estimate

$$|f(n)| \le \Big(\sup_{m>n} |g(m)| \Big) \exp\Big(\sum_{m=n+1}^{\infty} \hat{K}(n,m) \Big). \tag{2.112}$$

Proof. (of Theorem 2.25) The fact that $H - H_p$ is compact implies $\sigma_{ess}(H) = \sigma_{ess}(H_p)$. To prove the remaining claims it suffices to show the existence of solutions $u_\pm(\lambda,.)$ of $\tau u = \lambda u$ for $\lambda \in \sigma(H_p)$ satisfying

$$\lim_{|n|\to\infty} |u_\pm(\lambda,n) - u_{p,\pm}(\lambda,n)| = 0. \tag{2.113}$$

In fact, since $u_\pm(\lambda,.)$, $\lambda \in \sigma(H_p)$ are bounded and do not vanish near $\pm\infty$, there are no eigenvalues in the essential spectrum of H and invoking the principal of subordinacy (cf., [68], [69]) shows that the essential spectrum of H is purely absolutely continuous. Moreover, (2.113) with $\lambda = E_0$ implies that $H - E_0$ is non-oscillatory since we can assume (perhaps after flipping signs) $u_{p,\pm}(E_0,n) \ge \varepsilon > 0$, $n \in \mathbb{Z}$ and by Corollary 2.14 there are only finitely many eigenvalues below E_0. Similarly, (using Remark 2.15) there are only finitely many eigenvalues above E_{2g+1}. Applying Corollary 2.23 in each gap (E_{2j-1}, E_{2j}), $1 \le j \le g$ shows that the number of eigenvalues in each gap is finite as well.

It remains to show (2.113). Suppose $u_+(\lambda,.)$, $\lambda \in \sigma(H_p)$ satisfies (disregarding summability for a moment)

$$\begin{aligned} u_+(\lambda,n) &= \frac{a_p(n)}{a(n)} u_{p,+}(\lambda,n) \\ &\quad + \sum_{m=n+1}^{\infty} \frac{a_p(n)}{a(n)} K(\lambda,n,m) u_+(\lambda,m), \end{aligned} \tag{2.114}$$

with

$$\begin{aligned} K(\lambda,n,m) &= \frac{s_p(\lambda,n,m-1)}{a_p(m-1)} (a(m-1) - a_p(m-1)) \\ &+ \frac{s_p(\lambda,n,m+1)}{a_p(m+1)} (a(m) - a_p(m)) - \frac{s_p(\lambda,n,m)}{a_p(m)} (b(m) - b_p(m)), \end{aligned} \tag{2.115}$$

where $s_p(\lambda, ., m)$ is the solution of of $\tau_p u = zu$ satisfying the initial conditions $s_p(z, m, m) = 0$ and $s_p(z, m+1, m) = 1$. Then $u_+(\lambda, .)$ fulfills $\tau u = \lambda u$ and (2.113). Hence if we can apply Lemma 2.26 we are done. To do this we need an estimate for $K(\lambda, n, m)$ which again follows from Floquet theory

$$|s_p(\lambda, n, m)| \le M|n - m|, \qquad \lambda \in \sigma(H_p), \tag{2.116}$$

for some suitable constant $M > 0$. □

As anticipated, specializing to the case $a_p(n) = -1/2$, $b_p(n) = 0$ we obtain a corresponding result for the scattering case.

Corollary 2.27 *([42]) Suppose*

$$\sum_{n\in\mathbb{Z}} |n(1 + 2a(n))| < \infty, \quad \sum_{n\in\mathbb{Z}} |n\, b(n)| < \infty. \tag{2.117}$$

Then we have

$$\sigma_{ess}(H) = [-1, 1], \quad \sigma_p(H) \subseteq [\underline{c}, -1) \cup (1, \overline{c}]. \tag{2.118}$$

Moreover, the essential spectrum of H is purely absolutely continuous and the point spectrum of H is finite.

Corollary 2.27 is stated in [42] (for the case $a_p(n) = 1$ – but Lemma 2.2 plus a scaling transform takes care of that). Similar results can be obtained using the Birman-Schwinger principle.

Chapter 3

Spectral Deformations

3.1 Introduction

For a variety of reasons, techniques to insert and remove eigenvalues in spectral gaps of a given one-dimensional second-order differential (i.e., Sturm-Liouville) respectively difference (i.e., Jacobi) operator have recently attracted great interest. In fact, these techniques are vital in diverse fields such as the inverse scattering approach used by Deift and Trubowitz [17], supersymmetric quantum mechanics (cf. the literature cited, e.g., in [38]), level comparison theorems (see, e.g., [4]), in the construction of soliton solutions of the Korteweg-de Vries (KdV) and Toda hierarchies relative to general KdV and Toda background solutions (see, e.g., [6], [8], [15], [17], [19], Ch. 4, [25], [29], [34], [38], [51]-[56], [57], Sect. 6.6, [64]–[67]), and in connection with Bäcklund transformations for the KdV and Toda hierarchies (cf., e.g., [8], [20], [22], [26], [34], [36], [38], [58], [59], [75]).

Historically, methods of inserting eigenvalues in the case of differential operators go back to Jacobi [50], Darboux [14], Crum [13], Gel'fand and Levitan [30], Schmincke [65], and especially Deift [15]. Two particular such methods, the so called single commutation or Crum-Darboux method and the double commutation method, shortly to be described below, turned out to be of particular importance. The operator theoretic approach developed in [15] applies to the single commutation method and has been used in [15] to give a complete spectral characterization in the differential operator case. The double commutation method on the other hand required entirely different methods and was only recently solved in the differential operator case. A solution based on ODE techniques was given in [31] and most recently, a more general and at the same time greatly simplifying operator theoretic approach to a spectral characterization of the double commutation method appeared in [35].

Surprisingly, a complete spectral characterization of both the single and double commutation methods in the difference operator context is lacking in the literature thus far. Although special cases of the single commutation method with constant or algebro-geometric backgrounds have been discussed in [8], [16], [74], no treatment of general backgrounds is known to us. Moreover, with the exception of reference [74], where an eigenvalue is inserted into the spectral gap of a two-band periodic Jacobi operator with period 2, no general formulation of the double commutation method for finite difference operators seems to be available in the literature. The present chapter fills these gaps and provides a complete spectral characterization of the single commutation method (based on Deift's operator theoretic approach) in Sections 3.2 and 3.3 and develops the corresponding results for the double commutation method in Sections 3.2-3.6. Section 3.7 gives three applications of our results. The discrete analog of the FIT formula for the isospectral torus of periodic Schrödinger operators, thereby deriving an explicit realization of the isospectral torus of all algebro-geometric quasi-periodic finite-gap Jacobi operators, and the N-soliton solutions of the Toda and Kac-van Moerbeke equations on an arbitrary background solution using the single and double commutation methods.

In the remainder of this introduction we provide an informal discussion of commutation methods and restrict ourselves to the case of the whole line and bounded Jacobi operators (so we don't have to bother with domain considerations).

We first review the single commutation method [40]: Let $a, b \in \ell(\mathbb{Z})$ be two bounded, real-valued sequences satisfying

$$a(n) < 0, \qquad b(n) \in \mathbb{R}, \tag{3.1}$$

and introduce the corresponding Jacobi operator H in $\ell^2(\mathbb{Z})$

$$(Hf)(n) = a(n)f(n+1) + a(n-1)f(n-1) - b(n)f(n), \quad u \in \ell^2(\mathbb{Z}). \tag{3.2}$$

Next (cf. Lemma 2.2), assume the existence of two weak positive solutions $u_\pm(\lambda_1, n)$ of

$$Hu_\pm = \lambda_1 u_\pm, \qquad u_\pm(\lambda_1, n) > 0, \quad u_\pm(\lambda_1, n) \in \ell^2(\pm\mathbb{N}) \tag{3.3}$$

(implying $b(n) + \lambda_1 < 0$, i.e., $H - \lambda_1 \geq 0$). $u_\pm$ are the principal solutions as used, e.g., in [37]. Any positive solution can then be written as

$$u_{\sigma_1}(\lambda_1, n) = \frac{1+\sigma_1}{2} u_+(\lambda_1, n) + \frac{1-\sigma_1}{2} u_-(\lambda_1, n), \quad \sigma_1 \in [-1, 1]. \tag{3.4}$$

Now define the operator A_{σ_1} in $\ell^2(\mathbb{Z})$ by

$$(A_{\sigma_1} f)(n) = \rho_{o,\sigma_1}(n) f(n+1) + \rho_{e,\sigma_1}(n) f(n), \quad f \in \ell^2(\mathbb{Z}), \tag{3.5}$$

where

$$\rho_{o,\sigma_1}(n) = -\sqrt{-\frac{a(n) u_{\sigma_1}(\lambda_1, n)}{u_{\sigma_1}(\lambda_1, n+1)}}, \quad \rho_{e,\sigma_1}(n) = \sqrt{-\frac{a(n) u_{\sigma_1}(\lambda_1, n+1)}{u_{\sigma_1}(\lambda_1, n)}}. \tag{3.6}$$

We will always take the positive branch of all square roots involved. We note that ρ_{o,σ_1} and ρ_{e,σ_1} are bounded sequences as can be seen from

$$\left|\frac{a(n) u_{\sigma_1}(\lambda_1, n+1)}{u^1_{\sigma_1}(\lambda_1, n)}\right| + \left|\frac{a(n-1) u_{\sigma_1}(\lambda_1, n-1)}{u_{\sigma_1}(\lambda_1, n)}\right| = |b(n) + \lambda_1|. \tag{3.7}$$

The adjoint operator $A^*_{\sigma_1}$ of A_{σ_1} is given by

$$(A^*_{\sigma_1} f)(n) = \rho_{o,\sigma_1}(n-1) f(n-1) + \rho_{e,\sigma_1}(n) f(n), \quad f \in \ell^2(\mathbb{Z}), \tag{3.8}$$

and for the (positive self-adjoint) operator $A^*_{\sigma_1} A_{\sigma_1}$ one infers

$$A^*_{\sigma_1} A_{\sigma_1} = H - \lambda_1. \tag{3.9}$$

This shows that $(H - \lambda_1) \geq 0$ is a necessary condition for the existence of a positive solution of (3.3). We remark that this condition is also sufficient (see, e.g., [37], Theorem 2.8). Commuting $A^*_{\sigma_1}$ and A_{σ_1} (observing $(A^*_{\sigma_1})^* = A_{\sigma_1}$) yields a second positive self-adjoint bounded operator $A_{\sigma_1} A^*_{\sigma_1}$ and further the commuted operator

$$H_{\sigma_1} = A_{\sigma_1} A^*_{\sigma_1} + \lambda_1. \tag{3.10}$$

A straightforward calculation shows

$$(H_{\sigma_1} f)(n) = a_{\sigma_1}(n) f(n+1) + a_{\sigma_1}(n-1) f(n-1) - b_{\sigma_1}(n) f(n), \tag{3.11}$$

with

$$a_{\sigma_1}(n) = -\frac{\sqrt{a(n)a(n+1)u_{\sigma_1}(\lambda_1,n)u_{\sigma_1}(\lambda_1,n+2)}}{u_{\sigma_1}(\lambda_1,n+1)}, \tag{3.12}$$

$$b_{\sigma_1}(n) = a(n)\Big(\frac{u_{\sigma_1}(\lambda_1,n)}{u_{\sigma_1}(\lambda_1,n+1)} + \frac{u_{\sigma_1}(\lambda_1,n+1)}{u_{\sigma_1}(\lambda_1,n)}\Big) - \lambda_1. \tag{3.13}$$

As proven by Deift [15], the operators $H - \lambda_1$ and $H_{\sigma_1} - \lambda_1$, restricted to the orthogonal complements of their respective null-spaces, are unitarily equivalent. Specifically, we have

$$\begin{aligned} \sigma(H_{\sigma_1}) &= \begin{cases} \sigma(H) \cup \{\lambda_1\}, & \sigma_1 \in (-1,1) \\ \sigma(H), & \sigma_1 \in \{-1,1\} \end{cases}, & \sigma_{ac}(H_{\sigma_1}) &= \sigma_{ac}(H), \\ \sigma_p(H_{\sigma_1}) &= \begin{cases} \sigma_p(H) \cup \{\lambda_1\}, & \sigma_1 \in (-1,1) \\ \sigma_p(H), & \sigma_1 \in \{-1,1\} \end{cases}, & \sigma_{sc}(H_{\sigma_1}) &= \sigma_{sc}(H). \end{aligned} \tag{3.14}$$

Here $\sigma_p(.)$, $\sigma_{ac}(.)$, and $\sigma_{sc}(.)$ denote the the point spectrum (i.e., the set of eigenvalues), absolutely, and singularly continuous spectrum, respectively.

This method is known as the single commutation method [40] and we will give a complete spectral characterization of it in Sections 3.2 and 3.3.

Our next aim is to remove the condition that H is bounded from below and thereby introduce the double commutation method. Fix $\gamma_\pm > 0$ and define

$$\rho_{o,\gamma_\pm}(n) = \rho_{e,\pm 1}(n+1)\sqrt{\frac{c_{\gamma_\pm}(\lambda_1,n)}{c_{\gamma_\pm}(\lambda_1,n+1)}}, \tag{3.15}$$

$$\rho_{e,\gamma_\pm}(n) = \rho_{o,\pm 1}(n)\sqrt{\frac{c_{\gamma_\pm}(\lambda_1,n+1)}{c_{\gamma_\pm}(\lambda_1,n)}}, \tag{3.16}$$

where

$$c_{\gamma_\pm}(\lambda_1, n) = 1 + \gamma_\pm \sum_{j=\pm\infty}^{\substack{n+1\\n}} u_\pm(\lambda_1, j)^2, \tag{3.17}$$

and introduce corresponding operators $A_{\gamma_\pm}, A_{\gamma_\pm}^*$ in $\ell^2(\mathbb{Z})$ by

$$\begin{aligned} (A_{\gamma_\pm} f)(n) &= \rho_{o,\gamma_\pm}(n) f(n+1) + \rho_{e,\gamma_\pm}(n) f(n), & (3.18)\\ (A_{\gamma_\pm}^* f)(n) &= \rho_{o,\gamma_\pm}(n-1) f(n-1) + \rho_{e,\gamma_\pm}(n) f(n). & (3.19) \end{aligned}$$

A simple calculation shows that $A_{\gamma_\pm}^* A_{\gamma_\pm} = A_{\pm 1} A_{\pm 1}^*$ and hence

$$H_{\pm 1} = A_{\gamma_\pm}^* A_{\gamma_\pm} + \lambda_1. \tag{3.20}$$

Performing a second commutation yields the doubly commuted operator

$$H_{\gamma_\pm} = A_{\gamma_\pm} A_{\gamma_\pm}^* + \lambda_1. \tag{3.21}$$

Explicitly, one verifies

$$(H_{\gamma_\pm} f)(n) = a_{\gamma_\pm}(n) f(n+1) + a_{\gamma_\pm}(n-1) f(n-1) - b_{\gamma_\pm}(n) f(n), \tag{3.22}$$

with

$$\begin{aligned} a_{\gamma_\pm}(n) &= a(n+1) \frac{\sqrt{c_{\gamma_\pm}(\lambda_1, n) c_{\gamma_\pm}(\lambda_1, n+2)}}{c_{\gamma_\pm}(\lambda_1, n+1)}, & (3.23)\\ b_{\gamma_\pm}(n) &= b(n+1) \pm \gamma_\pm \Big(\frac{a(n) u_\pm(\lambda_1, n) u_\pm(\lambda_1, n+1)}{c_{\gamma_\pm}(\lambda_1, n)} \\ &\quad - \frac{a(n+1) u_\pm(\lambda_1, n+1) u_\pm(\lambda_1, n+2)}{c_{\gamma_\pm}(\lambda_1, n+1)} \Big). & (3.24) \end{aligned}$$

Now observe that $H_{\gamma_\pm}$ remains well-defined even if $u_\pm$ is no longer positive. This applies, in particular, in the case where $u_\pm(\lambda_1)$ has zeros and hence all intermediate operators $A_{\pm 1}, A_{\gamma_\pm}, H_{\pm 1}$, etc., become ill-defined. Thus to define $H_{\gamma_\pm}$ it suffices to assume the existence of a solution $u_\pm(\lambda_1)$ which is square summable near $\pm\infty$. This condition is much less restrictive than the existence of a positive solution (e.g., $\sigma(H) \neq \mathbb{R}$, i.e., the existence of a spectral gap for H around λ_1 is sufficient in this context).

One expects that formulas analogous to (3.14) will carry over to this more general setup. That this is actually the case will be shown in our principal Theorem 3.13 of Section 3.4. Hence the double commutation method (contrary to the single commutation method) enables one to insert eigenvalues not only below the spectrum of H but into arbitrary spectral gaps of H.

3.2 The Single Commutation Method

In this section we intend to give a detailed investigation of the single commutation method. We will assume a, b to satisfy (H.2.1) throughout Sections 3.2 and 3.3.

We shall consider (self-adjoint) Jacobi operators H associated with the difference expression

$$(\tau f)(n) = a(n)f(n+1) + a(n-1)f(n-1) - b(n)f(n), \tag{3.25}$$

in the Hilbert space $\ell^2(\mathbb{Z})$. As a preparation we prove

We start with operators associated with the difference expression (3.25) on the half axis $\pm\mathbb{N}$. For simplicity we will do most calculations only for $\ell^2(\mathbb{N})$. Let $u(\lambda_1)$ be a positive solution of $\tau u = \lambda_1 u$ and define

$$\rho_{o,+}(n) = -\sqrt{-\frac{a(n)u(\lambda_1, n+1)}{u(\lambda_1, n)}}, \tag{3.26}$$

$$\rho_{e,+}(n) = \sqrt{-\frac{a(n-1)u(\lambda_1, n-1)}{u(\lambda_1, n)}}, \quad n > 0. \tag{3.27}$$

Define the operator $\dot{A}_+$ on $\ell_0(\mathbb{N})$

$$(\dot{A}_+ f)(n) = \rho_{o,+}(n)f(n+1) + \rho_{e,+}(n)f(n), \quad f \in \ell_0(\mathbb{N}) \tag{3.28}$$

and denote its operator closure (in $\ell^2(\mathbb{N})$) by A_+. One verifies,

$$\mathfrak{D}(A_+) \subseteq \{f \in \ell^2(\mathbb{N}) | \rho_{o,+}(n)f(n+1) + \rho_{e,+}(n)f(n) \in \ell^2(\mathbb{N})\}. \tag{3.29}$$

The adjoint A_+^* of A_+ is then given by

$$(A_+^* f)(n) = \rho_{o,+}(n-1)f(n-1) + \rho_{e,+}(n)f(n), \tag{3.30}$$
$$\mathfrak{D}(A_+^*) = \{f \in \ell^2(\mathbb{N}) | f(0) = 0; \rho_{o,+}(n-1)f(n-1) + \rho_{e,+}(n)f(n) \in \ell^2(\mathbb{N})\}.$$

(The boundary condition $f(0) = 0$ is only introduced so that we don't have to specify $(A_+^* f)(1)$ separately.) Due to a well known result of von Neumann (see, e.g., [76], Theorem 5.39) the operator $A_+ A_+^*$ is a positive self-adjoint operator when defined naturally

$$\mathfrak{D}(A_+ A_+^*) = \{f \in \mathfrak{D}(A_+^*) | A_+^* f \in \mathfrak{D}(A_+)\}. \tag{3.31}$$

A simple calculation shows $A_+ A_+^* f = (\tau - \lambda_1) f$ and hence we may define

$$H_+ = A_+ A_+^* + \lambda_1, \quad \mathfrak{D}(H_+) \subseteq \{f \in \ell^2(\mathbb{N}) | f(0) = 0, \ \tau f \in \ell^2(\mathbb{N})\}, \tag{3.32}$$

where equality in the last relation is equivalent to τ being limit point ($l.p.$) at $+\infty$. Similarly one defines for $n < 0$

$$\rho_{o,-}(n) = -\sqrt{-\frac{a(n)u(\lambda_1, n)}{u(\lambda_1, n+1)}}, \quad \rho_{e,-}(n) = \sqrt{-\frac{a(n)u(\lambda_1, n+1)}{u(\lambda_1, n)}} \tag{3.33}$$

and operators A_-, and A_-^* in $\ell^2(-\mathbb{N})$ which satisfy $H_- = A_-^* A_- + \lambda_1$.

Commuting $A_\pm^*$ and $A_\pm$ yields a second positive self-adjoint operator $A_- A_-^*$, respectively $A_+^* A_+$, and further the commuted operators

$$H_{+,1} = A_+^* A_+ + \lambda_1, \qquad H_{-,1} = A_- A_-^* + \lambda_1. \tag{3.34}$$

The next theorem characterizes $H_{\pm,1}$ in terms of $H_\pm$, but first we need to introduce

Hypothesis H. 3.1 *Suppose $H_\pm$ satisfies one of the following spectral conditions.*
(i) $\sigma_{ess}(H_\pm) \neq \emptyset$.
(ii) $\sigma(H_\pm) = \sigma_d(H_\pm) = \{\lambda_{\pm,j}\}_{j\in J_\pm}$ with $\sum_{j\in J_\pm}(1+\lambda_{\pm,j}^2)^{-1} = \infty$.

Hypothesis (H.3.1) is satisfied if a, b are bounded near $\pm\infty$.

Either one of the conditions (i), (ii) implies that τ is *l.p.* at $\pm\infty$. This follows since otherwise the resolvent of $H_\pm$ would be a Hilbert-Schmidt operator contradicting (i), (ii). This further implies that the domain of $H_\pm$ is given by

$$\mathfrak{D}(H_\pm) = \{f \in \ell^2(\pm\mathbb{N}) | f(0) = 0,\ \tau f \in \ell^2(\pm\mathbb{N})\}. \tag{3.35}$$

Theorem 3.2 *Assume (H.2.1) and (H.3.1). Then the operators $H_{\pm,1}$ constructed above satisfy (H.2.1) and (H.3.1) and are given by*

$$\begin{aligned} (H_{\pm,1}f)(n) &= (\tau_{\pm,1}f)(n) \\ &= a_{\pm,1}(n)f(n+1) + a_{\pm,1}(n-1)f(n-1) - b_{\pm,1}(n)f(n), \quad (3.36) \\ \mathfrak{D}(H_{\pm,1}) &= \{f \in \ell^2(\pm\mathbb{N}) | f(0) = 0,\ \tau_{\pm,1}f \in \ell^2(\pm\mathbb{N})\}, \end{aligned}$$

with

$$\begin{aligned} a_{+,1}(n) &= -\frac{\sqrt{a(n-1)a(n)u(\lambda_1,n-1)u(\lambda_1,n+1)}}{u(\lambda_1,n)}, \quad n>0, \quad (3.37) \\ b_{+,1}(n) &= a(n-1)\Big(\frac{u(\lambda_1,n)}{u(\lambda_1,n-1)} + \frac{u(\lambda_1,n-1)}{u(\lambda_1,n)}\Big) - \lambda_1, \quad n>1, \\ b_{+,1}(1) &= a(0)\frac{u(\lambda_1,0)}{u(\lambda_1,1)} - \lambda_1, \quad (3.38) \end{aligned}$$

and

$$\begin{aligned} a_{-,1}(n) &= -\frac{\sqrt{a(n)a(n+1)u(\lambda_1,n)u(\lambda_1,n+2)}}{u(\lambda_1,n+1)}, \quad n<-1, \quad (3.39) \\ b_{-,1}(n) &= a(n)\Big(\frac{u(\lambda_1,n)}{u(\lambda_1,n+1)} + \frac{u(\lambda_1,n+1)}{u(\lambda_1,n)}\Big) - \lambda_1, \quad n<-1, \\ b_{-,1}(-1) &= a(-1)\frac{u(\lambda_1,0)}{u(\lambda_1,-1)} - \lambda_1. \quad (3.40) \end{aligned}$$

Moreover, $H_\pm - \lambda_1$ and $H_{\pm,1} - \lambda_1$ restricted to the orthogonal complements of their null-spaces are unitarily equivalent and hence

$$\begin{aligned} \sigma(H_{\pm,1})\backslash\{\lambda_1\} &= \sigma(H_\pm)\backslash\{\lambda_1\}, \quad \sigma_{ac}(H_{\pm,1}) = \sigma_{ac}(H_\pm), \\ \sigma_p(H_{\pm,1})\backslash\{\lambda_1\} &= \sigma_p(H_\pm)\backslash\{\lambda_1\}, \quad \sigma_{sc}(H_{\pm,1}) = \sigma_{sc}(H_\pm). \end{aligned} \tag{3.41}$$

Proof. The unitary equivalence follows from [15], Theorem 1 and clearly settles the spectral claims. Thus both $H_\pm$ and $H_{\pm,1}$ satisfy (H.3.1) and hence $\tau_\pm$ and $\tau_{\pm,1}$ are *l.p.* at $\pm\infty$. The rest are straightforward calculations. □

Next we turn to the case of the whole lattice $\ell^2(\mathbb{Z})$. We pick $\sigma_1 \in [-1, 1]$ and $\lambda_1 < \inf(\sigma(H))$. Further denote by $u_\pm(\lambda, n)$ (for $\lambda < \inf(\sigma(H))$) the solutions constructed in Lemma 2.2 and set

$$u_{\sigma_1}(\lambda_1, n) = \frac{1+\sigma_1}{2} u_+(\lambda_1, n) + \frac{1-\sigma_1}{2} u_-(\lambda_1, n). \tag{3.42}$$

Now define sequences

$$\rho_{o,\sigma_1}(n) = -\sqrt{-\frac{a(n)u_{\sigma_1}(\lambda_1, n)}{u_{\sigma_1}(\lambda_1, n+1)}}, \quad \rho_{e,\sigma_1}(n) = \sqrt{-\frac{a(n)u_{\sigma_1}(\lambda_1, n+1)}{u_{\sigma_1}(\lambda_1, n)}}, \tag{3.43}$$

and the corresponding operator A_{σ_1} (first on $\ell_0(\mathbb{Z})$ and then take the closure in $\ell^2(\mathbb{Z})$ as before) together with its adjoint $A^*_{\sigma_1}$,

$$(A_{\sigma_1} f)(n) = \rho_{o,\sigma_1}(n) f(n+1) + \rho_{e,\sigma_1}(n) f(n), \tag{3.44}$$

$$\mathfrak{D}(A_{\sigma_1}) \subseteq \{f \in \ell^2(\mathbb{Z}) | \rho_{o,\sigma_1}(n) f(n+1) + \rho_{e,\sigma_1}(n) f(n) \in \ell^2(\mathbb{Z})\},$$

$$(A^*_{\sigma_1} f)(n) = \rho_{o,\sigma_1}(n-1) f(n-1) + \rho_{e,\sigma_1}(n) f(n), \tag{3.45}$$

$$\mathfrak{D}(A^*_{\sigma_1}) = \{f \in \ell^2(\mathbb{Z}) | \rho_{o,\sigma_1}(n-1) f(n-1) + \rho_{e,\sigma_1}(n) f(n) \in \ell^2(\mathbb{Z})\}.$$

Again by von Neumann's result $A^*_{\sigma_1} A_{\sigma_1}$ is a positive self-adjoint operator when defined naturally by

$$\mathfrak{D}(A^*_{\sigma_1} A_{\sigma_1}) = \{f \in \mathfrak{D}(A_{\sigma_1}) | A_{\sigma_1} f \in \mathfrak{D}(A^*_{\sigma_1})\}. \tag{3.46}$$

A simple calculation shows $A^*_{\sigma_1} A_{\sigma_1} = \tau - \lambda_1$ and we hence may define

$$H = A^*_{\sigma_1} A_{\sigma_1} + \lambda_1, \qquad \mathfrak{D}(H) \subseteq \{f \in \ell^2(\mathbb{Z}) | \tau f \in \ell^2(\mathbb{Z})\}. \tag{3.47}$$

Commuting $A^*_{\sigma_1}$ and A_{σ_1} yields a second positive self-adjoint operator $A_{\sigma_1} A^*_{\sigma_1}$ and further the commuted operator

$$H_{\sigma_1} = A_{\sigma_1} A^*_{\sigma_1} + \lambda_1, \qquad \mathfrak{D}(H_{\sigma_1}) \subseteq \{f \in \ell^2(\mathbb{Z}) | \tau_{\sigma_1} f \in \ell^2(\mathbb{Z})\}, \tag{3.48}$$

where τ_{σ_1} is the difference expression corresponding to H_{σ_1}. The next theorem characterizes H_{σ_1} under Assumption (H.2.2) for H_+ and H_- implying that τ is *l.p.* at $\pm\infty$ and hence that

$$\mathfrak{D}(H) = \{f \in \ell^2(\mathbb{Z}) | \tau f \in \ell^2(\mathbb{Z})\}. \tag{3.49}$$

Theorem 3.3 *Assume (H.2.1) and (H.3.1). Then the operator* H_{σ_1}*,*

$$\begin{aligned}(H_{\sigma_1} f)(n) &= (\tau_{\sigma_1} f)(n)\\ &= a_{\sigma_1}(n)f(n+1) + a_{\sigma_1}(n-1)f(n-1) - b_{\sigma_1}(n)f(n), \\ \mathfrak{D}(H_{\sigma_1}) &= \{f \in \ell^2(\mathbb{Z}) | \tau_{\sigma_1} f \in \ell^2(\mathbb{Z})\},\end{aligned} \tag{3.50}$$

is self-adjoint. Moreover,

$$a_{\sigma_1}(n) = -\frac{\sqrt{a(n)a(n+1)u_{\sigma_1}(\lambda_1,n)u_{\sigma_1}(\lambda_1,n+2)}}{u_{\sigma_1}(\lambda_1,n+1)}, \tag{3.51}$$

$$b_{\sigma_1}(n) = a(n)\Big(\frac{u_{\sigma_1}(\lambda_1,n)}{u_{\sigma_1}(\lambda_1,n+1)} + \frac{u_{\sigma_1}(\lambda_1,n+1)}{u_{\sigma_1}(\lambda_1,n)}\Big) - \lambda_1 \tag{3.52}$$

and a_{σ_1}*,* b_{σ_1} *satisfy (H.2.1). The equation* $\tau_{\sigma_1} v = \lambda_1 v$ *has the positive solution*

$$v_{\sigma_1}(\lambda_1,n) = \frac{1}{\sqrt{-a(n)u_{\sigma_1}(\lambda_1,n)u_{\sigma_1}(\lambda_1,n+1)}} \tag{3.53}$$

which is an eigenfunction of H_{σ_1} *if and only if* $\sigma_1 \in (-1,1)$*.* $H - \lambda_1$ *and* $H_{\sigma_1} - \lambda_1$ *restricted to the orthogonal complements of their corresponding one-dimensional null-spaces are unitarily equivalent and hence*

$$\begin{aligned}\sigma(H_{\sigma_1}) &= \begin{cases}\sigma(H)\cup\{\lambda_1\}, & \sigma_1\in(-1,1)\\ \sigma(H), & \sigma_1\in\{-1,1\}\end{cases}, \quad & \sigma_{ac}(H_{\sigma_1}) &= \sigma_{ac}(H),\\ \sigma_p(H_{\sigma_1}) &= \begin{cases}\sigma_p(H)\cup\{\lambda_1\}, & \sigma_1\in(-1,1)\\ \sigma_p(H), & \sigma_1\in\{-1,1\}\end{cases}, \quad & \sigma_{sc}(H_{\sigma_1}) &= \sigma_{sc}(H).\end{aligned} \tag{3.54}$$

In addition, the sequence

$$(A_{\sigma_1}u)(z,n) = \frac{W_n(u_{\sigma_1}(\lambda_1), u(z))}{\sqrt{-a(n)u_{\sigma_1}(\lambda_1,n)u_{\sigma_1}(\lambda_1,n+1)}} \tag{3.55}$$

solves $\tau_{\sigma_1} u = zu$ *if* $u(z)$ *solves* $\tau u = zu$ *for some* $z \in \mathbb{C}$*, where*

$$W_n(u,v) = a(n)(u(n)v(n+1) - u(n+1)v(n)) \tag{3.56}$$

denotes the modified Wronskian. Moreover, one obtains

$$W_{\sigma_1,n}(Au(z), Av(z)) = (\lambda_1 - z)W_n(u(z), v(z)) \tag{3.57}$$

for solutions u, v *of* $\tau u = zu$*, where* $W_{\sigma_1,n}(u,v) = a_{\sigma_1}(n)(u(n)v(n+1) - u(n+1)v(n))$*. The resolvents of* H, H_{σ_1} *for* $z \in \mathbb{C}\backslash(\sigma(H)\cup\{\lambda_1\})$ *are related via*

$$(H_{\sigma_1} - z)^{-1} = \frac{1}{z-\lambda_1}\Big(1 - A_{\sigma_1}(H-z)^{-1}A_{\sigma_1}^*\Big) \tag{3.58}$$

or, in terms of Green's functions for $n \geq m$, $z \in \mathbb{C}\backslash(\sigma(H) \cup \{\lambda_1\})$,

$$G(z,n,m) = \frac{u_+(z,n)u_-(z,m)}{W_n(u_+(z),u_-(z))}$$

$$\textit{implies} \quad G_{\sigma_1}(z,n,m) = \frac{(A_{\sigma_1}u_+)(z,n)(-A_{\sigma_1}u_-)(z,m)}{(z-\lambda_1)W_n(u_+(z),u_-(z))}. \tag{3.59}$$

Furthermore, $u_{\sigma_1,\pm}(z,n)$, the principal solutions of $(H_{\sigma_1} - z)u = 0$ for $z < \lambda_1$, are given by

$$u_{\sigma_1,\pm}(z,n) = \pm A_{\sigma_1}u_\pm(z,n) = \frac{\mp W_n(u_{\sigma_1}(\lambda_1),u_\pm(z))}{\sqrt{-a(n)u_{\sigma_1}(\lambda_1,n)u_{\sigma_1}(\lambda_1,n+1)}}. \tag{3.60}$$

In addition, we have

$$\sum_{n\in\mathbb{Z}} v_{\sigma_1}(\lambda_1,n)^2 = \frac{4}{1-\sigma_1^2}W(u_-(\lambda_1),u_+(\lambda_1))^{-1}, \quad \sigma_1 \in (-1,1) \tag{3.61}$$

and, if $\tau u(\lambda) = \lambda u(\lambda)$, $u(\lambda,.) \in \ell^2(\mathbb{Z})$,

$$\sum_{n\in\mathbb{Z}}(A_{\sigma_1}u)(\lambda,n)^2 = (\lambda-\lambda_1)\sum_{n\in\mathbb{Z}}u(\lambda,n)^2. \tag{3.62}$$

Proof. The unitary equivalence together with equation (3.58) follow from [15], Theorem 1. That H_{σ_1} is *l.p.* at $\pm\infty$ follows upon looking at the restrictions $H_\pm$, $H_{\pm,1}$ and using Theorem 3.2. Equation (3.58) together with (3.57) imply (3.59). The facts concerning the point spectrum follow since $G_{\sigma_1}(z,n,n)$ has a pole at $z = \lambda_1$ if and only if $\sigma_1 \in (-1,1)$. (3.61) can be obtained by investigating the residue of $G_{\sigma_1}(z,n,n)$ at $z = \lambda_1$. The rest are straightforward calculations. □

Remark 3.4 *(i). Hypothesis (H.3.1) is only needed in Theorem 3.3 to characterize the domains of H and H_{σ_1} explicitly.*
(ii). Multiplying u_{σ_1} with a positive constant leaves all formulas and, in particular, H_{σ_1} invariant.
(iii). If H is bounded from above we can insert eigenvalues into the highest spectral gap, i.e., above the spectrum of H, upon considering $-H$. Then $\lambda > \sup(\sigma(H))$ implies that we don't have positive but rather alternating solutions and all our previous calculations carry over with minor changes.
(iv). We can weaken (H.2.1) by requiring $a(n) \neq 0$ instead of $a(n) < 0$. Everything stays the same with the only difference that $u_\pm$ are not positive but change sign in such a way that (1.76) stays positive. Moreover, the signs of $a_{\sigma_1}(n)$ can also be prescribed arbitrarily by altering the signs of ρ_{o,σ_1} and ρ_{e,σ_1}.
(v). The fact that $v_{\sigma_1} \in \ell^2(\mathbb{Z})$ if and only if $\sigma_1 \in (-1,1)$ gives an alternate proof of

$$\sum_{n=0}^{\pm\infty} \frac{1}{-a(n)u_{\sigma_1}(\lambda_1,n)u_{\sigma_1}(\lambda_1,n+1)} < \infty \text{ if and only if } \sigma_1 \in \begin{matrix}[-1,1) \\ (-1,1]\end{matrix} \tag{3.63}$$

(cf. [62] and [37], Lemma 2.10, Remark 2.11).

At the end of this section we will show some connections between the single commutation method and some other theories. We start with the Weyl-Titchmarsh theory and freely use the definitions of Appendices B and C.

Lemma 3.5 *Assume (H.2.1). The Weyl $\tilde{m}$-functions $\tilde{m}_{\pm,\sigma_1}(z)$ of H_{σ_1}, $\sigma_1 \in [-1,1]$ in terms of $\tilde{m}_\pm(z)$, the ones of H, read*

$$\tilde{m}_{\pm,\sigma_1}(z) = \frac{-u_{\sigma_1}(\lambda_1,1)}{a(1)u_{\sigma_1}(\lambda_1,2)}\Big(1 + \frac{(z-\lambda_1)\tilde{m}_\pm(z)}{1+\frac{a(0)u_{\sigma_1}(\lambda_1,0)}{u_{\sigma_1}(\lambda_1,1)}\tilde{m}_\pm(z)}\Big). \tag{3.64}$$

Proof. The above formulas are straightforward calculations using (3.59) and (1.47), (1.48). □

Finally we turn to scattering theory. In order to facilitate comparison with the standard literature on (inverse) scattering theory for second-order difference operators (cf. [10], [11], [23], [32], [42], [73]) we now assume

$$a(n) > 0,\ b(n) \in \mathbb{R}, \qquad n|1-2a(n)|, nb(n) \in \ell^1(\mathbb{Z}) \tag{3.65}$$

(cf. Remark 3.4). This implies

$$\sigma_{ac}(H) = [-1,1], \quad \sigma_{sc}(H) = \emptyset, \quad \sigma_p(H) = \{\lambda_j\}_{j\in J} \subseteq \mathbb{R}\backslash[-1,1], \tag{3.66}$$

where $J \subseteq \mathbb{N}$ is a suitable (finite) index set, and the existence of the so called Jost solutions $f_\pm(k,n)$,

$$\Big(\tau - \frac{k+k^{-1}}{2}\Big)f_\pm(k,n) = 0, \quad \lim_{n\to\pm\infty} k^{\mp n} f_\pm(k,n) = 1, \quad |k| \le 1. \tag{3.67}$$

Transmission $T(k)$ and reflection $R_\pm(k)$ coefficients are then defined via

$$T(k) f_\mp(k,n) = f_\pm(k^{-1},n) + R_\pm(k) f_\pm(k,n), \quad |k| = 1, \tag{3.68}$$

and the norming constants $\gamma_{\pm,j}$ corresponding to $\lambda_j \in \sigma_p(H)$ are given by

$$\gamma_{\pm,j}^{-1} = \sum_{n\in\mathbb{Z}} |f_\pm(k_j,n)|^2, \quad k_j = \lambda_j + \sqrt{\lambda_j^2 - 1} \in (-1,0),\ j \in J. \tag{3.69}$$

Lemma 3.6 *Suppose H satisfies* (2.117) *and let H_{σ_1} be constructed as in Theorem 3.3 with*

$$u_{\sigma_1}(\lambda_1,n) = \frac{1+\sigma_1}{2} f_+(k_1,n) + \frac{1-\sigma_1}{2} f_-(k_1,n). \tag{3.70}$$

Then the transmission $T_{\sigma_1}(k)$ and reflection coefficients $R_{\pm,\sigma_1}(k)$ of H_{σ_1} in terms of the corresponding scattering data $T(k), R_\pm(k)$ of H are given by

$$T_{\sigma_1}(k) = \frac{1-k\,k_1}{k-k_1} T(k), \quad R_{\pm,\sigma_1}(k) = k^{\pm1}\frac{k-k_1}{1-k\,k_1} R_\pm(k),\ \sigma_1 \in (-1,1), \tag{3.71}$$

$$T_{\sigma_1}(k) = T(k), \quad R_{\pm,\sigma_1}(k) = \frac{k_1^{\sigma_1} - k^{\mp 1}}{k_1^{\sigma_1} - k^{\pm 1}} R_\pm(k), \quad \sigma_1 \in \{-1, 1\}, \tag{3.72}$$

where $k_1 = \lambda_1 + \sqrt{\lambda_1^2 - 1} \in (-1, 0)$. *Moreover, the norming constants* $\gamma_{\sigma_1,\pm,j}$ *associated with* $\lambda_j \in \sigma_p(H_{\sigma_1})$ *in terms of* $\gamma_{\pm,j}$ *corresponding to* H *read*

$$\begin{aligned} \gamma_{\sigma_1,\pm,j} &= |k_j|^{\pm 1} \frac{1 - k_j k_1}{(k_j - k_1)} \gamma_{\pm,j}, \quad j \in J,\ \sigma_1 \in (-1, 1), \\ \gamma_{\sigma_1,\pm,1} &= \left(\frac{1 - \sigma_1}{1 + \sigma_1}\right)^{\pm 1} |1 - k_1^{\mp 2}|\, T(k_1), \quad \sigma_1 \in (-1, 1), \end{aligned} \tag{3.73}$$

$$\gamma_{\sigma_1,\pm,j} = |k_1^{\sigma_1} - k_j^{\mp 1}| \gamma_{\pm,j}, \quad j \in J,\ \sigma_1 \in \{-1, 1\}. \tag{3.74}$$

Proof. The claims follow easily after observing that up to normalization the Jost solutions of H_{σ_1} are given by $A_{\sigma_1} f_\pm(k, n)$ (compare (3.59)). □

3.3 Iteration of the Single Commutation Method

By choosing $\lambda_2 < \lambda_1$ and $\sigma_2 \in [-1, 1]$ we can define

$$u_{\sigma_1,\sigma_2}(\lambda_2, n) = \frac{1 + \sigma_2}{2} u_{\sigma_1,+}(\lambda_2, n) + \frac{1 - \sigma_2}{2} u_{\sigma_1,-}(\lambda_2, n) \tag{3.75}$$

and repeat the process of the previous section by defining $\rho_{o,\sigma_1,\sigma_2}$, $\rho_{e,\sigma_1,\sigma_2}$ and corresponding operators A_{σ_1,σ_2}, $A^*_{\sigma_1,\sigma_2}$ which satisfy

$$H_{\sigma_1} = A^*_{\sigma_1,\sigma_2} A_{\sigma_1,\sigma_2} - \lambda_2. \tag{3.76}$$

A further commutation then yields the operator

$$H_{\sigma_1,\sigma_2} = A_{\sigma_1,\sigma_2} A^*_{\sigma_1,\sigma_2} - \lambda_2 \tag{3.77}$$

associated with sequences a_{σ_1,σ_2}, b_{σ_1,σ_2}. The result after N steps is summarized in

Theorem 3.7 *Assume (H.2.1) and (H.3.1). Let* H *be as in Section 3.2 and choose*

$$\lambda_N < \ldots < \lambda_2 < \lambda_1 < \inf(\sigma(H)), \quad \sigma_\ell \in [-1, 1],\ 1 \le \ell \le N,\ N \in \mathbb{N}. \tag{3.78}$$

Then we have

$$a_{\sigma_1,\ldots,\sigma_N}(n) = -\sqrt{a(n)a(n+N)} \frac{\sqrt{C_n(u^1_{\sigma_1}, \ldots, u^N_{\sigma_N}) C_{n+2}(u^1_{\sigma_1}, \ldots, u^N_{\sigma_N})}}{C_{n+1}(u^1_{\sigma_1}, \ldots, u^N_{\sigma_N})}, \tag{3.79}$$

$$\begin{aligned} b_{\sigma_1,\ldots,\sigma_N}(n) &= -\lambda_N + a(n) \frac{C_{n+2}(u^1_{\sigma_1}, \ldots, u^{N-1}_{\sigma_{N-1}}) C_n(u^1_{\sigma_1}, \ldots, u^N_{\sigma_N})}{C_{n+1}(u^1_{\sigma_1}, \ldots, u^{N-1}_{\sigma_{N-1}}) C_{n+1}(u^1_{\sigma_1}, \ldots, u^N_{\sigma_N})} \\ &\quad + a(n+N-1) \frac{C_n(u^1_{\sigma_1}, \ldots, u^{N-1}_{\sigma_{N-1}}) C_{n+1}(u^1_{\sigma_1}, \ldots, u^N_{\sigma_N})}{C_{n+1}(u^1_{\sigma_1}, \ldots, u^{N-1}_{\sigma_{N-1}}) C_n(u^1_{\sigma_1}, \ldots, u^N_{\sigma_N})}, \end{aligned} \tag{3.80}$$

where

$$u^\ell_{\sigma_\ell}(n) = \frac{1+\sigma_\ell}{2} u_+(\lambda_\ell, n) + (-1)^{\ell+1} \frac{1-\sigma_\ell}{2} u_-(\lambda_\ell, n), \tag{3.81}$$

and C_n denotes the n-dimensional Casoratian

$$C_n(u_1, \dots, u_N) = \det\{u_i(n+j-1)\}_{1\le i,j\le N}. \tag{3.82}$$

Moreover, for $1 \le \ell \le N$, $\lambda < \lambda_\ell$

$$u_{\sigma_1,\dots,\sigma_\ell,\pm}(\lambda, n) = \frac{\pm\sqrt{\prod_{j=0}^{\ell-1}(-a(n+j))} C_n(u^1_{\sigma_1}, \dots, u^\ell_{\sigma_\ell}, u_\pm(\lambda))}{\sqrt{C_n(u^1_{\sigma_1}, \dots, u^\ell_{\sigma_\ell}) C_{n+1}(u^1_{\sigma_1}, \dots, u^\ell_{\sigma_\ell})}}, \tag{3.83}$$

are the principal solutions of $\tau_{\sigma_1,\dots,\sigma_\ell} u = \lambda u$ and

$$u_{\sigma_1,\dots,\sigma_\ell}(\lambda_\ell, n) = \frac{1+\sigma_\ell}{2} u_{\sigma_1,\dots,\sigma_{\ell-1},+}(\lambda_\ell, n) + \frac{1-\sigma_\ell}{2} u_{\sigma_1,\dots,\sigma_{\ell-1},-}(\lambda_\ell, n) \tag{3.84}$$

is used to define $H_{\sigma_1,\dots,\sigma_\ell}$. We also have

$$\rho_{o,\sigma_1,\dots,\sigma_N}(n) = -\sqrt{-a(n) \frac{C_{n+2}(u^1_{\sigma_1},\dots,u^{N-1}_{\sigma_{N-1}}) C_n(u^1_{\sigma_1},\dots,u^N_{\sigma_N})}{C_{n+1}(u^1_{\sigma_1},\dots,u^{N-1}_{\sigma_{N-1}}) C_{n+1}(u^1_{\sigma_1},\dots,u^N_{\sigma_N})}}, \tag{3.85}$$

$$\rho_{e,\sigma_1,\dots,\sigma_N}(n) = \sqrt{-a(n+N-1) \frac{C_n(u^1_{\sigma_1},\dots,u^{N-1}_{\sigma_{N-1}}) C_{n+1}(u^1_{\sigma_1},\dots,u^N_{\sigma_N})}{C_{n+1}(u^1_{\sigma_1},\dots,u^{N-1}_{\sigma_{N-1}}) C_n(u^1_{\sigma_1},\dots,u^N_{\sigma_N})}}. \tag{3.86}$$

The spectrum of $H_{\sigma_1,\dots,\sigma_N}$ is given by

$$\sigma(H_{\sigma_1,\dots,\sigma_N}) = \sigma(H) \cup \{\lambda_\ell \mid \sigma_\ell \in (-1,1),\ 1 \le \ell \le N\}. \tag{3.87}$$

Proof. It is enough to prove the formulas for $a_{\sigma_1,\dots,\sigma_N}(n)$ and $u_{\sigma_1,\dots,\sigma_N}(n)$, the remaining assertions then follow easily. We will use a proof by induction on N. They are valid for $N = 1$ and we need to show

$$u_{\sigma_1,\dots,\sigma_{N+1},\pm}(\lambda, n) = \frac{\sqrt{-a_{\sigma_1,\dots,\sigma_N}(n)} C_n(u_{\sigma_1,\dots,\sigma_N}(\lambda_N), u_{\sigma_1,\dots,\sigma_N,\pm 1}(\lambda))}{\pm\sqrt{u_{\sigma_1,\dots,\sigma_N}(\lambda_N, n) u_{\sigma_1,\dots,\sigma_N}(\lambda_N, n+1)}}, \tag{3.88}$$

$$\begin{aligned} a_{\sigma_1,\dots,\sigma_{N+1}}(n) &= \sqrt{a_{\sigma_1,\dots,\sigma_N}(n) a_{\sigma_1,\dots,\sigma_N}(n+1)} \times \\ & \frac{\sqrt{u_{\sigma_1,\dots,\sigma_N}(\lambda_N, n) u_{\sigma_1,\dots,\sigma_N}(\lambda_N, n+1)}}{u_{\sigma_1,\dots,\sigma_N}(\lambda_N, n+1)}. \end{aligned} \tag{3.89}$$

The first relation follows after a straightforward calculation using Sylvester's determinant identity (cf. [28], Sect. II.3)

$$\begin{aligned} & C_n(u^1_{\sigma_1}, \dots, u^N_{\sigma_N}, u_\pm(\lambda)) C_{n+1}(u^1_{\sigma_1}, \dots, u^{N+1}_{\sigma_{N+1}}) \\ & - C_{n+1}(u^1_{\sigma_1}, \dots, u^N_{\sigma_N}, u_\pm(\lambda)) C_n(u^1_{\sigma_1}, \dots, u^{N+1}_{\sigma_{N+1}}) \\ & = C_{n+1}(u^1_{\sigma_1}, \dots, u^N_{\sigma_N}) C_n(u^1_{\sigma_1}, \dots, u^{N+1}_{\sigma_{N+1}}, u_\pm(\lambda)), \end{aligned} \tag{3.90}$$

and the second is a simple calculation. □

Remark 3.8 *If $u(z,n)$ is any solution of $\tau u = zu$, $z \in \mathbb{C}$ define $u_{\sigma_1,\dots,\sigma_N}(z,n)$ as in (3.83) but with $\ell = N$ and $u_\pm(\lambda,n)$ replaced by $u(z,n)$. Then $u_{\sigma_1,\dots,\sigma_N}(z,n)$ solves $\tau_{\sigma_1,\dots,\sigma_N} u = zu$.*

Finally we extend Lemma 3.6 and assume for brevity $\sigma_\ell \in (-1,1)$.

Lemma 3.9 *Suppose H satisfies* (2.117) *and let $H_{\sigma_1,\dots,\sigma_N}$, $\sigma_\ell \in (-1,1,)$, $1 \le \ell \le N$ be constructed as in Theorem 3.7 with*

$$u^\ell_{\sigma_\ell}(n) = \frac{1+\sigma_\ell}{2} f_+(k_\ell,n) + (-1)^{\ell+1}\frac{1-\sigma_\ell}{2} f_-(k_\ell,n). \tag{3.91}$$

Then the transmission $T_{\sigma_1,\dots,\sigma_N}(k)$ and reflection coefficients $R_{\pm,\sigma_1,\dots,\sigma_N}(k)$ of the operator $H_{\sigma_1,\dots,\sigma_N}$ in terms of the corresponding scattering data $T(k), R_\pm(k)$ of H are given by

$$T_{\sigma_1,\dots,\sigma_N}(k) = \left(\prod_{\ell=1}^N \frac{1-k\,k_\ell}{k-k_\ell}\right) T(k), \tag{3.92}$$

$$R_{\pm,\sigma_1,\dots,\sigma_N}(k) = k^{\pm N}\left(\prod_{\ell=1}^N \frac{k-k_\ell}{1-k\,k_\ell}\right) R_\pm(k), \tag{3.93}$$

where $k_\ell = \lambda_\ell + \sqrt{\lambda_\ell^2 - 1} \in (-1,0)$, $1 \le \ell \le N$. Moreover, the norming constants $\gamma_{\sigma_1,\dots,\sigma_N,\pm,j}$ associated with $\lambda_j \in \sigma_p(H_{\sigma_1,\dots,\sigma_N})$ in terms of $\gamma_{\pm,j}$ corresponding to H read

$$\gamma_{\sigma_1,\dots,\sigma_N,\pm,j} = \left(\tfrac{1-\sigma_j}{1+\sigma_j}\right)^{\pm 1} |k_j|^{-2\mp(N-1)} \frac{\prod_{\ell=1}^N |1-k_jk_\ell|}{\prod_{\substack{\ell=1\\ \ell\neq j}}^N |k_j-k_l|} T(k_j), \quad 1 \le j \le N,$$

$$\gamma_{\sigma_1,\dots,\sigma_N,\pm,j} = |k_j|^{\pm N} \prod_{\ell=1}^N \frac{1-k_jk_\ell}{|k_j-k_\ell|}\gamma_{\pm,j}, \quad j \in J. \tag{3.94}$$

Proof. Observe that

$$\begin{aligned} u_{\sigma_1,\sigma_2}(\lambda_2,n) &= \frac{1+\sigma_2}{2} A_{\sigma_1} f_+(k_2,n) + \frac{1-\sigma_2}{2} A_{\sigma_1} f_-(k_2,n) \\ &= c\Big(\frac{1+\hat\sigma_2}{2} f_{\sigma_1,+}(k_2,n) + \frac{1-\hat\sigma_2}{2} f_{\sigma_1,-}(k_2,n)\Big), \end{aligned} \tag{3.95}$$

where $c > 0$ and $\sigma_2, \hat\sigma_2$ are related via

$$\frac{1+\hat\sigma_2}{1-\hat\sigma_2} = \frac{1}{k_2}\frac{1+\sigma_1}{1-\sigma_1}. \tag{3.96}$$

The claims now follow from Lemma 3.6 after extending this result by induction. □

3.4 The Double Commutation Method

In this section we provide a complete characterization of the double commutation method for Jacobi operators. We start with a linear transformation which turns out to be unitary when restricted to proper subspaces of our Hilbert space. We use this transformation to construct an operator H_{γ_1} from a given background operator H. This operator H_{γ_1} will be the doubly commuted operator of H as discussed in the Introduction. The results of Sections 4-6 appear to be without precedent.

Let $\mathfrak{H} = \ell^2(M_- - 1, M_+ + 1)$ be the underlying Hilbert space ($-\infty \le M_- < M_+ \le \infty$) and let $\psi(n)$ be a given real-valued sequence which is square summable near M_-. Choose a positive constant $\gamma > 0$ and define

$$c_\gamma(n) = 1 + \gamma \sum_{j=M_-}^{n} \psi(j)^2, \qquad n \ge M_-. \tag{3.97}$$

(We set in addition $c_\gamma(M_- - 1) = 1$ if M_- is finite.) Denote the set of sequences in $\ell(M_- - 1, M_+ + 1)$ which are square summable near M_- by $\mathfrak{H}_-$ and consider the following (linear) transformation

$$\begin{array}{rccl} U_\gamma : & \mathfrak{H}_- & \to & \mathfrak{H}_- \\ & f(n) & \mapsto & f_\gamma(n) = \sqrt{\frac{c_\gamma(n)}{c_\gamma(n-1)}} f(n) - \gamma \psi_\gamma(n) \sum\limits_{j=M_-}^{n} \psi(j) f(j). \end{array} \tag{3.98}$$

By inspection, the sequence f_γ is also square summable near M_- and the inverse transformation is given by

$$\begin{array}{rccl} U_\gamma^{-1} : & \mathfrak{H}_- & \to & \mathfrak{H}_- \\ & g(n) & \mapsto & \sqrt{\frac{d_\gamma(n)}{d_\gamma(n-1)}} g(n) + \gamma \psi(n) \sum\limits_{j=M_-}^{n} \psi_\gamma(j) g(j) \end{array}, \tag{3.99}$$

where

$$d_\gamma(n) = c_\gamma(n)^{-1} = 1 - \gamma \sum_{j=M_-}^{n} \psi_\gamma(j)^2, \qquad \psi_\gamma(n) = \frac{\psi(n)}{\sqrt{c_\gamma(n-1) c_\gamma(n)}}. \tag{3.100}$$

Lemma 3.10 *Define ψ_γ as in (3.100). Then $\psi_\gamma \in \mathfrak{H}$ and*

$$\|\psi_\gamma\|^2 = \frac{1}{\gamma}\Big(1 - \lim_{n \to M_+} c_\gamma(n)^{-1}\Big). \tag{3.101}$$

If P, P_γ denote the orthogonal projections onto the one-dimensional subspaces of $\mathfrak{H}$ spanned by ψ, ψ_γ (set $P = 0$ if $\psi \notin \mathfrak{H}$) the operator U_γ is unitary from $(1 - P)\mathfrak{H}$ onto $(1 - P_\gamma)\mathfrak{H}$.

Proof. For the claims concerning ψ we use

$$\sum_{j=M_-}^{n} |\psi_\gamma(j)|^2 = \frac{1}{\gamma} \sum_{j=M_-}^{n} \Big(\frac{1}{c_\gamma(j-1)} - \frac{1}{c_\gamma(j)}\Big) = \frac{1}{\gamma}\Big(1 - \frac{1}{c_\gamma(n)}\Big). \tag{3.102}$$

Next we note that

$$c_\gamma(n) \sum_{j=M_-}^{n} \psi_\gamma(j) f_\gamma(j) = \sum_{j=M_-}^{n} \psi(j) f(j) \tag{3.103}$$

and a direct calculation shows

$$\sum_{j=M_-}^{n} |f_\gamma(j)|^2 = \sum_{j=M_-}^{n} |f(j)|^2 - \frac{\gamma}{c_\gamma(n)} |\sum_{j=M_-}^{n} f(j)\psi(j)|^2. \tag{3.104}$$

This clearly proves the lemma if $\psi \in \mathfrak{H}$. Otherwise, i.e., if $\psi \notin \mathfrak{H}$, consider U_γ, U_γ^{-1} on the dense subspace $\ell_0((M_-, M_+))$ and take closures (cf., e.g., [76], Theorem 6.13). □

Using, e.g., the polarization identity, we further get

$$\sum_{j=M_-}^{n} \overline{g_\gamma(j)} f_\gamma(j) = \sum_{j=M_-}^{n} \overline{g(j)} f(j) - \frac{\gamma}{c_\gamma(n)} \sum_{j=M_-}^{n} \psi(j) f(j) \sum_{j=M_-}^{n} \psi(j) \overline{g(j)}. \tag{3.105}$$

Next we take two sequences a, b satisfying

Hypothesis H. 3.11 *Suppose*

$$a, b \in \ell(M_- - 1, M_+ + 1), \qquad a(n) \in \mathbb{R}\backslash\{0\},\ b(n) \in \mathbb{R} \tag{3.106}$$

and introduce the difference expression

$$(\tau f)(n) = a(n) f(n+1) + a(n-1) f(n-1) - b(n) f(n). \tag{3.107}$$

We want to consider a self-adjoint operator H associated with τ and separated boundary conditions at $M_\pm$ and assume the existence of a sequence $\psi(\lambda_1, n)$ of the following kind.

Hypothesis H. 3.12 *Suppose $\psi(\lambda)$ satisfies the following conditions.*
(i) $\psi(\lambda, n)$ is a real-valued solution of $\tau\psi(\lambda) = \lambda\psi(\lambda)$.
(ii) $\psi(\lambda, n)$ is square summable near M_- and fulfills the boundary condition (of H) at M_- (if any, i.e., if τ is l.c. at M_-).
(iii) $\psi(\lambda, n)$ also fulfills the boundary condition (of H) at M_+ if τ is l.c. at M_+ ($\psi(\lambda, n)$ is then an eigenfunction of H).

Sufficient conditions for the above function to exist are
(i) $\lambda \in \sigma_p(H)$, or
(ii) τ is *l.c.* at M_- but not at M_+, or
(iii) $\sigma(H) \neq \mathbb{R}$ (and $\lambda \in \mathbb{R}\backslash\sigma(H)$), or
(iv) $\sigma(H_-) \neq \mathbb{R}$ (and $\lambda \in \mathbb{R}\backslash\sigma(H_-)$), where H_- is a restriction of H to $\ell^2(M_- - 1, \hat{M} + 1)$ with $\hat{M} \in \mathbb{Z}$ and (for instance) a Dirichlet boundary condition at $\hat{M} + 1$.

It follows that H is explicitly given by

$$\begin{aligned}\mathfrak{D}(H) = \{f \in \mathfrak{H}|\ \tau f \in \mathfrak{H};\ & W_{M_--1}(\psi(\lambda_1), f) = 0 \text{ if } \tau \text{ is } l.c. \text{ at } M_-, \\ & W_{M_+}(\psi(\lambda_1), f) = 0 \text{ if } \tau \text{ is } l.c. \text{ at } M_+\}.\end{aligned} \tag{3.108}$$

We now use Lemma 3.10 with $\psi(n) = \psi(\lambda_1, n)$, $\gamma = \gamma_1$, $U_\gamma = U_{\gamma_1}$ to prove

Theorem 3.13 *Suppose (H.3.11) and (H.3.12) and let τ_{γ_1} be the difference expression*

$$(\tau_{\gamma_1} f)(n) = a_{\gamma_1}(n) f(n+1) + a_{\gamma_1}(n-1) f(n-1) - b_{\gamma_1}(n) f(n), \tag{3.109}$$

where

$$a_{\gamma_1}(n) = a(n) \frac{\sqrt{c_{\gamma_1}(\lambda_1, n-1) c_{\gamma_1}(\lambda_1, n+1)}}{c_{\gamma_1}(\lambda_1, n)}, \tag{3.110}$$

$$\begin{aligned} b_{\gamma_1}(n) &= b(n) + \gamma_1 \Big(\frac{a(n-1)\psi(\lambda_1, n-1)\psi(\lambda_1, n)}{c_{\gamma_1}(\lambda_1, n-1)} \\ &\quad - \frac{a(n)\psi(\lambda_1, n)\psi(\lambda_1, n+1)}{c_{\gamma_1}(\lambda_1, n)} \Big). \end{aligned} \tag{3.111}$$

Then the operator H_{γ_1} defined by

$$H_{\gamma_1} f = \tau_{\gamma_1} f, \tag{3.112}$$

$$\mathfrak{D}(H_{\gamma_1}) = \{f \in \mathfrak{H} | \tau_{\gamma_1} f \in \mathfrak{H}; W_{\gamma_1, M_--1}(\psi_{\gamma_1}(\lambda_1), f) = W_{\gamma_1, M_+}(\psi_{\gamma_1}(\lambda_1), f) = 0\},$$

where $W_{\gamma_1,n}(u, v) = a_{\gamma_1}(n)(u(n)v(n+1) - u(n+1)v(n))$, is self-adjoint and has the eigenfunction

$$\psi_{\gamma_1}(\lambda_1, n) = \frac{\psi(\lambda_1, n)}{\sqrt{c_{\gamma_1}(\lambda_1, n-1) c_{\gamma_1}(\lambda_1, n)}} \tag{3.113}$$

associated with the eigenvalue λ_1. If $\psi(\lambda_1) \not\in \mathfrak{H}$ (and hence τ is l.p. at M_+) we have

$$(1 - P_{\gamma_1}(\lambda_1)) H_{\gamma_1} = U_{\gamma_1} H U_{\gamma_1}^{-1} (1 - P_{\gamma_1}(\lambda_1)), \tag{3.114}$$

where U_{γ_1} is the unitary transformation of Lemma 3.10 and thus

$$\begin{aligned} \sigma(H_{\gamma_1}) &= \sigma(H) \cup \{\lambda_1\}, & \sigma_{ac}(H_{\gamma_1}) &= \sigma_{ac}(H), \\ \sigma_p(H_{\gamma_1}) &= \sigma_p(H) \cup \{\lambda_1\}, & \sigma_{sc}(H_{\gamma_1}) &= \sigma_{sc}(H). \end{aligned} \tag{3.115}$$

If $\psi(\lambda_1) \in \mathfrak{H}$ there is a unitary operator $\tilde{U}_{\gamma_1} = U_{\gamma_1} \oplus \sqrt{1 + \gamma_1 \|\psi(\lambda_1)\|^2} \mathbf{1}$ on $(1 - P_{\gamma_1}(\lambda_1))\mathfrak{H} \oplus P_{\gamma_1}(\lambda_1)\mathfrak{H}$ such that

$$H_{\gamma_1} = \tilde{U}_{\gamma_1} H \tilde{U}_{\gamma_1}^{-1} \tag{3.116}$$

and thus

$$\begin{aligned} \sigma(H_{\gamma_1}) &= \sigma(H), & \sigma_{ac}(H_{\gamma_1}) &= \sigma_{ac}(H), \\ \sigma_p(H_{\gamma_1}) &= \sigma_p(H), & \sigma_{sc}(H_{\gamma_1}) &= \sigma_{sc}(H). \end{aligned} \tag{3.117}$$

Proof. It suffices to prove

$$(1 - P_{\gamma_1}(\lambda_1))H_{\gamma_1} = U_{\gamma_1} H U_{\gamma_1}^{-1}(1 - P_{\gamma_1}(\lambda_1)). \tag{3.118}$$

Let f be a sequence which is square summable near M_- such that τf is also square summable near M_- and assume that f fulfills the boundary condition at M_-, if any. Then a straightforward calculation shows

$$\tau_{\gamma_1}(U_{\gamma_1} f) = U_{\gamma_1}(\tau f) \tag{3.119}$$

and we only have to check the boundary conditions at $M_\pm$. Equation (3.104) shows that τ_{γ_1} is *l.c.* at M_- if and only if τ is and that τ_{γ_1} is *l.c.* at M_+ if τ is. The formula

$$W_{\gamma_1,n}(\psi_{\gamma_1}(\lambda_1), U_{\gamma_1} f) = \frac{W_n(\psi(\lambda_1), f)}{c_{\gamma_1}(\lambda_1, n)} \tag{3.120}$$

shows that

$$W_{\gamma_1,M_- -1}(\psi_{\gamma_1}(\lambda_1), U_{\gamma_1} f) = 0, \qquad f \in \mathfrak{D}(H). \tag{3.121}$$

We further claim that

$$W_{\gamma_1,M_+}(\psi_{\gamma_1}(\lambda_1), U_{\gamma_1} f) = 0, \qquad f \in \mathfrak{D}(H). \tag{3.122}$$

This is clear if $\psi(\lambda_1) \in \mathfrak{H}$. Otherwise, i.e., if $\psi(\lambda_1) \notin \mathfrak{H}$, we use

$$\frac{W_n(\psi(\lambda_1), f)}{c_{\gamma_1}(\lambda_1, n)} = \frac{\sum_{j=M_-}^{n} \psi(\lambda_1, j)(\lambda_1 - \tau) f(j)}{c_{\gamma_1}(\lambda_1, n)}. \tag{3.123}$$

The right hand side tends to zero for $f \in \mathfrak{D}(H)$ as can be seen from (3.104) and the fact that U_{γ_1} is unitary. Combining (3.121) and (3.122) yields

$$(1 - P_{\gamma_1}(\lambda_1))U_{\gamma_1}\mathfrak{D}(H) \subseteq (1 - P_{\gamma_1}(\lambda_1))\mathfrak{D}(H_{\gamma_1}). \tag{3.124}$$

But $(1 - P_{\gamma_1}(\lambda_1))U_{\gamma_1}\mathfrak{D}(H)$ cannot be properly contained in $(1 - P_{\gamma_1}(\lambda_1))\mathfrak{D}(H_{\gamma_1})$ by the property of self-adjoint operators being maximally defined. □

Remark 3.14 *(i). By choosing $\lambda_1 \in \sigma_{ac}(H) \cup \sigma_{sc}(H)$ (provided the continuous spectrum is not empty and a solution satisfying (H.3.12) exists) we can use the double commutation method to construct operators with eigenvalues embedded in the continuous spectrum.*
(ii). If $M_+ = \infty$ and H has an eigenfunction $\psi(\lambda_1)$ one can remove this eigenfunction from the spectrum upon choosing $\gamma_1 = -\|\psi(\lambda_1)\|^{-2}$. The corresponding function $\psi_{\gamma_1}(\lambda_1)$ is then no longer in $\mathfrak{H}$, implying that τ_{γ_1} is l.p. at M_+.
(iii). Especially, removing an eigenvalue from an operator which is l.c. at ∞ yields an operator which is l.p.. Thus τ_{γ_1} is not necessary l.p. if τ is. Moreover, this shows that one cannot insert additional eigenvalues into an operator which is l.c. at M_+ (remove this eigenvalue again to obtain a contradiction).
(iv). The limiting case $\gamma_1 = \infty$ can be handled analogously producing a unitarily equivalent operator if $\psi(\lambda_1) \notin \mathfrak{H}$ and removes the eigenvalue λ_1 otherwise.

The previous theorem tells us only how to transfer solutions of $\tau u = zu$ into solutions of $\tau_{\gamma_1} v = zv$ if u is square summable near M_-. The following lemma treats the general case.

Lemma 3.15 *The sequence*

$$u_{\gamma_1}(z,n) = \frac{c_{\gamma_1}(\lambda_1,n)u(z,n) - \frac{\gamma_1}{z-\lambda_1}\psi(\lambda_1,n)W_n(\psi(\lambda_1),u(z))}{\sqrt{c_{\gamma_1}(\lambda_1,n-1)c_{\gamma_1}(\lambda_1,n)}}, \quad z \in \mathbb{C}\backslash\{\lambda_1\} \tag{3.125}$$

solves $\tau_{\gamma_1} u = zu$ if $u(z)$ solves $\tau u = zu$. If $u(z)$ is square summable near M_- and fulfills the boundary condition at M_- (if any) we have $u_{\gamma_1}(z,n) = (U_{\gamma_1} u)(z,n)$ justifying our notation. Furthermore, we note

$$\begin{aligned} |u_{\gamma_1}(z,n)|^2 &= |u(z,n)|^2 - \frac{\gamma_1}{|z-\lambda_1|^2} \times \\ & \Big(\frac{|W_n(\psi(\lambda_1),u(z))|^2}{c_{\gamma_1}(\lambda_1,n)} - \frac{|W_{n-1}(\psi(\lambda_1),u(z))|^2}{c_{\gamma_1}(\lambda_1,n-1)}\Big), \end{aligned} \tag{3.126}$$

and

$$W_{\gamma_1,n}(\psi_{\gamma_1}(\lambda_1),u_{\gamma_1}(z)) = \frac{W_n(\psi(\lambda_1),u(z))}{c_{\gamma_1}(\lambda_1,n)}. \tag{3.127}$$

Hence u_{γ_1} is square summable near M_+ if u is. If $\hat{u}_\gamma(\hat{z})$ is constructed analogously then

$$\begin{aligned} W_{\gamma_1,n}(u_{\gamma_1}(z),\hat{u}_{\gamma_1}(\hat{z})) &= W_n(u(z),\hat{u}(\hat{z})) + \frac{\gamma_1}{c_{\gamma_1}(\lambda_1,n)} \frac{z-\hat{z}}{(z-\lambda_1)(\hat{z}-\lambda_1)} \times \\ & W_n(\psi(\lambda_1),u(z))W_n(\psi(\lambda_1),\hat{u}(\hat{z})). \end{aligned} \tag{3.128}$$

Proof. All facts are tedious but straightforward calculations. □

Next we want to give some conditions implying the *l.p.* case of τ_{γ_1} at M_+, assuming $M_+ = \infty$. Let $M_- < \hat{M} < \infty$ and let H_+ denote a self-adjoint operator associated with τ on $(\hat{M}-1,\infty)$ and the boundary condition induced by $\psi(\lambda_1)$ at $\hat{M}$ (cf. equation (3.108)).

Hypothesis H. 3.16 *Suppose H_+ satisfies one of the following spectral conditions:*
(i). $\sigma_{ess}(H_+) \neq \emptyset$.
(ii). $\sigma(H_+) = \sigma_d(H_+) = \{\lambda_{+,j}\}_{j\in J_+}$ with $\sum_{j\in J_+}(1+\lambda_{+,j}^2)^{-1} = \infty$.

Clearly Hypothesis (H.3.16) is satisfied if a, b are bounded near ∞ (which is equivalent to H_+ being bounded) since then τ is *l.p.* at ∞.

Theorem 3.17 *Assume (H.3.11), (H.3.12), and (H.3.16). Then τ_{γ_1} is l.p. at $M_+ = \infty$.*

Proof. Let $\gamma_{1,+} = c_{\gamma_1}(\lambda_1, \hat{M})^{-1}\gamma_1$ and consider the doubly commuted operator $H_{+,\gamma_{1,+}}$ of H_+. Then $\tau_{\gamma_1}|_{(\hat{M},\infty)} = \tau_{\gamma_{1,+}}$ and $H_{+,\gamma_{1,+}}$ also satisfies (H.3.16). Hence τ_{γ_1} is *l.p.* at ∞ as claimed. □

Remark 3.18 *We can interchange the role of M_- and M_+ in this section by substituting $M_- \leftrightarrow M_+$, $\sum_{j=M_-}^{n} \rightarrow \sum_{j=n+1}^{M_+}$ and $\gamma_1 \rightarrow -\gamma_1$.*

Let $M_\pm = \pm\infty$ and H be a given Jacobi operator satisfying (2.117). Our next aim is to show how the scattering data of the operators H, H_{γ_1} are related, where H_{γ_1} is defined as in Theorem 3.13.

Lemma 3.19 *Let H be a given Jacobi operator satisfying (2.117). Then the doubly commuted operator H_{γ_1}, defined via $\psi(\lambda_1, n) = f_-(k_1, n)$, $\quad \lambda_1 = (k_1 + k_1^{-1})/2$ as in Theorem 3.13, has the transmission and reflection coefficients*

$$T_{\gamma_1}(k) = \mathrm{sgn}(k_1)\frac{k\,k_1 - 1}{k - k_1}T(k), \tag{3.129}$$

$$R_{-,\gamma_1}(k) = R_-(k), \qquad R_{+,\gamma_1}(k) = \left(\frac{k - k_1}{k\,k_1 - 1}\right)^2 R_+(k), \tag{3.130}$$

where k and z are related via $z = (k + k^{-1})/2$. Furthermore, the norming constants $\gamma_{-,j}$ corresponding to $\lambda_j \in \sigma_p(H)$, $j \in J$ (cf. (3.69)) remain unchanged except for an additional eigenvalue λ_1 with norming constant $\gamma_{-,1} = \gamma_1$ if $\psi(\lambda_1) \notin \mathfrak{H}$ respectively with norming constant $\tilde{\gamma}_{-,1} = \gamma_{-,1} + \gamma_1$ if $\psi(\lambda_1) \in \mathfrak{H}$ and $\gamma_{-,1}$ denotes the original norming constant of $\lambda_1 \in \sigma_p(H)$.

Proof. By Lemma 3.15 the Jost solutions $f_{\gamma_1,\pm}(k, n)$ are up to a constant given by

$$\frac{c_{\gamma_1}(\lambda_1, n-1)f_\pm(k, n) - \frac{\gamma_1}{z-\lambda_1}\psi(\lambda_1, n)W_{n-1}(\psi(\lambda_1), f_\pm(k))}{\sqrt{c_{\gamma_1}(\lambda_1, n-1)c_{\gamma_1}(\lambda_1, n)}}. \tag{3.131}$$

This constant is easily seen to be 1 for $f_{\gamma_1,-}(k, n)$. Thus we can compute $R_-(\lambda)$ using (3.128) (the second unknown constant cancels). The rest follows by a straightforward calculation. □

3.5 Double Commutation and Weyl–Titchmarsh Theory

In this section we want to reveal the connections between Weyl–Titchmarsh theory and the double commutation method. Without loss of generality we consider only the cases $\ell^2(\mathbb{N})$ and $\ell^2(\mathbb{Z})$. We start with the half-line $\mathbb{N}$ and freely use the notation employed in Appendices A–D.

Let H_+ be a self-adjoint operator associated with τ on $\mathbb{N}$ and a Dirichlet boundary condition at 0. Without loss of generality we assume $\psi(\lambda_1, 1) = 1$.

Theorem 3.20 *Assume (H.3.11), $\psi(\lambda_1, 1) = 1$ and let $m_+(z, 0)$, $m_{+,\gamma_1}(z, 0)$ denote the Weyl m-functions of H_+, H_{+,γ_1}. Then we have*

$$m_{+,\gamma_1}(z, 0) = \frac{1}{1+\gamma_1}\Big(m_+(z, 0) - \frac{\gamma_1}{z - \lambda_1}\Big). \tag{3.132}$$

If μ_+ and μ_{+,γ_1} denote the corresponding spectral functions of H_+ and H_{+,γ_1} it follows that

$$\mu_{+,\gamma_1}(\lambda) = \frac{1}{1+\gamma_1}\Big(\mu_+(\lambda) + \gamma_1 \Theta(\lambda - \lambda_1)\Big), \tag{3.133}$$

where $\Theta(.)$ denotes the (right continuous) step function

$$\Theta(x) = \left\{ \begin{array}{ll} 1, & x \geq 0 \\ 0, & x < 0 \end{array} \right. . \tag{3.134}$$

Proof. As in Section 1.2 we use the finite approximations $m_N(z, 0)$ and $m_{N,\gamma_1}(z, 0)$. If $\gamma_j(N)$, $\gamma_{j,\gamma_1}(N)$ are the corresponding norming constants we have

$$\gamma_{j,\gamma_1}(N) = \frac{1}{1+\gamma_1}\left\{ \begin{array}{ll} \gamma_j(N) + \gamma_1, & \lambda_j = \lambda_1 \\ \gamma_j(N), & \lambda_j \neq \lambda_1 \end{array} \right. . \tag{3.135}$$

This follows since $\psi(z, 0) = 0$, $\psi(z, 1) = 1$ implies $\psi_{\gamma_1}(z, 0) = 0$, $\psi_{\gamma_1}(z, 1) = (1 + \gamma_1)^{-1/2}$. Hence we infer

$$m_{N,\gamma_1}(z, 0) = \frac{1}{1+\gamma_1}\Big(m_N(z, 0) - \frac{\gamma_1}{z - \lambda_1}\Big) \tag{3.136}$$

and the theorem follows upon taking the limit $N \to \infty$. □

Remark 3.21 *If we transform the operator H_+ into it's diagonal form as in Section 1.3 the double commutation method gets particularly transparent: it corresponds to adding a step function to the spectral function. This approach can also be used to derive the unitary transformation stated in Section 2 in the following way. Take the spectral function μ_+ of a given Jacobi operator, switch to μ_{+,γ_1}, and compute the orthogonal polynomials with respect to this new measure (compare Section 1.3 and [1], Ch. 1). Now take a sequence $f(n)$ and its transform $F(z)$ and use (1.35) to obtain (3.98).*

Next we turn to operators in $\ell^2(\mathbb{Z})$. Without loss of generality we assume

$$\psi(\lambda_1, 0) = -\sin(\alpha), \quad \psi(\lambda_1, 0) = \cos(\alpha), \quad \alpha \in [0, \pi). \tag{3.137}$$

Theorem 3.22 *Assume (H.3.11) and let* $\tilde{m}_\pm(z,\alpha)$, $\tilde{m}_{\pm,\gamma_1}(z,\alpha)$ *denote the Weyl* $\tilde{m}$-*functions of* H, H_{γ_1} *as introduced in Section 1.2. Then we have*

$$\tilde{m}_{\pm,\gamma_1}(z,\tilde{\alpha}) = \frac{(1+\tilde{\gamma}_1(\cos(\alpha)^4 - \sin(\alpha)^4))^{-1/2}}{((1+\tilde{\gamma}_1\cos(\alpha)^2)(1-\tilde{\gamma}_1\sin(\alpha)^2))^{1/2}}\Big(\tilde{m}_\pm(z,\alpha) - \frac{\tilde{\gamma}_1}{z-\lambda_1}\Big), \quad (3.138)$$

where

$$\tilde{\gamma}_1 = \frac{\gamma_1}{c_{\gamma_1}(\lambda_1,0)}, \qquad \tan(\tilde{\alpha}) = \sqrt{\frac{c_{\gamma_1}(\lambda_1,1)}{c_{\gamma_1}(\lambda_1,-1)}}\tan(\alpha). \quad (3.139)$$

Proof. Consider the sequences

$$\phi_{\alpha,\gamma_1}(z,n), \quad \theta_{\alpha,\gamma_1}(z,n) - \frac{\tilde{\gamma}_1}{z-\lambda_1}\phi_{\alpha,\gamma_1}(z,n) \quad (3.140)$$

constructed from the fundamental system $\theta_\alpha(z,n)$, $\phi_\alpha(z,n)$ for τ (cf. (1.11)) as in Lemma 3.15. They form a fundamental system for τ_{γ_1} corresponding to the initial conditions associated with $\tilde{\alpha}$ up to constant multiples. Now use (3.127) to evaluate (1.13). □

The Weyl M-matrix and the corresponding spectral matrix can now be computed in a straightforward manner (cf. Section 1.4).

3.6 Iteration of the double commutation method

Finally we demonstrate how to iterate the double commutation method. We choose a given background operator H (with coefficients a, b satisfying (H.3.11)) and further $\gamma_1 > 0, \lambda_1 \in \mathbb{R}$. Next choose $\psi(\lambda_1)$ as in Hypothesis (H.3.12) to define the transformation U_{γ_1} and the operator H_{γ_1}. In the second step, we choose $\gamma_2 > 0, \lambda_2 \in \mathbb{R}$ and another function $\psi(\lambda_2)$ to define $\psi_{\gamma_1}(\lambda_2) = U_{\gamma_1}\psi(\lambda_2)$, a corresponding transformation U_{γ_1,γ_2}, and an operator H_{γ_1,γ_2}. Applying this procedure N-times results in

Theorem 3.23 *Assuming (H.3.11) let* H *be a given background Jacobi operator in* $\mathfrak{H} = \ell^2(M_- - 1, M_+ + 1)$ *and let* $\gamma_j > 0$, λ_j, $1 \le j \le N$ *be such that there exist corresponding solutions* $\psi(\lambda_j,n)$ *of* $\tau\psi = \lambda_j\psi$ *satisfying Hypothesis (H.3.12). We set* $\psi_{\gamma_1,\dots,\gamma_k}(\lambda_j) = U_{\gamma_1,\dots,\gamma_k}\cdots U_{\gamma_1}\psi(\lambda_j)$ *and define the following matrices* $(1 \le \ell \le N)$

$$C^\ell(n) = \left\{\delta_r(s) + \sqrt{\gamma_r\gamma_s}\sum_{m=M_-}^{n}\psi(\lambda_r,m)\psi(\lambda_s,m)\right\}_{1\le r,s\le\ell}, \quad (3.141)$$

$$C^\ell_{i,j}(n) = \left\{\begin{array}{cc} C^{\ell-1}(n)_{r,s} & r,s\le\ell-1 \\ \sqrt{\gamma_s}\sum_{m=M_-}^{n}\psi(\lambda_i,m)\psi(\lambda_s,m) & s\le\ell-1, r=\ell \\ \sqrt{\gamma_r}\sum_{m=M_-}^{n}\psi(\lambda_r,m)\psi(\lambda_j,m) & r\le\ell-1, s=\ell \\ \sum_{m=M_-}^{n}\psi(\lambda_i,m)\psi(\lambda_j,m) & r=s=\ell \end{array}\right\}_{1\le r,s\le\ell}, \quad (3.142)$$

$$\Psi^{\ell}(\lambda_j,n) = \left\{ \begin{array}{cc} C^{\ell}(n)_{r,s} & {\scriptstyle r,s\le \ell} \\ \sqrt{\gamma_s} \sum\limits_{m=M_-}^{n} \psi(\lambda_j,m)\psi(\lambda_s,m) & {\scriptstyle s\le\ell, r=\ell+1} \\ \sqrt{\gamma_r}\psi(\lambda_r,n) & {\scriptstyle r\le\ell, s=\ell+1} \\ \psi(\lambda_j,n) & {\scriptstyle r=s=\ell+1} \end{array} \right\}_{1\le r,s\le \ell+1}. \quad (3.143)$$

Then we have (set $C^0(n)=1$*)*

$$c_{\gamma_\ell}(\lambda_\ell,n) = 1+\gamma_\ell \sum_{m=M_-}^{n} \psi_{\gamma_1,\dots,\gamma_\ell}(\lambda_\ell,m)^2 = \frac{\det C^{\ell}(n)}{\det C^{\ell-1}(n)}, \quad (3.144)$$

and hence

$$\prod_{\ell=1}^{N} c_{\gamma_\ell}(\lambda_\ell,n) = \det C^N(n). \quad (3.145)$$

Moreover,

$$\sum_{m=M_-}^{n} \psi_{\gamma_1,\dots,\gamma_\ell}(\lambda_i,m)\psi_{\gamma_1,\dots,\gamma_\ell}(\lambda_j,m) = \frac{\det C^{\ell}_{i,j}(n)}{\det C^{\ell-1}(n)} \quad (3.146)$$

and

$$\psi_{\gamma_1,\dots,\gamma_\ell}(\lambda_j,n) = \frac{\det \Psi^{\ell}(\lambda_j,n)}{\sqrt{\det C^{\ell}(n-1)\det C^{\ell}(n)}}. \quad (3.147)$$

In addition, we get

$$\begin{aligned} a_{\gamma_1,\dots,\gamma_N}(n) &= a(n)\frac{\sqrt{\det C^N(n-1)\det C^N(n+1)}}{\det C^N(n)}, \quad (3.148) \\ b_{\gamma_1,\dots,\gamma_N}(n) &= b(n) - \sum_{\ell=1}^{N}\gamma_\ell \Big(a(n)\frac{\det\Psi^{\ell}(\lambda_\ell,n)\det\Psi^{\ell}(\lambda_\ell,n+1)}{\det C^{\ell-1}(n)\det C^{\ell}(n)} \\ &\quad - a(n-1)\frac{\det\Psi^{\ell}(\lambda_\ell,n-1)\det\Psi^{\ell}(\lambda_\ell,n)}{\det C^{\ell-1}(n-1)\det C^{\ell}(n-1)}\Big) \\ &= -\lambda_N + a(n)\frac{\det C^N(n-1)}{\det C^N(n)}\frac{\det\Psi^N(\lambda_N,n+1)}{\det\Psi^N(\lambda_N,n)} \\ &\quad + a(n-1)\frac{\det C^N(n)}{\det C^N(n-1)}\frac{\det\Psi^N(\lambda_N,n-1)}{\det\Psi^N(\lambda_N,n)}, \quad (3.149) \end{aligned}$$

the last equation only being valid if $\det\Psi^N(\lambda_N,n)\neq 0$ *(e.g., if* $\lambda_N \le \inf\sigma(H)$*). The spectrum of* $H_{\gamma_1,\dots,\gamma_N}$ *is given by*

$$\begin{array}{llll} \sigma(H_{\gamma_1,\dots,\gamma_N}) = \sigma(H)\cup\{\lambda_j\}_{j=1}^N, & \sigma_{ac}(H_{\gamma_1,\dots,\gamma_N}) = \sigma_{ac}(H), \\ \sigma_p(H_{\gamma_1,\dots,\gamma_N}) = \sigma_p(H)\cup\{\lambda_j\}_{j=1}^N, & \sigma_{sc}(H_{\gamma_1,\dots,\gamma_N}) = \sigma_{sc}(H). \end{array} \quad (3.150)$$

Moreover,

$$H_{\gamma_1,\dots,\gamma_N}(1-\sum_{j=1}^{N} P_{\gamma_1,\dots,\gamma_N}(\lambda_j))$$
$$= (U_{\gamma_1,\dots,\gamma_N}\cdots U_{\gamma_1}) H (U_{\gamma_1}^{-1}\cdots U_{\gamma_1,\dots,\gamma_N}^{-1})(1-\sum_{j=1}^{N} P_{\gamma_1,\dots,\gamma_N}(\lambda_j)), \quad (3.151)$$

where $P_{\gamma_1,\dots,\gamma_N}(\lambda_j)$ *denotes the projection onto the one-dimensional subspace of* $\mathfrak{H}$ *spanned by* $\psi_{\gamma_1,\dots,\gamma_N}(\lambda_j)$*.*

Proof. We start with (3.146). Using Sylvester's determinant identity (cf. [28], Sect. II.3) we obtain

$$\det C^{\ell-1}(n) \det C_{i,j}^{\ell+1}(n)$$
$$= \det C^{\ell}(n) \det C_{i,j}^{\ell}(n) - \gamma_\ell \det C_{\ell,j}^{\ell}(n) \det C_{i,\ell}^{\ell}(n), \quad (3.152)$$

which proves (3.146) together with a look at (3.105) by induction on N. Next, (3.144) easily follows from (3.146). Similarly,

$$\det C^{\ell}(n) \det \Psi^{\ell+1}(\lambda_j, n)$$
$$= \det C^{\ell+1}(n) \det \Psi^{\ell}(\lambda_j, n) - \gamma_\ell \det \Psi^{\ell}(\lambda_\ell, n) \det C_{j,\ell}^{\ell}(n), \quad (3.153)$$

and (3.99) prove (3.147). The rest follows in a straightforward manner. □

Remark 3.24 *(i). For a sequence* f*, which is square summable near* M_-*,* $f_{\gamma_1,\dots,\gamma_j} = U_{\gamma_1,\dots,\gamma_j}\cdots U_{\gamma_1} f$ *is given by substituting* $\psi(\lambda_j) \to f$ *in (3.147). Similarly we get the scalar product of* $f_{\gamma_1,\dots,\gamma_i}$ *and* $g_{\gamma_1,\dots,\gamma_j}$ *from (3.146) by substituting* $f \to \psi(\lambda_i)$ *and* $g \to \psi(\lambda_j)$ *in (3.142).*
(ii). Equation (3.147) can be rephrased as

$$(\gamma_1\psi_{\gamma_1,\dots,\gamma_\ell}(\lambda_1, n), \dots, \gamma_\ell\psi_{\gamma_1,\dots,\gamma_\ell}(\lambda_\ell, n)) =$$
$$\sqrt{\frac{\det C^{\ell}(n)}{\det C^{\ell}(n-1)}} (C^{\ell}(n))^{-1} (\gamma_1\psi(\lambda_1, n), \dots, \gamma_\ell\psi(\lambda_\ell, n)), \quad (3.154)$$

where $(C^{\ell}(n))^{-1}$ *is the inverse matrix of* $C^{\ell}(n)$*.*

Clearly Theorem 3.17 extends (by induction) to this more general situation.

Theorem 3.25 *Assume (H.3.11) and (H.3.16). Then* $\tau_{\gamma_1,\dots,\gamma_N}$ *is l.p. at* M_+*.*

Finally we also extend Lemma 3.19. For simplicity we assume $\psi(\lambda_j, n) \notin \mathfrak{H}$, $1 \le j \le N$.

Lemma 3.26 *Let H be a given Jacobi operator satisfying (2.117). Then $H_{\gamma_1,\dots,\gamma_N}$, defined via $\psi(\lambda_\ell, n) = f_-(k_\ell, n)$, $\lambda_\ell = (k_\ell + k_\ell^{-1})/2 \in \mathbb{R}\backslash\sigma(H_{\gamma_1,\dots,\gamma_{\ell-1}})$, $1 \le \ell \le N$ has the transmission and reflection coefficients*

$$T_{\gamma_1,\dots,\gamma_N}(k) = \prod_{\ell=1}^N \mathrm{sgn}(k_\ell)\tfrac{k\,k_\ell - 1}{k - k_\ell} T(k), \tag{3.155}$$

$$R_{-,\gamma_1,\dots,\gamma_N}(k) = R_-(k), \quad R_{+,\gamma_1,\dots,\gamma_N}(k) = \left(\prod_{\ell=1}^N \Big(\frac{k - k_\ell}{k\,k_\ell - 1}\Big)^2\right) R_+(k), \tag{3.156}$$

where $z = (k + k^{-1})/2$. Furthermore, the norming constants $\gamma_{-,j}$ corresponding to $\lambda_j \in \sigma_p(H)$, $j \in J$ (cf. (3.69)) remain unchanged and the additional eigenvalues λ_ℓ have norming constants $\gamma_{-,\ell} = \gamma_\ell$.

Remark 3.27 *Of special importance is the case $a(n) = 1/2$, $b(n) = 0$. Here we have $f_\pm(k, n) = k^{\pm n}$, $T(k) = 1$, and $R_\pm(k) = 0$. It is well known from inverse scattering theory that $R_\pm(k)$, $|k| = 1$ together with the point spectrum and corresponding norming constants uniquely determine $a(n), b(n)$. Hence we infer from Lemma 3.9 that $H_{\gamma_1,\dots,\gamma_N}$ constructed from $\psi(\lambda_\ell, n) = f_-(k_\ell, n)$ as in Theorem 3.23 and $H_{\sigma_1,\dots,\sigma_N}$ constructed from $u^\ell_{\sigma_\ell} = \frac{1+\sigma_\ell}{2} f_+(k_\ell, n) + (-1)^{\ell+1}\frac{1-\sigma_\ell}{2} f_-(k_\ell, n)$ as in Theorem 3.7 coincide if*

$$\gamma_j = \left(\frac{1 - \sigma_j}{1 + \sigma_j}\right)^{-1} |k_j|^{-1-N} \frac{\prod_{\ell=1}^N |1 - k_j k_\ell|}{\prod_{\substack{\ell=1\\ \ell\neq j}}^N |k_j - k_\ell|} T(k_j), \quad 1 \le j \le N. \tag{3.157}$$

For a direct proof compare [40].

3.7 Applications

First we state the discrete analogue of the FIT-formula derived in [24] for the isospectral torus of periodic Schrödinger operators. This yields an explicit realization of the isospectral torus of all algebro-geometric quasi-periodic finite-gap Jacobi operators.

Let $a(n), b(n)$ be given algebro-geometric quasi-periodic g-gap sequences characterized by the band-edges $E_0 < E_1 < \dots < E_{2g+1}$ and Dirichlet data $\{(\mu_j, \sigma_j)\}_{j=1}^g$ at the reference point $n = 0$ (cf. [8]), where $\mu_j \in [E_{2j-1}, E_{2j}]$ and $\sigma_j \in \{\pm\}$, $1 \le j \le g$. Then the spectrum of the associate Jacobi operator H is of the type

$$\begin{aligned} \sigma(H) = \sigma_{ac}(H) = \textstyle\bigcup_{n=1}^{g+1}[E_{2n-2}, E_{2n-1}],\\ \sigma_{sc}(H) = \sigma_p(H) = \emptyset. \end{aligned} \tag{3.158}$$

and (cf. (3.47))

$$\sigma(H_\pm) = \sigma(H) \cup \{\mu_j | \sigma_j = \pm,\ 1 \le j \le g\}. \tag{3.159}$$

Then considerations as in Theorem 3.7 readily yield that all other isospectral algebro-geometric g-gap sequences can be realized in the following way

$$a_{(\tilde{\mu}_1,\tilde{\sigma}_1),\dots,(\tilde{\mu}_g,\tilde{\sigma}_g)}(n) = -\sqrt{a(n-g)a(n-g+2)}\times \sqrt{\frac{C_{n-g}(\psi_{\sigma_1}(\mu_1),\psi_{-\tilde{\sigma}_1}(\tilde{\mu}_1),\dots,\psi_{\sigma_g}(\mu_g),\psi_{-\tilde{\sigma}_g}(\tilde{\mu}_g))}{C_{n-g+1}(\psi_{\sigma_1}(\mu_1),\psi_{-\tilde{\sigma}_1}(\tilde{\mu}_1),\dots,\psi_{\sigma_g}(\mu_g),\psi_{-\tilde{\sigma}_g}(\tilde{\mu}_g))}}\times \sqrt{\frac{C_{n-g+2}(\psi_{\sigma_1}(\mu_1),\psi_{-\tilde{\sigma}_1}(\tilde{\mu}_1),\dots,\psi_{\sigma_g}(\mu_g),\psi_{-\tilde{\sigma}_g}(\tilde{\mu}_g))}{C_{n-g+1}(\psi_{\sigma_1}(\mu_1),\psi_{-\tilde{\sigma}_1}(\tilde{\mu}_1),\dots,\psi_{\sigma_g}(\mu_g),\psi_{-\tilde{\sigma}_g}(\tilde{\mu}_g))}}, \tag{3.160}$$

$$\begin{aligned} b_{(\tilde{\mu}_1,\tilde{\sigma}_1),\dots,(\tilde{\mu}_g,\tilde{\sigma}_g)}(n) = {} & a(n-g)\frac{C_{n-g+2}(\psi_{\sigma_1}(\mu_1),\psi_{-\tilde{\sigma}_1}(\tilde{\mu}_1),\dots,\psi_{\sigma_g}(\mu_g))}{C_{n-g+1}(\psi_{\sigma_1}(\mu_1),\psi_{-\tilde{\sigma}_1}(\tilde{\mu}_1),\dots,\psi_{\sigma_g}(\mu_g))}\times \\ & \frac{C_{n-g}(\psi_{\sigma_1}(\mu_1),\psi_{-\tilde{\sigma}_1}(\tilde{\mu}_1),\dots,\psi_{\sigma_g}(\mu_g),\psi_{-\tilde{\sigma}_g}(\tilde{\mu}_g))}{C_{n-g+1}(\psi_{\sigma_1}(\mu_1),\psi_{-\tilde{\sigma}_1}(\tilde{\mu}_1),\dots,\psi_{\sigma_g}(\mu_g),\psi_{-\tilde{\sigma}_g}(\tilde{\mu}_g))} \\ & + a(n+1)\frac{C_{n-g}(\psi_{\sigma_1}(\mu_1),\psi_{-\tilde{\sigma}_1}(\tilde{\mu}_1),\dots,\psi_{\sigma_g}(\mu_g))}{C_{n-g+1}(\psi_{\sigma_1}(\mu_1),\psi_{-\tilde{\sigma}_1}(\tilde{\mu}_1),\dots,\psi_{\sigma_g}(\mu_g))}\times \\ & \frac{C_{n-g+1}(\psi_{\sigma_1}(\mu_1),\psi_{-\tilde{\sigma}_1}(\tilde{\mu}_1),\dots,\psi_{\sigma_g}(\mu_g),\psi_{-\tilde{\sigma}_g}(\tilde{\mu}_g))}{C_{n-g}(\psi_{\sigma_1}(\mu_1),\psi_{-\tilde{\sigma}_1}(\tilde{\mu}_1),\dots,\psi_{\sigma_g}(\mu_g),\psi_{-\tilde{\sigma}_g}(\tilde{\mu}_g))} - \tilde{\mu}_g, \end{aligned} \tag{3.161}$$

where $\psi_\pm(z,n)$ are the branches of the Baker-Akhiezer function associated with a, b (i.e., the solutions of $\tau\psi = z\psi$ which are square summable near $\pm\infty$) and the new sequences are associated with the new Dirichlet data $\{(\tilde{\mu}_j,\tilde{\sigma}_j)\}_{j=1}^g$ at the same reference point $n = 0$. Even though $\psi_\pm(z,n)$ is not necessarily positive as required in our Theorem 3.7, the above sequences can be shown to be well-defined by using the explicit theta-function representations for $\psi_\pm(z,n)$ (cf., e.g., [8]) as long as $\tilde{\mu}_j \in [E_{2j-1}, E_{2j}]$ and $\tilde{\sigma}_j \in \{\pm\}$, $1 \le j \le g$. In fact, consider the hyperelliptic Riemann surface K_g associated with the function

$$R_{2g+2}(z)^{1/2} = \prod_{j=0}^{2g+1} (z - E_j)^{1/2} \tag{3.162}$$

and branch points $E_0 < E_1 < \dots < E_{2g+1}$. A point $P \in K_g$ will be denoted by $P = (z, \pm R_{2g+2}(z)^{1/2})$ and we add two points $\infty_\pm \in K_g$ such that K_g is compact. Introduce

$$\underline{z}(P,n) = \underline{\hat{A}}_{P_0}(P) - \sum_{j=1}^{g} \underline{\hat{A}}_{P_0}(\hat{\mu}_j) + 2n\underline{\hat{A}}_{P_0}(\infty_+) - \underline{\hat{\Xi}}_{P_0}, \tag{3.163}$$

where $\underline{\hat{A}}_{P_0}$ is Abel's map with base point $P_0 = (E_0, 0)$ and $\underline{\hat{\Xi}}_{P_0}$ is the vector of Riemann constants (cf. [8] for more details). Then

$$a(n) = \tilde{a}[\theta(\underline{z}(\infty_+, n-1))\theta(\underline{z}(\infty_+, n+1))/\theta(\underline{z}(\infty_+, n))^2]^{1/2}, \tag{3.164}$$

$$
\begin{aligned}
b(n) &= -E_0 + \tilde{a}\frac{\theta(\underline{z}(\infty_+, n-1))\theta(\underline{z}(P_0, n+1))}{\theta(\underline{z}(\infty_+, n))\theta(\underline{z}(P_0, n))} \\
&\quad + \tilde{a}\frac{\theta(\underline{z}(\infty_+, n))\theta(\underline{z}(P_0, n-1))}{\theta(\underline{z}(\infty_+, n-1))\theta(\underline{z}(P_0, n))},
\end{aligned} \tag{3.165}
$$

where θ is Riemann's theta function associated with K_g and $\tilde{a}$ is a constant depending only on K_g (i.e.,on $\{E_j\}_{j=0}^{2g+1}$). Performing one single commutation at a point $Q = (z, \sigma R_{2g+2}(z)^{1/2}) \in K_g$ (i.e., choosing $\psi_\sigma(z, n)$ to perform the commutation) it is shown in [8], Chapter 9 that the new sequences are again given by (3.164), (3.165) if $\underline{z}(P, n)$ is replaced by

$$
\tilde{\underline{z}}(P, n) = \underline{z}(P, n) + \underline{\hat{A}}_{P_0}(Q) + \underline{\hat{A}}_{P_0}(\infty_+). \tag{3.166}
$$

As a consequence we note that for the standard procedure as in Theorem 3.3 (i.e., with $Q = (\lambda_1, \sigma_1 R_{2g+2}(\lambda_1)^{1/2})$, $\sigma_1 \in \{\pm 1\}$) the corresponding commuted operator H_{σ_1} is again quasi-periodic and isospectral to H.

Hence, choosing $Q = \hat{\mu}_j$ we obtain

$$
\tilde{\underline{z}}(P, n) = \underline{z}(P, n) + \underline{\hat{A}}_{P_0}(\hat{\mu}_j) + \underline{\hat{A}}_{P_0}(\infty_+) \tag{3.167}
$$

and the Dirichlet eigenvalue at $\hat{\mu}_j$ is formally replaced by one at ∞_- (since $\underline{\hat{A}}_{P_0}(\infty_-) = -\underline{\hat{A}}_{P_0}(\infty_+)$). The corresponding sequences are neither real-valued nor well-defined. To repair this we perform a second single commutation making the following choice $Q = (\tilde{\mu}_j, -\tilde{\sigma}_j R_{2g+2}(\tilde{\mu}_j)^{1/2})$. The resulting sequences $a_{(\tilde{\mu}_j, \tilde{\sigma}_j)}$, $b_{(\tilde{\mu}_j, \tilde{\sigma}_j)}$ are associated with

$$
\underline{z}_{(\tilde{\mu}_j, \tilde{\sigma}_j)}(P, n) = \underline{z}(P, n+1) + \underline{\hat{A}}_{P_0}(\hat{\mu}_j) - \underline{\hat{A}}_{P_0}((\tilde{\mu}_j, \tilde{\sigma}_j R_{2g+2}(\tilde{\mu}_j)^{1/2})) \tag{3.168}
$$

and are again real-valued. Moreover, we have replaced the Dirichlet eigenvalue (μ_j, σ_j) by $(\tilde{\mu}_j, \tilde{\sigma}_j)$ and we have shifted the reference point for the Dirichlet boundary condition by one (since $\underline{z}(P, n+1)$ and not $\underline{z}(P, n)$ occurs in (3.168)) whereas everything else remains unchanged. From Section 3 we know that $a_{(\tilde{\mu}_j, \tilde{\sigma}_j)}$, $b_{(\tilde{\mu}_j, \tilde{\sigma}_j)}$ are equivalently given by

$$
\begin{aligned}
a_{(\tilde{\mu}_j, \tilde{\sigma}_j)}(n+1) &= -\sqrt{a(n)a(n+2)} \times \\
&\quad \sqrt{\frac{C_n(\psi_{\sigma_j}(\mu_j), \psi_{-\tilde{\sigma}_j}(\tilde{\mu}_j)) C_{n+2}(\psi_{\sigma_j}(\mu_j), \psi_{-\tilde{\sigma}_j}(\tilde{\mu}_j))}{C_{n+1}(\psi_{\sigma_j}(\mu_j), \psi_{-\tilde{\sigma}_j}(\tilde{\mu}_j))^2}}, \qquad (3.169) \\
b_{(\tilde{\mu}_j, \tilde{\sigma}_j)}(n+1) &= a(n)\frac{\psi_{\sigma_j}(\mu_j, n+2) C_n(\psi_{\sigma_j}(\mu_j), \psi_{-\tilde{\sigma}_j}(\tilde{\mu}_j))}{\psi_{\sigma_j}(\mu_j, n+1) C_{n+1}(\psi_{\sigma_j}(\mu_j), \psi_{-\tilde{\sigma}_j}(\tilde{\mu}_j))} + \\
&\quad a(n+1)\frac{\psi_{\sigma_j}(\mu_j, n) C_{n+1}(\psi_{\sigma_j}(\mu_j), \psi_{-\tilde{\sigma}_j}(\tilde{\mu}_j))}{\psi_{\sigma_j}(\mu_j, n+1) C_n(\psi_{\sigma_j}(\mu_j), \psi_{-\tilde{\sigma}_j}(\tilde{\mu}_j))} - \tilde{\mu}_j, \qquad (3.170)
\end{aligned}
$$

where the $n+1$ on the left-hand-side takes the aforementioned shift of reference point into account. Thus, applying this procedure g times we can replace all Dirichlet eigenvalues proving (3.160), (3.161).

The reader might be puzzled by the fact that the Dirichlet eigenvalue $\hat{\mu}_j$ is shifted to ∞_- (as opposed to ∞_+) which seemingly distinguishes ∞_- from ∞_+. However, this apparent asymmetry between ∞_+ and ∞_- is related to our way of factoring H. If we would instead split up H as

$$H = \tilde{A}^*_{\sigma_j} \tilde{A}_{\sigma_j} + \mu_j, \tag{3.171}$$

where

$$(\tilde{A}_{\sigma_j}) f(n) = -\sqrt{-\frac{a(n-1)\psi_{\sigma_j}(\mu_j, n)}{\psi_{\sigma_j}(\mu_j, n-1)}} f(n-1) + \sqrt{-\frac{a(n-1)\psi_{\sigma_j}(\mu_j, n-1)}{\psi_{\sigma_j}(\mu_j, n)}} f(n), \tag{3.172}$$

with $\tilde{A}^*_{\sigma_j}$ being the adjoint of $\tilde{A}_{\sigma_j}$, the role of ∞_+ and ∞_- would be interchanged.

We stress again that (3.160), (3.161) represent an explicit realization of the isospectral torus of all algebro-geometric quasi-periodic g-gap Jacobi operators with spectrum (3.158).

Next we turn to bounded solutions $(a(n,t), b(n,t))$ of the Toda equations and construct N-soliton solutions on these (arbitrary) background solutions using the single commutation method.

The corresponding Jacobi operators $H(t)$ satisfy $\inf(\sigma(H(t))) = \inf(\sigma(H(0))) > -\infty$ for all $t \in \mathbb{R}$. Furthermore, this implies the existence of principal solutions $u_\pm(\lambda, n, t)$ which satisfy

$$\begin{aligned} H(t) u_\pm(\lambda, n, t) &= \lambda u_\pm(\lambda, n, t), & (3.173)\\ \frac{d}{dt} u_\pm(\lambda, n, t) &= P(t) u_\pm(\lambda, n, t), \quad (n,t) \in \mathbb{Z} \times \mathbb{R}, & (3.174) \end{aligned}$$

where the difference expression $P(t)$ associated with $(a(t), b(t))$ is defined by

$$(P(t)f)(n) = a(n,t) f(n+1) - a(n-1,t) f(n-1). \tag{3.175}$$

(3.173) and (3.174) then imply the Toda lattice equations,

$$\left.\begin{aligned} \frac{d}{dt} a(n,t) &= a(n,t)\Big(b(n,t) - b(n+1,t)\Big) \\ \frac{d}{dt} b(n,t) &= 2\Big(a(n-1,t)^2 - a(n,t)^2\Big) \end{aligned}\right., \quad (n,t) \in \mathbb{Z} \times \mathbb{R} \tag{3.176}$$

which are well-known to be equivalent to the Lax equation

$$\frac{d}{dt} H(t) - [P(t), H(t)] = 0, \quad t \in \mathbb{R} \tag{3.177}$$

(where $[.,.]$ denotes the commutator).

Next, let $H(t)$ be as above and choose

$$\lambda_N < \ldots < \lambda_1 < \inf(\sigma(H(0))), \quad \sigma_j \in [-1, 1], \quad 1 \le j \le N \in \mathbb{N}. \tag{3.178}$$

Then Theorem 3.7 implies

$$a_{\sigma_1,\dots,\sigma_N}(n,t) = -\sqrt{a(n,t)a(n+N,t)}\times \\ \frac{\sqrt{C_n(u^1_{\sigma_1},\dots,u^N_{\sigma_N})C_{n+2}(u^1_{\sigma_1},\dots,u^N_{\sigma_N})}}{C_{n+1}(u^1_{\sigma_1},\dots,u^N_{\sigma_N})}, \tag{3.179}$$

$$\begin{aligned} b_{\sigma_1,\dots,\sigma_N}(n,t) = & -\lambda_N \\ & + a(n,t)\frac{C_{n+2}(u^1_{\sigma_1},\dots,u^{N-1}_{\sigma_{N-1}})C_n(u^1_{\sigma_1},\dots,u^N_{\sigma_N})}{C_{n+1}(u^1_{\sigma_1},\dots,u^{N-1}_{\sigma_{N-1}})C_{n+1}(u^1_{\sigma_1},\dots,u^N_{\sigma_N})} \\ & + a(n+N-1,t)\frac{C_n(u^1_{\sigma_1},\dots,u^{N-1}_{\sigma_{N-1}})C_{n+1}(u^1_{\sigma_1},\dots,u^N_{\sigma_N})}{C_{n+1}(u^1_{\sigma_1},\dots,u^{N-1}_{\sigma_{N-1}})C_n(u^1_{\sigma_1},\dots,u^N_{\sigma_N})}, \end{aligned} \tag{3.180}$$

where

$$u^\ell_{\sigma_\ell}(n,t) = \frac{1+\sigma_\ell}{2}u_+(\lambda_\ell,n,t) + (-1)^{\ell+1}\frac{1-\sigma_\ell}{2}u_-(\lambda_\ell,n,t). \tag{3.181}$$

Moreover, for $\lambda < \lambda_N$,

$$u_{\sigma_1,\dots,\sigma_N,\pm}(\lambda,n,t) = \frac{\pm\sqrt{\prod_{j=0}^{N-1}(-a(n+j,t))}C_n(u^1_{\sigma_1},\dots,u^\ell_{\sigma_N},u_\pm(\lambda))}{\sqrt{C_n(u^1_{\sigma_1},\dots,u^N_{\sigma_N})C_{n+1}(u^1_{\sigma_1},\dots,u^N_{\sigma_N})}} \tag{3.182}$$

are the principal solutions of $\tau_{\sigma_1,\dots,\sigma_N}(t)u = \lambda u$ satisfying

$$\frac{d}{dt}u_{\sigma_1,\dots,\sigma_N,\pm}(\lambda,n,t) = P_{\sigma_1,\dots,\sigma_N}(t)u_{\sigma_1,\dots,\sigma_N,\pm}(\lambda,n,t), \tag{3.183}$$

where $P_{\sigma_1,\dots,\sigma_N}(t)$ is defined as in (3.175) with a replaced by $a_{\sigma_1,\dots,\sigma_N}$. We also have (cf. (3.85), (3.86))

$$\rho_{o,\sigma_1,\dots,\sigma_N}(n,t) = -\sqrt{-a(n,t)\tfrac{C_{n+2}(u^1_{\sigma_1},\dots,u^{N-1}_{\sigma_{N-1}})C_n(u^1_{\sigma_1},\dots,u^N_{\sigma_N})}{C_{n+1}(u^1_{\sigma_1},\dots,u^{N-1}_{\sigma_{N-1}})C_{n+1}(u^1_{\sigma_1},\dots,u^N_{\sigma_N})}}, \tag{3.184}$$

$$\rho_{e,\sigma_1,\dots,\sigma_N}(n,t) = \sqrt{-a(n+N-1,t)\tfrac{C_n(u^1_{\sigma_1},\dots,u^{N-1}_{\sigma_{N-1}})C_{n+1}(u^1_{\sigma_1},\dots,u^N_{\sigma_N})}{C_{n+1}(u^1_{\sigma_1},\dots,u^{N-1}_{\sigma_{N-1}})C_n(u^1_{\sigma_1},\dots,u^N_{\sigma_N})}}. \tag{3.185}$$

Finally, the sequences $a_{\sigma_1,\dots,\sigma_N}(n,t)$, $b_{\sigma_1,\dots,\sigma_N}(n,t)$ fulfill the Toda lattice equations (3.176) and the sequence

$$\rho_{\sigma_1,\dots,\sigma_N}(n,t) = \left\{ \begin{array}{ll} \rho_{e,\sigma_1,\dots,\sigma_N}(m,t), & n=2m \\ \rho_{o,\sigma_1,\dots,\sigma_N}(m,t), & n=2m+1 \end{array} \right., \tag{3.186}$$

fulfills the Kac–van Moerbeke lattice equation

$$\frac{d}{dt}\rho(n,t) = \rho(n,t)\Big(\rho(n+1,t)^2 - \rho(n-1,t)^2\Big). \tag{3.187}$$

At the end we derive the N-soliton solutions relative to an arbitrary Toda background solution $(a(t), b(t))$ using the double commutation method.

Denote by $\psi(\lambda, n, t)$ the solutions of $\tau(t)\psi = \lambda\psi$ which are square summable near $-\infty$ and satisfy

$$\frac{d}{dt}\psi(\lambda, n, t) = P(t)\psi(\lambda, n, t). \tag{3.188}$$

As in Theorem 3.23 we define the following matrices

$$C^N(n,t) = \left\{ \delta_r(s) + \sqrt{\gamma_r\gamma_s} \sum_{m=M_-}^{n} \psi(\lambda_r, m, t)\psi(\lambda_s, m, t) \right\}_{1\le r,s\le N}, \tag{3.189}$$

$$\Psi^N(\lambda_j, n, t) = \left\{ \begin{array}{cc} C^N(n,t)_{r,s} & {\scriptstyle r,s\le N} \\ \sqrt{\gamma_s} \sum\limits_{m=M_-}^{n} \psi(\lambda_j, m, t)\psi(\lambda_s, m, t) & {\scriptstyle s\le \ell, r=N+1} \\ \sqrt{\gamma_r}\psi(\lambda_r, n, t) & {\scriptstyle r\le\ell, s=N+1} \\ \psi(\lambda_j, n, t) & {\scriptstyle r=s=N+1} \end{array} \right\}_{1\le r,s\le N+1}. \tag{3.190}$$

Then the sequences

$$\begin{aligned} a_{\gamma_1,\dots,\gamma_N}(n,t) &= a(n,t)\frac{\sqrt{\det C^N(n-1,t)\det C^N(n+1,t)}}{\det C^N(n,t)}, & (3.191)\\ b_{\gamma_1,\dots,\gamma_N}(n,t) &= b(n,t) - \frac{1}{2}\frac{d}{dt}\ln\frac{\det C^N(n,t)}{\det C^N(n-1,t)}. & (3.192)\end{aligned}$$

satisfy the Toda lattice equations (3.176). Moreover,

$$\psi_{\gamma_1,\dots,\gamma_N}(\lambda_j, n, t) = \frac{\det \Psi^N(\lambda_j, n, t)}{\sqrt{\det C^N(n-1,t)\det C^N(n,t)}} \tag{3.193}$$

satisfies

$$\frac{d}{dt}\psi_{\gamma_1,\dots,\gamma_N}(\lambda_j, n, t) = P_{\gamma_1,\dots,\gamma_N}(t)\psi_{\gamma_1,\dots,\gamma_N}(\lambda_j, n, t), \tag{3.194}$$

where again $P_{\gamma_1,\dots,\gamma_N}(t)$ is defined as in (3.175) with a replaced by $a_{\gamma_1,\dots,\gamma_N}$.

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