Proseminar Partielle Differentialgleichungen

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Please see the lecture notes for further details.

1. Show Leibniz' rule
$$\partial_\alpha(f \cdot g) = \sum_{\beta\le\alpha} \binom{\alpha}{\beta} (\partial_\beta f)(\partial_{\alpha-\beta} g), \quad |\alpha| \le k.$$

2. Verify the Gauss–Green theorem (by computing both integrals) in the case $u(x) = x$ and $U = B_1(0) \subset \mathbb{R}^n$.

3. Let U is a bounded C^1 domain in $\mathbb{R}^n$ and set $\frac{\partial g}{\partial \nu} := \nu \cdot \partial g$. Verify Green's first identity
$$\int_U (f\Delta g + \partial f \cdot \partial g) d^n x = \int_{\partial U} f \frac{\partial g}{\partial \nu} dS$$
for $f \in C^1(\overline{U})$, $g \in C^2(\overline{U})$ and Green's second identity
$$\int_U (f\Delta g - g\Delta f) d^n x = \int_{\partial U} \left(f \frac{\partial g}{\partial \nu} - g \frac{\partial f}{\partial \nu} \right) dS$$
for $f, g \in C^2(\overline{U})$.

4. Find the general solution of the differential equation

$$x_1 u_{x_1} + \cdots + x_n u_{x_n} = c\,u$$

5. Show that for a conservation law with bounded initial conditions, the gradient of the solution remains bounded on bounded positive time intervals (as long as the solution exists) if F is convex and g is increasing. (Hint: Use implicit differentiation to find a formula for u_t).

6. Solve the Laplace equation with initial conditions $u(0,y) = 0$, $u_x(0,y) = y$.

7. Use the Cauchy–Kovalevskaya theorem to solve the wave equation

$$u_{tt} = u_{xx}$$

with initial conditions $u(0,x) = g(x)$, $u_t(0,x) = h(x)$ and establish d'Alembert's formula

$$u(t,x) = \frac{g(x+t)+g(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy.$$

(Hint: Introduce $v := u_x$, $w := u_t$ and compute the t derivatives of w.)

8. A function f is in the Gevrey class of order θ if for every $r > 0$, there are some constants M, a such that
$$|f^{(m)}(t)| \le M a^m (m!)^\theta, \qquad |t| < r.$$
Note that $\theta = 1$ gives the class of real analytic functions, while for $\theta > 1$ the function f will no longer be real analytic in general.
Show that if $\theta < 2$, then
$$u(t,x) = \sum_{m=0}^{\infty} \frac{f^{(m)}(t)}{(2m)!} x^{2m}$$
converges for all $x \in \mathbb{R}$ and defines a solution of the heat equation.

9. Consider the Fourier sine
$$f(x) = \sum_{n=1}^{\infty} s_n(f) \sin(n\pi x), \qquad s_n(f) = 2\int_0^1 \sin(n\pi x) f(x) dx,$$
and Fourier cosine series
$$f(x) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n(f) \cos(n\pi x), \qquad c_n(f) = 2\int_0^1 \cos(n\pi x) f(x) dx.$$
For given $k \in \mathbb{N}_0$, show that
$$\sum_{n=1}^{\infty} n^k |c_n(f)| < \infty, \qquad \sum_{n=1}^{\infty} n^k |s_n(f)| < \infty$$
if $f \in C^{k+1}([0,1], \mathbb{C})$ with $f^{(2j)}(0) = f^{(2j)}(1) = 0$ for $0 \le j \le k/2$ for the case of $s_n(f)$ and $f^{(2j+1)}(0) = f^{(2j+1)}(1) = 0$ for $0 \le j < k/2$ for the case of $c_n(f)$. (Hint: Use integration by parts to show
$$c_n(f') = 2((-1)^n f(1) - f(0)) + n\pi s_n(f)$$
and
$$s_n(f') = -n\pi c_n(f).$$
Now use that for $g \in C([0,1], \mathbb{C})$, both $s_n(g)$ and $c_n(g)$ are square summable (this is known as Parseval's theorem and you can take it for granted). Moreover, the sequence n^{-1} is also square summable and the product of two square summable is (absolutely) summable by the Cauchy–Schwarz inequality.)

10. Show that for $u \in C^1[0,1]$ with $u(0) = u(1) = 0$ we have the Poincaré inequality
$$\int_0^1 u(x)^2 dx \le C \int_0^1 u'(x)^2 dx$$
for some $C > 0$. (Hint: Insert $u(x) = \int_0^x u'(y)dy$ one the left. This gives the inequality with $C = \frac{1}{2}$.)

11. Solve the heat equation with Neumann boundary conditions $u_x(t,0) = u_x(t,1) = 0$. Show that the solution converges to the average temperature at an exponential rate. Show that the solution is unique.

12. Show that
$$\varphi(t) := \begin{cases} \mathrm{e}^{-1/t^2}, & t > 0, \\ 0, & t \le 0, \end{cases}$$
is in the Gevrey class of order $\theta = \frac{3}{2}$. (Hint: Use the Cauchy integral formula
$$\varphi^{(m)}(t) = \frac{m!}{2\pi\mathrm{i}} \oint_\gamma \frac{\mathrm{e}^{-z^{-2}}}{(z-t)^{m+1}} dz$$
with $\gamma = \{t + \frac{t}{2}\mathrm{e}^{\mathrm{i}\vartheta} | 0 \le \vartheta \le 2\pi\}$. You will need to find the minimum of $\mathrm{Re}(z^{-2})$ on this circle. You can either argue why this minimum is positive or compute it explicitly. In the latter case a CAS might be helpful. Also you can use that $n^n \le \mathrm{e}^n n!$.)

13. Show uniqueness for the heat equation with Robin boundary conditions provided $a_0, a_1 \ge 0$.

14. Find transformations which reduce
    - $u_t = u_{xx} + cu$
    - $u_t = u_{xx} - au_x$

    to the heat equation. (Hint: For the first multiply u by a suitable function. For the second equation switch to a moving frame $y = x - at$.)

15. Let $u \in C(\overline{U_T}) \cap C^{1;2}(U_T)$ solve
$$u_t = u_{xx} + f, \quad \begin{cases} u(0,x) = g(x), & x \in (0,1), \\ u(t,0) = a_0(t),\ u(t,1) = a_1(t), & t \in [0,T]. \end{cases}$$
Show
$$|u| \le \max_{[0,1]} |g| + \max_{[0,T]} |a_0| + \max_{[0,T]} |a_1| + T \max_{[0,T]\times[0,1]} |f|.$$
(Hint: Apply the maximum principle to $v := u - tF$, where F is a suitably chosen constant.)

16. Let
$$L := -\frac{d^2}{dx^2} + q(x), \qquad x \in (a,b),$$
defined for $f, g \in C^2(a,b)$ satisfying the boundary conditions
$$\cos(\alpha)y(a) = \sin(\alpha)y'(a), \quad \cos(\beta)y(b) = \sin(\beta)y'(b).$$
Show that for twice differentiable functions f, g satisfying the boundary conditions we have
$$\langle f, Lg \rangle = \langle Lf, g \rangle.$$
Use this to show that all eigenvalues E_n of L are real and eigenvectors corresponding to different eigenvalues are orthogonal.

17. Show that solutions of
$$\begin{aligned} &\frac{1}{c^2} u_{tt}(t,x) - u_{xx}(t,x) + q(x)u(t,x) = 0, \\ &u(0,x) = g(x), \qquad u_t(0,x) = h(x), \\ &u(t,0) = u(t,1) = 0. \end{aligned}$$
preserve the energy
$$E(t) = \frac{1}{2} \int_0^1 \big(c^{-2} u_t(t,x)^2 + u_x(t,x)^2 + q(x)u(t,x)^2 \big) dx.$$
Conclude that solutions are unique if $q(x) \ge 0$.

18. Prove Lemma 2.10 from the lecture notes.

19. Solve the wave equation with damping
$$u_{tt}(t,x) + 2\eta u_t = c^2 u_{xx}, \qquad 0 < \eta < c,$$
and Dirichlet boundary conditions on $[0,1]$. Show that the solutions converge to 0. Show that the energy is non-increasing and conclude that solutions are unique.

20. Solve the conservation law in $\mathbb{R}^2$ with velocity field $v(t,x) := x$.

21. Let $\Gamma = \{x \in \mathbb{R}^2 | x_1 = 0\}$ and solve
$$u_{x_1} u_{x_2} = 4u, \qquad u(0, x_2) = x_2^2.$$

22. Solve the Laplace equation with initial conditions $u(0,y) = y^2$, $u_x(0,y) = 0$.

23. Solve
$$u_t(t,x) = u_{xx}(t,x), \qquad u(t,0) = u(t,1) = 0, \qquad u(0,x) = 2\sin(\pi x)\cos(\pi x)$$

24. Explain how d'Alembert's formula can be used to obtain solutions which satisfy Neumann boundary conditions $u_x(t,0) = u_x(t,1) = 0$. Discuss what happens to a small bump traveling to the right.

25. Let ρ_1, ρ_2 be two solutions of

$$\rho'' + \frac{1}{r}\rho' - \frac{n^2}{r^2}\rho = -\lambda\rho$$

corresponding to λ_1, λ_2, respectively. Show that

$$\frac{d}{dr} r\big(\rho_1(r)\rho_2'(r) - \rho_1'(r)\rho_2(r)\big) = (\lambda_1 - \lambda_2) r\, \rho_1(r)\rho_2(r).$$

Conclude

$$\int_0^1 J_n(j_{n,k}r)J_n(j_{n,l}r)r\,dr = \begin{cases} \frac{1}{2}J_n'(j_{n,k})^2, & l = k, \\ 0, & l \neq k. \end{cases}$$

Note that $J_n'(j_{n,k}) \neq 0$ since if for a solution of a second order linear equation both the function and its derivative would vanish, it would be the zero solution.

26. Consider the vibrations of a chain of length 1 suspended at $x = 1$. Denote the displacement by $u(t,x)$. Then the motion is described by the equation

$$u_{tt}(t,x) = g\frac{\partial}{\partial x}x\frac{\partial}{\partial x}u(t,x), \quad x \in [0,1],$$

with boundary conditions $u(t,1) = 0$, where $g > 0$ is a constant. Apply separation of variables to find the eigenvalues and eigenfunctions.

27. Find the solution of the Laplace equation on the unit disc which satisfies $u(x,y) = x^2$ for $x^2 + y^2 = 1$. Give the solution in Cartesian coordinates.

28. Compute the Fourier transform of the following functions $f:\mathbb{R}\to\mathbb{C}$:

    (i) $f(x)=\chi_{(-1,1)}(x)$. (ii) $f(x)=\frac{\mathrm{e}^{-a|x|}}{a}$, $\mathrm{Re}(a)>0$.

29. Show that
$$\psi_n(x)=H_n(x)\mathrm{e}^{-\frac{x^2}{2}},$$
where $H_n(x)$ is the Hermite polynomial of degree n given by
$$H_n(x):=\mathrm{e}^{\frac{x^2}{2}}\left(x-\frac{d}{dx}\right)^n \mathrm{e}^{-\frac{x^2}{2}},$$
are eigenfunctions of the Fourier transform: $\hat{\psi}_n(k)=(-\mathrm{i})^n\psi_n(k)$.

30. Find the solution of the heat equation on $(0,\infty)$ with a Dirichlet boundary condition at 0. What about Neumann boundary conditions? (Hint: Reflection.)

31. Compute the energy of the fundamental solution Φ of the heat equation. Show that it does not decay exponentially, and conclude that there is no Poincaré inequality on $\mathbb{R}$.

32. Establish Huygens' principle: Suppose g,h are supported in $[a,b]\subset\mathbb{R}$. Then the solution of the wave equation has support in $\{(t,x)|x\in[a-ct,b+ct]\}$. If $\int_a^b h(x)dx=0$ the support is in $\{(t,x)|x\in[a-ct,b-ct]\cup[a+ct,b+ct]\}$.

33. Find the solution of the wave equation on $(0,\infty)$ with a Dirichlet boundary condition at 0. What about Neumann boundary conditions? (Hint: Reflection.)

34. Verify that
$$u(t,x) = \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} f(s,y)dy\, ds$$
is in C^2 and solves the inhomogeneous wave equation provided $f \in C^{0;1}(\mathbb{R}^2)$.

35. Let $\phi \in C^3$ with $\phi'(k_0) = 0$ and $\phi''(k_0) \neq 0$. Show that there is a local change of coordinates $\kappa \in C^2$ such that $\phi(k) - \phi(k_0) = \frac{\sigma}{2}\kappa(k-k_0)^2$, $\sigma := \operatorname{sign}(\phi''(k_0))$, holds in a neighborhood of k_0. In particular, $\kappa(0) = 0$, $\kappa'(0) = \sqrt{|\phi''(k_0)|}$, and $\kappa''(0) = \frac{\sigma}{3\sqrt{|\phi''(k_0)|}}\phi'''(k_0)$. Moreover, if $\phi \in C^4$, then $\kappa \in C^3$.

36. Use the integral representation for the Bessel function
$$J_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}(nt - \sin(t)x)} dt, \qquad n \in \mathbb{N}_0,$$
to establish the asymptotics
$$J_n(x) = \sqrt{\frac{2}{\pi x}}\left(\cos\left(x - \frac{\pi}{2}n - \frac{\pi}{4}\right) + O(x^{-1/2})\right).$$

37. Show that a positive harmonic function on $\mathbb{R}^n$ is constant. Moreover, a harmonic function on $\mathbb{R}^n$ is constant if it is bounded from above or from below. (Hint: Fix two points x, y and note that $B_r(x) \subset B_{r+d}(y)$ for $d := |x - y|$.)

38. Find all harmonic functions u in $\mathbb{R}^2$ such that $u_x(x, y) < u_y(x, y)$.

39. Show that if u is harmonic, then $\varphi(u)$ is subharmonic for every convex function $\varphi \in C(\mathbb{R})$.

40. Suppose f is integrable with compact support and $n \geq 3$. Then the Newton potential satisfies

$$u(x) = C\Phi(x) + O(|x|^{-n+1})$$

as $|x| \to \infty$, where $C := \int_{\mathbb{R}^n} f(y) d^n y$. (Hint: The inverse triangle inequality $||x| - |y|| \leq |x - y|$ might be useful.)

41. Let U be a bounded C^1 domain and let a partition of its boundary $\partial U = V_1 \cup V_2$ be given. Show that solutions $u \in C^2(\overline{U})$ of the mixed Dirichlet/Neumann problem

$$-\Delta u = f, \qquad u|_{V_1} = g_1, \quad \frac{\partial u}{\partial \nu}\Big|_{V_2} = g_2,$$

differ by at most a constant. Moreover, this constant is zero if V_1 is nonempty. (Hint: Green's first identity with both functions equal.)

42. Let G be the Green function of the unit ball. Compute $\int_{B_1(0)} G(x,y) d^n y$. (Hint: There is no need to do the integral.)

43. Prove the following Theorem:

Let U be a bounded domain and suppose L is uniformly elliptic with $c \ge 0$. Then the problem
$$Lu = f, \qquad u|_{\partial U} = g.$$
has at most one solution $u \in C^2(U) \cap C(\overline{U})$ for given $g \in C(\partial U)$, $f \in C(U)$. Moreover, there is a constant C depending only on U and L such that a solution satisfies
$$\max_{\overline{U}} |u| \le \max_{\partial U} |g| + C \sup_{U} |f|.$$

(Hint: Assume that U is within a strip $0 < x_1 < r$ and construct a supersolution using $\mathrm{e}^{\lambda x_1}$.)

44. Derive a Dirichlet principle for the elliptic operator $\tilde{L}$ in divergence form with $A \in C^1$ and $b = 0$.

45. Compute the Fourier transform of

$$|x|^2 \mathrm{e}^{-|x|^2/2}$$

in $\mathbb{R}^n$. (Hint: There is no need to compute integrals.)

46. Find a function f such that $\int_{\mathbb{R}} f(y)f(x-y)dy = \mathrm{e}^{-x^2}$.

47. Compute the Fourier transform of

$$\varphi_t(x) = \sqrt{\frac{\pi}{2}} \frac{\chi_{[0,t]}(|x|)}{|x|}, \qquad t \geq 0,$$

in $\mathbb{R}^3$. (Hint: Spherical coordinates.)

48. Suppose g is integrable. Show that solutions of the heat equation satisfy

$$\int_{\mathbb{R}^n} |u(t,x)| d^n x \leq \int_{\mathbb{R}^n} |g(x)| d^n x$$

with the inequality being strict unless g is of one sign.

49. Let $\rho_n(t) := \sqrt{2nt\log(\frac{r^2}{4\pi t})}$. Show

$$\frac{1}{2r^n}\int_{t-r^2/4\pi}^{t}\frac{\rho_n(t-s)}{t-s}\int_{|y-x|=\rho_n(t-s)} dS(y)ds = 1.$$

(Hint: The final integral can be evaluated in terms of the Gamma function — Problem A.5 in the text.)

50. Let L be elliptic and suppose $c(x) \ge -c_0$ with $c_0 \ge 0$. Show that if u solves $u_t + Lu \le 0$, then

$$u(t,x) \le \mathrm{e}^{c_0 t}\max_{(t,x)\in\Gamma_T} u(t,x), \qquad 0 \le t \le T.$$

(Hint: Find a transformation to reduce it to the case $c(x) \ge 0$. Then apply the results from the text.)

51. Find the solution of the wave equation with initial condition $g(x) = 0$, $h(x) = |x|^2$ in $\mathbb{R}^3$.

52. Derive a formula for the Fourier transform $\hat{u}(t,k)$ of a solution of the Klein–Gordon equation

$$u_{tt} = \Delta u - m^2 u, \qquad u(0) = g, \quad u_t(0) = h,$$

with $m > 0$.

53. Maxwell's equations in vacuum for the electric field $E(t,x)$ and the magnetic field $B(t,x)$ are given by

$$B_t = -\operatorname{curl} E, \quad \mu_0\varepsilon_0 E_t = \operatorname{curl} B, \quad \operatorname{div} E = 0, \quad \operatorname{div} B = 0,$$

where $\mu_0 > 0$ and $\varepsilon_0 > 0$ are the permeability and the permittivity of the vacuum, respectively. Show that both E and B satisfy the wave equation with $c = (\mu_0\varepsilon_0)^{-1/2}$. Here $\operatorname{curl} f := \nabla \times f$ is the infinitesimal circulation (also known as rotation) of a vector field f in $\mathbb{R}^3$.

54. Suppose g and h are supported in a ball of radius r. Use Kirchhoff's formula to show

$$|u(t,x)| \le \frac{Cr^2}{|t|}\big(\sup_{\mathbb{R}^3}|g| + \sup_{\mathbb{R}^3}|\nabla g| + \sup_{\mathbb{R}^3}|h|\big)$$

for $t \ge 1$. (Hint: What is the area of $\partial B_{|t|}(x) \cap B_r(0)$?)