Contents

1 Complex numbers and functions

1.1 Complex numbers

The quadratic equation $x^2 + 1 = 0$ has the two formal solutions $x_{1,2} = \pm\sqrt{-1}$, and Euler[1] writes in 1777 : " formulam $\sqrt{-1}$ littera i in posterum designabo."

Definition 1.1. We consider $\mathbb{C} := \mathbb{R}^2$ as a vector space over $\mathbb{R}$, with multiplication in $\mathbb{C}$: $(a,b) \in \mathbb{C},\ a,b \in \mathbb{R},\ (c,d) \in \mathbb{C},\ c,d \in \mathbb{R}$

$$(a,b)(c,d) = (ac - bd, bc + ad)$$

In this way $\mathbb{C}$ becomes a commutative field with zero element $(0,0)$ and unit element $(1,0)$; for $(a,b) \neq (0,0)$, we have

$$(a,b)^{-1} = \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right).$$

For $(a,b) \in \mathbb{C}$, we also write $a + ib$, this means that $(a,0)$ corresponds to the real number a and $(0,1)$ to the imaginary unit i.

The multiplication law from above stems from the formal multiplication

$$(a+ib)(c+id) = ac - bd + i(bc + ad).$$

For $z = (x,y)$, we will write

$$z = (x,y) = (x,0) + (0,1)(y,0) = x + iy,$$

where $x = \Re z$ is called real part of z and $y = \Im z$ imaginary part of z.

For $z = (x,y) = x + iy \in \mathbb{C}$, the complex number

$$\overline{z} = (x,-y) = x - iy$$

is called complex conjugate of z, and we have the following rules : $z, w \in \mathbb{C}$:

$$(z+w)^- = \overline{z} + \overline{w},\ \overline{zw} = \overline{z}\ \overline{w},\ \overline{\overline{z}} = z.$$

We define $|z|^2 = z\overline{z}$, $|z|$ is the absolute value of z. We have $|z| = \sqrt{x^2+y^2}$, for $z = x + iy$, and $|z| = |\overline{z}|$. In addiltion

$$\Re z = \frac{1}{2}(z + \overline{z}),\ \Im z = \frac{1}{2i}(z - \overline{z}),$$

1 Euler, Leonhard (1707–1783)

$$|zw| = |z||w|, \; |z+w| \leq |z| + |w|, \; ||z| - |w|| \leq |z-w|.$$

Polar representation.

Let $z = x + iy \neq 0$. we set $x = r\cos\theta$ and $y = r\sin\theta$, where $r = |z|$ is the absolute value of z and $\theta = \arg z$ the argument of z. We have $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$, which will become clear after introducing the complex exponential function, see section 1.10. Since $\cos\theta = \cos(\theta + 2k\pi)$ and $\sin\theta = \sin(\theta + 2k\pi)$, for $k \in \mathbb{Z}$, there are infinitely many values of θ corresponding to a single z. The principal argument $\operatorname{Arg} z$ of z is to be taken $-\pi < \operatorname{Arg} z \leq \pi$, in this way the polar representation of z becomes uniquely determined.

Examples. (a) $z = 2 + 2i, \; r = |z| = 2\sqrt{2}, \; \operatorname{Arg} z = \frac{\pi}{4}$;
(b) $z = 2 - 2i, \; r = |z| = 2\sqrt{2}, \; \operatorname{Arg} z = -\frac{\pi}{4}$.

Product of complex numbers.
Let $z = r(\cos\theta + i\sin\theta)$ und $w = s(\cos\phi + i\sin\phi)$. Then

$$\begin{aligned} zw &= rs[(\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\cos\theta\sin\phi + \sin\theta\cos\phi)] \\ &= rs(\cos(\theta + \phi) + i\sin(\theta + \phi)), \end{aligned}$$

where we used the addition rules for the cosine and sine function. Hence one has to multiply the absolute values and to add the angles. Concerning the principal argument one has to be careful in this connection as the following example shows: let $z = -1$ and $w = i$. Then $\operatorname{Arg}(-1) = \pi$ and $\operatorname{Arg}(i) = \pi/2$, but $\operatorname{Arg}((-1)i) = \operatorname{Arg}(-i) = -\pi/2 \neq \operatorname{Arg}(-1) + \operatorname{Arg}(i)$.

de Moivre's formula.
Let $n \in \mathbb{N}$. Then

$$z^n = r^n(\cos n\theta + i\sin n\theta) = r^n e^{in\theta},$$

which follows by induction.

Roots of complex numbers.
Let $n \in \mathbb{N}, \; z \neq 0, \; z = r(\cos\theta + i\sin\theta)$. Using de Moivre's formula we get an n-th root of z by

$$z^{1/n} = r^{1/n}(\cos\theta/n + i\sin\theta/n).$$

As $z = r(\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi))$, for $k \in \mathbb{Z}$, the expressions

$$r^{1/n}(\cos((\theta + 2k\pi)/n) + i\sin((\theta + 2k\pi)/n))$$

are also n-th roots of z.

There are n different n-th roots of z, with arguments

$$\frac{\theta}{n}, \frac{\theta + 2\pi}{n}, \dots, \frac{\theta + 2(n-1)\pi}{n}.$$

Riemann sphere and stereographic projection.
Let $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$ be the unit sphere in $\mathbb{R}^3$ with $N = (0, 0, 1)$ as north pole on S^2. In this context, S^2 is also called Riemann sphere. [2] We consider $\mathbb{C}$ as the equator plane of S^2. A point $z \in \mathbb{C}$ is associated with the intersection point P of S^2 with the ray joining the north pole with $z \in \mathbb{C}$. In this way we obtain a homeomorphism ϕ between $S^2 \setminus \{N\}$ and $\mathbb{C}$, given by

$$\phi(x_1, x_2, x_3) = \frac{1}{1 - x_3}(x_1 + ix_2),$$

with the inverse

$$\phi^{-1}(x_1 + ix_2) = \frac{1}{x_1^2 + x_2^2 + 1}(2x_1, 2x_2, x_1^2 + x_2^2 - 1).$$

One-point compactification.
By continuous continuation of ϕ to the whole of S^2 we obtain a topological homeomorphism between the compact space S^2 and the so-called extended plane (one-point compactification of $\mathbb{C}$) $\overline{\mathbb{C}}$:

$$\phi(N) := \infty \ , \ \phi^{-1}(\infty) := N \ , \phi(S^2) = \overline{\mathbb{C}}.$$

1.2 Some topological concepts

We explain some topological concepts and basic results, which will be important later on.

Definition 1.2. Let $(z_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{C}$. z_0 is called accumulation point of the sequence , if for each $\epsilon > 0$ the ball $K_\epsilon(z_0) = \{z : |z - z_0| < \epsilon\}$ contains infinitely many members of the sequence. z_0 is called limit of the sequence , if for each $\epsilon > 0$ there exists $n_\epsilon > 0$ such that

$$|z_n - z_0| < \epsilon \ , \ \forall n > n_\epsilon.$$

2 Riemann, Georg Friedrich Bernhard (1826–1866)

The sequence $(z_n)_{n=1}^{\infty}$ is called Cauchy sequence [3], if for each $\epsilon > 0$ there exists $n_\epsilon > 0$, such that

$$|z_n - z_m| < \epsilon \ , \ \forall n, m > n_\epsilon.$$

Each Cauchy sequence has a limit, $\mathbb{C}$ is complete.

Definition 1.3. Leti $G \subseteq \mathbb{C}$. G is open, if for each $z \in G$ there exists $\epsilon > 0$, such that $K_\epsilon(z) \subseteq G$. A set U is a neighborhood of the set M, if there exists an open set V such that $M \subset V \subset U$. $A \subseteq \mathbb{C}$ is called closed , if $\mathbb{C} \setminus A$ is open.

The union of arbitrarily many open sets is open; the intersection of finitely many open sets is open; the union of finitely many closed sets is closed; the intersection of arbitrarily many closed sets is closed.
Let $M \subseteq \mathbb{C}$ be an arbitrary set in $\mathbb{C}$.

$$M^\circ := \bigcup \{U : U \subseteq M, \ U \text{ open}\}$$

is called interior of M; M° is an open set, it is the largest open set which is contained in M.

$$\overline{M} := \bigcap \{A : A \supseteq M, \ A \text{ closed}\}$$

is called closure of M; $\overline{M}$ is a closed set, it is the smallest closed set which contains M.
The set $\partial M := \overline{M} \setminus M^\circ$ is called boundary of M. .
Let $M \subseteq \mathbb{C}$ be an arbitrary set in $\mathbb{C}$. A subset $U \subseteq M$ is called relatively open in M, if there is an open set O in $\mathbb{C}$ such that $U = O \cap M$.

A set $X \subseteq \mathbb{C}$ is called connected , if $X = X_1 \cup X_2$ with $X_1 \cap \overline{X}_2 = \overline{X}_1 \cap X_2 = \emptyset$ implies that one of the two sets X_1, X_2 is empty. An open connected subset $G \subseteq \mathbb{C}$ is called a domain in $\mathbb{C}$ domains are also pathwise connected , any two points can be joined by a continuous curve in G.
If $X \subseteq \mathbb{C}$ and $x \in X$, we denote by E_x the largest connected set in X containing the point x. The set E_x is called connected component of x.
A domain G in the complex plane is called simply connected if its complement with respect to the extended plane, $\overline{\mathbb{C}} \setminus G$, is connected.

Leti $N \subseteq M \subseteq \mathbb{C}$. We say that N is dense in M, if $\overline{N} \supseteq M$.
Let $N \subseteq M \subseteq \mathbb{C}$. We say that N is discrete in M, if for each $z \in M$ there exists a neighborhood U (an open disc with center z), such that $U \cap N$ contains at most finitely many elements of N.

3 Cauchy, Augustin Louis (1789–1857)

A subset $K \subseteq \mathbb{C}$ is called compact , if each open covering of K has a finite subcovering. K is compact in $\mathbb{C}$ if aad only if K is closed and bounded in $\mathbb{C}$ (i.e. there exists $C > 0$ such that $|z| \leq C, \ \forall z \in K$).
Let $M \subseteq \mathbb{C}$ be a subset of $\mathbb{C}$. U is relatively compact in M (we write $U \subset\subset M$), if $\overline{U} \subseteq M$ and $\overline{U}$ is compact.

Definition 1.4. Let $G \subseteq \mathbb{C}$ and $f : G \longrightarrow \mathbb{C}$ a function. we separate $f(z)$ in its real and imaginary part: $f(z) = u(z) + iv(z) = \Re f(z) + i\Im f(z)$, where $u, v : \mathbb{C} \longrightarrow \mathbb{R}$ are real-valued functions. The set $\{(z, w) : f(z) = w, \ z \in G\} \subseteq \mathbb{C}^2$ is called graph of f. The sets $\{z : \Re f(z) = \text{const.}\}$, $\{z : \Im f(z) = \text{const.}\}$, $\{z : |f(z)| = \text{const.}\}$ are the level lines of f.

Examples: a) $f(z) = z^2, \ \Re f(z) = x^2 - y^2, \ \Im f(z) = 2xy$. The level lines are circles and hyperbolas.
b) $f(z) = az, \ a \in \mathbb{C}, \ a \neq 0$. This is a rotation-dilation. We write $a = \alpha + i\beta$, then we have

$$az = \alpha x - \beta y + i(\beta x + \alpha y).$$

If one considers f as a mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$, one obtains

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \alpha x - \beta y \\ \beta x + \alpha y \end{pmatrix} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (\alpha^2 + \beta^2)^{1/2} \begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

where

$$\cos\gamma = \frac{\alpha}{(\alpha^2 + \beta^2)^{1/2}} \ , \ \sin\gamma = \frac{\beta}{(\alpha^2 + \beta^2)^{1/2}}.$$

Definition 1.5. Let $O \subseteq \mathbb{C}$ be an open subset of $\mathbb{C}$, and let $f : O \longrightarrow \mathbb{C}$ be a function. f is called continuous at $z_0 \in O$, if for each $\epsilon > 0$ there exists $\delta > 0$, such that

$$|f(z) - f(z_0)| < \epsilon \ \text{ for } \ |z - z_0| < \delta.$$

f is called continuous on a set M, if f is continuous in each point of M.

f is continuous in z_0 if and only if for each sequence $(z_n)_{n=1}^{\infty}$ with $\lim_{n\to\infty} z_n = z_0$ we have $\lim_{n\to\infty} f(z_n) = f(\lim_{n\to\infty} z_n) = f(z_0)$.
If f and g are continuous, then $f + g, \ f.g, \ \frac{f}{g}(g \neq 0)$ are continuous.
f is continuous, if and only if $\Re f$ und $\Im f$ are continuous.
If f is continuous, then $|f|$ is continuous.
If f is continuous on a compact set K, then

$$\sup_{z \in K} |f(z)| = \max_{z \in K} |f(z)| = |f(z_1)| \ , \ \text{for} \quad \text{some} \ \ z_1 \in K,$$

$$\inf_{z \in K} |f(z)| = \min_{z \in K} |f(z)| = |f(z_2)| \ , \ \text{for} \quad \text{some} \ z_2 \in K.$$

If, in addition, $f \neq 0$ on K, then there exists $\delta > 0$ such that

$$|f(z)| \geq \delta \ , \ \forall z \in K.$$

1.3 Holomorphic functions

Definition 1.6. Let $U \subseteq \mathbb{C}$ be an open set and $f : U \longrightarrow \mathbb{C}$ a function. f is called complex differentiable at $z_0 \in U$, if there exists a function $\Delta : U \longrightarrow \mathbb{C}$, which is continuous at z_0, such that

$$f(z) = f(z_0) + (z - z_0)\Delta(z), \ z \in U.$$

f is called holomorphic on U, if f is complex differentiable at each point of U, we write $f \in \mathcal{H}(U)$.
f is holomorphic at $z_0 \in U$, if there exists an open neighborhood U_0 of z_0, such that f is holomorphic on U_0.

Remark. If f is complex differentiable at z_0, we have

$$\frac{f(z) - f(z_0)}{z - z_0} = \Delta(z) \ ; \ \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \Delta(z_0) = f'(z_0).$$

The following theorem has the same proof as in the real case.

Theorem 1.7. *Let f and g be complex differentiable at z_0. Then $f + g$ and $f \cdot g$ are complex differentiable at z_0. In addition, λf is complex differentiable at z_0, where $\lambda \in \mathbb{C}$, and the following rules are valid:*

$$(f + g)' = f' + g' \ , \ (\lambda f)' = \lambda f' \ , \ (f \cdot g)' = f' \cdot g + f \cdot g'.$$

If $g(z_0) \neq 0$, then

$$\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2}.$$

Let $w_0 = f(z_0)$ and let h be complex differentiable at w_0. Then

$$(h \circ f)'(z_0) = h'(f(z_0))f'(z_0) \quad \textit{(chain rule).}$$

We will need some results about holomorphic functions derived from the Cauchy Theorem to show an inverse function theorem for holomorphic functions.

Theorem 1.8. *Suppose f is holomorphic on a domain G, $z_0 \in G$, and $f'(z_0) \neq 0$. Then there exists an open neighborhood U of z_0 with $U \subset G$ such that f is injective on U, the image $V = f(U)$ of U is open, and the inverse function*

$$f^{-1} : V \longrightarrow U$$

is holomorphic on V and satisfies

$$(f^{-1})'(f(z)) = 1/f'(z),\ z \in U.$$

For the proof see 2.36.

EXAMPLES: a) $f(z) = z^n$, $n \in \mathbb{N}$.

$$f'(z_0) = \lim_{z \to z_0} \frac{z^n - z_0^n}{z - z_0} = \lim_{z \to z_0} \frac{(z - z_0)(z^{n-1} + z^{n-2}z_0 + \cdots + zz_0^{n-2} + z_0^{n-1})}{z - z_0}$$
$$= nz_0^{n-1}.$$

b) $f(z) = \overline{z}$.

First we take the limit $z \to z_0$ parallel to the real axis: $z - z_0 = h \in \mathbb{R},\ h \to 0$

$$\frac{\overline{z} - \overline{z}_0}{z - z_0} = \frac{h}{h} = 1;$$

and now parallel to the imaginary axis: $z - z_0 = ih,\ h \in \mathbb{R},\ h \to 0$

$$\frac{\overline{z} - \overline{z}_0}{z - z_0} = \frac{-ih}{ih} = -1.$$

Hence $f(z) = \overline{z}$ is nowhere complex differentiable.

1.4 The Cauchy–Riemann equations

Here we explain the relationship between real and complex differentiable functions, which is expressed by the Cauchy–Riemann equations.

Definition 1.9. Let $U \subseteq \mathbb{C}$ be open and $g : U \longrightarrow \mathbb{R}$. g is real differentiable at $z_0 \in U$, if there exist functions $\Delta_1, \Delta_2 : U \longrightarrow \mathbb{R}$, continuous at z_0, such that

$$g(z) = g(z_0) + (x - x_0)\Delta_1(z) + (y - y_0)\Delta_2(z), \tag{1.1}$$

where $z = x + iy$ and $z_0 = x_0 + iy_0$.

We have that $\Delta_1(z_0) = \frac{\partial g}{\partial x}(z_0) = g_x(z_0)$, which is the partial derivative with respect to x, and $\Delta_2(z_0) = \frac{\partial g}{\partial y}(z_0) = g_y(z_0)$ is the partial derivative with respect to y. In order to see this we first choose $z = x + iy_0$, then (1.1) implies

$$\Delta_1(z) = \frac{g(z) - g(z_0)}{x - x_0}$$

and putting $x \to x_0$, we obtain the assertion about the partial derivative with respect to$x$. Choosing $z = x_0 + iy$ we get the assertion about the partial derivative with respect to y.

In addition we have that each real differentiable function at z_0 is also continuous at z_0.

Example 1.10. Let $z = x + iy$ and

$$u(z) = \begin{cases} \frac{xy}{|z|^2} & \text{falls } z \neq 0 \\ 0 & \text{falls } z = 0. \end{cases}$$

Then u fails to be continuous at $z = 0$, since

$$\lim_{n\to\infty} u(1/n, 1/n) = \lim_{n\to\infty} \frac{1/n^2}{2/n^2} = \frac{1}{2} \neq u(0,0) = 0.$$

But the partial derivatives exist and $u_x(0,0) = u_y(0,0) = 0$, since

$$u_x(0,0) = \lim_{h\to 0} \frac{u(h,0) - u(0,0)}{h} = 0.$$

Using the mean value theorem from real analysis one can show the following result: if u_x and u_y are continuous at z_0, then u is real differentiable at z_0.

Now we take a function f being complex differentiable at at z_0. For $h \in \mathbb{R}$ both of the limits

$$\lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h} \quad \text{and} \quad \lim_{h\to 0} \frac{f(z_0 + ih) - f(z_0)}{ih}$$

exist and are equal. Splitting f into real and imaginary part $f(z) = u(x,y) + iv(x,y)$ we obtain

$$\begin{aligned} f'(z_0) &= \lim_{h\to 0} \frac{u(x_0 + h, y_0) - u(x_0, y_0)}{h} + i \lim_{h\to 0} \frac{v(x_0 + h, y_0) - v(x_0, y_0)}{h} \\ &= \lim_{h\to 0} \frac{u(x_0, y_0 + h) - u(x_0, y_0)}{ih} + i \lim_{h\to 0} \frac{v(x_0, y_0 + h) - v(x_0, y_0)}{ih}. \end{aligned}$$

This implies that $u_x(z_0) + iv_x(z_0) = 1/i(u_y(z_0) + iv_y(z_0))$ and, again after taking real and imaginary parts,

$$\begin{aligned} u_x(z_0) &= v_y(z_0) \\ v_x(z_0) &= -u_y(z_0). \end{aligned}$$

This system of two partial differential equations for the functions u and v is called the Cauchy–Riemann diifferential equations.

Hence we have now shown the following

Theorem 1.11. *Let f be complex differentiable at z_0 and split the function into real and imaginary parts $f = u + iv$. Then*

$$\begin{aligned} u_x(z_0) &= v_y(z_0) \\ v_x(z_0) &= -u_y(z_0). \end{aligned}$$

In addition

$$f'(z_0) = u_x(z_0) + iv_x(z_0) = v_y(z_0) - iu_y(z_0).$$

Now we consider complex-valued functions.

Definition 1.12. Now let $f : U \longrightarrow \mathbb{C}$ be a complex-valued function We say that f is real differentiable at z_0, if there are functions $\Delta_1, \Delta_2 : U \longrightarrow \mathbb{C}$, being continuous at z_0 such that

$$f(z) = f(z_0) + (x - x_0)\Delta_1(z) + (y - y_0)\Delta_2(z), z \in U.$$

We have again

$$\Delta_1(z_0) = \frac{\partial f}{\partial x}(z_0) = f_x(z_0) \quad \text{und} \quad \Delta_2(z_0) = \frac{\partial f}{\partial y}(z_0) = f_y(z_0)\ .$$

Lemma 1.13. *The following assertions are equivalent:*

(1) $f : U \longrightarrow \mathbb{C}$ is real differentiable at z_0;

(2) real and imaginary part of $f = u + iv$ are real differentiable at z_0;

(3) there exist functions $A_1, A_2 : U \longrightarrow \mathbb{C}$ being continuous at z_0 such that

$$f(z) = f(z_0) + (z - z_0)A_1(z) + (z - z_0)^{-}A_2(z),$$

where

$$A_1(z_0) = \frac{1}{2}(f_x(z_0) - if_y(z_0)) \text{ and } A_2(z_0) = \frac{1}{2}(f_x(z_0) + if_y(z_0)).$$

Proof. For $f = u + iv$ we have:

$$\begin{aligned} f(z) &= f(z_0) + (x - x_0)\Delta_1(z) + (y - y_0)\Delta_2(z) \\ &= u(z_0) + (x - x_0)\Re\Delta_1(z) + (y - y_0)\Re\Delta_2(z) \\ &\quad + i[v(z_0) + (x - x_0)\Im\Delta_1(z) + (y - y_0)\Im\Delta_2(z)]. \end{aligned}$$

This shows that (1) and (2)are equivalent and that

$$f_x(z_0) = u_x(z_0) + iv_x(z_0) \text{ and } f_y(z_0) = u_y(z_0) + iv_y(z_0).$$

an easy computation shows that

$$\begin{aligned} f(z) &= f(z_0) + (x - x_0)\Delta_1(z) + (y - y_0)\Delta_2(z) \\ &= f(z_0) + [(x - x_0) + i(y - y_0)]\frac{1}{2}(\Delta_1(z) - i\Delta_2(z)) \\ &\quad + [(x - x_0) - i(y - y_0)]\frac{1}{2}(\Delta_1(z) + i\Delta_2(z)) \\ &= f(z_0) + (z - z_0)A_1(z) + (z - z_0)^{-}A_2(z), \end{aligned}$$

where

$$A_1(z) = \frac{1}{2}(\Delta_1(z) - i\Delta_2(z)) \text{ and } A_2(z) = \frac{1}{2}(\Delta_1(z) + i\Delta_2(z)).$$

□

The expressions for the functions A_1 and A_2 lead to the following

Definition 1.14. The Wirtinger–derivatives [4] are defined by

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \ , \ \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

Theorem 1.15. *The function $f : U \longrightarrow \mathbb{C}$ satisfies the Cauchy– Riemann differential equations at $z_0 \in U$ if and only if $\frac{\partial f}{\partial \overline{z}}(z_0) = 0$.*

Proof. Splitting f into real and imaginary part one obtains

$$\begin{aligned}\frac{\partial f}{\partial \overline{z}} &= \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left[u_x + iv_x + i(u_y + iv_y)\right] \\ &= \frac{1}{2}\left[u_x - v_y + i(v_x + u_y)\right].\end{aligned}$$

Hence

$$\frac{\partial f}{\partial \overline{z}} = 0 \Leftrightarrow u_x = v_y \ , \ u_y = -v_x .$$

□

Remark. The advantage of this concept is that the system of partial differential equations

$$\begin{aligned}u_x(z_0) &= v_y(z_0) \\ v_x(z_0) &= -u_y(z_0).\end{aligned}$$

can be written in one equation, namely

$$\frac{\partial f}{\partial \overline{z}}(z_0) = 0.$$

Theorem 1.16. *Let $f : U \longrightarrow \mathbb{C}$ and $z_0 \in U$.*
f is complex differentiable at z_0 if and only if f is real differentiable at z_0 and $\frac{\partial f}{\partial \overline{z}}(z_0) = 0$. In this case we have $f'(z_0) = \frac{\partial f}{\partial z}(z_0)$.

Proof. First suppose that $f(z) = f(z_0) + (z - z_0)\Delta(z)$, where Δ is continuous at z_0. We set $A_1 = \Delta, A_2 = 0$. Then $f(z) = f(z_0) + (z - z_0)A_1(z)$ and, by Lemma 1.13, we have that f is real differentiable. Since $A_2 = 0$, the Cauchy–Riemann differential equations are satisfied.

For the other direction, let f be real differentiable at z_0. Then, by 1.13, there exist functions A_1 and A_2, being continuous at z_0, such that

$$f(z) = f(z_0) + (z - z_0)A_1(z) + (z - z_0)^{-}A_2(z),$$

4 Wirtinger, Wilhelm (1865–1945)

and, by assumption,we have $A_2(z_0) = \frac{\partial f}{\partial \bar{z}}(z_0) = 0$. Let

$$\tilde{\Delta}(z) = \begin{cases} \frac{A_2(z)(z-z_0)^-}{z-z_0} & \text{für } z \neq z_0 \\ 0 & \text{für } z = z_0. \end{cases}$$

Since A_2 is continuous at z_0 and $A_2(z_0) = 0$ and $\left|\frac{(z-z_0)^-}{z-z_0}\right| = 1$, we have $\lim_{z \to z_0} \tilde{\Delta}(z) = 0$. Hence $\tilde{\Delta}$ is continuous at z_0.

Now let $\Delta = A_1 + \tilde{\Delta}$. Then Δ is continuous at z_0 and

$$\begin{aligned} f(z) &= f(z_0) + (z - z_0)A_1(z) + (z - z_0)^- A_2(z) \\ &= f(z_0) + (z - z_0)\left[A_1(z) + \frac{A_2(z)(z - z_0)^-}{z - z_0}\right] \\ &= f(z_0) + (z - z_0)(A_1(z) + \tilde{\Delta}(z)) \\ &= f(z_0) + (z - z_0)\Delta(z). \end{aligned}$$

Hence f is complex differentiable at z_0. □

The assumptions about differentiability can considerably be weakened:

Theorem 1.17 (Looman–Menchoff). *Let $f = u + iv : U \longrightarrow \mathbb{C}$. Suppose that all partial derivatives u_x, u_y, v_x, v_y exist and satisfy the Cauchy–Riemann differential equations on U. Then f is complex differentiable on U.*

For a proof see for instance (**?**).

EXAMPLES

(1) $f = u + iv,\ u(x, y) = x^2 - y^2,\ v(x, y) = 2xy$. f is real differentiable on $\mathbb{C}$ and since $u_x = 2x,\ u_y = -2y,\ v_x = 2y,\ v_y = 2x$, the Cauchy–Riemann differential equations are satisfied. Hence f is complex differentiable on $\mathbb{C}$ and holomorphic on $\mathbb{C}$. We have

$$f(z) = z^2 = x^2 - y^2 + 2ixy.$$

(2) $f = u + iv,\ u(x, y) = x^3 - 3xy^2,\ v(x, y) = 3x^2y - y^3$. Also in this case, the Cauchy–Riemann differential equations are satiafied and f is holomorphic on $\mathbb{C}$; we have $f(z) = z^3$.

(3)$f = u + iv,\ u(x, y) = e^x \cos y,\ v(x, y) = e^x \sin y$. The Cauchy–Riemann differential equations are satisfied and f is holomorphic on $\mathbb{C}$. Later on (see section 1.8.) we will see that f is the complex exponential function. We have

$$f(z) = e^z = e^{x+iy} = e^x(\cos y + i \sin y).$$

(4) $f = u + iv,\ u(x, y) = x^3y^2,\ v(x, y) = x^2y^3$. f is real differentiable on $\mathbb{C}$ and $u_x = 3x^2y^2,\ v_y = 3x^2y^2,\ u_y = 2x^3y,\ v_x = 2xy^3$. It follows that the Cauchy–Riemann differential equations are satisfied if and only if $(x^2 + y^2)xy = 0$. These are

exactly the points on the coordinate axes. Hence f is complex differentiable there but not holomorphic.

In the following we explain some formulas of the so-called Wirtinger calculus.

Remark. (1)We consider complex-valued functions which can be expressed by the complex conjugate variables z and $\overline{z}$. Computing the Wirtinger derivatives $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \overline{z}}$ of such functions one can take z and $\overline{z}$ as independent variables.

Examplel. $f(z) = |z|^2 = z\overline{z}$, $\frac{\partial f}{\partial z} = \overline{z}$, $\frac{\partial f}{\partial \overline{z}} = z$. f is complex differentiable at $z = 0$, but fails to be holomorphic in $z = 0$.

(2) $\frac{\partial}{\partial z}$, $\frac{\partial}{\partial \overline{z}}$ are $\mathbb{C}$–linear operators, i.e. for real differentiable functions $f, g : U \longrightarrow \mathbb{C}$ and for $c, d \in \mathbb{C}$ we have

$$\frac{\partial}{\partial z}(cf + dg) = c\frac{\partial f}{\partial z} + d\frac{\partial g}{\partial z} \ , \ \frac{\partial}{\partial \overline{z}}(cf + dg) = c\frac{\partial f}{\partial \overline{z}} + d\frac{\partial g}{\partial \overline{z}}.$$

(3)

$$\begin{aligned}
\frac{\partial}{\partial \overline{z}}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial}{\partial \overline{z}}\left(\frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)\right) &= \frac{1}{4}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \\
&= \frac{1}{4}\left(\frac{\partial^2 f}{\partial x^2} - i\frac{\partial^2 f}{\partial x \partial y} + i\frac{\partial^2 f}{\partial y \partial x} + \frac{\partial^2 f}{\partial y^2}\right) \\
&= \frac{1}{4}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) \\
&= \frac{1}{4}\Delta f
\end{aligned}$$

The differential operator Δ is called Laplace operator[5].

(4) Let $f, \varphi : U \longrightarrow \mathbb{C}$ real differentiable functions. The equation

$$\frac{\partial f}{\partial \overline{z}} = \varphi$$

is called the inhomogeneous Cauchy–Riemann differential equation . It corresponds to the system of the two partial differential equations

$$1/2(u_x - v_y) = \Re\varphi \ , \ 1/2(u_y + v_x) = \Im\varphi.$$

5 Laplace, Pierre Simon (1749–1827)

1.5 A geometric interpretation of the complex derivative

Definition 1.18. Let $U \subseteq \mathbb{C}$ and let $f : U \longrightarrow \mathbb{C}$ be real differentiable on U. We can consider f as a mapping from a subset of $\mathbb{R}^2$ to $\mathbb{R}^2$, if we separate f into real and imaginary part: $f(z) = u(x, y) + iv(x, y)$.

$$J_f = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = u_x v_y - u_y v_x.$$

J_f is the Jacobi determinant. [6] of f.

Remark. If f is complex differentiable, then the Cauchy–Riemann differential equations are valid: $u_x = v_y$, $u_y = -v_x$. Hence

$$J_f = u_x^2 + v_x^2.$$

By 1.11 we have $f' = u_x + iv_x$, which implies that $|f'|^2 = u_x^2 + v_x^2$ and

$$J_f = |f'|^2.$$

Definition 1.19. Let $f : U \longrightarrow \mathbb{C}$ be complex differentiable at $z_0 \in U$. Then

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0) - hf'(z_0)}{h} = 0.$$

Let

$$(Tf(z_0))(h) := f'(z_0)h \ , \ h \in \mathbb{C}.$$

$(Tf(z_0)) : \mathbb{C} \longrightarrow \mathbb{C}$ is called the tangential map of f at the point z_0. It is a $\mathbb{C}$–linear map.

Remark. If f is only real differentiable at z_0, then we have by 1.13

$$f(z) = f(z_0) + (z - z_0)A_1(z) + (z - z_0)^- A_2(z).$$

Take for $h = z - z_0$ and set

$$(Tf(z_0))(h) = \frac{\partial f}{\partial z}(z_0)h + \frac{\partial f}{\partial \bar{z}}(z_0)\overline{h}.$$

This mapping $(Tf(z_0)) : \mathbb{C} \longrightarrow \mathbb{C}$ is in general only $\mathbb{R}$–linear.

It is easy to show that f is complex differentiable at z_0 if and only if $(Tf(z_0)) : \mathbb{C} \longrightarrow \mathbb{C}$ is $\mathbb{C}$–linear.

Definition 1.20. Let $\mathcal{T} : \mathbb{C} \longrightarrow \mathbb{C}$ be a bijective $\mathbb{R}$–linear map. For $z = x + iy$ and $w = u + iv$ let $\langle z, w\rangle = \Re(z\overline{w}) = xu + yv$ the euklidean scalar product of the vector space $\mathbb{C} = \mathbb{R}^2$ over $\mathbb{R}$.

$\mathcal{T}$ is an angle preserving map, if

$$|z||w|\langle \mathcal{T}z, \mathcal{T}w\rangle = |\mathcal{T}z||\mathcal{T}w|\langle z, w\rangle \ \forall z, w \in \mathbb{C}.$$

6 Jacobi, Carl Gustav (1804–1851)

Remark. If φ is angle between z and w, then

$$\cos\varphi = \frac{\langle z, w\rangle}{|z||w|},$$

hence the upper assumption means that the angle between $\mathcal{T}z$ and $\mathcal{T}w$ coincides with the original angle.

Definition 1.21. Let $f : U \longrightarrow \mathbb{C}$ be real differentiable on U.

f is angle preserving at $z_0 \in U$, if the tangential map $(Tf(z_0)) : \mathbb{C} \longrightarrow \mathbb{C}$ is angle preserving.

f is called angle preserving on U, if f angle preserving at each point of U.

Theorem 1.22. *Let $f : U \longrightarrow \mathbb{C}$ be a holomorphic function on U and suppose that $f'(z) \neq 0$, $\forall z \in U$. Then f is angle preserving on U.*

Proof. By 1.19 we have $(Tf(z))(h) = f'(z)h$. We have to show that

$$|h||k|\langle (Tf(z))(h), (Tf(z))(k)\rangle = |(Tf(z))(h)||(Tf(z))(k)|\langle h, k\rangle \ , \ \forall h, k \in \mathbb{C}.$$

The left hand side is equal to

$$|h||k|\langle f'(z)h, f'(z)k\rangle = |h||k|\Re(f'(z)h\overline{f'(z)k}) = |h||k||f'(z)|^2\Re(h\overline{k}),$$

and the right hand side

$$|f'(z)h||f'(z)k|\Re(h\overline{k}) = |f'(z)|^2|h||k|\Re(h\overline{k}).$$

□

Let $\gamma : [a, b] \longrightarrow U \subseteq \mathbb{C}$ be a curve in $\mathbb{C}$ (see Chapter 2, section 1). We split into real– and imaginary part

$$t \mapsto \gamma(t) = x(t) + iy(t) \ , \ t \in [a, b].$$

γ is differentiablezierbar in $s \in (a, b)$, if the derivatives $x'(s)$ and $y'(s)$ exist. We set $\gamma'(s) = x'(s) + iy'(s)$.

Let $z = \gamma(s)$ and suppose that $\gamma'(s) \neq 0$. The map

$$t \mapsto z + \gamma'(s)\ t \ , \ t \in \mathbb{R}$$

the tangent to the curve γ in $z = \gamma(s)$.

If $f : U \longrightarrow \mathbb{C}$ is a holomorphic function, we can consider the image curve:

$$f \circ \gamma : [a, b] \longrightarrow \mathbb{C}.$$

We have

$$t \mapsto f(\gamma(t)) = u(x(t), y(t)) + iv(x(t), y(t)) \ , \ \gamma(t) = z = x(t) + iy(t).$$

Using the chain rule we obtain

$$\begin{aligned}(f\circ\gamma)'(s) &= u_x(z)x'(s)+u_y(z)y'(s)+i[v_x(z)x'(s)+v_y(z)y'(s)]\\ &= 1/2[u_x(z)+iv_x(z)-i(u_y(z)+iv_y(z))](x'(s)+iy'(s))\\ &\quad+1/2[u_x(z)+iv_x(z)+i(u_y(z)+iv_y(z))](x'(s)-iy'(s))\\ &= \frac{\partial f}{\partial z}(z)(x'(s)+iy'(s))+\frac{\partial f}{\partial \overline{z}}(z)(x'(s)-iy'(s)) \text{ (nach 1.14)}\\ &= (Tf(z))(\gamma'(s)) \text{ (nach 1.19)}.\end{aligned}$$

If $(f\circ\gamma)'(s)\neq 0$, Then the image curve has the tangent

$$t\mapsto f(z)+(f\circ\gamma)'(s)\ t = f(z)+(Tf(z))(\gamma'(s))\ t\ ,\ t\in\mathbb{R}$$

in the point $f(z)$.

Now let γ_1 and γ_2 be two curves through the point z. The angle between the two curves in the point z is given by the angle between the corresponding tangents in this point. The direction vectors of the tangents are $\gamma_1'(s)$ and $\gamma_2'(s)$. Hence the angle between the two curves in the point z is given by $\sphericalangle(\gamma_1'(s),\gamma_2'(s))$. The angle between the image curves in the point Punkt $f(z)$ is $\sphericalangle((Tf(z))(\gamma_1'(s)),(Tf(z))(\gamma_2'(s)))$. We suppose that f is holomorphic in z and $f'(z)\neq 0$. Then, by 1.22, we get

$$\sphericalangle(\gamma_1'(s),\gamma_2'(s)) = \sphericalangle((Tf(z))(\gamma_1'(s)),(Tf(z))(\gamma_2'(s))).$$

Example: $f(z)=z^2\ ,\ f'(z)=2z\ ,\ f'(z)\neq 0\ ,\ \forall z\in\mathbb{C}^*=\mathbb{C}\setminus\{0\}$. Then we have $f=u+iv$ with $u(x,y)=x^2-y^2$ and $v(x,y)=2xy$. The parallels to the y–Achse with equation $x=a$ are mapped by f to the parabolas $v^2=4a^2(a^2-u)$ with the common focus $(0,0)$. The parallels to the x–Achse with equation $y=b$ are mapped by f to the parabolas $v^2=4b^2(b^2+u)$ with common focus $(0,0)$. This follows by using the above formulas for u and v, plugging in the equations for the parallels and finally eliminating x and y. The angles between the curves $(90°)$ are preserved by the map f.

1.6 Uniform convergence

The concept of uniform convergence will play an important role in the investigation of power series and limits of sequences of holomorphic functions.

Definition 1.23. Let $f_n : U \longrightarrow \mathbb{C}$ be a sequence of functions and $A\subseteq U$. $(f_n)_{n=1}^{\infty}$ converges uniformly on A to a function f, if for $\epsilon>0$ there exists $n_\epsilon\in\mathbb{N}$, such that

$$|f_n(z)-f(z)|<\epsilon\ ,\ \forall n>n_\epsilon \text{ and } \forall z\in A.$$

A series $\sum_{n=1}^{\infty} f_n$ converges uniformly on A, if the sequence of its partial sums $\left(\sum_{n=1}^{N} f_n\right)_N$ converges uniformly on A.
We write $|f|_A := \sup_{z\in A} |f(z)|$.

Theorem 1.24. *Let $f_n : U \longrightarrow \mathbb{C}$ be a sequence of functions. The following assertions are equivalent:*

(1) $(f_n)_n$ converges uniformly on $A \subseteq U$;

(2) $(f_n)_n$ is a Cauchy sequence on A, i.e. $\forall \epsilon > 0 \;\; \exists n_\epsilon \in \mathbb{N}$, such that

$$|f_n - f_m|_A < \epsilon \quad , \forall m, n > n_\epsilon .$$

Proof. (1) $\Rightarrow$ (2): If $(f_n) \to f$ converges uniformly on A, then

$$|f_n - f_m|_A \leq |f_n - f|_A + |f - f_m|_A < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

for n and m large enough.

(2) $\Rightarrow$ (1): $\forall z \in A$ we have

$$|f_n(z) - f_m(z)| \leq |f_n - f_m|_A < \epsilon \;, \;\forall n, m > n_\epsilon .$$

For a fixed $z \in A$, the sequence $(f_n(z))_n$ is a Cauchy sequence in $\mathbb{C}$. Since $\mathbb{C}$ is complete, there exists the limit of this sequence:

$$\lim_{n\to\infty} f_n(z) = f(z)$$

(pointwise convergence). For an arbitrary $z \in A$, we choose $m_0 = m(z)$, such that $|f_m(z) - f(z)| < \epsilon \;, \;\forall m > m_0$. Then

$$|f_n(z) - f(z)| \leq |f_n(z) - f_m(z)| + |f_m(z) - f(z)| < \epsilon + \epsilon$$

$\forall n > n_\epsilon \;, \;\forall m > m_0$ and for arbitrary $z \in A$. □

Theorem 1.25. $\sum_{n=1}^{\infty} f_n$ *converges uniformly on A, if and only if for each $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{N}$, such that*

$$|f_{m+1}(z) + \cdots + f_n(z)| < \epsilon$$

for all $n > m \geq n_\epsilon$ and for all $z \in A$.

Proof. Since

$$f_{m+1}(z) + \cdots + f_n(z) = \sum_{k=1}^{n} f_k(z) - \sum_{k=1}^{m} f_k(z),$$

everything follows from Theorem 1.24. □

Theorem 1.26 (Weierstraß majorant criterion). [7] *Let $f_n : U \longrightarrow \mathbb{C}$ be a sequence of functions with*

$$\sup_{z\in A} |f_n(z)| = |f_n|_A \le M_n \ , \ M_n \ge 0.$$

Suppose that $\sum_{n=1}^{\infty} M_n < \infty$. Then the series $\sum_{n=1}^{\infty} f_n$ converges uniformly on A.

Proof. Let $\epsilon > 0$. There exists n_ϵ with

$$|\sum_{k=m+1}^{n} f_k(z)| \le \sum_{k=m+1}^{n} |f_k(z)| \le \sum_{k=m+1}^{n} M_k < \epsilon,$$

$\forall n > m \ge n_\epsilon$ and $\forall z \in A$; now we can apply Theorem 1.25 □

EXAMPLES. 1) $f_n(z) = z^n \ , \ n \in \mathbb{N}$. $f_n \to 0$ converges uniformly on each compact subset of $\mathbb{D} = \{z : |z| < 1\}$. If $K \subset \mathbb{D}$ is a compact subset of $\mathbb{D}$, then there exists $0 < r < 1$ with $K \subset D_r(0) = \{z : |z| < r\}$, and we have

$$|f_n(z)| = |z^n| \le r^n \to 0 \ , \ \forall z \in K.$$

2) $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ with uniform convergence on all compact subsets of $\mathbb{D}$. Let K be as above in example 1, then

$$\left|\sum_{n=0}^{\infty} z^n\right| \le \sum_{n=0}^{\infty} |z^n| \le \sum_{n=0}^{\infty} r^n < \infty,$$

$\forall z \in K$, and we can apply Theorem 1.26.

3) $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges uniformly on all compact subsets of $\mathbb{C}$. If K is a compact subset in $\mathbb{C}$, then there exits $N \in \mathbb{N}$ with $K \subset D_N(0)$. We assume to know the Taylor series expansion of the real exponential function and obtain

$$\left|\sum_{n=0}^{\infty} \frac{z^n}{n!}\right| \le \sum_{n=0}^{\infty} \left|\frac{z^n}{n!}\right| \le \sum_{n=0}^{\infty} \frac{N^n}{n!} = e^N,$$

$\forall z \in K$. Now we apply again Theorem 1.26.

1.7 Power series

Definition 1.27. Let $z_0 \in \mathbb{C}$ be a fixed point and $a_n \in \mathbb{C}$ for $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The expression

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

7 Weierstraß, Karl Theodor Wilhelm (1815–1897) is called formal power series at z_0 with coefficients a_n.

For two formal power series

$$P = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad \text{and} \qquad Q = \sum_{n=0}^{\infty} b_n (z - z_0)^n$$

we define the sum

$$P + Q = \sum_{n=0}^{\infty} (a_n + b_n)(z - z_0)^n,$$

and the product

$$P \cdot Q = \sum_{n=0}^{\infty} c_n (z - z_0)^n, \text{ where } c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1 + a_n b_0$$

the last expression is also called the Cauchy product.

Theorem 1.28. *If $s, M > 0$ are two constants with the property that*

$$|a_n| s^n \leq M \ , \ \forall n \in \mathbb{N}_0,$$

then the power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges uniformly and absolutely on each compact subset of $D_s(z_0) = \{z : |z - z_0| < s\}$.

Proof. If K is a compact subset of $D_s(z_0)$, then there exists a positive $r < s$ with $K \subset D_r(z_0)$. Set $q = r/s < 1$. Then we have

$$\sup_{z \in K} |a_n (z - z_0)^n| \leq \sup_{z \in D_r(z_0)} |a_n (z - z_0)^n| \leq |a_n| r^n = |a_n| s^n \frac{r^n}{s^n} \leq M q^n,$$

since $\sum_{n=0}^{\infty} M q^n < \infty$, the assertion follows from Theorem 1.26 □

Remark. If $|a_n| s^n \leq M \ , \ \forall n \in \mathbb{N}_0$, then the sequence $(|a_n| r^n)_n$ converges to 0 for each positive $r < s$.

Definition 1.29. Let $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ be a power series and

$$R := \sup\{t \geq 0 : (|a_n| t^n)_n \text{ is a bounded sequence}\}.$$

R is called radius of convergence of the power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$.

Theorem 1.30. *Let $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ be a power series with radius of convergence R. Then:*

(1) $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ is uniformly convergent on each compact subset of $D_R(z_0)$.

(2) $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ fails to be convergent in $\mathbb{C} \setminus \overline{D_R(z_0)}$.

Proof. (1) If $R = 0$, the assertion is trivial. If $R > 0$, then we have for an arbitrary positive $s < R$ that the sequence $(|a_n|s^n)_n$ is bounded. By Theorem 1.28, the power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges uniformly on each compact subset of $D_s(z_0)$, and as s was an arbitrary positive number $< R$, the first assertion of the Theorem follows.

(2) If $|z - z_0| > R$, then the sequence $(|a_n|\ |z - z_0|^n)_n$ is unbounded and the power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ fails to be convergent. □

In the following theorem we show that the radius of convergence can be computed in terms of the coefficients of the power series.

Theorem 1.31 (Cauchy–Hadamard). [8] *Let $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ be a power series with radius of convergence R. Then:*

$$R = \left(\limsup_{n\to\infty} |a_n|^{1/n}\right)^{-1} \qquad (1/0 = \infty \quad 1/\infty = 0).$$

Proof. Let $L = \left(\limsup_{n\to\infty} |a_n|^{1/n}\right)^{-1}$. We show that $L = R$.

Let $\epsilon > 0$ be an arbitrary positive number. For almost all $n \in \mathbb{N}$ we have: $|a_n|^{1/n} \le 1/(L-\epsilon)$. Hence $|a_n|(L-\epsilon)^n \le 1$,which implies that the sequence $(|a_n|(L-\epsilon)^n)_n$is bounded. So we have $L - \epsilon \le R$ and $L \le R$, as ϵ was arbitrary. Now suppose that $L < \infty$. In order to show that $R \le L$, it suffices to prove that

$$R \le s\ ,\ \forall s > L.$$

Let $L < s < \infty$. then $s^{-1} < L^{-1} = \limsup_{n\to\infty} |a_n|^{1/n}$, hence there exists an infinite subset $M \subseteq \mathbb{N}$ such that

$$s^{-1} < |a_m|^{1/m}\ ,\ \forall m \in M.$$

This implies $|a_m|s^m > 1$, $\forall m \in M$ and $(|a_n|s^n)_n$ fails to be a null-sequence. Therefore we must have $s \ge R$, because $s < R$ would imply that $(|a_n|s^n)_n$ is a null-sequence (see the remark from above).

If $L = \infty$, then we get $L = R$ from the first step of the proof. □

Examples.
(a) $\sum_{n=0}^{\infty} n^n z^n, \quad |a_n|^{1/n} = n$ and $R = 0$.
(b) $\sum_{n=0}^{\infty} z^n, \quad |a_n|^{1/n} = 1$ and $R = 1$.
(c) $\sum_{n=0}^{\infty} \frac{z^n}{n^n}, \quad |a_n|^{1/n} = 1/n$ and $R = \infty$.
(d) $\sum_{n=0}^{\infty} \frac{z^n}{n!}$, for this example we use the following

8 Hadamard, Jacques Solomon (1865–1963)

Theorem 1.32. *Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a power series with radius of convergence R, and suppose that $a_n \neq 0$ for almost all n. Then*

$$\liminf_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| \leq R \leq \limsup_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|.$$

If the limit $\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ exists, then

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|.$$

Example. $\sum_{n=0}^{\infty} \frac{z^n}{n!}$, $\frac{a_n}{a_{n+1}} = \frac{(n+1)!}{n!} = n+1$ and $R = \infty$.

Proof. Let $S = \liminf_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$. If $S = 0$, we have $S \leq R$. If $S > 0$, it suffices to show that $s \leq R$ for each $0 < s < S$. Since S is a $\liminf$, there exists $l \in \mathbb{N}$ such that

$$\left|\frac{a_n}{a_{n+1}}\right| > s, \ \forall n \geq l.$$

Hence $|a_n|s^{-1} > |a_{n+1}| \quad, \forall n \geq l$. Let $A = |a_l|s^l$. Then the last inequality implies that

$$|a_{l+1}|s^{l+1} < |a_l|s^{-1}s^{l+1} = |a_l|s^l = A,$$

iterating this argument one obtains

$$|a_{l+m}|s^{l+m} \leq A \quad, \forall m \in \mathbb{N}.$$

Hence the sequence $(|a_n|s^n)_n$ is bounded. Now we get from the defintion of the radius of convergence that $s \leq R$.

Now let $T = \limsup_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$. If $T = \infty$, we have $R \leq T$. If $T < \infty$, it remains to show that $t \geq R$ for each $t > T$. We get again an $l \in \mathbb{N}$ with

$$\left|\frac{a_n}{a_{n+1}}\right| < t, \ \forall n \geq l.$$

This implies that $|a_{n+1}| > |a_n|t^{-1} \quad, \forall n \geq l$. One can choose l such that $B = |a_l|t^l > 0$. Then

$$|a_{l+1}|t^{l+1} > |a_l|t^{-1}t^{l+1} = |a_l|t^l = B,$$

and by iteration $|a_{m+l}|t^{l+m} > B \quad, \forall m \in \mathbb{N}$. Hence the sequence $(|a_n|t^n)_n$ fails to be a null-sequence. Therefore we have $t \geq R$. □

1.8 Line integrals

Here we take up the complex integral calculus. We describe the basic properties of line integrals and its relationship to complex primitives for holomorphic functions.

Definition 1.33. Let $\gamma : [a,b] \longrightarrow U \subseteq \mathbb{C}$ be a continuous map. We call γ a curve in $\mathbb{C}$. We use γ to denote the map and γ^* for the set $\gamma^* = \{\gamma(t) : t \in [a,b]\}$. $\gamma(a)$ is called initial point and $\gamma(b)$ end point of the curve. $[a,b]$ is the parameter interval. If $\gamma(a) = \gamma(b)$, then γ is called a closed curve.
Let $a = s_0 < s_1 < s_2 < \cdots < s_n = b$ and $\gamma|_{[s_{j-1},s_j]}$ for $j = 1,\dots,n$ continuously differentiable (we will also say a $\mathcal{C}^1$ function). A piecewise continuously differentiable γ is called a path.
Let $f : U \longrightarrow \mathbb{C}$ be a continuous function and $\gamma : [a,b] \longrightarrow U$ a path in U.

$$\begin{aligned}\int_\gamma f(z)\,dz &= \int_a^b f(\gamma(t))\gamma'(t)\,dt := \sum_{j=0}^{n-1} \int_{s_j}^{s_{j+1}} f(\gamma(t))\gamma'(t)\,dt \\ &= \int_a^b \Re\left(f(\gamma(t))\gamma'(t)\right)\,dt + i\int_a^b \Im\left(f(\gamma(t))\gamma'(t)\right)\,dt\end{aligned}$$

is the line integral of f along γ.

Remark 1.34. (a) Let $\varphi : [a_1,b_1] \longrightarrow [a,b]$ be a bijective $\mathcal{C}^1$ function such that $\varphi' > 0$ everywhere on $[a_1,b_1]$, let $\gamma : [a,b] \longrightarrow U$ and $\gamma_1 := \gamma \circ \varphi : [a_1,b_1] \longrightarrow U$ be paths in U. Since we supposed that $\varphi' > 0$, we have $\varphi(a_1) = a$ und $\varphi(b_1) = b$, which means that the orientation in γ and γ_1 coincides.
For each continuous function $f : U \longrightarrow \mathbb{C}$ we have

$$\begin{aligned}\int_{\gamma_1} f(z)\,dz &= \int_{a_1}^{b_1} f(\gamma_1(t))\gamma_1'(t)\,dt = \int_{a_1}^{b_1} f(\gamma(\varphi(t)))\gamma'(\varphi(t))\varphi'(t)\,dt \\ &= \int_a^b f(\gamma(s))\gamma'(s)\,ds = \int_\gamma f(z)\,dz,\end{aligned}$$

where we substituted $s = \varphi(t)\ ,\ ds = \varphi'(t)dt$.
We say that the curves γ_1 and γ are equivalent. The line integrals are independent of the parametrization.

(b) Let $\gamma_1 : [a,b] \longrightarrow U$ and $\gamma_2 : [b,c] \longrightarrow U$ be paths in U with $\gamma_1(b) = \gamma_2(b)$. We define a new path $\gamma : [a,c] \longrightarrow U$ by $\gamma|_{[a,b]} = \gamma_1$ and $\gamma|_{[b,c]} = \gamma_2$ (composition of paths).

Then we have

$$\int_{\gamma} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz.$$

(c) Let $\gamma : [0,1] \longrightarrow U$ be a path. We define the inverse path γ_1 to γ by $\gamma_1(t) := \gamma(1-t)$, $t \in [0,1]$. Then $\gamma^* = \gamma_1^*$ and

$$\int_{\gamma_1} f(z)\,dz = -\int_{\gamma} f(z)\,dz.$$

This follows by the substitution $1-t=s$, $dt=-ds$ in the integral

$$\int_{\gamma_1} f(z)\,dz = \int_0^1 f(\gamma(1-t))(-\gamma'(1-t))\,dt = \int_1^0 f(\gamma(s))(-\gamma'(s))(-1)\,ds$$

$$= -\int_0^1 f(\gamma(s))\gamma'(s)\,ds = -\int_{\gamma} f(z)\,dz.$$

We write $\gamma_1 = \gamma^{-1}$.

(d) Let $\gamma : [a,b] \longrightarrow U$ be a path and $f : U \longrightarrow \mathbb{C}$ a continuous function. The length $L(\gamma)$ of the path γ is given by

$$L(\gamma) = \int_a^b |\gamma'(t)|\,dt.$$

We have

$$\left|\int_{\gamma} f(z)\,dz\right| \le L(\gamma) \max_{z\in\gamma^*} |f(z)|,$$

which is shown by the following estimate:

$$\left|\int_{\gamma} f(z)\,dz\right| = \left|\int_a^b f(\gamma(t))\gamma'(t)\,dt\right| \le \int_a^b |f(\gamma(t))|\;|\gamma'(t)|\,dt \le L(\gamma)\max_{z\in\gamma^*}|f(z)|.$$

Example 1.35. (1) Let $a \in \mathbb{C},\ r > 0$ and $\gamma(t) = a + r(\cos t + i\sin t),\ t \in [0, 2\pi]$ the positively oriented circle with center a and radius r (once passed through). Let $f : U \longrightarrow \mathbb{C}$ be a continuous function and suppose that $D_r(a) \subset U$.

$$\int_{\gamma} f(z)\,dz = r\int_0^{2\pi} f(a + r(\cos t + i\sin t))(-\sin t + i\cos t)dt.$$

Special cases : $a = 0,\ r = 1,\ f(z) = z$:

$$\int_{\gamma} z\,dz = i\int_{0}^{2\pi}(\cos 2t + i\sin 2t)\,dt = 0;$$

$f(z) = \overline{z}$:

$$\int_{\gamma} \overline{z}\,dz = i\int_{0}^{2\pi}(\cos^2 t + \sin^2 t)\,dt = i\int_{0}^{2\pi} dt = 2\pi i;$$

$f(z) = 1/z$:

$$\int_{\gamma} \frac{1}{z}\,dz = \int_{0}^{2\pi} \frac{-\sin t + i\cos t}{\cos t + i\sin t}\,dt = 2\pi i.$$

for arbitrary $a \in \mathbb{C}$ and $r > 0$:

$$\int_{\gamma} \frac{dz}{z-a} = r\int_{0}^{2\pi} \frac{-\sin t + i\cos t}{a + r(\cos t + i\sin t) - a}\,dt = ir\int_{0}^{2\pi} \frac{1}{r}\,dt = 2\pi i.$$

(2) Let $a, b \in \mathbb{C}$, $a \neq b$. The path $\gamma(t) = a + (b-a)t$, $t \in [0,1]$ describes the straight line segment $[a,b]$ joining the points a and b, $f : U \longrightarrow \mathbb{C}$, where $\gamma^* \subset U$.

$$\int_{\gamma} f(z)\,dz = (b-a)\int_{0}^{1} f(a + (b-a)t)\,dt.$$

If $a = -1$, $b = 1$ and $f(z) = z$:

$$\int_{\gamma} z\,dz = 2\int_{0}^{1}(-1 + 2t)\,dt = 2(-t + t^2)\Big|_0^1 = 0.$$

If γ_1 is the positively oriented semicircle between -1 and 1, then

$$\int_{\gamma_1} z\,dz = i\int_{\pi}^{2\pi}(\cos 2t + i\sin 2t)\,dt = 0.$$

Taking the function $f(z) = z$ we see that the line integral does not depend on the way between -1 and 1. But for $f(z) = \overline{z}$ we have

$$\int_{\gamma} \overline{z}\,dz = 2\int_{0}^{1}(-1 + 2t)\,dt = 0,$$

and

$$\int_{\gamma_1} \overline{z}\, dz = i \int_{\pi}^{2\pi} (\cos^2 t + \sin^2 t)\, dt = i\pi.$$

(3) Let $a, b, c \in \mathbb{C}$ and $\Delta = \Delta(a, b, c)$ be the triangle with vertices a, b, c, with the orientation: $a \to b \to c \to a$.

$$\int_{\partial\Delta} f(z)\, dz = \int_{[a,b]} f(z)\, dz + \int_{[b,c]} f(z)\, dz + \int_{[c,a]} f(z)\, dz.$$

If $\Delta' = \Delta(a, c, b)$ is the triangle with the orientation $a \to c \to b \to a$, then

$$\begin{aligned}
\int_{\partial\Delta'} f(z)\, dz &= \int_{[a,c]} f(z)\, dz + \int_{[c,b]} f(z)\, dz + \int_{[b,a]} f(z)\, dz \\
&= -\int_{[c,a]} f(z)\, dz - \int_{[b,c]} f(z)\, dz - \int_{[a,b]} f(z)\, dz \\
&= -\int_{\partial\Delta} f(z)\, dz.
\end{aligned}$$

1.9 Primitive functions

In the following we state an analogue to the fundamental theorem of calculus and show that the second statement of the fundamental theorem holds for holomorphic functions on convex domains.

Definition 1.36. Let $U \subseteq \mathbb{C}$be an open set and $f : U \longrightarrow \mathbb{C}$ a continuous function. f has a primitive function on U, if there exists a holomorphic function F on U such that $F' = f$ on U. F is called a primitive function for f.

Remark. If $U \subseteq \mathbb{C}$ is a connected set and $f : U \longrightarrow \mathbb{C}$ has two primitive functions F_1 und F_2 on U, then we have $F_1 = F_2 + C$, where C is a constant. This follows from $F_1' - F_2' = f - f = 0$.

Theorem 1.37. *Suppose that the continuous function $f : U \longrightarrow \mathbb{C}$ has a primitive function F on U. Let $z_0, z_1 \in U$ and γ an arbitrary path in U from z_0 to z_1. Then*

$$\int_{\gamma} f(z)\, dz = F(z_1) - F(z_0).$$

Proof. Let $\gamma : [a,b] \longrightarrow U$, $a = t_0 < t_1 < \cdots < t_n = b$ and γ a $\mathcal{C}^1$ map on $[t_{k-1}, t_k]$ for $k = 1, \dots n$. Then

$$\begin{aligned}\int_\gamma f(z)\,dz &= \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} f(\gamma(t))\gamma'(t)\,dt = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} F'(\gamma(t))\gamma'(t)\,dt \\ &= \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} (F\circ\gamma)'(t)\,dt = \sum_{k=1}^{n} [F(\gamma(t_k)) - F(\gamma(t_{k-1}))] \\ &= F(z_1) - F(z_0).\end{aligned}$$

□

Corollary 1.38. *Suppose that the continuous function $f : U \longrightarrow \mathbb{C}$ has a primitive on U, then*

$$\int_\gamma f(z)\,dz = 0,$$

for each closed path γ in U.

Remark. If $F \in \mathcal{H}(U)$ and F' is continuous (later we will see that we do not need the last assumption), then

$$\int_\gamma F'(z)\,dz = 0,$$

for each closed path γ in U.

Example 1.39. (a) The function $f(z) = z^n$, $n \in \mathbb{Z}$, $n \neq -1$ has $F(z) = \frac{z^{n+1}}{n+1}$ as primitive on $\mathbb{C}$ for $n \geq 0$ and on $\mathbb{C}^*$ for $n < -1$. We have

$$\int_{[z_0,z_1]} z^n\,dz = \frac{1}{n+1}(z_1^{n+1} - z_0^{n+1}) \ , \quad \int_\gamma z^n\,dz = 0,$$

for $n \neq -1$ and for each closed path γ with $0 \notin \gamma^*$.
The complex polynomial $p(z) = \sum_{k=0}^{n} a_k z^k$ has $\sum_{k=0}^{n} \frac{a_k}{k+1} z^{k+1}$ as primitive.

(b) For the function $f(z) = \overline{z}$, the line integral $\int_\gamma f(z)\,dz$ depends not only on the initial and endpoint of the path (see Example (2) in 1.35), hence, by 1.37, f has no primitive on any open subset of $\mathbb{C}$.

(c) If $\gamma(t) = \cos t + i \sin t$, $t \in [0, 2\pi]$, then

$$\int_\gamma \frac{dz}{z} = 2\pi i,$$

by 1.38, the function $f(z) = \frac{1}{z}$ has no primitive on $\mathbb{C}^*$.

Now we will prove the converse of Corollary 1.38.

Theorem 1.40. *Let $G \subseteq \mathbb{C}$ be a domain (open and connected) and $f : G \longrightarrow \mathbb{C}$ a continuous function. Suppose that for each closed path γ in G we have*

$$\int_\gamma f(z)\,dz = 0.$$

Then f has a primitive on G.

Proof. Fix $a \in G$ fix. For an arbitrary $z \in G$, we choose a path γ_z in G from a to z and we set

$$F(z) = \int_{\gamma_z} f(\zeta)\,d\zeta.$$

We will show that $F'(z_0) = f(z_0)$ for an arbitrary $z_0 \in G$.
If z is sufficiently close to z_0, then $[z_0, z] \subset G$ and the path γ, which is composed by γ_{z_0}, $[z_0, z]$ and γ_z^{-1}, is a closed path in G. Hence, by assumption, we have

$$0 = \int_\gamma f(\zeta)\,d\zeta = \int_{\gamma_{z_0}} f(\zeta)\,d\zeta + \int_{[z_0,z]} f(\zeta)\,d\zeta - \int_{\gamma_z} f(\zeta)\,d\zeta.$$

This implies

$$\begin{aligned} F(z) - F(z_0) &= \int_{\gamma_z} f(\zeta)\,d\zeta - \int_{\gamma_{z_0}} f(\zeta)\,d\zeta = \int_{[z_0,z]} f(\zeta)\,d\zeta \\ &= \int_0^1 f(z_0 + t(z - z_0))(z - z_0)\,dt = (z - z_0)A(z), \end{aligned}$$

where $A(z) = \int_0^1 f(z_0 + t(z - z_0))\,dt$, $A(z_0) = f(z_0)$. Hence

$$A(z) = \frac{F(z) - F(z_0)}{z - z_0}.$$

In order to show $F'(z_0) = f(z_0)$, it suffices to prove that A is continuous at z_0. For this aim we estimate

$$|A(z) - A(z_0)| \leq \max_{t \in [0,1]} |f(z_0 + t(z - z_0)) - f(z_0)|.$$

Since f is continuous at z_0, we get now the same for A. Hence F is a primitive of f on G.

We remark that the definition of the function F does not depend on the choice of the path γ_z. If $\tilde{\gamma_z}$ is a different path from a to z, then $\gamma_z \tilde{\gamma_z}^{-1}$ is a closed path in G and hence

$$\int_{\gamma_z} f(\zeta)\, d\zeta - \int_{\tilde{\gamma_z}} f(\zeta)\, d\zeta = \int_{\gamma_z \tilde{\gamma_z}^{-1}} f(\zeta)\, d\zeta = 0$$

which implies that

$$\int_{\gamma_z} f(\zeta)\, d\zeta = \int_{\tilde{\gamma_z}} f(\zeta)\, d\zeta.$$

□

If we assume something more about the domain G, we can considerably weaken the assumptions in Theorem 1.40.

Definition 1.41. The domain $G \subseteq \mathbb{C}$ is convex, if whenever two points z_0, z_1 belong to G, then the straight line segment $[z_0, z_1]$ joining the two points is contianed in G.

Theorem 1.42. *Let G be a convex domain in $\mathbb{C}$ and $f : G \longrightarrow \mathbb{C}$ a continuous function. Suppose that for each triangle $\Delta \subseteq G$*

$$\int_{\partial \Delta} f(z)\, dz = 0.$$

Then f has a primitive on G.

Proof. Fix $a \in G$ and let

$$F(z) = \int_{[a,z]} f(\zeta)\, d\zeta,$$

for $z \in G$. By assumption, the straight line segment $[a, z] \subset G$. If $z_0 \in G$, then the triangle Δ with vertices a, z, z_0 is contained in G and hence

$$0 = \int_{\partial \Delta} f(\zeta)\, d\zeta = \int_{[a,z_0]} f(\zeta)\, d\zeta + \int_{[z_0,z]} f(\zeta)\, d\zeta - \int_{[a,z]} f(\zeta)\, d\zeta.$$

Now we can continue as in the proof of Theorem 1.40. □

Remark. If G is not convex, the assertions of Theorem 1.42 are true at least in each convex neighborhood $U_z \subseteq G$ of an arbitrary point $z \in G$.

In the following we study the interchange of limit processes where uniform limits of sequences of functions and line integrals are involved.

Theorem 1.43. *Leti γ be a path in $\mathbb{C}$ and $(f_n)_n$ be a sequence of continuous functions on γ^*. Suppose that the sequence $(f_n)_n$ converges uniformly on γ^* to a function f. Then*

$$\lim_{n\to\infty} \int_\gamma f_n(z)\,dz = \int_\gamma \lim_{n\to\infty} f_n(z)\,dz = \int_\gamma f(z)\,dz.$$

Proof. By 1.34 (d) we have

$$\left|\int_\gamma f_n(z)\,dz - \int_\gamma f(z)\,dz\right| = \left|\int_\gamma (f_n(z) - f(z))\,dz\right| \le L(\gamma) \max_{z\in\gamma^*} |f_n(z) - f(z)|,$$

which implies the assertion. □

Remark. If the series $\sum_{n=0}^\infty f_n$ of continuous functions f_n converges uniformly on γ^*, then

$$\int_\gamma \left(\sum_{n=0}^\infty f_n(z)\right) dz = \sum_{n=0}^\infty \int_\gamma f_n(z)\,dz.$$

Theorem 1.44. *Let $P(z) = \sum_{n=0}^\infty a_n(z - z_0)^n$ be a power series with radius of convergence $R > 0$. Then P is holomorphic on $D_R(z_0)$ and*

$$P'(z) = \sum_{n=1}^\infty n a_n (z - z_0)^{n-1}.$$

(we can interchange summation and differentiation)

Proof. First we show the following assertion. Let R' be the radius of convergence of the power series $Q(z) = \sum_{n=1}^\infty n a_n (z - z_0)^{n-1}$. Then $R' \ge R$.
Without loss of generality we can assume that $z_0 = 0$. The power series $\sum_{n=0}^\infty n z^n$ has radius of convergence 1, i.e. $\forall \rho \in [0, 1)$ the sequence $(n\rho^n)_n$ is bounded. Now we have $|a_n z_1^n| \le M$, $\forall n \in \mathbb{N}$ $(M > 0)$, for an arbitrary $z_1 \in D_R(0)$, in addition we have

$$|n a_n z_2^{n-1}| = \frac{n}{|z_2|} \left|\frac{z_2}{z_1}\right|^n |a_n z_1^n| \le n \left|\frac{z_2}{z_1}\right|^n \frac{M}{|z_2|},$$

for $0 < |z_2| < |z_1|$. Since $\left|\frac{z_2}{z_1}\right| < 1$, the sequence $(|n a_n z_2^{n-1}|)_n$ is bounded. As $|z_2| < |z_1| < R$ were chosen arbitrarily, we get from the definition of the radius of convergence 1.29 that $R' \ge R$.

Now let γ be an arbitrary closed path in $D_{R'}(0)$. We will show that Q has P as a primitive on $D_{R'}(0)$:

$$\int_\gamma Q(z)\,dz = \int_\gamma \left(\sum_{n=1}^{\infty} na_n z^{n-1}\right) dz = \sum_{n=1}^{\infty} na_n \int_\gamma z^{n-1}\,dz = 0,$$

here we interchanged integration and summation (see 1.43) and used Example 1.39(a). By Theorem 1.40, this implies that there exists a primitive of Q on $D_{R'}(0)$. A primitive of Q is

$$\int_{[0,z]} Q(\zeta)\,d\zeta = \int_{[0,z]} \left(\sum_{n=1}^{\infty} na_n \zeta^{n-1}\right) d\zeta = \sum_{n=1}^{\infty} na_n \int_{[0,z]} \zeta^{n-1}\,d\zeta$$

$$= \sum_{n=1}^{\infty} na_n \frac{z^n}{n} = \sum_{n=1}^{\infty} a_n z^n,$$

where we used again 1.43 and Example 1.39 (a). Hence also

$$P(z) = a_0 + \sum_{n=1}^{\infty} a_n z^n$$

is a primitive of Q auf $D_{R'}(0)$. We have also seen that P converges on $D_{R'}(0)$, hence $R' = R$. □

Remark. We can apply 1.44 for P' to see that P' is holomorphic on $D_R(z_0)$. Iterating this argument we obtain the existence of derivatives of P of arbitrary order and the formula

$$P^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\dots(n-k+1)a_n(z-z_0)^{n-k} \quad , \quad a_n = \frac{P^{(n)}(z_0)}{n!}.$$

1.10 Elementary functions

Here we define the complex elementary functions by its power series and derive the most important properties including a definition of π and a proof of the Eulerian identity $e^{2\pi i} = 1$.

Definition 1.45. We define the complex exponential function by the power series $\exp z = e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$.

The series converges uniformly on all compact subsets of $\mathbb{C}$. By Theorem 1.44 we have

$$(e^z)' = \sum_{n=1}^{\infty} \frac{nz^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z,$$

also $(e^z)' = e^z$.

Theorem 1.46. *For $z, w \in \mathbb{C}$ we have $e^{z+w} = e^z e^w$.*

Proof. Let $f(z) = e^{-z}e^{z+w}$. Then

$$f'(z) = -e^{-z}e^{z+w} + e^{-z}e^{z+w} = 0 \ , \ \forall z \in \mathbb{C}.$$

f is holomorphic in $\mathbb{C}$, and since $f' = u_x + iv_x = v_y - iu_y = 0$, we obtain $u_x = u_y = v_x = v_y = 0$. Hence $f =$ const., and

$$e^{-z}e^{z+w} = f(0) = e^w,$$

setting $w = 0$, we get $e^{-z}e^z = e^0 = 1$. This implies

$$(e^z)^{-1} = e^{-z} \text{ and } e^{z+w} = e^z e^w.$$

$\square$

Remark. We know from the lat proof that $e^{-z}e^z = e^0 = 1$ for all $z \in \mathbb{C}$. Hence the exponential function has no zeroes.

Next we define the complex sine and cosine function again by power series.

Definition 1.47.

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}, \ \cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n},$$

the series converge uniformly on all compact subsets of $\mathbb{C}$.

Using the power series it is easy to show the following formulas

$$\cos z + i \sin z = e^{iz} \ , \ \cos(-z) = \cos z \ , \ \sin(-z) = -\sin z,$$
$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right) \ , \ \sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right),$$
$$(\sin z)' = \cos z \ , \ (\cos z)' = -\sin z.$$

For $z = x + iy$ we have by 1.46

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y),$$

since
$$|e^{iy}|^2 = e^{iy}\overline{e^{iy}} = (\cos y + i\sin y)(\cos y - i\sin y) = e^{iy}e^{-iy} = 1,$$
we get $|e^{iy}| = 1$ and $\cos^2 y + \sin^2 y = 1$ for all $y \in \mathbb{R}$. In addition
$$\Re e^z = e^x \cos y \ , \ \Im e^z = e^x \sin y \ , \ |e^z| = |e^x(\cos y + i\sin y)| = e^x = e^{\Re z},$$
since $|\cos y + i\sin y| = 1$.
The expression
$$e^z = e^x(\cos y + i\sin y)$$
can be interpreted as polar representation of the exponential function, and we have
$$\arg e^z = \Im z.$$

Next we will determine the zeroes of the sine and cosine function.
First we claim that there are positive real numbers x such that $\cos x = 0$. Suppose that this is not true. Then, since $\cos 0 = 1$, we have $\cos x > 0$ for each $x > 0$. Hence we have for the derivative $\sin' x = \cos x > 0$ and the function sin would be strictly increasing. As $\sin 0 = 0$, we would have $\sin x > 0$ for $x > 0$. This would imply that for $0 < x_1 < x_2$:
$$(x_2 - x_1)\sin x_1 < \int_{x_1}^{x_2} \sin t\, dt = \cos x_1 - \cos x_2 \leq 2.$$
Since we have $\sin x > 0$, the last inequality gives a contradiction if x_2 is sufficiently large.
The zero set of the continuous function cos is closed and $\cos 0 \neq 0$, hence there exists a smallest positive number x_0 with $\cos x_0 = 0$.
We define the number π by
$$\pi := 2x_0.$$
Then $\cos(\pi/2) = 0$ and since $\cos^2 x + \sin^2 x = 1$ this implies $\sin(\pi/2) = \pm 1$. As $\cos x > 0$ in $(0, \pi/2)$, the function sin is increasing in $(0, \pi/2)$. Hence $\sin(\pi/2) = 1$. This implies
$$\exp(\frac{\pi}{2}\, i) = i.$$
By 1.46, we get
$$\exp(\pi i) = -1 \quad , \quad \exp(2\pi i) = 1.$$
In this way we get that all real zeroes of the function cos are of the form $\{(2k+1)\pi/2 : k \in \mathbb{Z}\}$ and all real zeroes of the function sin are of the form $\{k\pi : k \in \mathbb{Z}\}$.
Suppose that $\cos z = 0$. Set $z = x + iy$ and recall that $2\cos z = e^{iz} + e^{-iz}$. It follows that $e^{2iz} + 1 = 0$, which implies $e^{-2y}\cos 2x = -1$, $e^{-2y}\sin 2x = 0$. Hence $\cos z = 0$ implies that z must be real and cos has no other zeroes. Also sin has no other zeroes.

Remark. (a) The mapping $\varphi : t \mapsto e^{it}$ is a homomorphism between the additive group $(\mathbb{R}, +)$ and the multiplicative group $S^1 = \{z \in \mathbb{C} : |z| = 1\}$, since we have

$$\varphi(t_1 + t_2) = \varphi(t_1)\varphi(t_2)\ ,\ t_1, t_2 \in \mathbb{R}.$$

(b) The mapping $\psi : z \mapsto e^z$ is a homomorphism between the additive group $(\mathbb{C}, +)$ and the multiplicative group $(\mathbb{C}^* = \mathbb{C} \setminus \{0\}, .)$:

$$\psi(z_1 + z_2) = \psi(z_1)\psi(z_2)\ ,\ \forall z_1, z_2 \in \mathbb{C}.$$

ψ is surjective : let $w \in \mathbb{C}^*$, each $z = x + iy$ with $x = \log|w|$ and $y = \arg w$ is the preimage of w, since we have $e^z = e^x e^{iy} = e^{\log|w|} e^{i \arg w} = |w| e^{i \arg w}$, where the last expression corresponds to the polar representation of w.
ψ fails to be injective. $\mathrm{Ker}\psi = \{z : \psi(z) = 1\} = \{2\pi i k : k \in \mathbb{Z}\} = 2\pi i\mathbb{Z}$, since $1 = e^z = e^x(\cos y + i \sin y)$ implies $x = 0$ and $y = 2\pi k\ ,\ k \in \mathbb{Z}$.
The exponential function is periodic with period $2\pi i$, i.e. $\exp(z + 2\pi i) = \exp(z)$.
(c) Each strip $\{z \in \mathbb{C} : a \le \Im z < a + 2\pi\}\ ,\ a \in \mathbb{R}$, is mapped onto $\mathbb{C}^*$ by the exponential function. This follows from (b).

In the following we introduce some other elementary functions related to the exponential function.

Definition 1.48.

$$\tan z = \frac{\sin z}{\cos z}\ ,\ z \neq (k + 1/2)\pi\ ,\ k \in \mathbb{Z}$$
$$\cot z = \frac{\cos z}{\sin z}\ ,\ z \neq k\pi\ ,\ k \in \mathbb{Z}.$$

We have

$$\tan z = \frac{1}{i}\frac{e^{2iz} - 1}{e^{2iz} + 1}\ ,\ \cot z = i\frac{e^{2iz} + 1}{e^{2iz} - 1},$$
$$(\tan z)' = \frac{1}{\cos^2 z}\ ,\ (\cot z)' = -\frac{1}{\sin^2 z}.$$

Definition 1.49.

$$\cosh z = \frac{1}{2}(e^z + e^{-z})\ , \sinh z = \frac{1}{2}(e^z - e^{-z}).$$

We have

$$\cosh z = \cos iz\ ,\ \sinh z = \frac{1}{i}\sin iz\ ,\ \cos z = \cosh iz\ ,\ \sin z = \frac{1}{i}\sinh iz.$$

Now we investigate the inverse functions of the elementary functions introduced above.

Definition 1.50. For each $z \in \mathbb{C}^*$ there are infinitely many $w \in \mathbb{C}$ with $e^w = z$. Each of these values w is called a logarithm of z.
Each logarithm has the form

$$w = \log|z| + i\arg z.$$

Two logarithms of z differ by an entire multiple of $2\pi i$.
Let $\mathbb{C}^- = \mathbb{C} \setminus \{z \in \mathbb{C} : \Im z = 0\ ,\ \Re z \leq 0\}$ be the slit plane. For each $z \in \mathbb{C}^-$ we have a unique representation of $z = |z|e^{i\varphi}$ with $-\pi < \varphi < \pi$. We define

$$\mathrm{Log} z := \log|z| + i\varphi\ ,\ z \in \mathbb{C}^-$$

as the principal branch of the logarithm.

Example $\mathrm{Log}\, i = i\pi/2$

Theorem 1.51. *Let $G_0 = \{z \in \mathbb{C} : -\pi < \Im z < \pi\}$. Then the exponential function $exp : G_0 \longrightarrow \mathbb{C}^-$ is holomorphic and bijective, the inverse function is the principal branch of the logarithm, it is a map from $\mathbb{C}^-$ onto G_0, which is also holomorphic and bijective.*

Proof. For $z = x + iy \in G_0$ we observe that $w = e^z = e^x e^{i\Im z}$ belongs to the domain of the principal branch of the logarithm, since $|w| = e^x > 0$ and $\arg w = \Im z$. Then we have

$$\mathrm{Log}(\exp z) = x + i\Im z = x + iy = z.$$

If $w \in \mathbb{C}^-$, then

$$\exp(\mathrm{Log} w) = \exp(\log|w| + i\arg w) = |w|e^{i\arg w} = w.$$

Hence exp and Log are inverse functions to each other, hence both of them are bijective.
Since $(\exp z)' = \exp z \neq 0\ , \forall z \in \mathbb{C}$, it follows from the differentiation rule for inverse functions (see 1.8) that the principal branch of the logarithm Log is holomorphic on $\mathbb{C}^-$, and we have

$$(\mathrm{Log}\ w)' = \frac{1}{w}\ ,\ w \in \mathbb{C}^-.$$

□

Remark. One has to be careful when using the functional equation which is valid for the real logarithm. The following lines show that one must stay in the domain of the branch of the logarithm when using the functional equation:

$$\log(-1) = \log(i \cdot i) = \log i + \log i = i\pi/2 + i\pi/2 = i\pi,$$

but

$$0 = \log 1 = \log((-1)(-1)) = \log(-1) + \log(-1) = 2\log(-1) \Rightarrow \log(-1) = 0.$$

Definition 1.52. Let $z \in \mathbb{C}^-$ and $a \in \mathbb{C}$. We define

$$z^a := \exp(a\mathrm{Log}z).$$

The chain rule 1.7 shows that

$$(z^a)' = az^{a-1}.$$

EXAMPLES. $1^a = \exp(a\,\mathrm{Log}\,1) = e^0 = 1\ ,\ \forall a \in \mathbb{C}$.
The function $z \mapsto z^{1/2} = \exp(1/2\,\mathrm{Log}\,z)$ for $z \in \mathbb{C}^-$ is the principal branch of the complex root function.
In addition we have

$$i^i = \exp(i\,\mathrm{Log}\,i) = \exp(i^2\pi/2) = \exp(-\pi/2) = 0,208...$$

and

$$\sqrt{i} = i^{1/2} = \exp(1/2\,\mathrm{Log}\,i) = \exp(i\pi/4) = \frac{\sqrt{2}}{2}(1+i).$$

Also here one has to be careful when using arithmetic rules like

$$(z^\alpha)^\beta = z^{\alpha\beta},$$

which is to be understood as an equality of sets. Otherwise it can lead to absurd conclusions: let $z \neq 0$ and set $z = e^\alpha$ for a certain $\alpha \in \mathbb{C}$; let $\beta = \alpha/2\pi i$. Then $z = e^\alpha = e^{2\pi i\beta} = (e^{2\pi i})^\beta = 1^\beta = 1$.

1.11 Exercises

1) Determine the real and imaginary part and the absolute value of

$$\frac{2}{1-3i},\quad (1+i\sqrt{3})^6,\quad \left(\frac{1+i}{1-i}\right)^5,\quad \left(\frac{1+i\sqrt{3}}{1-i}\right)^4.$$

2) Determine all complex numbers z, for which $\overline{z} = z^2$.

3) Show that for $|z| = r > 0:$

$$\Re z = \frac{1}{2}\left(z + \frac{r^2}{z}\right),\quad \Im z = \frac{1}{2i}\left(z - \frac{r^2}{z}\right).$$

4) Prove the identity

$$|z_1+z_2|^2 + |z_1-z_2|^2 = 2(|z_1|^2+|z_2|^2),$$

and explain its geometric meaning.

5) Determine the absolute value and the principal argument of

$$2.718 - 3.010i, \quad \frac{3+2i}{5i-4}, \quad 3\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right).$$

6) Let $z_0 = \frac{\sqrt{3}+i}{-1-i}$. Compute z_0^{123}.

7) Compute all eighth roots of $256(\cos 80^\circ + i\sin 80^\circ)$.

8) Determine all real numbers x and y, for which the following equations hold:
(i) $2(x+iy) = (x+iy)^2$,
(ii) $|2-(x-iy)| = x+iy$,
(iii) $\frac{x-iy}{x+iy} = i$.

9) Give a geometric description of the sets of all points determined by the following relations:
(i) $|z-2+3i| < 5$,
(ii) $\Im z \geq \Re z$,
(iii) $|z-i| + |z-1| = 2$,
(iv) $\Re z = |z-2|$.

10) Let $|w| < 1$. Show:

$$\left|\frac{z-w}{\overline{w}z-1}\right| < 1, \quad \text{for } |z| < 1,$$

and

$$\left|\frac{z-w}{\overline{w}z-1}\right| = 1, \quad \text{for } |z| = 1.$$

11) Let $f(z) = U(r,\theta) + iV(r,\theta)$, where $z = r(\cos\theta + i\sin\theta)$. Let f be complex differentiable in $z_0 \neq 0$. Show that in z_0 we have

$$\frac{\partial U}{\partial r} = \frac{1}{r}\frac{\partial V}{\partial \theta} \quad \text{and} \quad \frac{\partial V}{\partial r} = -\frac{1}{r}\frac{\partial U}{\partial \theta}.$$

These are the Cauchy–Riemann differential equations in polar coordinates.

12) Let $f(z) = u(x,y) + iv(x,y)$ and suppose that f satisfies the Cauchy–Riemann differential equations in an open set $G \subseteq \mathbb{C}$. Let $\overline{G}$ the set in $\mathbb{C}$ obtained by reflection of G on the real axis, i.e. $(x,y) \in \overline{G}$, if $(x,-y) \in G$. Define a function g on $\overline{G}$ by

$$g(z) = \overline{f(\overline{z})}, \quad z \in \overline{G}.$$

Show that g satisfies the Cauchy–Riemann differential equations in $\overline{G}$.

13) Suppose that the function f satisfies the Cauchy–Riemann differential equations for $|z| < R$. Define g by

$$g(z) = \overline{f(R^2/\overline{z})}, \quad |z| > R.$$

Show that g satisfies the Cauchy–Riemann differential equations for $|z| > R$. (Use polar coordinates!)

14) Let $a, b, c \in \mathbb{R}$. For $x, y \in \mathbb{R}$ and $z = x + iy$ set $P(z) = ax^2 + 2bxy + cy^2$. Find a necessary and sufficient condition for the existence of a holomorphic function $f \in \mathcal{H}(\mathbb{C})$ such that $\Re(f) = P$.

15) Suppose that f is holomorphic in $\mathbb{C}$ holomorph and real-valued. Show that f is constant.

16) Let $f(z) = |z|^4 + (\Im z)^2$. Compute : f_z and $f_{\overline{z}}$.

17) Let f be a real differentiable function. Prove :

$$\frac{\partial f}{\partial z} = \overline{\frac{\partial \overline{f}}{\partial \overline{z}}}\ , \quad \frac{\partial f}{\partial \overline{z}} = \overline{\frac{\partial \overline{f}}{\partial z}}.$$

18) Let f be a real-valued, real differentiable function. Show

$$\frac{\partial f}{\partial z} = \overline{\frac{\partial f}{\partial \overline{z}}}.$$

19) Prove the following chain rules:

$$\frac{\partial (g \circ f)}{\partial z} = \frac{\partial g}{\partial w}\frac{\partial f}{\partial z} + \frac{\partial g}{\partial \overline{w}}\frac{\partial \overline{f}}{\partial z},$$

$$\frac{\partial (g \circ f)}{\partial \overline{z}} = \frac{\partial g}{\partial w}\frac{\partial f}{\partial \overline{z}} + \frac{\partial g}{\partial \overline{w}}\frac{\partial \overline{f}}{\partial \overline{z}}.$$

20) Let φ be a differentiable function of the real variable t. Show that :

$$\frac{d(f \circ \varphi)}{dt} = \frac{\partial f}{\partial z}\frac{d\varphi}{dt} + \frac{\partial f}{\partial \overline{z}}\frac{d\overline{\varphi}}{dt}.$$

21) Let

$$f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)\ , \quad z \in \mathbb{C} \setminus \{0\}.$$

Show that f is angle preserving on $\mathbb{C} \setminus \{-1, 0, 1\}$. Determine the images under f of the circles $|z| = r < 1$ and of the rays $z = ct, 0 < t < 1, |c| = 1$ and use these image curves in order to check the angle preservation of f.

22) Let

$$f_n(z) = \frac{1}{1 + az^n}\quad , a \neq 0.$$

Show that the sequence (f_n) converges to 1 uniformly on each compact subset of the open unit disc $D_1(0)$ and that (f_n) converges to 0 uniformly on each compact subset of $\mathbb{C} \setminus D_r(0)$, for $r > 1$.

23) Show that the series

$$\sum_{n=1}^{\infty} \frac{z^n}{(1-z^n)(1-z^{n+1})}$$

converges to $z(1-z)^{-2}$ uniformly on each compact subset of $D_1(0)$ and that it converges to $(1-z)^{-2}$ uniformly on each compact subset of $\mathbb{C} \setminus \overline{D_1(0)}$.

24) Where is the series

$$\sum_{n=1}^{\infty} \frac{z^n}{1-z^n},$$

uniformly convergent?

25) Let R_1 and R_2 be the radii of convergence of the power series

$$\sum_{n=0}^{\infty} a_n z^n \quad \textit{and} \quad \sum_{n=0}^{\infty} b_n z^n.$$

Show that the radius of convergence R of

$$\sum_{n=0}^{\infty} a_n b_n z^n$$

satisfies $R \geq R_1 R_2$; the radius of convergence R' of

$$\sum_{n=0}^{\infty} \frac{a_n}{b_n} z^n, \qquad b_n \neq 0, \quad n \in \mathbb{N}_0$$

satisfies $R' \leq R_1/R_2$; and the radius of convergence R_0 of

$$\sum_{n=0}^{\infty} (a_n b_0 + a_{n-1} b_1 + \cdots + a_0 b_n) z^n$$

satisfies $R_0 \geq \min(R_1, R_2)$.

26) Determine the radius of convergence of the following power series

$$\sum_{n=0}^{\infty} (-1)^n (z+i)^n, \qquad \sum_{n=0}^{\infty} (-1)^n 2^n z^{2n+2}, \qquad \sum_{n=1}^{\infty} n^{-1/2} z^n,$$

$$\sum_{n=1}^{\infty} \frac{2^n z^n}{(2n)!}, \qquad \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} z^n, \qquad \sum_{n=1}^{\infty} \frac{n!}{n^n} z^n,$$

$$\sum_{n=0}^{\infty} 2^{-n} z^{2^n}, \qquad \sum_{n=2}^{\infty} 2^{\log n} z^n, \qquad \sum_{n=0}^{\infty} (n + a^n) z^n,$$

$$\sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{n!} z^{2n}, \qquad \sum_{n=1}^{\infty} n^{-1} z^{3^n}, \qquad \sum_{n=1}^{\infty} \left[\frac{n^2}{(n+1)(n+2)}\right]^n z^n.$$

27) Suppose that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

has radius of convergence $R > 0$. Prove that

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}, \quad 0 < r < R.$$

Suppose that f is also bounded on $D_R(0)$, i.e.

$$|f(z)| \leq M, z \in D_R(0),$$

for some constant $M > 0$. Show that in this case

$$\sum_{n=0}^{\infty} |a_n|^2 R^{2n} \leq M^2.$$

Let $r < R$ and $M(r) = \sup_{0 \leq \theta \leq 2\pi} |f(re^{i\theta})|$. Prove that

$$|a_n| \leq r^{-n} M(r), \quad n \in \mathbb{N}_0.$$

28) Compute the following line integrals:

$$\int_{\gamma_1} \overline{z} \, dz, \quad \int_{\gamma_2} \overline{z} \, dz, \quad \int_{\gamma_3} z \, dz,$$

$$\int_{\gamma_4} \frac{dz}{z}, \quad \int_{\gamma_5} \overline{z} \, dz,$$

where γ_1 is the polygon from $(1,0)$ to $(0,0)$ and then to $(0,1)$, and γ_2 is the polygon from $(0,0)$ to $(1,1)$. γ_3 is the square with vertices $(0,0), (1,0), (1,1), (0,1)$ passed through once in positive direction, γ_4 is the unit circle $|z| = 1$ passed through once in positive direction, and γ_5 is a quater of a circle with vertices $(0,0), (2,0), (0,2)$ passed through once in positive direction.

29) Let Log denote the principal branch of the complex logarithm. Determine, for which $z, w \in \mathbb{C}$ one has

$$\mathrm{Log}(zw) = \mathrm{Log}\, z + \mathrm{Log}\, w.$$

30) Let $z \neq 0$. Determine $\alpha, \beta \in \mathbb{C}$ such that

$$(z^{\alpha})^{\beta} = z^{\alpha\beta}.$$

31) Show that the complex sine function sin maps the strip

$$\{z = x + iy : -\pi/2 < x < \pi/2, y \in \mathbb{R}\}$$

bijectively onto the region G obtained from the plane by deleting the two intervals $(-\infty, -1]$ and $[1, \infty)$. Determine the inverse function of sin on G in terms of the principal branch of the complex logarithm.

32) Determine the real and imaginary part of $\tan z$ and $\cot z$.

33) Show that for $z = x + iy$ one has

$$|\tan z|^2 = \frac{\sin^2 x + \sinh^2 y}{\cos^2 x + \sinh^2 y} \quad \text{and} \quad |\tanh z|^2 = \frac{\sinh^2 x + \sin^2 y}{\sinh^2 x + \cos^2 y}.$$

34) Compute the sum of the geometric series

$$\sum_{k=0}^{n} e^{ki\theta}$$

conclude from the result the sums of

$$\sum_{k=1}^{n} \sin k\theta \text{ and } \frac{1}{2} + \sum_{k=1}^{n} \cos k\theta.$$

2 Cauchy's theorem and Cauchy's formula

2.1 Winding numbers

The boundary of a domain in $\mathbb{C}$ may be unusually complicated. A natural assumption is that the boundary consists of one or several closed curves. For this purpose the Jordan curve theorem will be relevant. We recall some topological concepts.

Remark. Let $U \subseteq \mathbb{C}$ be an open set, $p, q \in U$. We say p is equivalent to q ($p \sim q$), if there exists a curve in U joining p with q. It is easily seen that $\sim$ defines an equivalence relation, the associated equivalence classes are the connected components of U. For $p \in U$ let U_p denote the equivalence class containing p. U_p is the largest connected subset of U containing p.

Theorem 2.1 (Jordan curve theorem). [1] *Let $\gamma : [a, b] \longrightarrow \mathbb{C}$ be a closed Jordan curve in $\mathbb{C}$, i.e. $\gamma(a) = \gamma(b)$, but $\gamma(s) \neq \gamma(t)$ for any $s, t \in (a, b), s \neq t$. Then the open set $\mathbb{C} \setminus \gamma^*$ has two connected components, a bounded one and an unbounded one. The bounded component is called the interior of γ, the unbounded one is called the exterior of γ. The bounded one is simply connected and γ^* is the boundary of each of the components.*

The proof is lengthy and difficult, see (1).

The next result leads to the concept of the winding number.

Theorem 2.2. *Let γ be a closed path and let $\Omega = \mathbb{C} \setminus \gamma^*$. Let*

$$\operatorname{Ind}_\gamma(z) := \frac{1}{2\pi i} \int_\gamma \frac{d\zeta}{\zeta - z} \ , \ z \in \Omega.$$

Then $\operatorname{Ind}_\gamma$ is an integer-valued function on Ω, this means $\operatorname{Ind}_\gamma(\Omega) \subseteq \mathbb{Z}$, and $\operatorname{Ind}_\gamma$ is constant on each connected component of Ω, in addition, $\operatorname{Ind}_\gamma(z) = 0$ for each z belonging to the unbounded connected component of Ω.
$\operatorname{Ind}_\gamma(z)$ is called the winding number of γ with respect to z.

Remark. The set γ^* is compact, hence there exists a disc D such that $\gamma^* \subset D$. The set $\mathbb{C} \setminus D$ is connected and is therefore contained in a connected component of Ω. Hence Ω has exactly one unbounded connected component.

In order to prove the theorem from above we need the following

1 Jordan, Camille (1838–1921)

Lemma 2.3. *Let $\phi, \psi : [a,b] \longrightarrow \mathbb{C}$ be two curves in $\mathbb{C}$. Let $\Omega \subset \mathbb{C}$ be an open subset of $\mathbb{C}$ such that $\Omega \cap \phi^* = \emptyset$. Let*

$$f(z) = \int_a^b \frac{\psi(t)}{\phi(t) - z}\, dt \ , \ z \in \Omega.$$

Then f is a holomorphic function on Ω.

Remark. Later on we will prove another general result on a holomorpphic parameter integral, see 2.39.

Proof. We consider an arbitrary $w \in \Omega$. then there exists $r > 0$ such that $D_r(w) \subseteq \Omega$. we will show that

$$f(z) = \sum_{n=0}^{\infty} c_n (z-w)^n,$$

where the sum is uniformly convergent on each compact subset of $D_r(w)$. Then, by 1.44, then the function f is holomorphic on $D_r(w)$ and as $w \in \Omega$ was an arbitrary point, we conclude that f is holomorphic on Ω.
For this aim we take $z \in D_{r/2}(w)$ and observe that the denominator in the integral can be written as

$$\begin{aligned} \frac{1}{\phi(t)-z} &= \frac{1}{\phi(t)-w} \, \frac{\phi(t)-w}{\phi(t)-w-(z-w)} = \frac{1}{\phi(t)-w} \, \frac{1}{1 - \frac{z-w}{\phi(t)-w}} \\ &= \frac{1}{\phi(t)-w} \sum_{n=0}^{\infty} \left(\frac{z-w}{\phi(t)-w} \right)^n = \sum_{n=0}^{\infty} \frac{(z-w)^n}{(\phi(t)-w)^{n+1}}, \end{aligned}$$

since $|\phi(t) - w| > r \ , \ \forall t \in [a,b]$ and $\Omega \cap \phi^* = \emptyset$, the last sum converges uniformly for all $t \in [a,b]$.
Hence

$$\begin{aligned} f(z) &= \int_a^b \left[\sum_{n=0}^{\infty} \frac{(z-w)^n}{(\phi(t)-w)^{n+1}} \right] \psi(t)\, dt = \sum_{n=0}^{\infty} \int_a^b \frac{(z-w)^n}{(\phi(t)-w)^{n+1}} \psi(t)\, dt \\ &= \sum_{n=0}^{\infty} \left[\int_a^b \frac{\psi(t)}{(\phi(t)-w)^{n+1}}\, dt \right] (z-w)^n = \sum_{n=0}^{\infty} c_n (z-w)^n, \end{aligned}$$

where

$$|c_n| = \left| \int_a^b \frac{\psi(t)}{(\phi(t)-w)^{n+1}}\, dt \right| \le \frac{b-a}{r^{n+1}} \max_{t \in [a,b]} |\psi(t)|.$$

Now the series

$$f(z) = \sum_{n=0}^{\infty} c_n (z-w)^n$$

converges uniformly for z in an arbitrary compact subset of $D_r(w)$, because we can estimate

$$\sum_{n=0}^{\infty} |c_n| \, |(z-w)^n| \le C \sum_{n=0}^{\infty} \rho^n,$$

for some $\rho < 1$. $\square$

Proof of Theorem 2.2. Since

$$\mathrm{Ind}_\gamma(z) = \frac{1}{2\pi i} \int_\gamma \frac{d\zeta}{\zeta - z} = \frac{1}{2\pi i} \int_a^b \frac{\gamma'(s)}{\gamma(s) - z} \, ds \ , \ z \in \Omega,$$

we have $\mathrm{Ind}_\gamma(z) \in \mathbb{Z} \Leftrightarrow \Phi(b) = 1$, where

$$\Phi(t) = \exp\left[\int_a^t \frac{\gamma'(s)}{\gamma(s) - z} \, ds \right],$$

and we used the fact that

$$\frac{w}{2\pi i} \in \mathbb{Z} \Leftrightarrow e^w = 1,$$

see 1.47. Now we compute the logarithmic derivative of $\Phi(t)$:

$$\frac{\Phi'(t)}{\Phi(t)} = \frac{\gamma'(t)}{\gamma(t) - z},$$

excluding the finitely many points $a = s_0 < s_1 < \cdots < s_n = b$, where γ possibly fails to be differentiable. In addition we have

$$\Phi'(t)(\gamma(t) - z) - \Phi(t)\gamma'(t) = 0,$$

hence

$$\left(\frac{\Phi(t)}{\gamma(t) - z} \right)' = \frac{\Phi'(t)(\gamma(t) - z) - \Phi(t)\gamma'(t)}{(\gamma(t) - z)^2} = 0$$

except for finitely many points. On the other hand, the function

$$t \mapsto \frac{\Phi(t)}{\gamma(t) - z}$$

is continuous on $[a, b]$, which follows from the fundamental theorem of calculus, hence it must be constant on $[a, b]$. As $\Phi(a) = e^0 = 1$, it follows that

$$\frac{1}{\gamma(a) - z} = \frac{\Phi(a)}{\gamma(a) - z} = \frac{\Phi(t)}{\gamma(t) - z} \ , \ t \in [a, b]$$

and

$$\Phi(t) = \frac{\gamma(t) - z}{\gamma(a) - z} \ , \ t \in [a, b].$$

Since γ is a closed path, we have $\gamma(a) = \gamma(b)$, and hence $\Phi(b) = 1$, and $\operatorname{Ind}_\gamma(z) \in \mathbb{Z}$. By Lemma 2.3, the function

$$z \mapsto \operatorname{Ind}_\gamma(z) = \frac{1}{2\pi i} \int_a^b \frac{\gamma'(s)}{\gamma(s) - z} \, ds$$

is holomorphic on Ω. Since the image of connected sets under continuous maps is again connected, we obtain that the function $\operatorname{Ind}_\gamma$ is constant on the connected components of Ω.

From

$$\operatorname{Ind}_\gamma(z) = \frac{1}{2\pi i} \int_a^b \frac{\gamma'(s)}{\gamma(s) - z} \, ds$$

we conclude that $|\operatorname{Ind}_\gamma(z)| < 1$, if $|z|$ is sufficiently large. Hence $\operatorname{Ind}_\gamma(z) = 0$ on the unbounded connected component of Ω. $\square$

In the following we explain why $\operatorname{Ind}_\gamma(z)$ is called a winding number. For this purpose let

$$\lambda(t) = \int_a^t \frac{\gamma'(s)}{\gamma(s) - z} \, ds.$$

Then $\lambda(b) = 2\pi i \ \operatorname{Ind}_\gamma(z)$ and hence $\Im\lambda(b) = 2\pi \ \operatorname{Ind}_\gamma(z)$. Using the same notation as in the proof from above we get

$$\exp(\lambda(t)) = \Phi(t) = \frac{\gamma(t) - z}{\gamma(a) - z}$$

and $\arg\Phi(b) = \Im\lambda(b) = 2\pi \ \operatorname{Ind}_\gamma(z)$. Now we set $z = 0$. We have $\arg\Phi(a) = 0$ and

$$\arg\Phi(t) = \arg\gamma(t) - \arg\gamma(a).$$

This means, if t runs through the interval $[a, b]$, the expression $\operatorname{Ind}_\gamma(0)$ counts how many times γ turns around $z = 0$.

For a circle we get

Theorem 2.4. *Let $\gamma(t) = re^{2\pi i t} + a \ , \ t \in [0, 1] \ , \ a \in \mathbb{C}$. Then*

$$\operatorname{Ind}_\gamma(z) = \begin{cases} 1 & \text{for} \quad |z - a| < r \\ 0 & \text{for} \quad |z - a| > r. \end{cases}$$

Proof. From example (1) in 1.35 we know that

$$\frac{1}{2\pi i}\int_{\gamma}\frac{d\zeta}{\zeta - a} = 1.$$

Now we can apply 2.2 to finish the proof. □

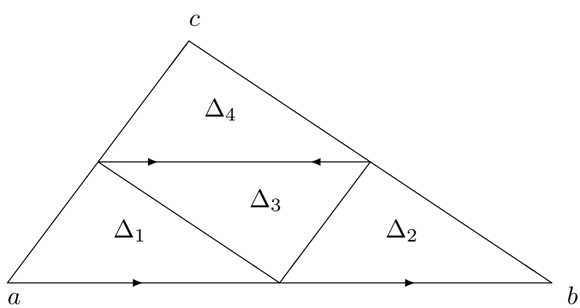
2.2 The Theorem of Cauchy–Goursat and Cauchy's Formula

The following result was first formulated by C. Goursat in full generality with a small gap in its proof (see ()), which was closed by A. Pringsheim one year later (()). Pringsheim's proof, which uses the method of shrinking triangles, became the classical one, still usual nowadays. Later on, we will prove a more general result, which is derived from Stokes' Theorem from real analysis (see section). The Theorem of Cauchy–Goursat and Cauchy's Formula yield powerful methods for the study of holomorphic functions, having no similar counterpart in real analysis.

Theorem 2.5. *Let $\Omega \subseteq \mathbb{C}$ be an open subset and fix a point $p \in \Omega$. Let $f : \Omega \longrightarrow \mathbb{C}$ be a continuous function and suppose that $f \in \mathcal{H}(\Omega \setminus \{p\})$. Let Δ be an open triangle with $\overline{\Delta} \subset \Omega$. Then*

$$\int_{\partial\Delta} f(z)\,dz = 0.$$

Proof. (by A. Pringsheim)
First we suppose that $p \notin \Delta$.
We divide Δ into 4 area equal triangles $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ as in the figure from below.



Then

$$\left|\int_{\partial\Delta} f(z)\,dz\right| = \left|\sum_{j=1}^{4}\int_{\partial\Delta_j} f(z)\,dz\right| \leq 4\max_{1\leq j\leq 4}\left|\int_{\partial\Delta_j} f(z)\,dz\right|.$$

The paths in the interior of the large triangle cancel each other out.
We denote the tringle where the maximum is attained by $\Delta^{(1)}$ and repeat the procedure from above for the triangle $\Delta^{(1)}$ instead of Δ obtaining a triangle $\Delta^{(2)}$, and so on.
In this way we get a sequence of triangles $\Delta \supset \Delta^{(1)} \supset \Delta^{(2)} \supset \dots$ and after n steps the inequality

$$\left|\int_{\partial\Delta} f(z)\,dz\right| \leq 4^n \left|\int_{\partial\Delta^{(n)}} f(z)\,dz\right|.$$

Let $L(\partial\Delta^{(n)})$ be the circumference of $\partial\Delta^{(n)}$. Then we have

$$L(\partial\Delta^{(n)}) = 2^{-1}L(\partial\Delta^{(n-1)}) = \cdots = 2^{-n}L(\partial\Delta).$$

Next we show that there exists a uniquely determined point $z_0 \in \bigcap_{n=1}^{\infty}\Delta^{(n)}$.
For this aim we choose a sequence $(z_n)_n$ with $z_n \in \Delta^{(n)}$, $n \in \mathbb{N}$. Since Δ is compact, there exists a limit point $z_0 \in \Delta$ of the sequnce $(z_n)_n$, in addition, z_0 also belongs to $\Delta^{(m)}$, $m \in \mathbb{N}$, which follows from the fact that the sequence $(z_n)_{n=m+1}^{\infty} \subset \Delta^{(m)}$. Hence we have $z_0 \in \bigcap_{n=1}^{\infty}\Delta^{(n)}$. z_0 is uniquely determined, since $L(\partial\Delta^{(n)}) \to 0$ as $n \to \infty$.
The function f is complex differentiable in z_0. Hence, by 1.6, we can write

$$f(z) = f(z_0) + (z - z_0)(f'(z_0) + B(z)),$$

wherei B is continuous in z_0 and $B(z_0) = 0$. The function $z \mapsto f(z_0) + (z - z_0)f'(z_0)$ has a primitive (by formal integration), hence, by Corollary 1.38,

$$\int_{\partial\Delta^{(m)}} (f(z_0) + (z - z_0)f'(z_0))\,dz = 0 \quad \forall m \in \mathbb{N}.$$

Now we have

$$\begin{aligned}\left|\int_{\partial\Delta^{(m)}} f(z)\,dz\right| &= \left|\int_{\partial\Delta^{(m)}} (z - z_0)B(z)\,dz\right| \leq L(\partial\Delta^{(m)}) \max_{z\in\partial\Delta^{(m)}}[|z - z_0|\ |B(z)|]\\ &\leq [L(\partial\Delta^{(m)})]^2 \max_{z\in\partial\Delta^{(m)}} |B(z)|,\end{aligned}$$

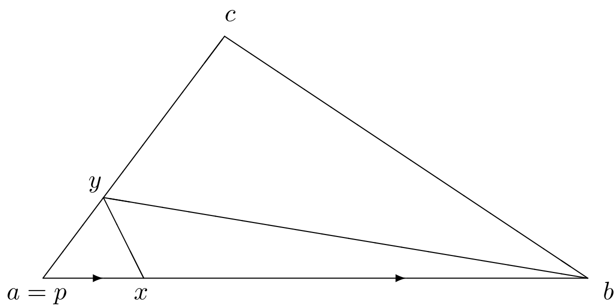
where we estimated $|z - z_0|$, $z \in \partial\Delta^{(m)}$ by the circumference $L(\partial\Delta^{(m)})$.
From this we get for the original integral

$$\left|\int_{\partial\Delta} f(z)\,dz\right| \leq 4^m\ (2^{-m})^2[L(\partial\Delta)]^2 \max_{z\in\partial\Delta^{(m)}} |B(z)| = [L(\partial\Delta)]^2 \max_{z\in\partial\Delta^{(m)}} |B(z)|.$$

As B is continuous in z_0 and $B(z_0) = 0$, the expression $\max_{z \in \partial\Delta^{(m)}} |B(z)|$ tends to 0 as $m \to \infty$, and hence

$$\int_{\partial\Delta} f(z)\, dz = 0.$$

If p is a corner point of Δ, for instance $p = a$, then we divide Δ as in the figure from below



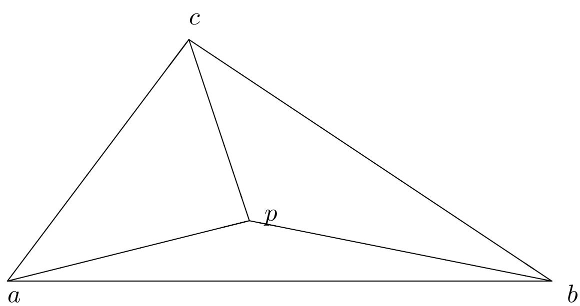
For the integral we obtain

$$\int_{\partial\Delta} = \int_{\partial\{a,x,y\}} + \int + \ldots,$$

where we only have to handle the first summand, because the other integrals do not contain the point p and are therefore 0 by the first step of the proof. For the first summand we can approach x and y arbitrarily close to a, by which

$$\int_{\partial\{a,x,y\}} f(z)\, dz \to 0$$

as f is continuous.
If p lies in the interior of Δ, we can use the following decomposition of the triangle



to reduce everyting to the second step of the proof. $\square$

Corollary 2.6. *Let Ω a convex open subset of $\mathbb{C}$, $p \in \Omega$ and let $f : \Omega \longrightarrow \mathbb{C}$ be a continuous function with $f \in \mathcal{H}(\Omega \setminus \{p\})$. Then f has a primitive on Ω and*

$$\int_\gamma f(z)\,dz = 0,$$

for each closed path γ in Ω.

Proof. By 2.5, we have

$$\int_{\partial\Delta} f(z)\,dz = 0$$

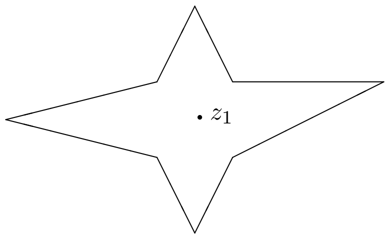
for each triangle Δ in Ω. Now, by 1.42, there exists a primitive of f on Ω and finally, by 1.38.

$$\int_\gamma f(z)\,dz = 0$$

for each closed path in Ω. □

Definition 2.7. An open subset $M \subseteq \mathbb{C}$ is called a starshaped domain, if there exists a point $z_1 \in M$ (called a centre), such that for each $z \in M$ the straight line segment $[z_1, z]$ joining the points z_1 and z is contained in M.

EXAMPLES.
(a) The figure from below shows a starshaped domain with centre z_1.



(b) $\mathbb{C}^-$: here the centre is to be chosen on the positive semi axis.

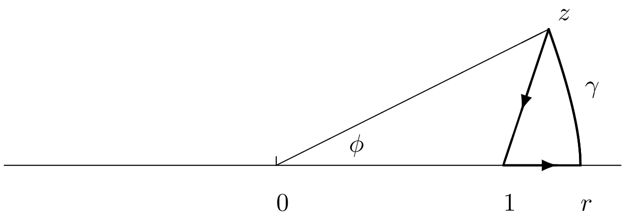
Theorem 2.8. *Let G be a starshaped domain, let c be a centre for G and $f \in \mathcal{H}(G)$. Then f has a primitive on G and*

$$\int_\gamma f(z)\,dz = 0,$$

for each closed path γ in G.

Proof. Theorem 1.42 is also valid for starshaped domains as is easily seen. Hence we can use the same proof as in 2.6. □

Example. We consider $\mathbb{C}^-$ with 1 as centre, we write $z = re^{i\phi} \in \mathbb{C}^-$, and take the closed path γ, as shown in the figure from below



By 2.8, we have

$$\int_\gamma \frac{d\zeta}{\zeta} = 0.$$

In addition, we can define $z \mapsto \int_{[1,z]} \frac{d\zeta}{\zeta}$ as a primitive for $z \mapsto 1/z$ on $\mathbb{C}^-$. Then

$$\int_{[1,z]} \frac{d\zeta}{\zeta} = \int_1^r \frac{dt}{t} + \int_0^\phi \frac{ire^{it}}{re^{it}}\, dt = \log r + i\phi.$$

In this way we have found the principal branch of the logarithm.

In the following we study integral representations of holomorphic functions. Cauchy's formula is the prototype of an integral representation. It will allow us to estimate the size of the holomorphic function involved, to show that all derivatives of holomorphic functions are again holomorphic and to obtain power series expansions of holomorphic functions.

Theorem 2.9 (Cauchy's formula). *Let Ω be a convex domain in $\mathbb{C}$, let γ be a closed path in Ω, and let $z \in \Omega$, $z \notin \gamma^*$. Suppose that $f \in \mathcal{H}(\Omega)$. Then*

$$f(z)\ \mathrm{Ind}_\gamma(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

Proof. Let

$$g(\zeta) = \begin{cases} \dfrac{f(\zeta) - f(z)}{\zeta - z} & , \zeta \neq z \\ f'(z) & , \zeta = z. \end{cases}$$

Then g is continuous on Ω and $g \in \mathcal{H}(\Omega \setminus \{z\})$. Hence g satisfies the assumptions of 2.6. So we have

$$\frac{1}{2\pi i} \int_\gamma g(\zeta)\, d\zeta = 0.$$

Finally we get

$$\frac{1}{2\pi i} \int_\gamma \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta - f(z) \frac{1}{2\pi i} \int_\gamma \frac{d\zeta}{\zeta - z} = 0.$$

$\square$

We mention an important special case. Let $\gamma(t) = z + re^{it}$, $t \in [0, 2\pi]$, $f \in \mathcal{H}(D_R(z))$, $R > r$. Then $\mathrm{Ind}_\gamma(z) = 1$ and

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

In addition, we obtain the important estimate

$$|f(z)| = \frac{1}{2\pi} \left| \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta \right| \leq \frac{1}{2\pi}\ 2\pi r \max_{\zeta \in \gamma^*} \left| \frac{f(\zeta)}{\zeta - z} \right| = \max_{\zeta \in \gamma^*} |f(\zeta)|\,.$$

Plugging in for the line integral we obtain

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z + re^{it})}{re^{it}}\ ire^{it}\, dt = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{it})\, dt,$$

which is the mean value property of holomorphic functions.

Examples. Using Cauchy's formula we compute the following line integral:

$$I = \int_\gamma \frac{e^z}{(z-1)(z-2)}\, dz.$$

(a) $G = D_{3/4}(0)$, $f(z) = \frac{e^z}{(z-1)(z-2)}$, $f \in \mathcal{H}(G)$, $\gamma(t) = \frac{e^{it}}{2}$, $t \in [0, 2\pi] : I = 0$, by 2.6

(b) $G = D_{11/6}(0)$, $f(z) = \frac{e^z}{z-2}$, $f \in \mathcal{H}(G)$, $\gamma(t) = \frac{3e^{it}}{2}$, $t \in [0, 2\pi]$: by 2.9, we have

$$f(1) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - 1}\, d\zeta = \frac{1}{2\pi i} \int_\gamma \frac{e^\zeta}{(\zeta - 2)(\zeta - 1)}\, d\zeta$$

Hence $I = -2\pi i e$.

(c) $G = D_4(0)$, $f(z) = e^z$, $f \in \mathcal{H}(G)$, $\gamma(t) = 3e^{it}$, $t \in [0, 2\pi]$: by 2.9 we have

$$I = \int_\gamma \frac{e^z}{z-2}\,dz - \int_\gamma \frac{e^z}{z-1}\,dz = \int_\gamma \frac{f(z)}{z-2}\,dz - \int_\gamma \frac{f(z)}{z-1}\,dz = 2\pi i(e^2 - e).$$

Theorem 2.10 (Taylor series expansion). [2] *Let $\Omega \subseteq \mathbb{C}$ be an open set and let $a \in \Omega$, $R > 0$ be such that $D_R(a) \subseteq \Omega$. Suppose that $f \in \mathcal{H}(\Omega)$. Then*

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n,$$

where the sum converges uniformly on all compact subsets of $D_R(a)$ and

$$a_n = \frac{f^{(n)}(a)}{n!} \ , \ n = 0, 1, 2, \ldots,$$

a_n are the Taylor coefficients of f in the expansion around the point a.

Proof. Let $\gamma(t) = a + re^{it}$, $r < R$, $t \in [0, 2\pi]$. Then $\mathrm{Ind}_\gamma(z) = 1$, $\forall z \in D_r(a)$ (see 2.4). By 2.9, we have

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(\gamma(t))\gamma'(t)}{\gamma(t) - z}\,dt \ , \ \forall z \in D_r(a).$$

As in the proof of 2.3 we expand the last integral into a power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n,$$

where

$$a_n = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(\gamma(t))\gamma'(t)}{(\gamma(t) - a)^{n+1}}\,dt = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - a)^{n+1}}\,d\zeta$$

and the series converges uniformly on all compact subsets of $D_r(a)$. Since r was an arbitrary number such that $r < R$, we obtain the same assertion for $D_R(a)$.
By 1.44, we obtain the formula for the Taylor coefficients. □

The proof of the last theorem contains the following important results:

Theorem 2.11. *If $f \in \mathcal{H}(\Omega)$, then $f^{(n)} \in \mathcal{H}(\Omega)$, $\forall n \in \mathbb{N}$, f has complex derivatives of arbitrary order.*

2 Taylor, Brook (1685–1731)

Theorem 2.12. *Under the same assumptions as in* 2.10 *and for* $\gamma(t) = a + re^{it}$, $r < R$, $t \in [0, 2\pi]$, *we have*

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - a)^{n+1}}\, d\zeta \ , \ n \in \mathbb{N}.$$

We are now able to prove the converse of Cauchy's theorem.

Theorem 2.13 (Morera's theorem). [3] *Let* G *be an open set in* $\mathbb{C}$ *and let* $f : G \longrightarrow \mathbb{C}$ *be continuous on* G. *Suppose that*

$$\int_{\partial\Delta} f(z)\, dz = 0$$

for all solid triangles Δ *in* G. *Then* $f \in \mathcal{H}(G)$.

Proof. Let $V \subseteq G$ be an arbitrary convex subset of G. By 1.42, there exists a primitive F of f on V, we have that $F \in \mathcal{H}(V)$ and by 2.11, also that $F' = f \in \mathcal{H}(V)$. Since V was an arbitrary convex subset of G, we conclude that $f \in \mathcal{H}(G)$. $\square$

2.3 Important consequences of Cauchy's theorem

In the following we study the zero sets of holomorphic functions. Also in this context there is no analogue in the theory of real differentiable functions.

Theorem 2.14. *Let* $\Omega \subseteq \mathbb{C}$ *be a domain and* $f \in \mathcal{H}(\Omega)$. *We define the zero set of* f *by*

$$Z(f) = \{a \in \Omega : f(a) = 0\}.$$

Then $Z(f) = \Omega$ *or* $Z(f)$ *is a discreet subset of* Ω *(i.e.* $Z(f)$ *has no limit point in* Ω*). In the second case, for each* $a \in Z(f)$ *there exists a uniquely determined integer* $m_a \in \mathbb{N}$ *such that*

$$f(z) = (z - a)^{m_a} g(z),$$

where $g \in \mathcal{H}(\Omega)$ *and* $g(a) \neq 0$.
We say that a *is a zero of order* m_a *of* f.
In addition the zero set $Z(f)$ *is at most countably infinite.*

3 Morera, Giacinto (1856–1909)

Proof. Let A the set of all limit points of $Z(f)$ in Ω. Since f is continuous, we have $A \subseteq Z(f)$.
Fix $a \in Z(f)$. fix. There exists $r > 0$, such that $D_r(a) \subseteq \Omega$. By 2.10, we can expand f into a Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \quad , \ z \in D_r(a).$$

Now we can distinguish between two cases.
1.) All Taylor coefficients $a_n = 0$, which implies that $f \equiv 0$ in $D_r(a)$. Hence $D_r(a) \subseteq A$, i.e. a is an interior point of A.
2.) There exists a minimal $m \in \mathbb{N}$ with $a_m \neq 0$. Now we define a function g by

$$g(z) = \begin{cases} (z-a)^{-m} f(z) & \text{for } z \in \Omega \setminus \{a\} \\ a_m & \text{for } z = a. \end{cases}$$

Then we have $f(z) = (z-a)^m g(z)$, for all $z \in \Omega \setminus \{a\}$ und $g \in \mathcal{H}(\Omega \setminus \{a\})$. But as $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ in $D_r(a)$, we get

$$g(z) = \sum_{n=0}^{\infty} a_{n+m} (z-a)^n \quad , z \in D_r(a).$$

Hence $g \in \mathcal{H}(D_r(a))$ and, by the definition of g, we obtain $g \in \mathcal{H}(\Omega)$. Since $g(a) = a_m \neq 0$ and since g is continuous, there exists an open neighborhood U of a such that $g(z) \neq 0$ for all $z \in U$, which implies that $f(z) \neq 0$, for all $z \in U \setminus \{a\}$. Now we have shown that a is an isolated point in $Z(f)$, i.e. there exists a neighborhood U of a, containing no other point of $Z(f)$.
In summary, we can state the following: if $a \in A$, then all $a_n = 0$, because otherwise a would be an isolated point (by 2.); so, by 1., we have $D_r(a) \subseteq A$ and A is an open set. But A is the set of all limit points of $Z(f)$ and is therefore also closed in Ω, and the set $B = \Omega \setminus A$ is open.We have $\Omega = A \cup B$, which is a union of two disjoint open sets. We assumed that Ω is connected, hence $\Omega = A$ or $A = \emptyset$. If $A = \Omega$we have $f \equiv 0$ on Ω; if $A = \emptyset$ we get that $Z(f)$ is discreet in Ω. In this case, there are at most finitely many points of $Z(f)$ in each compact subset of Ω, otherwise a limit point of $Z(f)$ would belong to this compact subset and $A \neq \emptyset$. We can write Ω as a countable union of compact subsets of Ω :

$$\Omega = \bigcup_{n=1}^{\infty} (D_n(0) \cap \Omega_{1/n})^{-},$$

where

$$\Omega_{1/n} = \{z \in \Omega : \operatorname{dist}(z, \partial\Omega) > 1/n\},$$

hence $Z(f)$ is at most countably infinite. $\square$

The following results are easy but important consequences, which are also referred to as the identity principles.

Theorem 2.15. *Let Ω be a domain in $\mathbb{C}$, let $f, g \in \mathcal{H}(\Omega)$ and suppose that $f = g$ on a subset M of Ω, which has a limit point in M. Then $f = g$ on the whole of Ω.*

Proof. Let $\phi = f - g$. Then $\phi(z) = 0$ for all $z \in M$ and 2.14 iimplies that $\phi(z) = 0$ on the whole of Ω. □

Remark. If $f = g$ on a non-empty open subset of Ω, then $f = g$ on the whole of Ω. If Ω is not connected, we cannot draw this conclusion: let $\Omega = D_1(0) \cup D_1(3)$ and

$$f(z) = \begin{cases} 1, & z \in D_1(0) \\ 0, & z \in D_1(3). \end{cases}$$

Then $f \in \mathcal{H}(\Omega)$, $f = 0$ on the open subset $D_1(3)$ of Ω, but f is not identically zero on Ω.

Corollary 2.16. *Let G be a domain in $\mathbb{C}$, let $f \in \mathcal{H}(G)$ and suppose that $f^{(n)}(a) = 0,\ n = 0, 1, 2, \dots,$ for some $a \in G$. Then $f \equiv 0$ on G.*

Proof. Let $r > 0$ be such that $D_r(a) \subseteq G$. Then, by 2.10 , f can be expanded into a Taylor series

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n$$

for $z \in D_r(a)$. Since $f^{(n)}(a) = 0,\ n = 0, 1, 2, \dots,$ we have $f \equiv 0$ on $D_r(a)$. Now 2.15 implies that $f \equiv 0$ on G. □

Remark. The above result is false for $\mathcal{C}^\infty$– functions on $\mathbb{R}$. Let

$$f(x) = \begin{cases} e^{-1/x^2}, & x \in \mathbb{R},\ x \neq 0 \\ 0, & x = 0. \end{cases}$$

It follows that $f \in \mathcal{C}^\infty(\mathbb{R})$ and $f^{(n)}(0) = 0\ ,\ n = 0, 1, 2, \dots,$ but f is not identically zero on $\mathbb{R}$.

Definition 2.17. Let $U \subseteq \mathbb{C}$ be open and $f \in \mathcal{H}(U)$. We say that f has a holomorphic logarithm on U, if there exists a function $g \in \mathcal{H}(U)$ such that $f = \exp(g)$ on U.
We say that f has a holomorphic m-the root on U, if there exists a function $q \in \mathcal{H}(U)$ such that $q^m = f$ on U.

Remark. If $\exp(g) = f$, then $f \neq 0$ on U. In addition we have $f' = g' \exp(g) = g' f$. Hence $g' = f'/f$. This expression is called the logarithmic derivative of f.

Theorem 2.18. *Let G be a domain in $\mathbb{C}$, and $f \in \mathcal{H}(G)$ with $f \neq 0$ on G. Then the following assertions are equivalent:*
(1) f has a holomorphic logarithm on G;
(2) f'/f has a primitive on G.

Proof. Suppose (1). Then we have $g' = f'/f$. Hence f'/f has a primitive on G. Now suppose that (2) holds. Let $F \in \mathcal{H}(G)$ a primitive of f'/f on G. Set $h = f \exp(-F)$. Then

$$h' = f' \exp(-F) - fF' \exp(-F) = f' \exp(-F) - \frac{ff'}{f} \exp(-F) = 0.$$

Since $h \in \mathcal{H}(G)$ and $h' = 0$ on G, it follows that h is constant on G, i.e. $f = a \exp(F)$ for some $a \in \mathbb{C}$, $a \neq 0$. Now there exists $b \in \mathbb{C}$ with $e^b = a$. We define $\phi = F + b$, and observe that $\phi \in \mathcal{H}(G)$ and

$$\exp(\phi) = \exp(F + b) = a \exp(F) = f.$$

□

Theorem 2.19. *Let G be a star-shaped domain in $\mathbb{C}$, let $f \in \mathcal{H}(G)$, $f \neq 0$ on G. Then f has a holomorphic logarithm and a holomorphic m-th root on G.*

Proof. By 2.18, it suffices to show that f'/f has a primitive on G. Now we use 2.8 : if c is a centre for G, then

$$g(z) = \int_{[c,z]} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta + b\ ,\ f(c) = e^b$$

is a primitive of f'/f on G and we have $\exp(g) = f$.
$q = \exp(g/m)$ is the desired holomorphic m–th root. □

2.4 Isolated singularities

Definition 2.20. Let $\Omega \subseteq \mathbb{C}$ be an open set and $a \in \Omega$. We say that a function $f \in \mathcal{H}(\Omega \setminus \{a\})$ has an isolated singularity at a. If one can define f at a such that $f \in \mathcal{H}(\Omega)$, we say that a is a removable singularity of f..

Theorem 2.21. *Let $f \in \mathcal{H}(\Omega \setminus \{a\})$ and denote by $D_r'(a) = \{z : 0 < |z-a| < r\}$ the punctured disk. Suppose that there exists a constant $M > 0$ such that $|f(z)| \leq M$ for all $z \in D_r'(a)$. Then f has a removable singularity at a.*

Proof. Let

$$h(z) = \begin{cases} (z-a)^2 f(z), & z \in \Omega \setminus \{a\} \\ 0, & z = a. \end{cases}$$

We claim that $h \in \mathcal{H}(\Omega)$. For this aim we compute

$$h'(a) = \lim_{z \to a} \frac{h(z) - h(a)}{z - a} = \lim_{z \to a} \frac{(z-a)^2 f(z)}{z-a} = \lim_{z \to a} (z-a) f(z) = 0,$$

where we used that f is bounded on $D_r'(a)$. By 2.10, we can expand h into a Taylor series in $D_r(a)$.

$$h(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \quad , \quad z \in D_r(a).$$

Now we define $f(a) = a_2$. Since $h(z) = (z-a)^2 f(z)$ in $D_r(a)$ we get

$$f(z) = \sum_{n=0}^{\infty} a_{n+2} (z-a)^n,$$

where the series converges in $D_r(a)$. Hence $f \in \mathcal{H}(D_r(a))$, and also that $f \in \mathcal{H}(\Omega)$. $\square$

We can now characterize all possible isolated singularities of holomorphic functions.

Theorem 2.22. *Let $\Omega \subseteq \mathbb{C}$ be an open set, let $a \in \Omega$, and $f \in \mathcal{H}(\Omega \setminus \{a\})$. We distinguish between three possible cases:*

(a) f has a removable singularity at a;

(b) there exist $c_1, \ldots, c_m \in \mathbb{C}$, $m \in \mathbb{N}$, $c_m \neq 0$, such that

$$f(z) - \sum_{k=1}^{m} \frac{c_k}{(z-a)^k}$$

has a removable singularity at a; in this case a is called a pole of order m of f;

$$\sum_{k=1}^{m} \frac{c_k}{(z-a)^k}$$

is called main part of f;

(c) for all $r > 0$ with $D_r(a) \subseteq \Omega$, the image $f(D_r'(a))$ is dense in $\mathbb{C}$, i.e. for all $w \in \mathbb{C}$ and for all $\epsilon > 0$ there exists $z \in D_r'(a)$ such that $|f(z) - w| < \epsilon$; equivalently:

for all $w \in \mathbb{C}$ *there exists a sequence* $(z_n)_n$ *in* $\mathbb{C}$ *with* $\lim_{n\to\infty} z_n = a$, *such that* $\lim_{n\to\infty} f(z_n) = w$; *in this case, a is called an essential singularity of f.*

Remark. The assertion from (c) is known as the Casorati–Weierstraß Theorem [4] .

Proof. We will show that cases (a) or (b) are valid, if (c) fails.
Suppose that (c) fails: then there exist $r > 0$ and $\delta > 0$ and there exists some $w \in \mathbb{C}$ such that

$$|f(z) - w| > \delta,$$

for all $z \in D'_r(a)$.
Set

$$g(z) = \frac{1}{f(z) - w} \quad , \; z \in D'_r(a).$$

Then $g \in \mathcal{H}(D'_r(a))$ and

$$|g(z)| < \frac{1}{\delta} \quad , \quad \forall\, z \in D'_r(a),$$

hence g is bounded on $D'_r(a)$, so g has a removable singularity at a, see 2.21 .
Now we have two cases:

1.) $g(a) \neq 0$: then there exist $0 < s < r$ and there exists $\rho > 0$ with $|g(z)| \geq \rho$, for all $z \in D_s(a)$. Hence

$$|f(z)| = \left| \frac{1}{g(z)} + w \right| \leq \frac{1}{\rho} + |w|,$$

for all $z \in D'_s(a)$, which means that f is bounded on $D'_s(a)$. By 2.21, f has a removable singularity at a.

2.) g has a zero of order m , $m \in \mathbb{N}$ at a. (see 2.14). Therefore

$$g(z) = (z - a)^m g_1(z) \quad , \quad \forall z \in D_r(a),$$

where $g_1 \in \mathcal{H}(D_r(a))$ and $g_1(a) \neq 0$. Since

$$g(z) = \frac{1}{f(z) - w} = (z - a)^m g_1(z)$$

on $D'_r(a)$, we even have that $g_1(z) \neq 0 \;\; \forall\, z \in D_r(a)$. Now we set $h = 1/g_1$.Then $h \in \mathcal{H}(D_r(a))$ and

$$f(z) - w = (z - a)^{-m} h(z),$$

for all $z \in D'_r(a)$.

4 Casorati, Felice (1835–1890)

We expand h into a Taylor series on $D_r(a)$

$$h(z) = \sum_{n=0}^{\infty} b_n (z-a)^n,$$

where $b_0 = h(a) \neq 0$. Setting $c_k = b_{m-k}$, $k = 1, \dots, m$, we have $c_m = b_0 \neq 0$ and

$$\begin{aligned} f(z) - w &= (z-a)^{-m} \left(c_m + c_{m-1}(z-a) + \dots + c_1(z-a)^{m-1} + b_m(z-a)^m + \dots\right) \\ &= \frac{c_m}{(z-a)^m} + \frac{c_{m-1}}{(z-a)^{m-1}} + \dots + \frac{c_1}{z-a} + b_m + b_{m+1}(z-a) + \dots . \end{aligned}$$

Hence

$$f(z) - \sum_{k=1}^{m} \frac{c_k}{(z-a)^k} = w + b_m + b_{m+1}(z-a) + \dots ,$$

where the right hand side is a holomorphic function for $z \in D_r(a)$. □

Example 2.23. (a) Let $f(z) = \sin z/z$, $z \in \mathbb{C} \setminus \{0\}$. Then $f \in \mathcal{H}(\mathbb{C} \setminus \{0\})$, since $\sin 0 = 0$ and $\cos 0 = (\sin z)'|_{z=0}$ we get from 2.14 that there exists $g \in \mathcal{H}(\mathbb{C})$ such that $\sin z = zg(z)$. Hence f has a removable singulariy at $z = 0$ and $f \in \mathcal{H}(\mathbb{C})$. Since $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - + \dots$, we have $f(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - + \dots .$

(b) Let $f(z) = \frac{1}{\sin z}$. Then $f \in \mathcal{H}(D_1'(0))$. Since $\sin z = zg(z)$, $g \in \mathcal{H}(\mathbb{C})$, $g(0) \neq 0$, there exists $r > 0$ such that $g(z) \neq 0$ for all $z \in D_r(0)$. Setting $h(z) = 1/g(z)$ we get $h \in \mathcal{H}(D_r(0))$. Hence

$$f(z) = \frac{1}{\sin z} = \frac{h(z)}{z} = \frac{1}{z}\left(\sum_{n=0}^{\infty} b_n z^n\right),$$

where we used the Taylor series expansion of h on $D_r(0)$ and that $h(0) = b_0 \neq 0$. Therefore

$$f(z) - \frac{b_0}{z} = b_1 + b_2 z + \dots ,$$

where the right hand side is a holomorphic function on $D_r(0)$. This means that $f(z) = \frac{1}{\sin z}$ has a pole of order 1 at $z = 0$.

(c) Let $f(z) = \exp(1/z)$. Then $f \in \mathcal{H}(\mathbb{C}^*)$. For $n \in \mathbb{N}$ we get that $f(1/n) \to \infty$ as $n \to \infty$. But $|f(i/n)| = 1$, for all $n \in \mathbb{N}$. Hence the assertions (a) and (b) from 2.22 are not valid. Hence f has an essential singularity at $z = 0$.

(d) In a similar way one shows that $f(z) = \exp(-1/z^2)$ has an essential singularity at $z = 0$. But $f|_{\mathbb{R}} \in \mathcal{C}^{\infty}(\mathbb{R})$.

2.5 The maximum principle and Cauchy's estimates

In this section we prove some useful inequalities related to the absolute value of a holomorphic function and its derivatives.

Theorem 2.24. *Let $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$ be a holomorphic function on $D_R(a)$. Let $0 < r < R$. Then*

$$\sum_{n=0}^{\infty} |c_n|^2 r^{2n} = \frac{1}{2\pi}\int_0^{2\pi} |f(a+re^{i\theta})|^2 \, d\theta.$$

Proof. We write $z - a = re^{i\theta}$, then

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} |f(a+re^{i\theta})|^2 \, d\theta &= \frac{1}{2\pi}\int_0^{2\pi} \left(\sum_{n=0}^{\infty} c_n r^n e^{in\theta}\right)\left(\sum_{n=0}^{\infty} \overline{c_n} r^n e^{-in\theta}\right) d\theta \\
&= \sum_{n=0}^{\infty} |c_n|^2 r^{2n}.
\end{aligned}$$

□

Definition 2.25. A function $f \in \mathcal{H}(\mathbb{C})$ is called an entire function.

Theorem 2.26 (Liouville's theorem). [5] *Each bounded entire function is constant.*

Proof. Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be a bounded entire function, i.e. there exists a constant $M > 0$ such that $|f(z)| \le M$, for all $z \in \mathbb{C}$. Then, by 2.24 , we have

$$\sum_{n=0}^{\infty} |c_n|^2 r^{2n} \le M^2,$$

for all $r > 0$, hence $c_n = 0$ for all $n \in \mathbb{N}$. □

Theorem 2.27 (Maximum principle). *Let Ω be a domain in $\mathbb{C}$ and $f \in \mathcal{H}(\Omega)$. Let $a \in \Omega$ be an arbitrary point in Ω and $r > 0$ such that $D_r(a) \subset\subset \Omega$, i.e. $\overline{D_r(a)} \subset \Omega$. Then*

$$|f(a)| \le \max\{|f(a+re^{i\theta})| : 0 \le \theta \le 2\pi\},$$

with equality if and only if f is constant on Ω.

5 Liouville, Joseph (1809–1882)

Remark. The assertion from above implies that the function $z \mapsto |f(z)|$ has no local maximum in Ω.

Proof. If

$$\max\{|f(a + re^{i\theta})| : 0 \le \theta \le 2\pi\} \le |f(a)|,$$

then 2.24 implies that the function $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$ satisfies

$$\sum_{n=0}^{\infty} |c_n|^2 r^{2n} = \frac{1}{2\pi} \int_0^{2\pi} |f(a + re^{i\theta})|^2 \, d\theta \le |f(a)|^2 = |c_0|^2.$$

Hence $c_n = 0$ for all $n \in \mathbb{N}$, and $f \equiv f(a)$ on $D_r(a)$. Since Ω is connected, we get from 2.15 that $f \equiv f(a)$ on the whole of Ω. □

Corollary 2.28. *Let Ω be a bounded domain in $\mathbb{C}$, and let $f \in \mathcal{H}(\Omega)$ be a non-constant function, which is continuous on $\overline{\Omega}$. Then the function $|f|$ attains its maximum on $\partial\Omega$.*

Proof. Suppose that there exists $z_0 \in \Omega$ with $\max_{z \in \overline{\Omega}} |f(z)| = |f(z_0)|$. Then there exists $r > 0$ such that $D_r(z_0) \subset\subset \Omega$. By 2.27, we conclude that

$$|f(z_0)| < \max_{0 \le \theta \le 2\pi} |f(z_0 + re^{i\theta})|,$$

where we used that f is non-constant. So we arrive at a contradiction. □

At this point we can give a proof of the fundamental theorem of algebra.

Theorem 2.29 (fundamental theorem of algebra). *Let $n \in \mathbb{N}$ and let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree n with complex coefficients $a_0, a_1, \ldots, a_{n-1} \in \mathbb{C}$. Then p has exactly n zeros (some of which may be counted according to its multiplicities).*

Proof. First we prove the following assertion:

$$|p(0)| = |a_0| < |p(re^{i\theta})|$$

for each $\theta \in [0, 2\pi]$ and each $r > 1 + 2|a_0| + |a_1| + \cdots + |a_{n-1}|$.
This assertion implies, that all zeros of p are contained in the disk $\overline{D_R(0)}$ where $R = 1 + 2|a_0| + |a_1| + \cdots + |a_{n-1}|$. In order to prove this assertion we take $r > R$. Then $r > 1$ and we get

$$\begin{aligned} 2|a_0| + |a_1|r + \cdots + |a_{n-1}|r^{n-1} &\le r^{n-1}(2|a_0| + |a_1| + \cdots + |a_{n-1}|) \\ &< r^{n-1} r = r^n. \end{aligned} \tag{2.1}$$

In addition we have

$$r^n - |p(re^{i\theta})| \le |\, |r^n e^{in\theta}| - |p(re^{i\theta})|\, | \le |r^n e^{in\theta} - p(re^{i\theta})|$$

$$= |r^n e^{in\theta} - r^n e^{in\theta} - a_{n-1}r^{n-1}e^{i(n-1)\theta} - \cdots - a_0| \le |a_{n-1}|r^{n-1} + \cdots + |a_1|r + |a_0|,$$

hence

$$r^n - (|a_{n-1}|r^{n-1} + \cdots + |a_0|) \le |p(re^{i\theta})|. \tag{2.2}$$

Now we use (2.1) and (2.2) to obtain :

$$|a_0| = 2|a_0| + |a_1|r + \cdots + |a_{n-1}|r^{n-1} - (|a_0| + |a_1|r + \cdots + |a_{n-1}|r^{n-1})$$

$$< r^n - (|a_0| + |a_1|r + \cdots + |a_{n-1}|r^{n-1}) \le |p(re^{i\theta})|.$$

To prove the fundamental theorem of algebra, we suppose that p has no zero.Then $f = 1/p$ is an entire non-constant function and the assertion we just proved implies that

$$|f(re^{i\theta})| < |f(0)|$$

for all sufficiently large r, which contradicts 2.27. Hence there exists $z_1 \in \mathbb{C}$ with $p(z_1) = 0$ and by 2.14 we have

$$p(z) = (z - z_1)^m \; q(z)$$

for some $m \in \mathbb{N}$ and a polynomial q with $\mathrm{grad}(q) < \mathrm{grad}(p)$. (Expand p into a Taylor series around the point z_1.)
Induction on the degree of the polynomial concludes the proof. □

The following estimate is related to the absolute value of a holomorphic function and its derivatives.

Theorem 2.30 (Cauchy's estimates). *Let $\Omega \subseteq \mathbb{C}$ be an open subset, $a \in \Omega$, $r > 0$, $D_r(a) \subset\subset \Omega$, $f \in \mathcal{H}(\Omega)$. Then for $n \in \mathbb{N}$ we have*

$$|f^{(n)}(a)| \le \frac{n!}{r^n}\; M_{r,a}(f),$$

where $M_{r,a}(f) = \max_{z\in\partial D_r(a)} |f(z)|$.

Proof. By 2.12, we have

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int\limits_{\partial D_r(a)} \frac{f(\zeta)}{(\zeta - a)^{n+1}}\, d\zeta$$

and hence

$$|f^{(n)}(a)| \le \frac{n!}{2\pi}\; 2\pi r\; \frac{M_{r,a}(f)}{r^{n+1}}.$$

□

In the following we address the problem under which conditions the limit of a sequence of holomorphic functions is again holomorphic. We point out that we use the whole theory we have developed up to now. This result will also be fundamental for the study of topological vector spaces of holomorphic functions and gives a first occasion to use the terminology of functional analysis.

Theorem 2.31 (Weierstraß' theorem). *Let Ω be an open subset of $\mathbb{C}$, $f_n \in \mathcal{H}(\Omega)$, $n \in \mathbb{N}$. Suppose that the sequence $(f_n)_n$ converges uniformly on all compact subsets of Ω to a function f. Then $f \in \mathcal{H}(\Omega)$ and, for each $k \in \mathbb{N}$, the sequence $f_n^{(k)}$ converges to $f^{(k)}$ uniformly on all compact subsets of Ω.*
If $f_n \in \mathcal{H}(\Omega)$ is a Cauchy sequence in the sense of uniform convergence on all compact subsets of Ω, i.e. for each compact subset $K \subset \Omega$ and for each $\epsilon > 0$ there exists $N_{\epsilon,K} > 0$ such that

$$\sup_{z\in K} |f_n(z) - f_m(z)| < \epsilon$$

for all $n, m > N_{\epsilon,K}$, Then there exists a holomorphic function $f \in \mathcal{H}(\Omega)$ with $\lim_{n\to\infty} f_n = f$ uniformly on all compact subsets of Ω.
We say that the space $\mathcal{H}(\Omega)$ is complete in the sense of uniform convergence on all compact subsets of Ω.

Proof. f is the uniform limit of continuous functions and therefore again continuous. Let Δ be a solid triangle in Ω. We get from 2.5

$$\int_{\partial\Delta} f_n(z)\, dz = 0,$$

and, since $\partial\Delta$ is compact,

$$\int_{\partial\Delta} f(z)\, dz = 0.$$

Apply 2.13 , to get that $f \in \mathcal{H}(\Omega)$.
If K is a compact subset of Ω, then $s = \mathrm{dist}(K, \Omega^c) > 0$ and, for $r = s/3$, we have

$$K \subset \bigcup_{z\in K} D_r(z) = U.$$

The closure $\overline{U}$ is a compact subset of Ω and we can apply 2.30

$$|f_n'(z) - f'(z)| \le \frac{1}{r} \max_{w\in U} |f_n(w) - f(w)| \quad , \quad \forall\, z \in K,$$

hence f_n' converges to f' uniformly on K.
If $f_n \in \mathcal{H}(\Omega)$ is a Cauchy sequence, there exists a limit function f, which is at least continuous on Ω (see reelle Analysis). The first part of the theorem implies that $f \in \mathcal{H}(\Omega)$. $\square$

Remark. There are sequences of $\mathcal{C}^\infty$–functions on $\mathbb{R}$, which converge uniformly to nowhere differentiable functions (see

Remark. We already mentioned that the space $\mathcal{H}(\Omega)$, endowed with the topology of uniform convergence on compact subsets of Ω is a complete topological vector space (2.31). The topology on $\mathcal{H}(\Omega)$ can be described by a metric:
consider a so-called compact exhaustion of Ω, which is a sequence $(K_j)_j$ of compact subsets of Ω such that $K_j \subset \overset{\circ}{K}_{j+1}$, $j \in \mathbb{N}$ und $\bigcup_{j=1}^\infty K_j = \Omega$.
For instance, one can take

$$K_j = \{z : \operatorname{dist}(z, \Omega^c) \geq 1/j\} \cap \overline{D_j(0)} \quad , \quad j \in \mathbb{N}.$$

Now one defines a sequence of seminorms

$$\|f\|_j = \sup_{z \in K_j} |f(z)| \ , \ f \in \mathcal{H}(\Omega) \ , \ j \in \mathbb{N},$$

and we have $\|f\|_j \leq \|f\|_{j+1}$. The metric on $\mathcal{H}(\Omega)$ is now defined by:

$$d(f,g) = \sum_{j=1}^\infty 2^{-j} \ \frac{\|f-g\|_j}{1 + \|f-g\|_j} \ , \ f, g \in \mathcal{H}(\Omega).$$

The metric d generates the original topology of uniform convergence on all compact subsets of Ω on $\mathcal{H}(\Omega)$. (see Exercises)
$\mathcal{H}(\Omega)$ is a Fréchet space, a complete, metrizable topological vector space.

2.6 Open mappings

Definition 2.32. Let $U \subseteq \mathbb{C}$ be an open set and $\phi : U \longrightarrow \mathbb{C}$ a function. ϕ is called open, if $\phi(V)$ is open, for each open subset $V \subseteq U$.

Remark. Recall that $\phi : U \longrightarrow \mathbb{C}$ is continuous, if $\phi^{-1}(O)$ is open, for each open subset $O \subseteq \mathbb{C}$. If ϕ is open and invertible, then ϕ^{-1} is continuous.

Example. Let $\phi : \mathbb{R} \longrightarrow \mathbb{R}$, $\phi(x) = x^2$. We have $\phi((-1,1)) = [0,1)$. Hence ϕ is not open.

Theorem 2.33 (minimum principle). *Let $U \subseteq \mathbb{C}$ be open and $f \in \mathcal{H}(U)$, let $c \in U$ and let V a disk with center c and $\overline{V} \subset U$. Suppose that*

$$\min_{z \in \partial V} |f(z)| > |f(c)|.$$

Then f has a zero in V.

Proof. Suppose that f has no zero in V. Our assumption implies that there exists an open neighborhood V_1 of $\overline{V}$ such that $V_1 \subseteq U$, and that f has no zero on V_1. Set $g(z) = 1/f(z)\ \ ,\ z \in V_1$. Then $g \in \mathcal{H}(V_1)$, and by 2.27 we have

$$|f(c)|^{-1} = |g(c)| \leq \max_{z \in \partial V} |g(z)| = \left[\min_{z \in \partial V} |f(z)|\right]^{-1},$$

and we arrive at a contradiction. □

Theorem 2.34 (Open mapping theorem). *Let $U \subseteq \mathbb{C}$ be an open set and $f \in \mathcal{H}(U)$. Suppose that there is no open subset of U where f is constant. Then f is an open mapping.*

Proof. Let $O \subseteq U$ be open and $c \in O$. We have to show that $f(O)$ contains an open disk with center $f(c)$. Without loss of generality we can suppose that $f(c) = 0$, otherwise we could consider the function $z \mapsto f(z) - f(c)$ instead of f. We supposed that f is not constant in any neighborhood of c. We claim that there exists a disk V with center c such that $\overline{V} \subset O$ and $0 \notin f(\partial V)$. (If for each disk V with center c and $\overline{V} \subset O$ there exists $z_0 \in \partial V$ with $f(z_0) = 0$, then, by 2.15, $f \equiv 0$ in some neighborhood of c, which contradicts our assumption on f.)
Now we set $2\delta = \min_{z \in \partial V} |f(z)| > 0$ and $D = D_\delta(0)$. We will show that $D \subseteq f(O)$. For this aim let $b \in D$ be an arbitrary point. Let $|b| < \delta$ and hence

$$|f(z) - b| \geq |f(z)| - |b| > \delta,$$

for all $z \in \partial V$, so we get

$$\min_{z \in \partial V} |f(z) - b| \geq \delta > |b| = |f(c) - b|.$$

Now we can apply the minimum principle 2.33 for the function $z \mapsto f(z) - b$, and get $z' \in V$ with $f(z') - b = 0$. Hence $f(z') = b$ and finally $b \in f(O)$. □

Remark. Let $\pi_m(z) = z^m\ \ ,\ m \in \mathbb{N}$. Then π_m is an open mapping. Each $w \neq 0$ is the image under π_m of exactly m different points $z_k\ ,\ k = 1, \ldots, m$, i.e. $\pi_m(z_k) = w\ ,\ k = 1, \ldots m$. For $w = re^{i\theta}$ we have $z_k = r^{1/m} e^{i(\theta + 2k\pi)/m}\ \ ,\ k = 1, \ldots, m$. The point $w = 0$ has only $z = 0$ as its preimage, it is a so-called branching point.
We will show that each non-constant holomorphic function is locally of the form $\pi_m \circ \phi + c$, where ϕ is an invertible holomrphic function and c is a constant.

Lemma 2.35. *Let $\Omega \subseteq \mathbb{C}$ be open and $f \in \mathcal{H}(\Omega)$. Define*

$$g(z, w) = \begin{cases} \frac{f(z) - f(w)}{z - w} & ,\ z \neq w,\ z, w \in \Omega \\ f'(z) & ,\ z = w \in \Omega. \end{cases}$$

Then $g : \Omega \times \Omega \longrightarrow \mathbb{C}$ is continuous on $\Omega \times \Omega$.

Proof. It suffices to show continuity on the diagonal $\{(a,a) : a \in \Omega\}$.
For this aim, fix $a \in \Omega$ and let $r > 0$ be such that $D_r(a) \subseteq \Omega$ and

$$|f'(\zeta) - f'(a)| < \epsilon \ \ , \ \forall\ \zeta \in D_r(a).$$

For $z, w \in D_r(a)$ we set $\zeta(t) = (1-t)z + tw$, $t \in [0,1]$, which describes the straight line from z to w. It is clear that $\zeta(t) \in D_r(a)$, $\forall\ t \in [0,1]$. Now we compute the integral

$$\int_0^1 f'(\zeta(t))\,dt = \frac{1}{-z+w}\int_0^1 f'(\zeta(t))(-z+w)\,dt = \frac{1}{-z+w}\int_0^1 \frac{df}{d\zeta}\frac{d\zeta}{dt}\,dt$$

$$= \frac{1}{-z+w} \quad f(\zeta(t))|_0^1 = \frac{f(z)-f(w)}{z-w}.$$

Hence we get

$$|g(z,w) - g(a,a)| = \left|\frac{f(z)-f(w)}{z-w} - f'(a)\right| = \left|\int_0^1 [f'(\zeta(t)) - f'(a)]\,dt\right|$$

$$\leq \sup_{t\in[0,1]} |f'(\zeta(t)) - f'(a)| \leq \epsilon.$$

□

First we prove a result about invertible holomorphic functions.

Theorem 2.36. *Let $\Omega \subseteq \mathbb{C}$ be open, $\phi \in \mathcal{H}(\Omega)$, $z_0 \in \Omega$ and $\phi'(z_0) \neq 0$.*
Then there exists an open neighborhood V of z_0, $V \subseteq \Omega$ such that
(1) $\phi|_V$ is injective,
(2) the function $\psi : \phi(V) \longrightarrow V$ defined by $\psi(\phi(z)) = z$, $z \in V$, is holomorphic on $W = \phi(V)$, ϕ has a holomorphic inverse on V.

Proof. By 2.35, there exists an open neighborhood $V \subseteq \Omega$ of z_0 such that

$$|\phi(z_1) - \phi(z_2)| \geq \frac{1}{2}|\phi'(z_0)||z_1 - z_2| \quad ,\forall\ z_1, z_2 \in V,$$

for this aim one has to choose V in such a way that

$$\left|\frac{\phi(z_1)-\phi(z_2)}{z_1 - z_2}\right| \geq \frac{|\phi'(z_0)|}{2}.$$

If $z_1, z_2 \in V$, $z_1 \neq z_2$, then $\phi(z_1) \neq \phi(z_2)$ and ϕ is injective on V.

Since $\phi'(z_0) \neq 0$, we can choose V also such that $\phi'(z) \neq 0$, $\forall\ z \in V$. By assertion (1), each $w \in W = \phi(V)$ has a uniquely determined $z \in V$ with $\phi(z) = w$.

Now let $z, z_1 \in V$ and $w, w_1 \in W$ be chosen such that $\phi(z) = w$, $\phi(z_1) = w_1$ and $\psi(w) = z$, $\psi(w_1) = z_1$. Then we have

$$\frac{\psi(w) - \psi(w_1)}{w - w_1} = \frac{z - z_1}{\phi(z) - \phi(z_1)};$$

if $w \to w_1$, then $z \to z_1$ and so the left hand side converges to $\psi'(w_1)$ as $w \to w_1$ and the right hand side converges at the same time to $1/\phi'(z_1)$. Hence we have

$$\psi'(w_1) = \frac{1}{\phi'(z_1)}$$

and since $\phi' \neq 0$ on V, we obtain $\psi \in \mathcal{H}(W)$. □

Now we are able to show the local form of a holomorphic function as indicated above.

Theorem 2.37. *Let Ω be a domain in $\mathbb{C}$, $f \in \mathcal{H}(\Omega)$ non-constant. $z_0 \in \Omega$ and $w_0 = f(z_0)$. Let m be the order of the zero z_0 of the function $z \mapsto f(z) - w_0$.*
Then there exists an open neighborhod $V \subseteq \Omega$ of z_0 and a function $\phi \in \mathcal{H}(V)$ such that
(1) $f(z) = w_0 + [\phi(z)]^m$, $\forall\, z \in V$;
(2) ϕ is invertible on V.

Remark. On V, we have $f - w_0 = \pi_m \circ \phi$, where $\pi_m(z) = z^m$. Hence f is an m–to–1 mapping on $V \setminus \{z_0\}$.

Proof. We can take a convex open neighborhood V of z_0 such that $f(z) \neq w_0$, $\forall\, z \in V \setminus \{z_0\}$, otherwise we could find a sequence $(z_n)_n$ in V such that $\lim_{n\to\infty} z_n = z_0$ and $f(z_n) = w_0$, $\forall\, n \in \mathbb{N}$, then, by 2.15, we would get that $f \equiv w_0$ on V, which is excluded by our assumption on f.
Now we can apply 2.14 to obtain that $f(z) - w_0 = (z - z_0)^m g(z)$, $\forall z \in V$, where $g \in \mathcal{H}(V)$ and $g \neq 0$ on V.
By 2.19, g has a holomorphic logarithm on V, i.e. there exists $h \in \mathcal{H}(V)$ such that $\exp(h) = g$ on V. Now we set

$$\phi(z) = (z - z_0)\exp(h(z)/m).$$

Then

$$[\phi(z)]^m = (z - z_0)^m \exp(h(z)) = (z - z_0)^m g(z) = f(z) - w_0.$$

In addition we have that

$$\phi'(z) = \exp(h(z)/m) + (z - z_0)h'(z)/m \exp(h(z)/m),$$

and since $\exp(h(z_0)/m) \neq 0$, we get $\phi'(z_0) \neq 0$. By taking a possibly smaller V we can also get that $\phi' \neq 0$ on V. The rest of the proof now follows from 2.36. □

Theorem 2.38. *Let Ω be a domain in $\mathbb{C}$, $f \in \mathcal{H}(\Omega)$. Suppose that f is injective on Ω. Then $f' \neq 0$ on Ω and f has a holomorphic inverse .*

Proof. If $f'(z_0) = 0$ for some $z_0 \in \Omega$, then f is an m–to–1 mapping in a punctured neighborhood of z_0, see 2.37, where $m > 1$. As $f'(z_0) = 0$, we arrive at a contradiction.

□

Remark. The converse of the last theorem is false.

Example: $f(z) = e^z$, $f'(z) = e^z \neq 0$ on $\mathbb{C}$. But the exponential function is not injective on $\mathbb{C}$.

2.7 Holomorphic parameter integrals

In this section we prove a result about holomorphic parameter dependence of integrals, which will be very useful for many applications later on. A first result in this direction was already used for the Taylor series expansion of holomorphic functions, see 2.3. Now we use general properties of L^1-functions, see for instance (**?**).

Theorem 2.39. *Let $\Omega \subset \mathbb{C}$ be open and let (X, μ) be a measure space with a positive measure μ. Let*

$$L^1(\mu) = \{g : X \longrightarrow \mathbb{C} \ \ \textit{messbar} \ : \int\limits_X |g|\, d\mu < \infty\}.$$

Suppose that the function $f : \Omega \times X \longrightarrow \mathbb{C}$ has the following properties:
(i) $f(z, .) \in L^1(\mu)$ for all $z \in \Omega$;
(ii) for all $x \in X$, the function $f(., x) : \Omega \longrightarrow \mathbb{C}$ is holomorphic;
(iii) for each disk $K \subset \Omega$ there exists an integrable non-negative function g_K on X, such that for all $z \in K$ we have : $|f(z, .)| \leq g_K$ μ-almost everywhere.
Then the function $F : \Omega \longrightarrow \mathbb{C}$, defined by

$$F(z) = \int\limits_X f(z, x)\, d\mu(x) \ , \ z \in \Omega,$$

is holomorphic on Ω. For all integers $n \geq 0$, the function

$$\frac{\partial^n f}{\partial z^n}(z, .)$$

is integrable on X, and for $z \in \Omega$ one has

$$F^{(n)}(z) = \int\limits_X \frac{\partial^n f}{\partial z^n}(z,x)\, d\mu(x).$$

Proof. Let $a \in \Omega$ and choose $r > 0$ such that $K := \overline{D_{2r}(a)} \subset \Omega$. Then, by Cauchy's formula 2.9, we have for all $z \in D_{2r}(a)$

$$f(z,x) = \frac{1}{2\pi i} \int\limits_{\partial D_{2r}(a)} \frac{f(\zeta,x)}{\zeta - z}\, d\zeta.$$

Hence, for all $z, w \in D_r(a)$, $z \neq w$ we obtain

$$\frac{F(z) - F(w)}{z - w} = \int\limits_X \frac{1}{2\pi i} \int\limits_{\partial D_{2r}(a)} \frac{f(\zeta,x)}{(\zeta - z)(\zeta - w)}\, d\zeta\, d\mu(x).$$

Now let $(w_k)_k$ be a sequence in $D_r(a)$ with $\lim_{k\to\infty} w_k = z$, where $w_k \neq z$ for all k; define

$$\varphi_k(z,x) := \frac{1}{2\pi i} \int\limits_{\partial D_{2r}(a)} \frac{f(\zeta,x)}{(\zeta - z)(\zeta - w_k)}\, d\zeta.$$

Then

$$|\varphi_k(z,.)| \leq \frac{4\pi r g_K(.)}{2\pi r^2} = \frac{2}{r} g_K(.) \quad \mu - \text{almost everywhere},$$

since $|\zeta - a| = 2r$ and $w_k, z \in D_r(a)$.
In addition we have

$$\varphi_k(z,.) = \frac{f(z,.) - f(w_k,.)}{z - w_k},$$

since

$$\frac{f(\zeta,x)}{\zeta - z} - \frac{f(\zeta,x)}{\zeta - w_k} = \frac{(z - w_k) f(\zeta,x)}{(\zeta - z)(\zeta - w_k)}$$

and we can apply Cauchy's formula for φ_k. Hence $\varphi_k(z,.)$is a measurable function. Considering the limit $w_k \to z$, we observe that $f(\zeta,x)/[(\zeta - z)(\zeta - w_k)]$ tends to $f(\zeta,x)/(\zeta - z)^2$ uniformly for $\zeta \in \partial D_{2r}(a)$. Hence we can interchange limit and integration and obtain

$$\lim_{k\to\infty} \varphi_k(z,x) = \frac{1}{2\pi i} \int\limits_{\partial D_{2r}(a)} \frac{f(\zeta,x)}{(\zeta - z)^2}\, d\zeta = \frac{\partial f}{\partial z}(z.x),$$

where we used Cauchy's formula for the first derivative of f with respect to z.
Now we apply the dominated convergence theorem (see (**?**)) and get the desired assertion for $n = 1$. Using Cauchy's formula for the higher derivatives we obtain the general result. ☐

2.8 Complex differential forms

Here we recall Stokes' theorem for 1-forms on $\mathbb{R}^2$, see for instance (**?**). As $\mathbb{R}^2$ is identified with $\mathbb{C}$, we use a complex notion of differential forms which is compatible with the Wirtinger derivatives.

Definition 2.40. We define the differential dx as a linear mapping from $\mathbb{R}^2 \cong \mathbb{C}$ to $\mathbb{R}$,

$$dx : z \mapsto (dx)_z = x \quad , \quad z = x + iy,$$

analogously $(dy)_z = y$. If $f : M \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ is a real differentiable function, we define

$$(df)_{z_0} := \frac{\partial f}{\partial x}(z_0)\ dx + \frac{\partial f}{\partial y}(z_0)\ dy$$

as the complete differential of f at the point z_0. More general, we say that

$$\alpha = f\ dx + g\ dy$$

is a 1-form , where f, g are functions. If h is another function, we define

$$h\alpha := hf\ dx + hg\ dy.$$

In particular, we have for $f(z) = z$ and $f(z) = \overline{z}$:

$$dz = dx + idy \ \text{ and } \ d\overline{z} = dx - idy.$$

Recall the Wirtinger derivatives with respect to z and $\overline{z}$:

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad bzw. \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$

and a simple computation shows that

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \overline{z}}d\overline{z}.$$

If f is a holomorphic function, we have $df = \frac{\partial f}{\partial z}dz$, since $\frac{\partial f}{\partial \overline{z}} = 0$.
The 2-form $dx \wedge dy : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is the alternating 2-linear form

$$dx \wedge dy(\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}) = \begin{vmatrix} \xi_1 & \eta_1 \\ \xi_2 & \eta_2 \end{vmatrix} = \xi_1\eta_2 - \xi_2\eta_1.$$

If f is a function,

$$\omega = f(dx \wedge dy)$$

is a general 2-form. The following rules are valid

$$(f\ dx + g\ dy) \wedge (f_1\ dx + g_1\ dy) = (fg_1 - gf_1)dx \wedge dy \quad , \quad dx \wedge dy = -dy \wedge dx,$$

$$dz \wedge d\overline{z} = -2i\ dx \wedge dy = -2id\lambda(z),$$

where $d\lambda$ is the Lebesgue measure in $\mathbb{C} \cong \mathbb{R}^2$,

$$dx \wedge dx = dy \wedge dy = dz \wedge dz = d\overline{z} \wedge d\overline{z} = 0.$$

The differential of a 1-form $\alpha = f\ dx + g\ dy$ is defined by

$$d\alpha := df \wedge dx + dg \wedge dy = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\ dx \wedge dy.$$

If $\alpha = F\ dz + G\ d\overline{z}$, we have

$$d\alpha = \left(\frac{\partial G}{\partial z} - \frac{\partial F}{\partial \overline{z}}\right)\ dz \wedge d\overline{z}.$$

In addition, $d(df) = 0$ and if f is real differentiable, then

$$d(f\ dz) = -\frac{\partial f}{\partial \overline{z}}\ dz \wedge d\overline{z}.$$

Example 2.41. Let z be a fixed point and let f be a real differentiable function, let ω be the following 1-form

$$\omega(\zeta) = \frac{1}{2\pi i}\frac{f(\zeta)}{\zeta - z}\ d\zeta \quad \text{für} \quad \zeta \neq z.$$

Then

$$\begin{aligned} d\omega(\zeta) &= -\frac{1}{2\pi i}\left[\frac{\partial f/\partial\overline{\zeta}}{\zeta - z} + f(\zeta)\frac{\partial}{\partial\overline{\zeta}}(\zeta \mapsto \frac{1}{\zeta - z})\right]\ d\zeta \wedge d\overline{\zeta} \\ &= -\frac{1}{2\pi i}\frac{\partial f/\partial\overline{\zeta}}{\zeta - z}\ d\zeta \wedge d\overline{\zeta}, \end{aligned}$$

since $\zeta \mapsto \frac{1}{\zeta - z}$ is a holomorphic function.

Let G be an open in $\mathbb{C}$ and $k \in \mathbb{N} \cup \{\infty\}$. $\mathcal{C}^k(G)$ denotes the space of k times continuously differentiable (in the real sense) functions on G. We also write $\mathcal{C}(G)$ instead of $\mathcal{C}^0(G)$. The space $\mathcal{C}^k(\overline{G})$ is the space of all functions f in $\mathcal{C}^k(G)$ such that all derivatives of f up to order k extend continuously to $\overline{G}$.

Theorem 2.42 (Stokes' theorem). *Let G be a domain in $\mathbb{C}$. Let ∂G consist of a positively oriented path and let $\omega \in \mathcal{C}^1(\overline{G})$ be a continuously differentiable 1-form on $\overline{G}$. Then*

$$\int_{\partial G} \omega = \int_G d\omega.$$

For a proof see (**?**).

2.9 The inhomogeneous Cauchy formula

Now we apply Stokes' theorem to prove a more general version of Cauchy's formula, which will be a useful tool for the study of the inhomogeneous Cauchy-Riemann equations.

Theorem 2.43. *Let G be a bounded domain in $\mathbb{C}$ with piecewise smooth positively oriented boundary ∂G. Let $f \in \mathcal{C}^1(\overline{G})$. Then, for $z \in G$, we have*

$$f(z) = \frac{1}{2\pi i} \int\limits_{\partial G} \frac{f(\zeta)}{\zeta - z}\, d\zeta + \frac{1}{2\pi i} \int\limits_{G} \frac{(\partial f/\partial \overline{\zeta})(\zeta)}{\zeta - z}\, d\zeta \wedge d\overline{\zeta}.$$

Remark. If $f \in \mathcal{H}(G)$, we have $(\partial f/\partial \overline{\zeta})(\zeta) = 0$, $\forall \zeta \in G$ and hence

$$f(z) = \frac{1}{2\pi i} \int\limits_{\partial G} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

In this sense, 2.43 is a generalization of Cauchy's formula.

Proof. Fix $z \in G$ and choose $r > 0$ such that $D_r(z) \subseteq G$. We remove the disk $D_r(z)$ from G and define $G_r = G \setminus \overline{D_r(z)}$, the boundary of G_r consists of the positively oriented boundary of G and of the negatively oriented circle κ_r. Walking on ∂G_r, the domain G_r lies always on the left hand side.
For $\zeta \in G_r$ we define the 1-form

$$\omega(\zeta) = \frac{1}{2\pi i} \frac{f(\zeta)}{\zeta - z}\, d\zeta,$$

we can apply Stokes' theorem 2.42 and obtain from 2.41

$$\frac{1}{2\pi i} \int\limits_{\partial G_r} \frac{f(\zeta)}{\zeta - z}\, d\zeta = -\frac{1}{2\pi i} \int\limits_{G_r} \frac{(\partial f/\partial \overline{\zeta})(\zeta)}{\zeta - z}\, d\zeta \wedge d\overline{\zeta}.$$

Now we take the limit $r \to 0$. First we show that the integral

$$\int\limits_{D_s(0)} \frac{1}{|z|}\, dz \wedge d\overline{z}$$

exists. For this purpose we use polar coordinates and get

$$\int\limits_{D_s(0)} \frac{1}{|z|}\, dz \wedge d\overline{z} = -2i \int\limits_{D_s(0)} \frac{1}{|z|}\, dx \wedge dy = -2i \int\limits_0^s \int\limits_0^{2\pi} \frac{1}{r} r\, dr d\phi,$$

and the last integral exists.
This implies that the function $\zeta \mapsto \frac{(\partial f/\partial\overline{\zeta})(\zeta)}{\zeta - z}$ is absolutely integrable on G. Hence the integral

$$\int\limits_{G_r} \frac{(\partial f/\partial\overline{\zeta})(\zeta)}{\zeta - z}\, d\zeta \wedge d\overline{\zeta} \quad \text{tends to} \quad \int\limits_{G} \frac{(\partial f/\partial\overline{\zeta})(\zeta)}{\zeta - z}\, d\zeta \wedge d\overline{\zeta},$$

as $r \to 0$. Here we used again the dominated convergence theorem.
Now we consider the line integral of Stokes' theorem and get

$$\frac{1}{2\pi i} \int\limits_{\partial G_r} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{2\pi i} \int\limits_{\partial G} \frac{f(\zeta)}{\zeta - z}\, d\zeta + \frac{1}{2\pi i} \int\limits_{\kappa_r} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

We have

$$\begin{aligned}
\frac{1}{2\pi i} \int\limits_{\kappa_r} \frac{f(\zeta)}{\zeta - z}\, d\zeta &= \frac{1}{2\pi i} \int\limits_{\kappa_r} \frac{f(z)}{\zeta - z}\, d\zeta + \frac{1}{2\pi i} \int\limits_{\kappa_r} \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta \\
&= f(z) \cdot \frac{1}{2\pi i} \int\limits_{\kappa_r} \frac{1}{\zeta - z}\, d\zeta + \frac{1}{2\pi i} \int\limits_{\kappa_r} \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta \\
&= -f(z) + \frac{1}{2\pi i} \int\limits_{\kappa_r} \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta,
\end{aligned}$$

and hence

$$\left| \int\limits_{\kappa_r} \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta \right| \le 2\pi r \max_{\zeta \in \kappa_r^*} \left| \frac{f(\zeta) - f(z)}{\zeta - z} \right| = 2\pi \max_{\zeta \in \kappa_r^*} |f(\zeta) - f(z)| \to 0,$$

as $r \to 0$, since f is continuous.
So the line integral

$$\frac{1}{2\pi i} \int\limits_{\partial G_r} \frac{f(\zeta)}{\zeta - z}\, d\zeta \quad \text{tends to} \quad \frac{1}{2\pi i} \int\limits_{\partial G} \frac{f(\zeta)}{\zeta - z}\, d\zeta - f(z),$$

as $r \to 0$, and we arrive at the desired result. □

2.10 General versions of Cauchy's Theorem and Cauchy's Formula

It will be convenient to consider integrals over sums of paths. This leads to the concepts of chains and cycles.

Definition 2.44. Let $\Omega \subseteq \mathbb{C}$ be open and let $\gamma_1, \ldots, \gamma_n$ be paths in Ω. Let $\Gamma^* = \bigcup_{j=1}^n \gamma_j^*$ and define

$$\tilde{\gamma}_j(f) := \int\limits_{\gamma_j} f(z)\, dz,$$

for $f \in \mathcal{C}(\Gamma^*)$. $\tilde{\gamma}_j : \mathcal{C}(\Gamma^*) \longrightarrow \mathbb{C}$ can be seen as a linear functional on $\mathcal{C}(\Gamma^*)$. We set $\tilde{\Gamma} = \tilde{\gamma}_1 + \cdots + \tilde{\gamma}_n$, and denote by $\Gamma = \gamma_1 + \cdots + \gamma_n$ the formal sum of the paths $\gamma_1, \ldots, \gamma_n$. We define

$$\int\limits_{\Gamma} f(z)\, dz = \tilde{\Gamma}(f) = \sum_{j=1}^{n} \int\limits_{\gamma_j} f(z)\, dz,$$

for $f \in \mathcal{C}(\Gamma^*)$. Γ is called a chain in Ω. If all paths $\gamma_1, \ldots, \gamma_n$ are closed, we call Γ a cycle in Ω.

Remark. (a) Chains and cycles cam be represented as sums of paths in many ways.
(b) By $-\Gamma$ we denote the cycle, where each path γ_j , $j = 1, \ldots, n$ is replaced by its opposite path, for $f \in \mathcal{C}(\Gamma^*)$ we have

$$\int\limits_{-\Gamma} f(z)\, dz = -\int\limits_{\Gamma} f(z)\, dz.$$

(c) If Γ_1 and Γ_2 are chains or cycles, we can form the sum $\Gamma = \Gamma_1 + \Gamma_2$ and have

$$\int\limits_{\Gamma} f(z)\, dz = \int\limits_{\Gamma_1} f(z)\, dz + \int\limits_{\Gamma_2} f(z)\, dz \quad , \; f \in \mathcal{C}(\Gamma_1^* \cup \Gamma_2^*).$$

Definition 2.45. Let $\Gamma = \gamma_1 + \cdots + \gamma_n$ be a cycle in Ω and $\alpha \notin \Gamma^*$. We define the index of α with respect to Γ by

$$\mathrm{Ind}_\Gamma(\alpha) := \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{dz}{z - \alpha}.$$

Obviously we have

$$\mathrm{Ind}_\Gamma(\alpha) = \sum_{j=1}^{n} \mathrm{Ind}_{\gamma_j}(\alpha).$$

Theorem 2.46 (Homology–version of Cauchy's theorem). *Let $\Omega \subseteq \mathbb{C}$ be an arbitrary open set, $f \in \mathcal{H}(\Omega)$, let Γ be a cycle in Ω such that $\mathrm{Ind}_\Gamma(\alpha) = 0$ for every $\alpha \notin \Omega$. Then*

$$f(z)\, \mathrm{Ind}_\Gamma(z) = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{f(w)}{w - z}\, dw \quad , \; \forall\, z \in \Omega \setminus \Gamma^*,$$

in addition

$$\int_\Gamma f(w)\,dw = 0.$$

If Γ_0 *and* Γ_1 *are cycles in* Ω *such that*

$$\mathrm{Ind}_{\Gamma_0}(\alpha) = \mathrm{Ind}_{\Gamma_1}(\alpha) \quad , \ \forall\ \alpha \notin \Omega$$

then

$$\int_{\Gamma_0} f(z)\,dz = \int_{\Gamma_1} f(z)\,dz.$$

Proof. (Dickson 1969)(**?**)
Let

$$g(z,w) = \begin{cases} \frac{f(z)-f(w)}{z-w} & ,\ z \neq w,\ z,w \in \Omega \\ f'(z) & ,\ z = w \in \Omega. \end{cases}$$

By 2.35, $g : \Omega\times\Omega \longrightarrow \mathbb{C}$ is continuous. The first assertion of the theorem is equivalent to the statement $h(z) = 0\ ,\ \forall\ z \in \Omega \setminus \Gamma^*$, where

$$h(z) = \frac{1}{2\pi i}\int_\Gamma g(z,w)\,dw,$$

because

$$\frac{1}{2\pi i}\int_\Gamma \frac{f(z)-f(w)}{z-w}\,dw = \frac{1}{2\pi i}\ f(z)\int_\Gamma \frac{dw}{z-w} - \frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{z-w}\,dw$$
$$= -f(z)\,\mathrm{Ind}_\Gamma(z) + \frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{w-z}\,dw.$$

First we show: $h \in \mathcal{H}(\Omega)$.
g is uniformly continuous on every compact subset of $\Omega\times\Omega$. Hence, if $z \in \Omega$ and $z_n \to z$ in Ω,then $g(z_n,w) \to g(z,w)$ uniformly for $w \in \Gamma^*$, which is a compact subset. So we get

$$\lim_{n\to\infty} h(z_n) = \frac{1}{2\pi i}\lim_{n\to\infty}\int_\Gamma g(z_n,w)\,dw$$
$$= \frac{1}{2\pi i}\int_\Gamma \lim_{n\to\infty} g(z_n,w)\,dw = \frac{1}{2\pi i}\int_\Gamma g(z,w)\,dw = h(z),$$

and h is continuous on Ω; limit and integral can be interchanged, because of uniform convergence.

Now let Δ be an arbitrary closed triangle in Ω. Then Fubini's theorem implies

$$\int_{\partial\Delta} h(z)\,dz = \frac{1}{2\pi i}\int_{\partial\Delta}\left(\int_{\Gamma} g(z,w)\,dw\right)dz = \frac{1}{2\pi i}\int_{\Gamma}\left(\int_{\partial\Delta} g(z,w)\,dz\right)dw.$$

For w fixed, the function $z \mapsto g(z,w)$ has a removable singularity at $z = w$ (see 2.21), so it is holomorphic on Ω and, by 2.5, we have

$$\int_{\partial\Delta} g(z,w)\,dz = 0 \quad , \ \forall\, w \in \Omega,$$

hence

$$\int_{\partial\Delta} h(z)\,dz = 0,$$

and, by 2.13, $h \in \mathcal{H}(\Omega)$.

Next we prove that $h(z) = 0$, $\forall\, z \in \Omega \setminus \Gamma^*$.
For this aim leti $\Omega_1 = \{z \in \mathbb{C} : \mathrm{Ind}_\Gamma(z) = 0\}$, we define

$$h_1(z) = \frac{1}{2\pi i}\int_{\Gamma}\frac{f(w)}{w-z}\,dw,$$

then $h_1 \in \mathcal{H}(\Omega_1)$, and for $z \in \Omega \cap \Omega_1$

$$h(z) = \frac{1}{2\pi i}\, f(z)\int_{\Gamma}\frac{dw}{z-w} - \frac{1}{2\pi i}\int_{\Gamma}\frac{f(w)}{z-w}\,dw = 0 + \frac{1}{2\pi i}\int_{\Gamma}\frac{f(w)}{w-z}\,dw = h_1(z).$$

Hence the function

$$\phi(z) = \begin{cases} h(z) & , \ z \in \Omega \\ h_1(z) & , \ z \in \Omega_1 \end{cases}$$

is holomorphic on $\Omega \cup \Omega_1$.
By assumption, we have $\mathrm{Ind}_\Gamma(\alpha) = 0$, $\forall\, \alpha \notin \Omega$, hence $\Omega^c \subseteq \Omega_1$ and $\Omega \cup \Omega_1 = \mathbb{C}$.
So $\phi \in \mathcal{H}(\mathbb{C})$ is an entire function.
The set Ω_1 contains the unbounded connected component of $\mathbb{C} \setminus \Gamma^*$, since the index is always zero there (see 2.2). Therefore

$$\lim_{|z|\to\infty}\phi(z) = \lim_{|z|\to\infty} h_1(z) = \frac{1}{2\pi i}\lim_{|z|\to\infty}\int_{\Gamma}\frac{f(w)}{w-z}\,dw = 0,$$

and Liouville's theorem 2.26 implies $\phi \equiv 0$, in particular $h(z) = 0$, $\forall\, z \in \Omega \setminus \Gamma^*$.

Finally we have to show that $\int_\Gamma f(z)\,dz = 0$. Let $a \in \Omega \setminus \Gamma^*$ and $F(z) := (z-a)f(z)$. From the first assertion of the theorem we get for F and $z = a$ that

$$0 = F(a)\ \mathrm{Ind}_\Gamma(a) = \frac{1}{2\pi i}\int_{\Gamma}\frac{F(w)}{w-a}\,dw = \frac{1}{2\pi i}\int_{\Gamma} f(w)\,dw.$$

For the last assertion of the theorem we pick two cycles Γ_0 and Γ_1 with

$$\mathrm{Ind}_{\Gamma_0}(\alpha) = \mathrm{Ind}_{\Gamma_1}(\alpha) \quad , \forall\ \alpha \notin \Omega.$$

Define $\Gamma = \Gamma_0 - \Gamma_1$, then $\mathrm{Ind}_\Gamma(\alpha) = 0 \quad , \forall\ \alpha \notin \Omega$ and, by the first part of the theorem,

$$0 = \int_\Gamma f(z)\,dz = \int_{\Gamma_0} f(z)\,dz - \int_{\Gamma_1} f(z)\,dz.$$

□

Remark. (a) If $\mathrm{Ind}_\Gamma(\alpha) = 0 \quad , \forall\ \alpha \notin \Omega$, the cycle Γ is called null-homologous in Ω.
(b) 2.46 is a generalization of 2.9 : if Ω is convex γ a closed path in Ω, then $w \mapsto 1/(w-\alpha)$ is holomorphic on Ω for every $\alpha \notin \Omega$ and, by 2.6, we obtain

$$\mathrm{Ind}_\gamma(\alpha) = \frac{1}{2\pi i}\int_\gamma \frac{dw}{w-\alpha} = 0.$$

Therefore the assumptions of 2.46 are satisfied, hence 2.46 implies 2.9.

Example. Let $\Omega = \mathbb{C} \setminus (D_{1/2}(-2) \cup D_{1/2}(0) \cup D_{1/2}(2))$, and let $\gamma_1(t) = -2 + \frac{3}{4}e^{it}$, $\gamma_2(t) = \frac{3}{4}e^{it}$, $\gamma_3(t) = 2 + \frac{3}{4}e^{it}$, $\Gamma(t) = 6e^{it}$, $t \in [0, 2\pi]$. Define $\gamma = \gamma_1 + \gamma_2 + \gamma_3$. Then

$$\mathrm{Ind}_\gamma(\alpha) = \mathrm{Ind}_\Gamma(\alpha) \quad , \forall\ \alpha \notin \Omega,$$

hence, by 2.46, we have

$$\int_\gamma f(z)\,dz = \int_\Gamma f(z)\,dz,$$

for each $f \in \mathcal{H}(\Omega)$.
In the following we discuss another topological concept which is important for Cauchy's theorem.

Definition 2.47. Let $\Omega \subseteq \mathbb{C}$ and $\gamma_0,\ \gamma_1 : [0,1] \longrightarrow \Omega$ be closed curves. γ_0 and γ_1 are Ω– homotopic, if there is a continuous mapping

$$H : [0,1] \times [0,1] \longrightarrow \Omega$$

such that

$$H(s,0) = \gamma_0(s) \quad , \forall\ s \in [0,1]\ ;\ H(s,1) = \gamma_1(s) \quad , \forall\ s \in [0,1]$$

and

$$H(0,t) = H(1,t) \quad , \forall\ t \in [0,1].$$

Set $\gamma_t(s) = H(s,t)$ for a fixed $t \in [0,1]$ and $s \in [0,1]$. Since $H(0,t) = H(1,t)$, the curves γ_t are also closed. We get a one-parameter familiy γ_t , $t \in [0,1]$ of closed curves connecting γ_0 and γ_1.

If a closed curve γ_0 is Ω–homotopic to a constant curve (consisting of just one point), we say that γ_0 null-homotopic in Ω.

A domain Ω is said to be simply connected, if every closed curve in Ω is null-homotopic in Ω.

EXAMPLE. Let Ω be a convex domain and let γ be a closed curve in Ω. Fix $z_0 \in \Omega$. Define

$$H(s,t) = (1-t)\gamma(s) + tz_0 \ , \ s,t \in [0,1].$$

Then $H(s,0) = \gamma(s)$ and $H(s,1) = z_0$, $\forall\ s \in [0,1]$. Since $\gamma(0) = \gamma(1)$,we have

$$H(0,t) = (1-t)\gamma(0) + tz_0 = (1-t)\gamma(1) + tz_0 = H(1,t) \ , \ \forall\ t \in [0,1].$$

For a fixed s, the expression $H(s,t) = (1-t)\gamma(s) + tz_0$, $t \in [0,1]$ describes the straight line from $\gamma(s)$ to z_0, which is contained in Ω, as Ω is convex.
Hence $H(s,t) \in \Omega$, $\forall\ s,t \in [0,1]$ and γ is null-homotopic in Ω.. Therefore Ω is simply connected.

Lemma 2.48. *Let γ_0 and γ_1 be closed paths in $\mathbb{C}$. Let $\alpha \in \mathbb{C}$ be a complex number such that*

$$|\gamma_0(s) - \gamma_1(s)| < |\alpha - \gamma_0(s)| \quad , \ \forall\ s \in [0,1].$$

Then $\mathrm{Ind}_{\gamma_0}(\alpha) = \mathrm{Ind}_{\gamma_1}(\alpha)$.

Proof. The assumption implies that $\alpha \notin \gamma_0^*$ and $\alpha \notin \gamma_1^*$.
We set

$$\gamma(s) = \frac{\gamma_1(s) - \alpha}{\gamma_0(s) - \alpha} \quad , \ s \in [0,1],$$

then we get for the derivatives

$$\gamma'(s) = \frac{(\gamma_0(s) - \alpha)\gamma_1'(s) - (\gamma_1(s) - \alpha)\gamma_0'(s)}{(\gamma_0(s) - \alpha)^2}$$

and

$$\begin{aligned} \frac{\gamma'(s)}{\gamma(s)} &= \frac{(\gamma_0(s) - \alpha)\gamma_1'(s) - (\gamma_1(s) - \alpha)\gamma_0'(s)}{(\gamma_0(s) - \alpha)^2} \ \frac{\gamma_0(s) - \alpha}{\gamma_1(s) - \alpha} \\ &= \frac{\gamma_1'(s)}{\gamma_1(s) - \alpha} - \frac{\gamma_0'(s)}{\gamma_0(s) - \alpha}. \end{aligned}$$

By assumption we have

$$|1 - \gamma(s)| = \left| \frac{\gamma_0(s) - \gamma_1(s)}{\gamma_0(s) - \alpha} \right| < 1,$$

hence $\gamma^* \subset D_1(1)$ and, by 2.2, $\mathrm{Ind}_\gamma(0) = 0$. This implies

$$0 = \mathrm{Ind}_\gamma(0) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z} = \frac{1}{2\pi i}\int_0^1 \frac{\gamma'(s)}{\gamma(s)}\,ds$$

$$= \frac{1}{2\pi i}\int_0^1 \frac{\gamma_1'(s)}{\gamma_1(s)-\alpha}\,ds - \frac{1}{2\pi i}\int_0^1 \frac{\gamma_0'(s)}{\gamma_0(s)-\alpha}\,ds = \mathrm{Ind}_{\gamma_1}(\alpha) - \mathrm{Ind}_{\gamma_0}(\alpha).$$

□

Theorem 2.49 (Homotopy version of Cauchy's theorem). *Let Ω be a domain in $\mathbb{C}$, let Γ_0 and Γ_1 closed paths in Ω, which are Ω–homotopic. Let $\alpha \notin \Omega$. Then*

$$\mathrm{Ind}_{\Gamma_0}(\alpha) = \mathrm{Ind}_{\Gamma_1}(\alpha)$$

and, by 2.46*,*

$$\int_{\Gamma_0} f(z)\,dz = \int_{\Gamma_1} f(z)\,dz\ ,\ \forall\ f \in \mathcal{H}(\Omega).$$

Proof. Let H be a homotopy function between zwischen Γ_0 and Γ_1. The difficulty of the proof relies on the fact that the one-parameter family $\Gamma_t(s) = H(s,t)$ does not consist of necessarily piece-wise differentiable curves. We will apporximate the curves Γ_t by suitable paths, in our case by polygonal closed paths.
Fix $\alpha \notin \Omega$. Since H is uniformly continuous on the compact set $[0,1]\times[0,1]$, there exists $\epsilon > 0$ such that

$$|\alpha - H(s,t)| > 2\epsilon\ ,\ \forall\ s,t \in [0,1]; \tag{2.3}$$

and there exists $n \in \mathbb{N}$ such that

$$|H(s,t) - H(s',t')| < \epsilon/2, \tag{2.4}$$

falls $|s-s'| + |t-t'| < 1/n$.
Now we define the approximating polygonal paths: for $k = 0,1,\ldots,n$ and $s \in [0,1]$ let

$$\gamma_k(s) = H\left(\frac{j}{n},\frac{k}{n}\right)\ (ns+1-j) + H\left(\frac{j-1}{n},\frac{k}{n}\right)\ (j-ns),$$

für $j-1 \le ns \le j\ ,\ j = 1,\ldots,n$.
It is easily seen that the curves $\gamma_0, \gamma_1, \ldots, \gamma_n$ are closed. There are also piece-wise differentiable, since the variable s only appears as a linear term, and not as an argument of the function H. By (2.4) and from the definition of γ_k we get

$$\left|\gamma_k(s) - H\left(s, \frac{k}{n}\right)\right| < \epsilon, \tag{2.5}$$

for $s \in [0,1]$ and $k = 0, 1, \ldots, n$.
If $j - 1 \le ns \le j$, we have $j - ns \le 1$, hence

$$\begin{aligned}
&\left|\gamma_k(s) - H\left(s, \frac{k}{n}\right)\right| \\
&= \left|H\left(\frac{j}{n}, \frac{k}{n}\right)\ (ns + 1 - j) + H\left(\frac{j-1}{n}, \frac{k}{n}\right)\ (j - ns) - H\left(s, \frac{k}{n}\right)\right| \\
&\le (j - ns)\left|H\left(\frac{j}{n}, \frac{k}{n}\right) - H\left(\frac{j-1}{n}, \frac{k}{n}\right)\right| + \left|H\left(\frac{j}{n}, \frac{k}{n}\right) - H\left(s, \frac{k}{n}\right)\right| \\
&< \epsilon/2 + \epsilon/2 = \epsilon.
\end{aligned}$$

In particular, for $k = 0$

$$|\gamma_0(s) - H(s,0)| = |\gamma_0(s) - \Gamma_0(s)| < \epsilon$$

and for $k = n$

$$|\gamma_n(s) - H(s,1)| = |\gamma_n(s) - \Gamma_1(s)| < \epsilon.$$

Now we get from (2.3) and (2.5)

$$\begin{aligned}
|\alpha - \gamma_k(s)| &= |\alpha - H(s, k/n) - (\gamma_k(s) - H(s, k/n))| \\
&\ge |\alpha - H(s, k/n)| - |\gamma_k(s) - H(s, k/n)| \\
&> 2\epsilon - \epsilon = \epsilon,
\end{aligned}$$

for $s \in [0,1]$ and $k = 0, 1, \ldots, n$. Using (2.4) and the definition of γ_k we get

$$|\gamma_{k-1}(s) - \gamma_k(s)| < \epsilon,$$

for $s \in [0,1]$ and $k = 1, \ldots, n$.
If $j - 1 \le ns \le j$, we have $ns + 1 - j \le 1$, therefore

$$\begin{aligned}
|\gamma_{k-1}(s) - \gamma_k(s)| \le (ns + 1 - j)&\left|H\left(\frac{j}{n}, \frac{k-1}{n}\right) - H\left(\frac{j}{n}, \frac{k}{n}\right)\right| \\
+ (j - ns)&\left|H\left(\frac{j-1}{n}, \frac{k-1}{n}\right) - H\left(\frac{j-1}{n}, \frac{k}{n}\right)\right| \\
< \epsilon/2 + \epsilon/2 = \epsilon.&
\end{aligned}$$

Since $|\alpha - \gamma_k(s)| > \epsilon$ and $|\gamma_{k-1}(s) - \gamma_k(s)| < \epsilon$, we obtain

$$|\gamma_{k-1}(s) - \gamma_k(s)| < |\alpha - \gamma_k(s)|\ ,$$

$\forall\ s \in [0,1]$ and $k = 1, \ldots, n$.
Similarily

$$|\gamma_0(s) - \Gamma_0(s)| < |\alpha - \Gamma_0(s)| \quad \text{and} \quad |\gamma_n(s) - \Gamma_1(s)| < |\alpha - \Gamma_1(s)|,$$

$\forall\ s \in [0, 1]$.
Now we can apply 2.48 for the pairs

$$(\Gamma_0, \gamma_0),\ (\gamma_0, \gamma_1),\ \ldots,\ (\gamma_{n-1}, \gamma_n),\ (\gamma_n, \Gamma_1)$$

and get

$$\mathrm{Ind}_{\Gamma_0}(\alpha) = \mathrm{Ind}_{\gamma_0}(\alpha) = \mathrm{Ind}_{\gamma_1}(\alpha) = \ldots \mathrm{Ind}_{\gamma_{n-1}}(\alpha) = \mathrm{Ind}_{\gamma_n}(\alpha) = \mathrm{Ind}_{\Gamma_1}(\alpha).$$

□

Corollary 2.50. *(a) Let Ω be a simply connected domain in $\mathbb{C}$ and let γ be a closed path in Ω. Then*

$$\int_\gamma f(z)\, dz = 0,$$

for each $f \in \mathcal{H}(\Omega)$.
(b) Let $\Omega \subseteq \mathbb{C}$ be an open set and let $a \in \Omega$. Let γ be a closed path in $\Omega \setminus \{a\}$, which isnull-homotopic in Ω. Then

$$f^{(n)}(a)\,\mathrm{Ind}_\gamma(a) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{n+1}}\, dz\ ,\ n \in \mathbb{N},$$

for each $f \in \mathcal{H}(\Omega)$.

Proof. (b) For $n = 1$, fthe assertion follows from 2.46. Finally, differentiation in the integral with respect to the variable a gives the desired result. □

Remark. If Ω is a domain and γ a closed null-homotopic path in Ω, then

$$\int_\gamma \frac{dz}{z-\alpha} = 0\ ,\ \forall\ \alpha \notin \Omega,$$

this follows from 2.49, since the function $z \mapsto 1/(z-\alpha)$ is holomorphic and γ is null-homotopic in Ω. This implies that γ is also null-homologous. Every null-homotopic path is null-homologous. The converse is false (see Exercises).
But if *every* closed path in Ω is null-homologous, then every closed path is also null-homotopic and Ω is simply connected (see Chapter 4).

We conclude this chapter with two useful results about the index of a path.

Definition 2.51. Let G be a domain in $\mathbb{C}$ and $\gamma : [0,1] \longrightarrow \mathbb{C}$ a path. We say, γ runs in G from boundary to boundary, if
(1) there exists $t_1, t_2 \in [0,1]$, $\;t_1 < t_2$, $\;\gamma(t_1), \gamma(t_2) \in \partial G$, $\;\gamma(t_1) \neq \gamma(t_2)$;
(2) $\gamma(t) \in G$, for $t_1 < t < t_2$;
(3) $\gamma(t) \notin \overline{G}$, for $t \in [0,1]$ but $t \notin [t_1, t_2]$;
(4) $G \backslash \gamma^*$ has exactly two connected components and $\gamma^* \cap G$ belongs to the boundary of both of these components.

Remark. If γ is injective and smooth and $z_0 \in \gamma^*$ is an arbitrary point on γ, there exists a neighborhood U of z_0, such that γ runs in U from boundary to boundary.

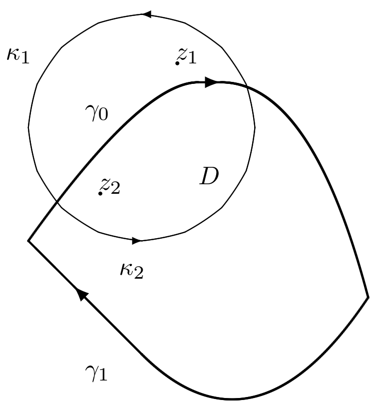
Theorem 2.52. *Let γ be a closed path in $\mathbb{C}$ and let D be a disk. Suppose that γ runs in D from boundary to boundary. Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and $a = \gamma(t_1)$, $, b = \gamma(t_2)$, $a, b \in \partial D$, and $\gamma|_{[t_1,t_2]} = \gamma_0$. Let D_1, D_2 denote the two connected components of $D \setminus \gamma^*$. Suppose that D_1 lies left of γ.*
Then

$$\mathrm{Ind}_\gamma(z_1) = \mathrm{Ind}_\gamma(z_2) + 1,$$

for $z_1 \in D_1$ and $z_2 \in D_2$.

Remark. Let γ in $\mathbb{C}$ be an arbitrary path. The index $\mathrm{Ind}_\gamma(z) = 0$ for z in the unbounded component of $\mathbb{C} \setminus \gamma^*$. Now one can use the theorem from above, in order to compute the indices of γ one after the other.

Proof. We separate $\gamma = \gamma_0 + \gamma_1$, and ∂D in κ_1 and κ_2 (positively oriented), $\kappa_1^* \subset \partial D_1$, $\kappa_2^* \subset \partial D_2$.



Then

$$\mathrm{Ind}_{-\kappa_1+\gamma_1}(z_1) = \mathrm{Ind}_{-\kappa_1+\gamma_1}(z_2)$$

and hence

$$\operatorname{Ind}_{\kappa_1}(z_1) - \operatorname{Ind}_{\kappa_1}(z_2) = \operatorname{Ind}_{\gamma_1}(z_1) - \operatorname{Ind}_{\gamma_1}(z_2),$$

where we used the notation

$$\operatorname{Ind}_{\kappa_j}(z_j) = \frac{1}{2\pi i}\int\limits_{\kappa_j} \frac{d\zeta}{\zeta - z_j},$$

also for paths which are not closed.
z_2 belongs to the unbounded component of $\mathbb{C} \setminus (\kappa_1 + \gamma_0)^*$, so we have

$$\operatorname{Ind}_{\kappa_1+\gamma_0}(z_2) = 0.$$

Similarily

$$\operatorname{Ind}_{\kappa_2-\gamma_0}(z_1) = 0.$$

This gives

$$\begin{aligned}\operatorname{Ind}_{\gamma}(z_1) - \operatorname{Ind}_{\gamma}(z_2) &= \operatorname{Ind}_{\gamma_0}(z_1) - \operatorname{Ind}_{\gamma_0}(z_2) + \operatorname{Ind}_{\gamma_1}(z_1) - \operatorname{Ind}_{\gamma_1}(z_2)\\ &= \operatorname{Ind}_{\gamma_0}(z_1) - \operatorname{Ind}_{\gamma_0}(z_2) + \operatorname{Ind}_{\kappa_1}(z_1) - \operatorname{Ind}_{\kappa_1}(z_2)\\ &= \operatorname{Ind}_{\gamma_0}(z_1) + \operatorname{Ind}_{\kappa_1}(z_1)\\ &= \operatorname{Ind}_{\gamma_0}(z_1) + \operatorname{Ind}_{\kappa_1}(z_1) + \operatorname{Ind}_{\kappa_2-\gamma_0}(z_1)\\ &= \operatorname{Ind}_{\kappa_1+\kappa_2}(z_1) = 1.\end{aligned}$$

□

Lemma 2.53. *Let $A \subset \mathbb{C}$ be a compact subset and $U \supset A$ an open set. Then there exists a cycle Γ in $U \setminus A$ such that*

$$\operatorname{Ind}_{\Gamma}(a) = 1\ ,\ \forall\ a \in A \text{ and } \operatorname{Ind}_{\Gamma}(z) = 0\ ,\ \forall\ z \notin U.$$

Proof. 1.) First we suppose that A is connected. Let $\delta > 0$ be such that $0 < 2\delta < \operatorname{dist}(A, \partial U)$. We use a lattice parallel to the axes with mesh width δ and positively oriented lattice squares. Since A is compact, there exist finitely many lattice squares $Q_1, \dots, Q_n$ with $Q_j \cap A \neq \emptyset\ ,\ j = 1, \dots, n$. Let Γ_j be the boundary cycle of Q_j and define

$$\Gamma = \Gamma_1 + \cdots + \Gamma_n.$$

Let $a \in A$ be an arbitrary point in the compact set A. Without loss of generality we can suppose that $a \in Q_1^\circ$. Then

$$\operatorname{Ind}_{\Gamma}(a) = \sum_{j=1}^{n} \operatorname{Ind}_{\Gamma_j}(a) = \operatorname{Ind}_{\Gamma_1}(a) = 1.$$

If a belongs to ∂Q_j for some j, then a belongs to the interior of four adjacent squares of the lattice, and we get the same result for $\mathrm{Ind}_\Gamma(a)$.
Now we modify Γ in the following way: we take only line segments $[p,q]$, which are line segments of exactly one square of our collection, these are line segments $[p,q]$ with $[p,q] \cap A = \emptyset$; line segments with non-empty intersection with A are passed through in both directions and drop out. The modified boundary cycle is again denoted by Γ.
Now we have $\Gamma^* \cap A = \emptyset$ and, by the choice of the mesh width, that $\Gamma^* \subset U$. Hence $\Gamma^* \subset U \setminus A$. As A is connected, we get from 2.2 that $\mathrm{Ind}_\Gamma(a) = 1\ ,\ \forall\, a \in A$.
If $z \notin U$, it follows that $\mathrm{Ind}_{\Gamma_j}(z) = 0\ ,\ j = 1, \dots, n$ and $\mathrm{Ind}_\Gamma(z) = 0$.

2.) A has finitle many connected components $A_1, \dots A_N$.
Now we choose points $a_j \in A_j\ ,\ j = 1, \dots, N$ and a lattice parallel to the axes with mesh width $\delta > 0$, where

$$2\delta < \min\{\mathrm{dist}(A, \partial U), |a_j - a_k|\ j \neq k\},$$

and we also choose the lattice such that different a_j belong to the interior of different squares of the lattice.
Then $\mathrm{Ind}_\Gamma(a_j) = 1$ and hence $\mathrm{Ind}_\Gamma(a) = 1$ for every jedes $a \in A_j\ ,\ j = 1, \dots, N$. Everything else can now be reduced to the first case.

3.) In the general case, A could have infinitely many connected components. For every $z \in A$ there exists an open square $Q(z)$ with line segments parallel to the axes such that $z \in Q(z) \subset\subset U$; since A is compact, finitely many of these squares cover A. We denote them by $Q_1, \dots, Q_m$. Now let

$$A_0 = \bigcup_{j=1}^{m} \overline{Q}_j \subset U.$$

A_0 is compact and has only finitely many connected components Since $A \subset A_0 \subset U$, it suffices to prove the assertion for A_0, which follows from 2.). $\square$

2.11 Laurent series and meromorphic functions

We now study holomorphic functions on annuli and obtain their canonical representations as Laurent series.

Theorem 2.54. *Leti $D_{r,R}(a) = \{z \in \mathbb{C} : r < |z - a| < R\}$ be an annulus, we define $D_{0,R}(a) = D_R(a) \setminus \{a\}$ and $D_{r,\infty}(a) = \{z : |z - a| > r\}$. Let $f \in \mathcal{H}(D_{r,R}(a))$, and* *define* $U_1 = D_{r,\infty}(a)$, $U_2 = D_R(a)$. *Then there exist functions* $f_1 \in \mathcal{H}(U_1)$ *and* $f_2 \in \mathcal{H}(U_2)$ *such that*

$$f = f_1 + f_2, \quad \textit{auf} \ \ U_1 \cap U_2 = D_{r,R}(a).$$

The function f_1 *can be chosen with the property* $\lim_{|z|\to\infty} f_1(z) = 0$. *In this way* f_1 *and* f_2 *are uniquely determined.*

Proof. Let $r < \rho < R$ and define

$$f_{2,\rho}(z) = \frac{1}{2\pi i} \int\limits_{\gamma_\rho} \frac{f(\zeta)}{\zeta - z}\, d\zeta,$$

where $\gamma_\rho(t) = a + \rho e^{2\pi i t}$, $t \in [0,1]$. The function in the integral is holomorphic for z in $D_\rho(a)$, hence $f_{2,\rho} \in \mathcal{H}(D_\rho(a))$. By 2.49, we have $f_{2,\rho}(z) = f_{2,\rho'}(z)$ on $D_\rho(a)$ for $r < \rho < \rho' < R$.
For $z \in U_2 = D_R(a)$ and $\max\{r, |z-a|\} < \rho < R$, we define

$$f_2(z) = \frac{1}{2\pi i} \int\limits_{\gamma_\rho} \frac{f(\zeta)}{\zeta - z}\, d\zeta,$$

where the integral is independent of ρ as long as $r < \rho < R$. Hence $f_2 \in \mathcal{H}(U_2)$.
For $z \in U_1 = \{z : |z-a| > r\}$ and $r < \sigma < \min\{R, |z-a|\}$, we define

$$f_1(z) = -\frac{1}{2\pi i} \int\limits_{\gamma_\sigma} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

Similarly we get $f_1 \in \mathcal{H}(U_1)$, and from the definition of f_1 we derive immediately that $\lim_{|z|\to\infty} |f_1(z)| = 0$.
For $z \in D_{r,R}(a)$, we choose ρ and σ such that

$$r < \sigma < |z-a| < \rho < R$$

and define the cycle $\Gamma = \gamma_\rho - \gamma_\sigma$.
It follows thatt $\mathrm{Ind}_\Gamma(\alpha) = 0$, $\forall\ \alpha \notin D_{r,R}(a)$ and $\mathrm{Ind}_\Gamma(z) = 1$. Now we apply 2.46 and obtain

$$f(z) = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{2\pi i} \int\limits_{\gamma_\rho} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i} \int\limits_{\gamma_\sigma} \frac{f(\zeta)}{\zeta - z}\, d\zeta = f_1(z) + f_2(z).$$

It remains to show uniqueness of the representation $f = f_1 + f_2$. For this aim let $f = g_1 + g_2$ another representation with $g_1 \in \mathcal{H}(U_1)$ and $g_2 \in \mathcal{H}(U_2)$, as well as $\lim_{|z|\to\infty} |g_1(z)| = 0$.

Then we have $f_1 - g_1 = g_2 - f_2$ on $U_1 \cap U_2$. We define

$$h = \begin{cases} f_1 - g_1 & \text{auf } U_1 \\ g_2 - f_2 & \text{auf } U_2 \end{cases}$$

and get a holomorphic function on $U_1 \cup U_2 = \mathbb{C}$.
Hence $h \in \mathcal{H}(\mathbb{C})$ and $\lim_{|z|\to\infty} |h(z)| = 0$. So h is a bounded entire function. By Liouville's theorem 2.26, it follows that $h \equiv 0$, and $f_1 = g_1$ as well as $f_2 = g_2$. □

Remark. f_1 is called principal part of f.
Since f_2 is holomorphic on $D_R(a)$, it can be expanded as a Taylor series

$$f_2(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \quad , \ z \in D_R(a).$$

Let $F(w) = a + 1/w$. Then F is a biholomorphic mapping (i.e. holomorphic in bpth directions) from $D'_{1/r}(0) = \{w : 0 < |w| < 1/r\}$ to $U_1 = \{w : |w-a| > r\}$. Hence, the composed function $f_1 \circ F \in \mathcal{H}(D'_{1/r}(0))$.
Since $\lim_{|z|\to\infty} |f_1(z)| = 0$, we have $\lim_{w\to 0}(f_1 \circ F)(w) = 0$. Hence $f_1 \circ F$ has a removable singularity at $w = 0$ (see 2.21), hence also ist $f_1 \circ F \in \mathcal{H}(D_{1/r}(0))$ and we can expand as a Taylor series around $w = 0$:

$$(f_1 \circ F)(w) = \sum_{n=1}^{\infty} b_n w^n,$$

where the series converges uniformly on $\overline{D_{1/\rho}(0)}$ $\rho > r$.
For $w = 1/(z-a)$, we have $F(w) = z$ and

$$f_1(z) = \sum_{n=-1}^{-\infty} a_n (z-a)^n,$$

where $a_{-n} = b_n$ and the series converges uniformly on $\mathbb{C} \setminus \overline{D_\rho(a)}$, $\rho > r$.

Theorem 2.55. *Let $f \in \mathcal{H}(D_{r,R}(a))$. Then f can be represented in the form*

$$f(z) = \sum_{n=-1}^{-\infty} a_n (z-a)^n + \sum_{n=0}^{\infty} a_n (z-a)^n,$$

[6] *which is the Laurent series of f in $D_{r,R}(a)$, and the series converges uniformly on all compact subsets of $D_{r,R}(a)$. The Laurent coefficients a_n are given by*

$$a_n = \frac{1}{2\pi i} \int\limits_{\gamma_\rho} \frac{f(z)}{(z-a)^{n+1}}\, dz \quad , \ n \in \mathbb{Z}$$

6 Laurent, Pierre Alphonse (1813–1854) *where* $\gamma_\rho(t) = a + \rho e^{2\pi i t}$ $, t \in [0,1]$ $,$ $r < \rho < R$.

Proof. It remains to prove the formula for the Laurent coefficients a_n. We have

$$(z-a)^{-n-1} f(z) = \sum_{k=-1}^{-\infty} a_{k+n+1}(z-a)^k + \sum_{k=0}^{\infty} a_{k+n+1}(z-a)^k$$

with uniform convergence on γ_ρ^*. Hence integration term by term yields

$$\int\limits_{\gamma_\rho} \frac{f(z)}{(z-a)^{n+1}}\, dz = a_n \int\limits_{\gamma_\rho} \frac{dz}{z-a} = 2\pi i a_n,$$

all summands are zero, except for $k = -1$. □

Example 2.56. 1) Let

$$f(z) = \frac{1}{z(z-i)^2}.$$

(a) Laurent series expansion in $D_1'(0) = \{z : 0 < |z| < 1\}$:

$$\frac{1}{z(z-i)^2} = -\frac{1}{z}\,\frac{1}{(1-z/i)^2} = -\frac{1}{z}\,\sum_{n=0}^{\infty}(n+1)\left(\frac{z}{i}\right)^n = -\frac{1}{z} + i\sum_{n=0}^{\infty}\frac{n+2}{i^n}\,z^n.$$

(b) Laurent series expansion in $D_{1,\infty}(0) = \{z : |z| > 1\}$:

$$\frac{1}{z(z-i)^2} = \frac{1}{z^3}\,\frac{1}{(1-i/z)^2} = \sum_{n=-3}^{-\infty} i^{-n-1}(n+2)z^n.$$

(c) Laurent series expansion in $D_{0,1}(i) = \{z : 0 < |z-i| < 1\}$:

$$\frac{1}{z(z-i)^2} = \frac{-i}{(z-i)^2} + \frac{1}{z-i} - \frac{1}{z} = \frac{-i}{(z-i)^2} + \frac{1}{z-i} - \frac{i}{1-i(z-i)} = \dots,$$

where the last term can be written as the sum of a geometric series.
2) Consider the Laurent series

$$\sum_{n=1}^{\infty} z^{-n} + \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}$$

with infinitely many negative powers of z. The first summand converges to $\frac{1}{z-1}$ for $|\, z\,| > 1$, and the second summand converges to $\frac{1}{2-z}$ for $|\, z\,| < 2$. So the whole series converges to the function $f(z) = \frac{1}{z-1} + \frac{1}{2-z}$ on the annulus $D_{1,2}(0)$. The function f is holomorphic at $z = 0$, altough the Laurent series in the annulus $D_{1,2}(0)$ has infinitely many negative powers of z.

Remark. We return to the characterization of isolated singularities 2.22, and can easily show that a function f has a pole of order k in a point a if and only if the Laurent expansion of f in the *punctured disk* $D'_r(a)$ has the form

$$f(z) = a_{-k}(z-a)^{-k} + \dots + a_{-1}(z-a)^{-1} + a_0 + a_1(z-a) + \dots,$$

where $a_{-k} \neq 0$.
In addition, it follows that f has an essential singularity in a if and only if the Laurent expansion of f in the *punctured disk* $D'_r(a)$ has infinitely many terms of the form $a_{-k}(z-a)^{-k}$, where $k > 0$ and $a_{-k} \neq 0$.

Definition 2.57. Let $U \subseteq \mathbb{C}$ be an open set. A function f is called meromorphic on U, if there exists a discrete subset P of U such that $f \in \mathcal{H}(U \setminus P)$ and f has poles in P.
$\mathcal{M}(U)$ denotes the set of all meromorphic functions on U.

EXAMPLES. 1) Let $U = D_1(0)$ and $f(z) = 1/z$. Then $f \in \mathcal{M}(U)$.
Every rational function p/q, where p and q are polynomials, belongs to $\mathcal{M}(\mathbb{C})$.
2) $\tan z = \frac{\sin z}{\cos z}$ belongs to $\mathcal{M}(\mathbb{C})$. It is easily seen that $\tan z$ has infinitely many poles.
3) Let $f, g \in \mathcal{H}(U)$, suppose that $g \not\equiv 0$ on every connected component of U. Then $f/g \in \mathcal{M}(U)$. (see 2.14)

Theorem 2.58. *Let $f \in \mathcal{M}(U)$. Then, for each $a \in U$, there exists an open neighborhood V of a, and $g, h \in \mathcal{H}(V)$ such that $f = g/h$ on V.*

Proof. If a is not a pole of f, we put $g = f$ and $h \equiv 1$ and take $V = U \setminus P_f$, where P_f denotes the set of all poles of f. Then $f = g/h$ on V and $g, h \in \mathcal{H}(V)$.
If a is a pole of order $m > 0$ of f, then, by 2.22, there exist complex numbers $c_1, \dots, c_m$ $(c_m \neq 0)$, such that the function

$$f(z) - \sum_{k=1}^{m} \frac{c_k}{(z-a)^k} = \phi(z)$$

has a removable singularity in a. Hence

$$f(z) = \phi(z) + \sum_{k=1}^{m} \frac{c_k}{(z-a)^k} = \frac{1}{(z-a)^m} \left[(z-a)^m \phi(z) + \sum_{k=1}^{m} c_k (z-a)^{m-k} \right],$$

where the expression in brackets is holomorphic in a neighborhood of a. Denote this expression by g, then we have $g(a) = c_m \neq 0$ and $f(z) = g(z)/(z-a)^m$ in a suitable neighborhood of a. □

Later we will be able to show that $f = g/h$ globally on U.

Theorem 2.59. *Let $f \in \mathcal{M}(U)$ and let a be a pole of f. Then*

$$\lim_{z \to a} |f(z)| = \infty,$$

i.e. for each compact subset $K \subset \mathbb{C}$ there exists $\delta > 0$ with $f(D'_\delta(a)) \subseteq \mathbb{C} \setminus K$.

Proof. From the proof of the last theorem we get

$$f(z) = \frac{1}{(z-a)^m} \left[(z-a)^m \phi(z) + \sum_{k=1}^{m} c_k (z-a)^{m-k} \right],$$

where $c_m \neq 0$. The limit $z \to a$ yields the desired result. $\square$

Theorem 2.60. *Let Ω be a domain in $\mathbb{C}$. Then $\mathcal{M}(\Omega)$ is a field with respect to pointwise addition and multiplication of functions.*

2.12 The Residue Theorem

The residue theorem is not only a generalization of the homology version of Cauchy's theorem when the function f has singularities, it enables us also to evaluate definite real integrals which are certainly not solvable by methods of real analysis and to count the number of zeroes and poles of meromorphic functions.

Definition 2.61. Let $U \subseteq \mathbb{C}$ be open and f a holomorphic function on U except for isolated singularities. Let a be an isolated singularity of f. Then there exists $r > 0$, such that f can be expanded in $D'_r(a)$ as a Laurent series

$$f(z) = \sum_{n=-1}^{-\infty} c_n (z-a)^n + \sum_{n=0}^{\infty} c_n (z-a)^n,$$

the coefficient $c_{-1} = \operatorname{Res}(f; a)$ is called the residue of f at a.

Remark. If one computes the line integral

$$\int_{\gamma_s} f(z)\, dz,$$

where $\gamma_s(t) = a - se^{it},\ t \in [0, 2\pi]$ and $0 < s < r$, one observes that the integral can be computed term by term in the Laurent series expansion and that only one term is left over, namely

$$\int\limits_{\gamma_s} \frac{c_{-1}}{z-a}\, dz = 2\pi i c_{-1},$$

which motivates the notion of a residue.

Theorem 2.62 (Residue Theorem). *Let $U \subseteq \mathbb{C}$ be open and f a holomorphic function on U except for isolated singularities. Denote by S_f the set of all singularities of f in U. Let Γ be a cycle in $U \setminus S_f$ such that* $\operatorname{Ind}_\Gamma(\alpha) = 0 \quad \forall\ \alpha \notin U$. *Then*

$$\frac{1}{2\pi i} \int\limits_\Gamma f(z)\, dz = \sum_{a \in S_f} \operatorname{Res}(f; a) \operatorname{Ind}_\Gamma(a).$$

Proof. First we show that the set $B = \{a \in S_f : \operatorname{Ind}_\Gamma(a) \neq 0\}$is finite, which implies that the sum in the theorem is a finite sum. For this aim let $W = \mathbb{C} \setminus \Gamma^*$. The index $\operatorname{Ind}_\Gamma$ is constant on every connected components V of W. If V is unbounded or if $V \cap (\mathbb{C} \setminus U) \neq \emptyset$, then $\operatorname{Ind}_\Gamma(\alpha) = 0\quad,\ \forall\ \alpha \in V$, by our assumption that $\operatorname{Ind}_\Gamma(\alpha) = 0\ \ \forall\ \alpha \notin U.$
S_f has no limit point in U, the limit points of S_f can only be on the boundary of U, therefore limit points can belong to the unbounded component of W or to a component V with $V \cap (\mathbb{C} \setminus U) \neq \emptyset$. We have $\operatorname{dist}(\Gamma^*, \partial U) > 0$, hence B must be finite.
Let $B = \{a_1, a_2, \ldots, a_n\}$, and let Q_j be the principal parts of f in the Laurent expansion around $a_j\ ,\ j = 1, \ldots, n$. Define

$$g = f - (Q_1 + Q_2 + \cdots + Q_n),$$

(if $B = \emptyset$, set $g = f$) then g has removable singularities in the points of B, and $g \in \mathcal{H}(U_0)$, where $U_0 = U \setminus (S_f \setminus B)$. Now $\operatorname{Ind}_\Gamma(\alpha) = 0\ ,\ \forall \alpha \notin U_0$, and, by 2.46, we have

$$\int\limits_\Gamma g(z)\, dz = 0$$

and, by the definition of g, we get

$$\begin{aligned}\frac{1}{2\pi i} \int\limits_\Gamma f(z)\, dz &= \frac{1}{2\pi i} \sum_{k=1}^{n} \int\limits_\Gamma Q_k(z)\, dz = \sum_{k=1}^{n} \operatorname{Res}(Q_k; a_k) \operatorname{Ind}_\Gamma(a_k) \\ &= \sum_{k=1}^{n} \operatorname{Res}(f; a_k) \operatorname{Ind}_\Gamma(a_k).\end{aligned}$$

$\square$

Remark. If U is open and convex and Γ is a closed, positively oriented path without double points in U and f is holomorphic in U except for isolated singularities, then

$$\frac{1}{2\pi i}\int_{\Gamma} f(z)\,dz = \sum \operatorname{Res}(f;a_k),$$

where the sum is taken over all singularities of f in the interior of Γ.

For the applications it will be convenient to know about some simple rules how to compute residues.

Theorem 2.63. *Let $U \subseteq \mathbb{C}$ be open and let f and g be holomorphic on U except for isolated singularities. Then*
(a)

$$\operatorname{Res}(f+g;a) = \operatorname{Res}(f;a) + \operatorname{Res}(g;a)$$

and for $\alpha_1, \alpha_2 \in \mathbb{C}$

$$\operatorname{Res}(\alpha_1 f + \alpha_2 g;a) = \alpha_1 \operatorname{Res}(f;a) + \alpha_2 \operatorname{Res}(g;a).$$

(b) *If z_0 is a pole of first order of f, we have*

$$\operatorname{Res}(f;z_0) = \lim_{z\to z_0}[(z-z_0)f(z)].$$

(c) *If g is holomorphic in z_0 and f has a pole of first order in z_0, we have*

$$\operatorname{Res}(fg;z_0) = g(z_0)\operatorname{Res}(f;z_0).$$

(d) *If h is holomorphic in z_0 und z_0 is a simple zero of h, we have*

$$\operatorname{Res}(1/h;z_0) = 1/h'(z_0).$$

(e) *If z_0 is a pole of order n of f, we have*

$$\operatorname{Res}(f;z_0) = \frac{1}{(n-1)!}\lim_{z\to z_0}\left\{\frac{d^{n-1}}{dz^{n-1}}[(z-z_0)^n f(z)]\right\}.$$

Proof. (a) Follows from the Laurent expansion of f and g around a.

(b) The Laurent expansion of f around z_0 has the form

$$f(z) = \frac{c_{-1}}{z-z_0} + \sum_{n=0}^{\infty} c_n(z-z_0)^n,$$

hence

$$(z-z_0)f(z) = c_{-1} + \sum_{n=0}^{\infty} c_n(z-z_0)^{n+1},$$

taking the limit $z \to z_0$, the infinite series disappears.

(c) We have

$$g(z) = g(z_0) + \sum_{n=1}^{\infty} b_n (z - z_0)^n \quad , \quad f(z) = \frac{c_{-1}}{z - z_0} + \sum_{n=0}^{\infty} c_n (z - z_0)^n,$$

this implies

$$f(z)g(z) = \frac{g(z_0)c_{-1}}{z - z_0} + \sum_{n=0}^{\infty} d_n (z - z_0)^n.$$

(d) $1/h$ has a pole of first order in z_0. Hence, by (b),

$$\mathrm{Res}(1/h; z_0) = \lim_{z \to z_0} \frac{z - z_0}{h(z)} = \lim_{z \to z_0} \frac{z - z_0}{h(z) - h(z_0)} = 1/h'(z_0).$$

(e) We have

$$f(z) = \frac{c_{-n}}{(z - z_0)^n} + \cdots + \frac{c_{-1}}{z - z_0} + \sum_{k=0}^{\infty} c_k (z - z_0)^k$$

which implies that

$$(z - z_0)^n f(z) = c_{-n} + c_{-n+1}(z - z_0) + \cdots + c_{-1}(z - z_0)^{n-1} + \sum_{k=0}^{\infty} c_k (z - z_0)^{k+n}.$$

Differentiating $(n-1)$ times we get the desired result. □

Theorem 2.64 (Rouché's Theorem). [7] *Let Ω be a domain in $\mathbb{C}$ and $f \in \mathcal{M}(\Omega)$ a meromorphic function. Let γ be a closed, null-homologous path in Ω. Suppose that f has no zeros and no poles on γ^* and that $\mathrm{Ind}_\gamma(\alpha) = 1$ or $= 0 \ \ \forall\, \alpha \in \mathbb{C} \setminus \gamma^*$. Let $\Omega_1 = \{z \in \Omega : \mathrm{Ind}_\gamma(z) = 1\}$ and N_f the number of zeros of f in Ω_1 and let P_f be the number of poles of f in Ω_1. Then*

$$N_f - P_f = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz = \mathrm{Ind}_\Gamma(0),$$

where $\Gamma = f \circ \gamma$.
In addition, let $g_1, g_2 \in \mathcal{H}(\Omega)$ be holomorphic functions on Ω such that

$$|g_1(z) - g_2(z)| < |g_1(z)| \quad , \ \forall\, z \in \gamma^*.$$

Then $N_{g_2} = N_{g_1}$.

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7 Rouché, Eugéne (1832–1910)

Proof. Let $\phi = f'/f$. Then $\phi \in \mathcal{M}(\Omega)$. Now let a be a zero of f of order $m(a)$. Then, by 2.14,

$$f(z) = (z-a)^{m(a)}h(z),$$

where h is holomorphic in a neighborhood of a and $h \neq 0$ there. We obtain

$$\phi(z) = \frac{m(a)(z-a)^{m(a)-1}h(z) + (z-a)^{m(a)}h'(z)}{(z-a)^{m(a)}h(z)} = \frac{m(a)}{z-a} + \frac{h'(z)}{h(z)},$$

where the second summand is holomorphic in a neighborhood of a. Hence $\operatorname{Res}(\phi; a) = m(a)$.

If b is a pole of f of order $p(b)$, we obtain from the Laurent expansion of f around b that

$$f(z) = (z-b)^{-p(b)}k(z),$$

where k is holomorphic in a neighborhood of b and $k \neq 0$ there. An analogous computation as above shows that $\operatorname{Res}(\phi; b) = -p(b)$.

Let $A = \{a \in \Omega_1 : f(a) = 0\}$ and B the set of all poles of f in Ω_1. Then, by 2.62,

$$\frac{1}{2\pi i}\int\limits_{\gamma} \frac{f'(z)}{f(z)}\,dz = \sum_{a\in A} \operatorname{Res}(\phi; a) + \sum_{b\in B} \operatorname{Res}(\phi; b) = \sum_{a\in A} m(a) - \sum_{b\in B} p(b) = N_f - P_f.$$

The chain rule implies that

$$\operatorname{Ind}_\Gamma(0) = \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{dz}{z} = \frac{1}{2\pi i}\int\limits_{0}^{1} \frac{f'(\gamma(s))}{f(\gamma(s))}\gamma'(s)\,ds = \frac{1}{2\pi i}\int\limits_{\gamma} \frac{f'(z)}{f(z)}\,dz = N_f - P_f.$$

Our assumption $|g_1(z) - g_2(z)| < |g_1(z)| \;\;, \;\forall\; z \in \gamma^*$, implies that g_2 has no zeros on γ^*. Let $\Gamma_1 = g_1 \circ \gamma$ and let $\Gamma_2 = g_2 \circ \gamma$. Then

$$|\Gamma_1(s) - \Gamma_2(s)| < |\Gamma_1(s)| \;\;, \;\forall\; s \in [0,1].$$

Now we apply 2.48 and the first part of the proof to obtain

$$N_{g_1} = \operatorname{Ind}_{\Gamma_1}(0) = \operatorname{Ind}_{\Gamma_2}(0) = N_{g_2}.$$

$\square$

Example. Let $g(z) = z^4 - 4z + 2$. How many zeros has g in $D_1(0)$?
On $|z| = 1$we have : $|z|^4 = 1 < 2 \leq |-4z+2|$. Set $f(z) = -4z + 2$, then on $|z| = 1$ we have

$$|f(z) - g(z)| = |-4z + 2 - z^4 + 4z - 2| = |z^4| < |-4z+2| = |f(z)|.$$

f has exactly one zero in $D_1(0)$, namely $z_0 = 1/2$, henc, by 2.64, we get that g also has exactly one zero in $D_1(0)$. hat.

Example 2.65. Applications of the Residue Theorem

First we compute the real definite integrals

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2-2x+2}\,dx \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{\sin x}{x^2-2x+2}\,dx.$$

For this purpose we consider the line integral

$$\int_{\gamma_R} \frac{e^{iz}}{z^2-2z+2}\,dz = \int_{-R}^{R} \frac{e^{ix}}{x^2-2x+2}\,dx + \int_{\Gamma_R} \frac{e^{iz}}{z^2-2z+2}\,dz \tag{2.6}$$

where γ_R is the path consisting of the line segment on the real axis from $-R$ to R and the semicircle Γ_R from R to $-R$ in the upper halfplane, we take $R > 3$.
The function $e^{iz}/(z^2-2z+2)$ has poles of first order at the points $1+i$ and $1-i$. The point $1+i$ lies in the interior of γ and we use 2.63 (b) to compute the residue of this function at $1+i$:

$$\lim_{z\to 1+i} \frac{e^{iz}(z-1-i)}{z^2-2z+2} = \lim_{z\to 1+i} \frac{e^{iz}}{z-1+i} = -\frac{ie^{-1+i}}{2}.$$

By the Residue Theorem 2.62 we have

$$\int_{\gamma_R} \frac{e^{iz}}{z^2-2z+2}\,dz = 2\pi i(-ie^{-1+i})/2 = \pi e^{-1+i}. \tag{2.7}$$

If $z = x+iy$ lies on the semicircle Γ_R from R to $-R$ in the upper half plane, we have $y \geq 0$, hence $|e^{iz}| = e^{-y} \leq 1$, and we can estimate

$$\left|\frac{e^{iz}}{z^2-2z+2}\right| \leq \frac{1}{R^2-2R-2},$$

for $z \in \Gamma_R^*$. This implies

$$\left|\int_{\Gamma_R} \frac{e^{iz}}{z^2-2z+2}\,dz\right| \leq \frac{\pi R}{R^2-2R-2},$$

and we obtain

$$\int_{\Gamma_R} \frac{e^{iz}}{z^2-2z+2}\,dz \to 0 \quad \text{as} \quad R \to \infty.$$

By (2.6) and (2.7), we get

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{e^{ix}}{x^2-2x+2}\,dx = \pi e^{-1+i} = \frac{\pi(\cos 1 + i \sin 1)}{e}.$$

Taking real and imaginary part on each side we finally obtain

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 - 2x + 2}\, dx = \frac{\pi \cos 1}{e} \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{\sin x}{x^2 - 2x + 2}\, dx = \frac{\pi \sin 1}{e}.$$

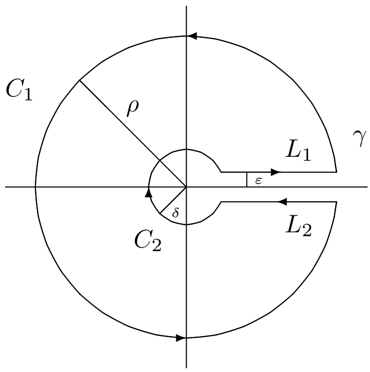
In the next example we consider two polynomials P and Q with grad $Q \geq$ grad $P+2$. Suppose that $Q(x) \neq 0 \;\; \forall\, x > 0$ and that $Q(0) = 0$ is a simple zero. Let $0 < \alpha < 1$ and $R = P/Q$. Using the Residue Theorem we will compute the real definite integral

$$\int_0^{\infty} x^{\alpha} R(x)\, dx,$$

where $x^{\alpha} = \exp(\alpha \log x)$.
For this purpose we choose the star-shaped domain $\Omega = \mathbb{C} \setminus \{x \in \mathbb{R} : x \geq 0\}$. By 2.19, there exists a branch g of the logarithm on Ω (it is not the principal branch) such that for fix $x > 0$:

$$\lim_{y \to 0^+} g(x + iy) = \log x \quad , \quad \lim_{y \to 0^+} g(x - iy) = \log x + 2\pi i.$$

For $\delta, \epsilon > 0$ small and $\rho > 0$ large, choose the following closed path γ in Ω :



By the Residue Theorem 2.62 we have

$$\frac{1}{2\pi i} \int_{\gamma} e^{\alpha g(z)} R(z)\, dz = \sum_a \operatorname{Res}(f; a), \tag{2.8}$$

where $f(z) = e^{\alpha g(z)} R(z)$ and the sum is taken over all poles of f in the interior of γ.
Observe that

$$|e^{\alpha g(z)}| = e^{\alpha \Re g(z)} = e^{\alpha \log |z|} = |z|^{\alpha}$$

and

$$|R(z)| \leq \frac{M}{|z|} \quad , \ M > 0$$

in a neighborhood of zero, since we supposed that Q has a zero of order ≤ 1 at zero, and choose $\delta > 0$ small enough that the last estimate holds on C_2^*.
Now we have

$$|\int_{C_2} f(z)\, dz| \leq 2\pi\delta \max_{z \in C_2^*} (|z|^\alpha |R(z)|) \leq 2\pi\delta\delta^\alpha M/\delta = 2\pi M \delta^\alpha.$$

Hence

$$\lim_{\delta \to 0} |\int_{C_2} f(z)\, dz| = 0. \tag{2.9}$$

On the other hand

$$|R(z)| \leq \frac{M'}{|z|^2} \quad , \ M' > 0$$

for $|z|$ large enough, since grad $Q \geq$ grad $P + 2$.
Hence we get for $\rho > 0$ large enough that

$$|\int_{C_1} f(z)\, dz\,| \leq 2\pi\rho \max_{z \in C_1^*} (|z|^\alpha |R(z)|) \leq 2\pi\rho^{\alpha+1} M'/\rho^2 = 2\pi M' \rho^{\alpha-1},$$

and since $0 < \alpha < 1$, we obtain

$$\lim_{\rho \to \infty} |\int_{C_1} f(z)\, dz| = 0. \tag{2.10}$$

Now fix δ and ρ, and take the limit $\epsilon \to 0$

$$\begin{aligned}\int_{L_1} f(z)\, dz + \int_{L_2} f(z)\, dz &\to \int_\delta^\rho e^{\alpha \log x} R(x)\, dx - \int_\delta^\rho e^{\alpha(\log x + 2\pi i)} R(x)\, dx \\ &= (1 - e^{2\pi i \alpha}) \int_\delta^\rho e^{\alpha \log x} R(x)\, dx.\end{aligned}$$

Hence we obtain from (2.9) and (2.10)

$$\int_\gamma f(z)\, dz = \int_{C_1} + \int_{L_1} + \int_{L_2} + \int_{C_2} \to (1 - e^{2\pi i \alpha}) \int_0^\infty e^{\alpha \log x} R(x)\, dx,$$

where we took first the limit $\epsilon \to 0$, then $\delta \to 0$ and finally $\rho \to \infty$.

Taking these limits, the sum on the right hand side of (2.8) must now we taken over all poles of f in Ω. This implies

$$\int_0^\infty x^\alpha R(x)\,dx = \frac{2\pi i}{1-e^{2\pi i\alpha}} \sum_{a\in\Omega} \operatorname{Res}(f;a). \tag{2.11}$$

Now let $R(x) = 1/(1+x^2)$. By (2.11) we obtain

$$\int_0^\infty \frac{x^\alpha}{1+x^2}\,dx = \frac{2\pi i}{1-e^{2\pi i\alpha}} \left[\operatorname{Res}(z^\alpha/(1+z^2);i) + \operatorname{Res}(z^\alpha/(1+z^2);-i)\right].$$

Using 2.63 we can compute the residues

$$\operatorname{Res}(z^\alpha/(1+z^2);i) = \lim_{z\to i}\left[(z-i)\frac{z^\alpha}{(z+i)(z-i)}\right] = \frac{i^\alpha}{2i} = \frac{1}{2}\,e^{i(\alpha-1)\pi/2},$$

$$\operatorname{Res}(z^\alpha/(1+z^2);-i) = \lim_{z\to -i}\left[(z+i)\frac{z^\alpha}{(z+i)(z-i)}\right] = \frac{(-i)^\alpha}{-2i} = \frac{1}{2}\,e^{i(\alpha-1)3\pi/2},$$

and get finally

$$\int_0^\infty \frac{x^\alpha}{1+x^2}\,dx = \frac{2\pi i}{1-e^{2\pi i\alpha}}\,\frac{1}{2}\left[e^{i(\alpha-1)\pi/2} + e^{i(\alpha-1)3\pi/2}\right] = \frac{\pi}{2}\,\frac{1}{\cos(\alpha\pi/2)}.$$

Example 2.66. We compute the inverse Fourier transform of $x \mapsto \frac{\sin x}{x}$: for $t \in \mathbb{R}$ we compute

$$\lim_{A\to\infty} \int_{-A}^{A} \frac{\sin x}{x}\,e^{itx}\,dx.$$

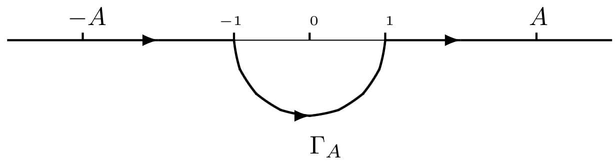
The function $z \mapsto \frac{\sin z}{z}$ has a removable singularity at 0, therefore

$$\psi(z) = \frac{\sin z}{z}\,e^{izt} = \frac{1}{2i}\,\frac{e^{iz(1+t)} - e^{iz(-1+t)}}{z}$$

is an entire function. Hence, by 2.6, we obtain

$$\int_{-A}^{A} \frac{\sin x}{x}\,e^{itx}\,dx = \int_{\Gamma_A} \psi(z)\,dz,$$

where the path Γ_A is shown in the figure from below.



Now set

$$\frac{1}{\pi}\phi_A(s) = \frac{1}{2\pi i}\int_{\Gamma_A} \frac{e^{isz}}{z}\,dz.$$

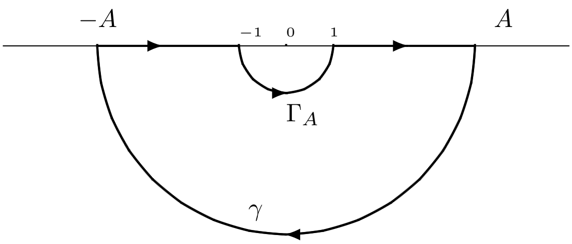
Then

$$\int_{-A}^{A} \frac{\sin x}{x}\, e^{itx}\,dx = \phi_A(t+1) - \phi_A(t-1).$$

The function $z \mapsto e^{isz}/z$ has a pole of first order at 0 with residue 1. Hence, by the Residue Theorem 2.62, we have

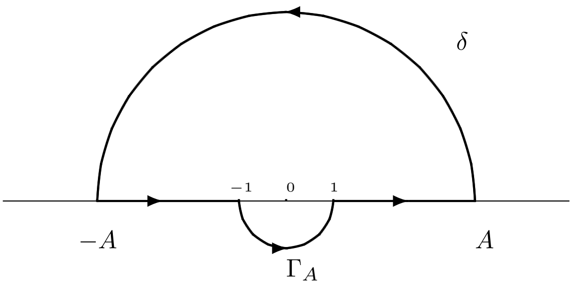
$$\frac{1}{2\pi i}\int_{\gamma} \frac{e^{isz}}{z}\,dz = 0,$$

where the path γ is shown in the figure from below.



In addition

$$\frac{1}{2\pi i}\int_{\delta} \frac{e^{isz}}{z}\,dz = 1.$$



Hence we get

$$\frac{1}{2\pi i}\int_{\gamma}\frac{e^{isz}}{z}\,dz = \frac{1}{\pi}\phi_A(s) + \frac{1}{2\pi i}\int_{0}^{-\pi}\exp(isAe^{i\theta})\,\frac{iAe^{i\theta}}{Ae^{i\theta}}\,d\theta = 0,$$

which implies

$$\frac{1}{\pi}\phi_A(s) = \frac{1}{2\pi}\int_{-\pi}^{0}\exp(isAe^{i\theta})\,d\theta, \tag{2.12}$$

and

$$\frac{1}{2\pi i}\int_{\delta}\frac{e^{isz}}{z}\,dz = \frac{1}{\pi}\phi_A(s) + \frac{1}{2\pi}\int_{0}^{\pi}\exp(isAe^{i\theta})\,d\theta = 1,$$

and finally

$$\frac{1}{\pi}\phi_A(s) = 1 - \frac{1}{2\pi}\int_{0}^{\pi}\exp(isAe^{i\theta})\,d\theta. \tag{2.13}$$

If s and $\sin\theta$ have the same signature,

$$\left|\exp(isAe^{i\theta})\right| = \exp(-sA\sin\theta) \to 0$$

as $A \to \infty$. By the dominated convergence theorem, we obtain from (2.12) and (2.13)

$$\lim_{A\to\infty}\phi_A(s) = \begin{cases}\pi\ , & s > 0\\ 0\ , & s < 0.\end{cases}$$

Again from (2.12) or (2.13) we get $\phi_A(0) = \pi/2$.
Hence

$$\begin{aligned}\lim_{A\to\infty}\int_{-A}^{A}\frac{\sin x}{x}\,e^{itx}\,dx &= \lim_{A\to\infty}[\phi_A(t+1) - \phi_A(t-1)]\\ &= \chi(t) = \begin{cases}\pi\ , & -1 < t < 1\\ \pi/2\ , & t = \pm 1\\ 0\ , & |t| > 1\end{cases}\end{aligned}$$

The Fourier transform of χ is

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\chi(t)e^{-itx}\,dt = \frac{\sin x}{x}.$$

2.13 Exercises

35) Compute the following line integrals:

$$\int_{\gamma_1} (z-a)^k \, dz,$$

where $a \in \mathbb{C},\ k \in \mathbb{Z}$ and γ_1 is the unit circle $|z| = 1$ passed through once in positive direction.

36) Let $\gamma(t) = e^{it},\ 0 \le t \le 2\pi$. Compute the following line integrals and compare the results with the assertions of Cauchy's Theorem:

$$\int_{\gamma} (\overline{z})^2 \, dz, \quad \int_{\gamma} z^{-2} \, dz,$$

where the annulus $A = \{z : \frac{1}{2} < |z| < 2\}$ is the corresponding domain.

37) Let $\gamma(t) = re^{it},\ r > 0,\ 0 \le t \le 2\pi$, and let $a, b \in \mathbb{C}$ such that $|a| < r < |b|$. Show that

$$\int_{\gamma} \frac{dz}{(z-a)(z-b)} = \frac{2\pi i}{a-b}.$$

38) Let γ, a, b be as in Exercise 37), $m, n \in \mathbb{N}$. Compute :

$$\int_{\gamma} \frac{dz}{(z-a)^m (z-b)^n}.$$

39) Let $\gamma(t) = e^{it},\ 0 \le t \le 2\pi$. Compute :

$$\int_{\gamma} \frac{e^{iz}}{z^2} \, dz, \quad \int_{\gamma} \frac{\sin z}{z^3} \, dz.$$

40) Let $\gamma(t) = e^{it},\ 0 \le t \le 2\pi$ and $n \in \mathbb{N}$. Compute:

$$\int_{\gamma} \frac{e^z - e^{-z}}{z^n} \, dz, \quad \int_{\gamma} \frac{dz}{(z-1/2)^n}.$$

41) Let $\gamma(t) = 1 + \frac{1}{2}e^{it},\ 0 \le t \le 2\pi,\ n \in \mathbb{N}_0$. Compute:

$$\int_{\gamma} \frac{\mathrm{Log}(z)}{z^n} \, dz.$$

42) Let $\gamma(t) = -1 + e^{it},\ 0 \le t \le 2\pi$. Compute:

$$\int_\gamma \frac{dz}{(z+1)(z-1)^3}.$$

43) Let $\gamma(t) = 2e^{it},\ 0 \le t \le 2\pi$. Compute:

$$\int_\gamma \frac{\sin z}{z+i}\, dz.$$

44) Let $\gamma(t) = \frac{1}{2}e^{it},\ 0 \le t \le 2\pi$. Compute:

$$\int_\gamma \frac{e^{1-z}}{z^3(1-z)}\, dz.$$

45) Compute:

$$\int_\gamma \frac{z^2+1}{z(z^2+4)}\, dz,$$

where $\gamma(t) = re^{it},\ 0 \le t \le 2\pi$, first for $0 < r < 2$ and then for $2 < r < \infty$.

46) Prove:

$$\int_0^{2\pi} \cos(\cos\theta)\cosh(\sin\theta)\, d\theta = 2\pi.$$

Hint: use the mean value property for the function $f(z) = \cos z$.

47)(a) Let $U \subseteq \mathbb{C}$ be an open and L a straight line, suppose that $f : U \longrightarrow \mathbb{C}$ is continuous and holomorphic on $U \setminus L$. Show that f is holomorphic on the whole of U. (Use Morera's Theorem!)
(b) Let G be a domain in $\mathbb{C}$, which is symmetric with respect to the real axis, i.e. if $z \in G$ then $\overline{z} \in G$). Let

$$f : \{z \in G : \Im z \ge 0\} \longrightarrow \mathbb{C}$$

be continuous, suppose that f is holomorphic on $\{z \in G : \Im z > 0\}$ and has real values on $\{z \in G : \Im z = 0\}$. Show that

$$\tilde{f}(z) = \begin{cases} f(z), & \text{für } \Im z \ge 0 \\ \overline{f(\overline{z})}, & \text{für } \Im z < 0 \end{cases}$$

is a holomorphic function on G. (Schwarz's reflection principle)

48) Examine which functions can be holomorphically extended into the point 0 :

$$z \cot z, \quad \frac{z}{e^z - 1}, \quad z^2 \sin\frac{1}{z}.$$

49) If f is holomorphic in $\{z : |z| > R\}$ and $f(\frac{1}{z})$ has an isolated singularity at 0, it is said that f has an isolated singularity at ∞.
Determine the type of isolated singularities (possibly also at ∞) of the following functions:

$$\frac{1}{z-z^3},\quad \frac{z^5}{(1-z)^2},\quad \frac{e^z}{1+z^2},\quad \frac{1-e^z}{1+e^z},$$

$$\exp\left(\frac{z}{1-z}\right),\quad (e^z-1)^{-1}\exp\left(\frac{1}{1-z}\right),\quad \exp\left(\tan\frac{1}{z}\right),\quad \sin\left(\cos\frac{1}{z}\right)^{-1}.$$

50) Let $a \in \mathbb{C}$, $R > 0$ and $f \in \mathcal{H}(D'_R(a))$ and suppose that a is an essential singularity of f. Let g be a non-constant entire function.
(i) Show that the closure of $g(\mathbb{C})$ equals to $\mathbb{C}$.
(ii) Prove that a is an essential singularity of $g \circ f$.

51) Let $a \in \mathbb{C}$, $R > 0$ and $f \in \mathcal{H}(D'_R(a))$ such that $\Re f(z) \geq 0$ for each $z \in D'_R(a)$.
(i) Show that a is not an essential singularity of f.
(ii) Prove that f can in fact be extended to a holomorphic function on $D_R(a)$.

52) Expand the following functions as power series around z_0 :

$$e^z,\ z_0 = \pi i;\quad \frac{2z+1}{(z^2+1)(z+1)^2},\ z_0 = 0;\quad \frac{1}{(z-i)^3},\ z_0 = -i$$

and determine the radius of convergence.

53) As Exercise 52) for:

$$(\cosh z)^2,\ z_0 = 0;\quad \frac{1}{az+b}, b \neq 0,\ z_0 = 0;$$

$$\int_{[0,z]} e^{\zeta^2}\, d\zeta,\ z_0 = 0;\quad \int_{[0,z]} \frac{\sin\zeta}{\zeta}\, d\zeta,\ z_0 = 0.$$

54) Let $f(z) = z/(e^z - 1)$. Expand f as a power series around $z_0 = 0$, and set

$$f(z) = \sum_{k=0}^{\infty} \frac{a_k}{k!} z^k.$$

Determine the radius of convergence and show that

$$0 = a_0 + \binom{n+1}{1} a_1 + \cdots + \binom{n+1}{n} a_n.$$

Use the fact that $f(z) + \frac{1}{2}z$ is an even function, in order to show that $a_k = 0$, for k odd and $k > 1$.

55) Let $a_n, n \in \mathbb{N}_0$ be as in Exercise 54). The numbers $B_{2n} = (-1)^{n-1} a_{2n}, n \geq 1$, are called the Bernoulli numbers. Compute $B_2, B_4, \ldots, B_{10}$.

56) Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a function in $\mathcal{H}(D_1(0))$ such that

$$|f(z)|(1 - |z|) \leq 1, \quad z \in D_1(0).$$

Prove that for $n \in \mathbb{N}$:

$$|a_n| \leq \left(1 + \frac{1}{n}\right)^n (n+1) < e(n+1).$$

57) Let f be an entire function such that

$$|f(z)| \leq A + B|z|^k,$$

for $z \in \mathbb{C}$, where A, B, k are positive constants. Show that f is a polynomial .

58) An entire function is called transcendental, if it has an essential singularity at ∞. Let f be a transcendental entire function and let $M(r) = \max\{|f(z)| : |z| = r\}$. Show that:

$$\lim_{r \to \infty} \frac{\log M(r)}{\log r} = \infty.$$

59) Let $a \in \mathbb{C}$, $R > 0$ and $f \in \mathcal{H}(D'_R(a))$. Suppose that a is a pole of f. Let g be a transcendental entire function. Show that a is an essential singularity of $g \circ f$.

60) Let $f \in \mathcal{H}(D_R(0))$ be non-constant. Show that the function $r \mapsto M(r) = \sup_{|z|=r} |f(z)|$ is strictly increasing for $r \in (0, R)$.

61) Let f be an entire function, α a zero of f and $z \in \mathbb{C}$. Show that:

$$|f(z)| \leq 2|z - \alpha| \sup\{|f(w)| : \ |z - w| = 1\},$$

for all $z \in \mathbb{C}$.

62) Let f be a holomorphic on $D_{r_1,r_2}(0) = \{z : r_1 < |z| < r_2\}$, suppose that f is continuous on $\overline{D_{r_1,r_2}(0)}$, let $M_k = \sup_{|z|=r_k} |f(z)|, k = 1, 2$, and $M(r) = \sup_{|z|=r} |f(z)|$. Show that

$$\log M(r) \leq \frac{\log r_2 - \log r}{\log r_2 - \log r_1} \log M_1 + \frac{\log r - \log r_1}{\log r_2 - \log r_1} \log M_2.$$

(Hadamard's Three Circle Theorem)
We say that $\log M(r)$ is a convex function of $\log r$.
Hint: consider the function $[f(z)]^p z^{-q}$, where p, q are integers, and use the maximum principle.

63) Which of the following domains are simply connected?

$$\mathbb{C} \setminus \{0\}; \mathbb{C} \setminus [0, 1]; \mathbb{C} \setminus \{x : x \leq 0\}.$$

64) Which of the following domains are simply connected?

(1) $\mathbb{C} \setminus \{(x,y) : x = 0, |y| \leq 1\} \setminus \{(x,y) : x > 0, y = \sin \frac{1}{x}\}$;
(2) the complement of an Archimedean spiral around 0 :

$$\mathbb{C}^* \setminus \{z : z = e^{t(1+i)}, t \in \mathbb{R}\};$$

(3) $G = \{(x,y) : 0 < x < 1, 0 < y < 1\} \setminus \bigcup_{n=1}^{\infty}\{(x,y) : x = \frac{1}{n}, 0 < y \leq \frac{1}{2}\}$;
(4) $\{z : |z| < 1\} \setminus \{\frac{1}{n} : n = 2, 3, \dots\} \setminus \{0\}$.

65) Find a suitable domain G, and a path in G, which is null-homologous, but not null-homotopic in G.

66) Determine the Laurent expansion of the function

$$f(z) = \frac{(z^2-1)}{(z+2)(z+3)}$$

in
(1) $\{z : 2 < |z| < 3\}$;
(2) $\{z : |z| > 3\}$.

67) Determine the Laurent expansion of the function

$$f(z) = \frac{1}{(z-a)(z-b)} \quad , 0 < |a| < |b|,$$

in
(1) $\{z : |a| < |z| < |b|\}$;
(2) $\{z : |z| > b\}$.

68) Determine the Laurent expansion of the function

$$f(z) = \left[\frac{z}{(z-1)(z-2)}\right]^{1/2} \quad , \Im f(3/2) > 0$$

im Kreisring $\{z : 1 < |z| < 2\}$.

69) Compute the residues of the following functions at the given points:

$$\frac{z}{(2-3z)(4z+3)} \text{ in } \frac{2}{3}, -\frac{3}{4} \; ; \; \frac{e^{z-1}}{e^z - 1} \text{ in } 0 \; ; \; \frac{e^{i\pi z}}{16 - z^4} \text{ in } 2 \; ;$$

$$\frac{\sin z}{1 - 2\cos z} \text{ in } \frac{\pi}{3} \; ; \; \frac{\cos^2 z}{(2\pi - z)^3} \text{ in } 2\pi \; ; \; z \tan z \text{ in } \frac{\pi}{2} \; ; \; \frac{z+1}{(z^2+4)^2} \text{ in } 2i \; .$$

70) Let G be a domain in $\mathbb{C}$, which is symmetric with respect to the real axis. Let $f \in \mathcal{M}(G)$ and suppose that f is real-valued on the real axis. Show that:

$$\overline{\operatorname{Res}(f; z)} = \operatorname{Res}(f; \overline{z}), z \in G.$$

71) Compare the residue of the function f at a simple pole $z = a \neq 0$ with the residue of the function $zf(z^2)$ at the point $z = a^{1/2}$.

72) Suppose that the function f has an isolated residue at ∞. The residue of f at ∞ is defined by

$$2\pi i \operatorname{Res}(f; \infty) = \int_\gamma f(z)\, dz,$$

where γ is a negatively oriented circle containing all other singularities f, this means that ∞ lies on the left side of γ.
Prove: if f is holomorphic on $\overline{\mathbb{C}}$ except for isolated singularities, then the sum of all residues of f is zero.

73) Suppose that the function f has an isolated singularity at ∞.
Let $g(z) = -z^{-2} f(1/z)$. Show that: $\operatorname{Res}(f; \infty) = \operatorname{Res}(g; 0)$.

74) Compute the residues at ∞ of the following functions:

$$f(z) = z^n \ , n \in \mathbb{Z} \ ; \ g(z) = \frac{z^2+3}{5z^4 - 7z^2 + 6z} \ ; \ h(z) = \frac{2z-3}{z^2}.$$

75) How many zeros has the function $f(z) = z^8 - 4z^5 + z^2 - 1$ in $D = \{z : |z| < 1\}$?

76) How many zeros has the function $g(z) = 2iz^2 + \sin z$ in the rectangular $R = \{(x, y) : |x| \leq \pi/2, |y| \leq 1\}$?

77) Prove the following theorem (Hurwitz' Theorem): Let G be a domain in $\mathbb{C}$ and $(f_n)_n$ a sequence of holomorphic functions on G without zeros in G, which converges uniformly on all compact subsets of G to $f \in \mathcal{H}(G)$. Then $f \equiv 0$ or f has no zeros in G. [8] Hint: use Rouché's Theorem .
What are the properties of the sequence $f_n(z) = e^z/n$ with respect to this theorem?

78) Let G be a domain in $\mathbb{C}$ and let $f \in \mathcal{H}(G)$ be the limit of a sequence $f_n \in \mathcal{H}(G)$, uniformly on all compact subsets of G. Show that the zeors of f are limits of sequences of zeros of the functions f_n.
Find an example of a limit point of zeros of the functions f_n at the boundary of G which is not necessarily a zeor of f.

79) Let $R(x, y)$ be a rational function of two variables such that

$$R(\cos t, \sin t)$$

is defined for all $t \in \mathbb{R}$. Show that:

$$\int_0^{2\pi} R(\cos t, \sin t)\, dt = 2\pi \sum_{|z|<1} \operatorname{Res}\left(\frac{1}{\zeta} R\left(\frac{1}{2}\left(\zeta + \frac{1}{\zeta}\right), \frac{1}{2i}\left(\zeta - \frac{1}{\zeta}\right)\right); z\right).$$

8 Hurwitz, Adolf (1859-1919)

Compute the following definite integrals using the formula from above:

$$\int_0^{\pi} \frac{dt}{a+\cos t}\ ,\ a>1;\ \int_0^{\pi/2} \frac{dt}{1+\sin^2 t}\ ;$$

$$\int_0^{2\pi} \frac{\sin^2 t}{1-2a\cos t+a^2}\,dt\ ,\ a\in\mathbb{R}\ ;\ \int_0^{2\pi} \frac{\cos^2 2t}{1-2a\cos t+a^2}\,dt\ ,\ -1<a<1.$$

80) Let $R(z)$ be a rational function having no poles on $\mathbb{R}$ and suppose that the degree of the denominator is larger than the degree of the numerator.Then

$$\int_{-\infty}^{+\infty} R(x)e^{ix}\,dx = 2\pi i \sum_{\Im z>0} \operatorname{Res}(R(\zeta)e^{i\zeta}; z).$$

Hint: choose positive r_1, r_2, s so large that all poles of R in the upper halfspace lie in the rectangular $[r_2, r_2+is, -r_1+is, -r_1, r_2]$ and use the Residue Theorem for integration along the boundary of this rectangular. Finally take the limits $r_1, r_2 \to \infty$.

Compute the following definite integrals using the formula from above:

$$\int_0^{\infty} \frac{\cos x}{a^2+x^2}\,dx\ ,\ a>0\ ;\ \int_0^{\infty} \frac{\cos ax}{(x^2+b^2)^2}\,dx\ ,\ a,b>0.$$

81) Compute

$$\int_{-\infty}^{\infty} \frac{e^{itx}}{1+x^2}\,dx, \quad t\in\mathbb{R}.$$

82) Let α be a complex number, $|\alpha|\neq 1$. Compute

$$\int_0^{2\pi} \frac{d\theta}{1-2\alpha\cos\theta+\alpha^2},$$

by integration of $(z-\alpha)^{-1}(z-1/\alpha)^{-1}$ along the unit circle.

83) Let G be a domain in $\mathbb{C}$ and $f\in\mathcal{H}(G)$, let $z_1, z_2, \dots \in G$, set $\omega_0(z)\equiv 1$ and

$$\omega_k(z) = \prod_{j=1}^{k} (z-z_j).$$

Let γ be a closed path in G without double points and such that the points $z_1, \dots, z_n$ belong to the interior of γ.

Prove that:

$$\mathcal{L}_{n-1}(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\omega_n(\zeta)} \frac{\omega_n(\zeta) - \omega_n(z)}{\zeta - z}\, d\zeta, \quad z \notin \gamma^\star$$

is a polynomial of degree $n-1$ with the property

$$\mathcal{L}_{n-1}(z_j) = f(z_j),\ j = 1, \dots, n.$$

Use the Residue Theorem to show that

$$\mathcal{L}_{n-1}(z) = \sum_{j=1}^{n} \frac{f(z_j)}{\omega_n'(z_j)} \frac{\omega_n(z)}{z - z_j}$$

(*Lagrange interpolation*).
Put $R_n(z) = f(z) - \mathcal{L}_{n-1}(z)$ and show that

$$R_n(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z} \frac{\omega_n(z)}{\omega_n(\zeta)}\, d\zeta.$$

Finally prove that

$$\mathcal{L}_{n-1}(z) = \sum_{j=0}^{n-1} \left[\frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\omega_{j+1}(\zeta)}\, d\zeta \right] \omega_j(z)\ ,\ z \in G$$

(*Newton interpolation*)[9].

9 Newton, Isaac (1643–1727)

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Index