On Global Attraction to Solitary Waves for the Klein-Gordon Equation Coupled to Nonlinear Oscillator

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Abstract The long-time asymptotics is analyzed for all finite energy solutions to a model $\mathbf{U}(1)$-invariant nonlinear Klein-Gordon equation in one dimension, with the nonlinearity concentrated at a point. Our main result is that each finite energy solution converges as $t \to \pm\infty$ to the set of "nonlinear eigenfunctions" $\psi(x)e^{-i\omega t}$.

Résumé. ATTRACTION GLOBALE VERS DES ONDES SOLITAIRES POUR L'ÉQUATION DE KLEIN-GORDON COUPLÉ À UN OSCILLATEUR NON LINÉAIRE . On s'intéresse aux solutions d'énergie finie d'une équation non linéaire de Klein-Gordon $\mathbf{U}(1)$-invariante monodimensionnelle, avec une non linéarité ponctuelle, et on analyse leur comportement asymptotique aux temps longs. Le principal résultat que nous avons obtenu est que toute solution d'énergie finie converge pour $t \to \pm\infty$ vers un ensemble de "fonctions propres non linéaires" $\psi(x)e^{-i\omega t}$.

1 Introduction

We consider the global attractor, that is, the attracting set for all finite energy solutions to a model system. For the first time, we prove that in a particular $\mathbf{U}(1)$-invariant dispersive Hamiltonian system the global attractor is finite-dimensional and is formed by solitary waves. The investigation is inspired by Bohr's quantum transitions ("quantum jumps"). Namely, according to Bohr's postulates, an unperturbed electron lives forever in a *quantum stationary state* $|E\rangle$ that has a definite value E of the energy. Under an external perturbation, the electron can jump from one state to another: $|E_-\rangle \longmapsto |E_+\rangle$. The postulate suggests the dynamical interpretation of the transitions as long-time attraction

$$\Psi(t) \longrightarrow |E_\pm\rangle, \qquad t \to \pm\infty \tag{1.1}$$

for any trajectory $\Psi(t)$ of the corresponding dynamical system, where the limiting states $|E_\pm\rangle$ generally depend on the trajectory. Then the quantum stationary states should be viewed as the points of the *global attractor* $\mathcal{S}$ which is the set of all limiting states. See Figure 1. Similar convergence to a global attractor is well-known for dissipative systems, like Navier-Stokes equations (see [BV92, Tem97]). In this context, the global attractor is formed by the *static stationary states*, and the corresponding asymptotics (1.2) only holds for $t \to +\infty$ (and with $\omega_+ = 0$).
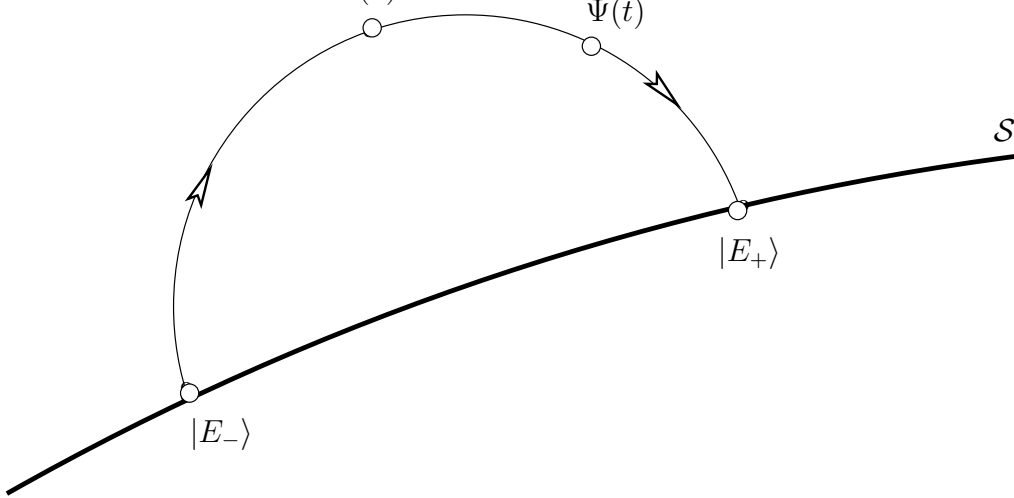
Figure 1: Attraction of any trajectory $\Psi(t)$ to the set of solitary waves as $t \to \pm\infty$.

Following de Broglie's ideas, Schrödinger identified the stationary states $|E\rangle$ as the solutions of the wave equation that have the form $\psi(x,t) = \phi_\omega(x)e^{-i\omega t}$, where $\omega = E/\hbar$, and $\hbar$ is Planck's constant. Then the attraction (1.1) takes the form of the long-time asymptotics

$$\psi(x,t) \sim \psi_\pm(x,t) = \phi_{\omega_\pm}(x)e^{-i\omega_\pm t}, \qquad t \to \pm\infty, \tag{1.2}$$

that hold for each finite energy solution. Our main impetus for considering this problem was the natural question whether dispersive Hamiltonian systems could, in the same spirit, possess finite dimensional global attractors, and whether such attractors are formed by the solitary waves. We prove such a global attraction for a model nonlinear Klein-Gordon equation

$$\ddot{\psi}(x,t) = \psi''(x,t) - m^2\psi(x,t) + \delta(x)F(\psi(0,t)), \qquad x \in \mathbb{R}. \tag{1.3}$$

Here $m > 0$, $\psi(x,t)$ is a continuous complex-valued wave function, and F is a nonlinearity. The dots stand for the derivatives in t, and the primes for the derivatives in x. We assume that equation (1.3) is $\mathbf{U}(1)$-invariant; that is, $F(e^{i\theta}\psi) = e^{i\theta}F(\psi)$, $\theta \in \mathbb{R}$.

Let $\mathcal{S}$ be the set of all functions $\phi_\omega(x) \in H^1(\mathbb{R})$ with $\omega \in \mathbb{C}$, so that $\phi_\omega(x)e^{-i\omega t}$ is a solution to (1.3). Our main result is the following long-time asymptotics (cf. (1.2)) for *nonlinear polynomial functions* $F(\psi)$:

$$\psi(\cdot,t) \longrightarrow \mathcal{S}, \qquad t \to \pm\infty, \tag{1.4}$$

where the convergence holds in local energy seminorms. In the linear case, when $F(\psi) = a\psi$ with $a \in \mathbb{R}$, there is no global attraction to $\mathcal{S}$ if $a > 0$, although the attraction holds if $a \leq 0$ (see Remark 2.5). Although we proved the attraction (1.4) to $\mathcal{S}$, we have not proved the attraction to a particular point of $\mathcal{S}$, falling short of proving (1.2). Hypothetically, a solution can be drifting along $\mathcal{S}$, keeping asymptotically close to it, but never stopping at a single point of $\mathcal{S}$. Let us comment on related earlier results:

i) The asymptotics of type (1.1) with $|E_\pm\rangle = 0$ were discovered in the scattering theory [Str68, MS72, Kla82, GV85]. In this case, the attractor $\mathcal{S}$ consists of the zero solution only, and the asymptotics mean well-known *local energy decay*.

2

for a number of nonlinear wave problems. There the attractor $\mathcal{S}$ is the set of all *static* stationary states. Let us mention that this set could be infinite and contain continuous components.

iii) First results on the asymptotics of type (1.2), with $\omega_\pm \neq 0$ were obtained for nonlinear $\mathbf{U}(1)$-invariant Schrödinger equations in the context of asymptotic stability. This establishes asymptotics of type (1.2) but only for solutions close to the solitary waves, proving the existence of a *local attractor*. This was first done in [SW90], and then developed in [BP92, SW99, Cuc01] and others.

Let us mention that the global attraction (1.2) for equation (1.3) with $m = 0$ follows from [Kom91]; In that case, $\omega_\pm = 0$. Our proofs for the case $m > 0$ are quite different from the approach used in [Kom91], and are based on a nonlinear spectral analysis of *omega-limit trajectories* at $t \to \pm\infty$.

2   Main Results

We consider the Cauchy problem for the equation (1.3) We define $\Psi(t) = \begin{bmatrix} \psi(x,t) \\ \pi(x,t) \end{bmatrix}$ and write the Cauchy problem in the vector form:

$$\dot{\Psi}(t) = \begin{bmatrix} 0 & 1 \\ \partial_x^2 - m^2 & 0 \end{bmatrix} \Psi(t) + \delta(x) \begin{bmatrix} 0 \\ F(\psi) \end{bmatrix}, \qquad \Psi|_{t=0} = \Psi_0 \equiv \begin{bmatrix} \psi_0 \\ \pi_0 \end{bmatrix}. \tag{2.1}$$

Definition 2.1.   *(i)* $\mathcal{E}$ is the Hilbert space of the states $\Psi = (\psi(x), \pi(x))$, with the norm

$$\|\Psi\|_{\mathcal{E}}^2 := \|\psi'\|_{L^2}^2 + \|\psi\|_{L^2}^2 + \|\pi\|_{L^2}^2, \qquad \text{where } L^2 = L^2(\mathbb{R}).$$

(ii) $\mathcal{E}_F$ is the space $\mathcal{E}$ endowed with the Fréchet topology defined by the seminorms

$$\|\Psi\|_{\mathcal{E},R}^2 := \|\psi'\|_{L_R^2}^2 + \|\psi\|_{L_R^2}^2 + \|\pi\|_{L_R^2}^2, \qquad \text{where } L_R^2 = L^2(-R,R), \quad R > 0.$$

We assume that the oscillator force F admits a real-valued potential: $F(\psi) = -\nabla U(\psi)$, $\psi \in \mathbb{C}$, where $U \in C^2(\mathbb{C})$, and the gradient is taken with respect to $\operatorname{Re}\psi$ and $\operatorname{Im}\psi$. Then equation (2.1) formally can be written as a Hamiltonian system. We assume that the potential $U(\psi)$ is $\mathbf{U}(1)$-invariant, where $\mathbf{U}(1)$ stands for the unitary group $e^{i\theta}$, $\theta \in \mathbb{R} \bmod 2\pi$: Namely, we assume that there exists $u \in C^2(\mathbb{R})$ such that $U(\psi) = u(|\psi|^2)$, $\psi \in \mathbb{C}$.

Theorem 2.2. *Assume that $U(\psi) \geq A - B|\psi|^2$, where $A, B \in \mathbb{R}$ and $B < m$. Then for every $\Psi_0 \in \mathcal{E}$ the Cauchy problem (2.1) has a unique solution $\Psi(t) = (\psi(x,t), \pi(x,t)) \in C(\mathbb{R}, \mathcal{E})$. The energy is conserved:*

$$\frac{1}{2} \int_{\mathbb{R}} \left(|\pi(x,t)|^2 + |\psi'(x,t)|^2 + m^2|\psi(x,t)|^2 \right) dx + U(\psi(0,t)) = \text{const}, \qquad t \in \mathbb{R},$$

and a priori bound $\|\Psi(t)\|_{\mathcal{E}} \leq C(\|\Psi_0\|_{\mathcal{E}})$ holds for $t \in \mathbb{R}$.

Definition 2.3.   *(i)* The solitary waves of equation (2.1) are solutions of the form

$$\Psi(t) = \Phi_\omega e^{-i\omega t}, \qquad \text{where } \omega \in \mathbb{C}, \quad \Phi_\omega = \begin{bmatrix} \phi_\omega(x) \\ -i\omega\phi_\omega(x) \end{bmatrix}, \quad \phi_\omega \in H^1(\mathbb{R}). \tag{2.2}$$

(ii) The solitary manifold is the set $\mathbf{S} = \{\Phi_\omega : \omega \in \mathbb{C}\}$ of all amplitudes Φ_ω.

3

are related by the *linear dispersion relation* $\omega^2 = m^2 - \kappa^2$ and the *coupling identity* $2\kappa C = F(C)$. Thus, $\mathbf{S}$ is generically a two-dimensional real submanifold of $\mathcal{E}$ that can be parametrized by the corresponding complex amplitudes C.

Theorem 2.4. *Let the nonlinearity* $F(\psi)$ *satisfy* $F(\psi) = -\nabla U(\psi)$, *where*

$$U(\psi) = \sum_{n=0}^{N} u_n |\psi|^{2n}, \quad N \geq 2; \quad u_n \in \mathbb{R}, \quad u_N > 0. \tag{2.3}$$

Then for any $\Psi_0 \in \mathcal{E}$ *the solution* $\Psi(t) \in C(\mathbb{R}, \mathcal{E})$ *to the Cauchy problem* (2.1) *with* $\Psi(0) = \Psi_0$ *converges to the set* $\mathbf{S}$ *in the space* $\mathcal{E}_F$:

$$\Psi(t) \xrightarrow{\mathcal{E}_F} \mathbf{S}, \quad t \to \pm\infty. \tag{2.4}$$

The assumption (2.3) that the nonlinearity is polynomial is crucial in our argument: It will allow to apply the Titchmarsh convolution theorem. Under this assumption, nonzero solitary waves (2.2) correspond only to real values of $\omega \in (-m, m)$.

Remark 2.5. In the linear case, when $F(\psi) = a\psi$ and $a > 0$, the equation admits two linearly independent solutions $\psi_{\pm}(x, t) = e^{-a|x|/2} e^{-i\omega_{\pm} t}$ with $\omega_{\pm} = \pm\sqrt{m^2 - a^2/4}$ if $m \neq a/2$, and $e^{-m|x|}$, $te^{-m|x|}$ if $m = a/2$. Hence the global attraction (2.4) fails because of the superposition principle. For $a \leq 0$ we have $\mathbf{S} = \{0\}$, and the attraction (2.4) holds.

Strategy of the proof of Theorem 2.4 For the Klein-Gordon equation with $m > 0$, the dispersive relation $\omega^2 = k^2 + m^2$ results in the group velocities $v = \omega'(k) = k/\sqrt{k^2 + m^2}$, so every velocity $0 \leq |v| < 1$ is possible. This complicates considerably the investigation of the energy propagation, so the approach of [Kom91] built on the fact that the group velocity was $|v| = 1$ no longer works.

We prove the absolute continuity of the spectrum of the solution for $|\omega| > m$. This observation is similar to the well-known Kato Theorem. The proof is not obvious and relies on the complex Fourier-Laplace transform and the Wiener-Paley arguments.

We then split the solution into two components: Dispersive and bound, with the frequencies $|\omega| > m$ and $\omega \in [-m, m]$, respectively. The dispersive component is an oscillatory integral of plane waves, while the bound component is a superposition of exponentially decaying functions. The stationary phase argument leads to a local decay of the dispersive component, due to the absolute continuity of its spectrum. This reduces the long-time behavior of the solution to the behavior of the bound component.

Next, we establish the spectral representation for the bound component. For this, we need to know an optimal regularity of the corresponding spectral measure; We have found out that the spectral measure belongs to the space of *quasimeasures* which are Fourier transforms of bounded continuous functions. The spectral representation implies compactness in the space of quasimeasures, which in turn leads to the existence of *omega-limit trajectories* at $t \to \pm\infty$.

Further, we prove that an omega-limit trajectory itself satisfies the nonlinear equation (1.3), and this implies the crucial spectral inclusion: The spectrum of the nonlinear term is included in the spectrum of the omega-limit trajectory. We then reduce the spectrum of this limiting trajectory to a single harmonic $\omega_{\pm} \in [-m, m]$ using the Titchmarsh convolution theorem [Tit26] (see also [Hör90, Theorem 4.3.3]). In turn, this means that any omega-limit trajectory lies in the manifold $\mathbf{S}$ of the solitary waves, proving that $\mathbf{S}$ is the global attractor.

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