Ann. Henri Poincaré 5 (2004) 1117 – 1135

1424-0637/04/061117-19
DOI 10.1007/s00023-004-0193-5

Annales Henri Poincaré

Soliton-Type Asymptotics for the Coupled Maxwell-Lorentz Equations

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Abstract. We consider a Maxwell field translation invariantly coupled to a single charge. This Hamiltonian system admits soliton-type solutions, where the charge and the co-moving field travel with constant velocity. We prove that a solution of finite energy converges, in suitable local energy seminorms, to a certain soliton-type solution in the long time limit $t \to \pm\infty$.

1 Introduction

We prove soliton-type asymptotics for the Maxwell-Lorentz system of the Maxwell field coupled to a relativistic charge: each finite energy solution converges to a soliton in a long-time limit. This is a generalization of the result [9] where a similar asymptotics is proved for a scalar field. This also strengthens the result [1], where an orbital stability of solitons is proved for the Maxwell-Lorentz system with a non-relativistic charge. The generalizations have required a considerable development of methods [1, 9]. In particular, we exploit strong Huygen's principle for the Maxwell-Lorentz equations and develop the Hamiltonian approach for canonical transformations of the equations.

We consider a single relativistic charge coupled to the Maxwell field. If $q(t) \in \mathbb{R}^3$ denotes the position of the charge at a time t, then the coupled Maxwell-Lorentz equations read

$$\begin{aligned} &\nabla \cdot E(x,t) = \rho(x-q(t)), && \dot{E}(x,t) = \nabla \wedge B(x,t) - \rho(x-q(t))\dot{q}(t), \\ &\nabla \cdot B(x,t) = 0, && \dot{B}(x,t) = -\nabla \wedge E(x,t), \\ &\dot{q}(t) = \frac{p(t)}{\sqrt{1+p^2(t)}}, && \dot{p}(t) = \int [E(x,t) + \dot{q}(t) \wedge B(x,t)]\rho(x-q(t))\, d^3x. \end{aligned} \tag{1.1}$$

Here and below all derivatives are understood in the sense of distributions. The last line is the Lorentz force equation and the first two lines are the inhomogeneous

*Supported partly by Austrian Science Foundation (FWF) Project (P16105-N05).
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Maxwell equations. $E(x,t)$, $B(x,t)$ is the Maxwell field, ρ is the charge distribution of the particle, on which we will comment below. We use units such that the velocity of light $c = 1$, $\varepsilon_0 = 1$, and the mechanical mass of the charge $m = 1$.

We consider all finite energy solutions to the equations (1.1). An appropriate phase space will be introduced below, but first we note that the energy integral

$$H(E,B,q,p) = (1+p^2)^{1/2} + \frac{1}{2}\int \Big(|E(x)|^2 + |B(x)|^2\Big) d^3x \tag{1.2}$$

and the total momentum integral

$$\mathbf{P}(E,B,q,p) = p + \int E(x) \wedge B(x)\, d^3x \tag{1.3}$$

are conserved along sufficiently smooth solution trajectories of (1.1). It is then natural to choose as the phase space the set of all finite energy states. In fact, (1.1) can be put into Hamiltonian form. In the canonical coordinates the energy H is then the Hamiltonian of the system.

The charge distribution ρ is real-valued, sufficiently smooth, radially symmetric, and of compact support,

$$\rho,\ \nabla\rho \in L^2(\mathbb{R}^3), \qquad \rho(x) = \rho_r(|x|), \quad \rho(x) = 0 \text{ for } |x| \ge R_\rho > 0. \tag{C}$$

As noted in [6, 8, 9] an additional important assumption is the Wiener condition

$$\hat\rho(k) = \int e^{ikx}\rho(x) d^3x \neq 0 \ \text{ for } \ k \in \mathbb{R}^3. \tag{W}$$

It ensures that all modes of the Maxwell field couple to the charge. In particular the total charge $\overline{\rho} = \hat\rho(0) \neq 0$. Charge distributions satisfying both (W) and (C) are constructed in [6, Section 10].

We will investigate the long-time behavior of all finite energy solutions to (1.1). A set of asymptotic solutions corresponds to the charge travelling with a uniform velocity, v. Up to translation they are of the form

$$S_v(t) = (E_v(x - vt), B_v(x - vt), vt, p_v) \tag{1.4}$$

with an arbitrary velocity $v \in V := \{v \in \mathbb{R}^3 : |v| < 1\}$, where

$$\begin{aligned} E_v(x) = -\nabla\phi_v(x) + v\cdot\nabla\ A_v(x),\ \ \phi_v(x) = \int \frac{\rho(y)d^3y}{4\pi|v(y-x)_\| + \lambda(y-x)_\perp|}, \\ B_v(x) = \nabla \wedge A_v(x), \quad A_v(x) = v\phi_v(x), \quad p_v = \frac{v}{\sqrt{1-v^2}}. \end{aligned} \tag{1.5}$$

Here $\lambda = \sqrt{1-v^2}$ and we set $x = vx_\| + x_\perp$, where $x_\| \in \mathbb{R}$ and $v \perp x_\perp \in \mathbb{R}^3$ for $x \in \mathbb{R}^3$. Below we call the solutions of type (1.4), (1.5) "solitons". Let us note that in [1] solitons are studied for a non-relativistic charge. Then they exist only for a finite range of energies. In our case of relativistic charge, we consider all finite energy solutions, an energy of a soliton is arbitrary.

Let us discuss and summarize now our main results, the precise theorems to be stated in the following section. One of the important issues of our paper is the relaxation of acceleration $\ddot{q}(t) \to 0$ as $t \to \pm\infty$ established in [8]. Our main contribution is that in this case the velocity has the limits

$$\lim_{t\to\pm\infty} \dot{q}(t) = v_\pm \tag{1.6}$$

together with co-moving travelling fields of type (1.4). Namely, the fields are asymptotically Coulomb travelling waves in the sense

$$(E(q(t)+x,t), B(q(t)+x,t)) - (E_{v_\pm}(x), B_{v_\pm}(x)) \to 0 \text{ as } t \to \pm\infty. \tag{1.7}$$

Since the energy is conserved, the convergence here is in the sense of suitable local norms.

Soliton-like asymptotics of type (1.7) are proved in [12] for some translation invariant 1D completely integrable equations and in [3] for translation invariant 1D nonlinear reaction systems. Soliton-like asymptotics are also proved for small perturbations of soliton-like solutions to 1D nonlinear translation invariant Schrödinger equations [2]. For a scalar field such kind of asymptotics was studied in [9]. For the Maxwell-Lorentz system (1.1) this asymptotics is proved in [7] under the condition $\|\rho\|_{L^2} \ll 1$. Under the Wiener condition (W) without the smallness condition, this asymptotics is proved for the first time in the present paper.

Let us note that a system of type (1.1), with a non-relativistic charge, has been considered by Bambusi and Galgani [1], where an *orbital stability* of the solitons is proved. We extend the orbital stability to the relativistic charge, and furthermore, prove a *global attraction* of all solutions to the soliton manifold, which means, in particular, its *asymptotic stability*. Similar global attraction is proved in [9] for a scalar field instead of the Maxwell Field. The extension to the Maxwell-Lorentz system (1.1) is not straightforward since it requires a detailed analysis of corresponding Hamiltonian structure and an extension of the strong Huygen's principle.

Let us give a general idea of our strategy. We transfer to Hamiltonian variables. In the case of non-relativistic charge the Hamiltonian structure is used in [1]. In the present paper we use essentially the Hamiltonian structure for the case of relativistic charge (for a scalar field this is done in [9]). Since the total momentum $\mathbf{P}$, see (1.3), is conserved, we obtain the new reduced Hamiltonian depending on $\mathbf{P}$ as on a parameter and with the cyclic variable conjugate to $\mathbf{P}$. We make this by a transfer to a moving frame as in [1, 9]. Here we prove for the first time that this transfer corresponds to a canonical transformation of the Maxwell-Lorentz equations. We will prove that the soliton with the same total momentum $\mathbf{P}$ is a critical point and the global minimum of the reduced Hamiltonian. Thus, initial data close to the soliton must remain close forever by conservation of energy, which translates into the orbital stability of soliton-type solutions. Note that for a general class of nonlinear wave equations with symmetries such orbital stability of soliton-like solutions is proved in [5]. Our argument here combine the Lyapunov function method of [1] and [9].

Orbital stability by itself is not enough. It only ensures that initial data close to a soliton remain so and does not yield the convergence of $\dot{q}(t)$, even less the convergence (1.7). Thus we need an additional, not quite obvious argument which combines the limit $\ddot{q}(t) \to 0$ as $t \to \infty$, from [8], with the orbital stability in order to establish the long time asymptotics. As an essential input we exploit retarded field part of the solution, apply the strong Huygen's principle for the Maxwell-Lorentz equations, and estimate oscillations of Maxwell-Lorentz Hamiltonian along solutions to the perturbed Maxwell-Lorentz system, cf. Section 4.

2 Main results

We first define a suitable phase space. A point in phase space is referred to as state. Let L^2 denote the real Hilbert space $L^2(\mathbb{R}^3, \mathbb{R}^3)$ with the norm $|\cdot|$. We introduce the Hilbert spaces $\mathcal{F} = L^2 \oplus L^2$ and $\mathcal{L} = \mathcal{F} \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$ with finite norms

$$\|(E(x), B(x))\|_{\mathcal{F}} = |E| + |B| \text{ and } \|Y\|_{\mathcal{L}} = |E| + |B| + |q| + |p| \quad \text{for } Y = (E(x), B(x), q, p) \in \mathcal{L}. \tag{2.1}$$

$\mathcal{L}$ is the space of finite energy states. The energy functional H is continuous on the space $\mathcal{L}$. On $\mathcal{F}$ and $\mathcal{L}$ we define the local energy seminorms by

$$\|(E(x), B(x)\|_R = |E|_R + |B|_R \text{ and } |\!|\!|Y|\!|\!|_R = |E|_R + |B|_R + |q| + |p| \quad \text{for } Y = (E(x), B(x), q, p) \tag{2.2}$$

for every $R > 0$, where $|\cdot|_R$ is the norm in $L^2(B_R)$, B_R the ball $\{x \in \mathbb{R}^3 : |x| < R\}$. Let us denote by $\mathcal{F}_F$, $\mathcal{L}_F$ the spaces $\mathcal{F}$, $\mathcal{L}$ equipped with the Fréchet topology induced by these seminorms. Note that the spaces $\mathcal{L}$ and $\mathcal{L}_F$, $\mathcal{F}_F$ are metrizable, but $\mathcal{L}_F$, $\mathcal{F}_F$ are not complete.

The system (1.1) is overdetermined. Therefore its actual phase space is a nonlinear sub-manifold of the linear space $\mathcal{L}$.

Definition 2.1

i) *The phase space $\mathcal{M}$ for Maxwell-Lorentz equations* (1.1) *is the metric space of states $(E(x), B(x), q, p) \in \mathcal{L}$ satisfying the constraints,*

$$\nabla \cdot E(x) = \rho(x - q) \text{ and } \nabla \cdot B(x) = 0 \text{ for } x \in \mathbb{R}^3. \tag{2.3}$$

The metric on $\mathcal{M}$ is induced through the embedding $\mathcal{M} \subset \mathcal{L}$.

ii) *$\mathcal{M}^\sigma$ for $0 \le \sigma \le 1$ is the set of the states $(E(x), B(x), q, p) \in \mathcal{M}$ such that $\nabla E(x)$,$\nabla B(x)$ are L^∞_{loc} outside the ball B_{R^0} with some $R^0 = R^0(Y) > 0$ and*

$$|E(x)| + |B(x)| + |x|\Big(|\nabla E(x)| + |\nabla B(x)|\Big) \le C^0 |x|^{-1-\sigma} \text{ for } |x| > R^0. \tag{2.4}$$

In the sequel we consider the space $\mathcal{M}$ endowed with Fréchet topology induced through the embedding $\mathcal{M} \subset \mathcal{L}_F$.

Remarks. i) $\mathcal{M}$ is a complete metric space, a nonlinear sub-manifold of $\mathcal{L}$. The spaces $\mathcal{M}$ and $\mathcal{M}$ endowed with Fréchet topology are metrizable.

ii) $\mathcal{M}^1$ is dense in $\mathcal{M}$, [8, Lemma 7.4]. On the other hand, since the total charge $\overline{\rho} = \hat{\rho}(0) \neq 0$, $\mathcal{M}^\sigma = \emptyset$ for $\sigma > 1$ because of the Gauss theorem. By the same reason $\operatorname{supp} E(x)$ cannot be a compact set in contrast to $\operatorname{supp} B(x)$.

Let us write the system (1.1) as a dynamical equation on $\mathcal{M}$

$$\dot{Y}(t) = F(Y(t)) \text{ for } t \in \mathbb{R}, \tag{2.5}$$

where $Y(t) = (E(x,t), B(x,t), q(t), p(t))) \in \mathcal{M}$.

Proposition 2.2 *Let* (C) *hold and* $Y^0 = (E^0(x), B^0(x), q^0, p^0) \in \mathcal{M}$. *Then*

i) *The system* (1.1) *has a unique solution* $Y(t) = (E(x,t), B(x,t), q(t), p(t)) \in C(\mathbb{R}, \mathcal{M})$ *with* $Y(0) = Y^0$.

ii) *The energy is conserved, i.e.,*

$$H(Y(t)) = H(Y^0) \;\; for \; t \in \mathbb{R}. \tag{2.6}$$

iii) *The total momentum is conserved, i.e.,*

$$\mathbf{P}(Y(t)) = \mathbf{P}(Y^0) \;\; for \; t \in \mathbb{R}. \tag{2.7}$$

iv) *The estimate holds,*

$$|\dot{q}(t)| \leq \overline{v} < 1, \;\; t \in \mathbb{R}. \tag{2.8}$$

We refer to [8], where the statements i), ii), iv) are proved. The conservation (2.7) follows from the last equation of (3.15) of the present paper.

In [8] also the following preliminary result on asymptotics is proved.

Proposition 2.3 ([8]). *Let* (C), (W) *hold. Let* $Y(t) \in C(\mathbb{R}, \mathcal{M})$ *be the solution of the Maxwell-Lorentz equations* (1.1) *with initial state* $Y^0 \in \mathcal{M}^\sigma$ *with some* $\sigma > 1/2$. *Then*

$$\ddot{q}(t) \to 0 \;\; as \; t \to \pm\infty, \tag{2.9}$$

$$(E(q(t) + \cdot, t), B(q(t) + \cdot, t)) - (E_{v(t)}(\cdot), B_{v(t)}(\cdot)) \xrightarrow{\mathcal{F}_F} 0 \;\; as \; t \to \pm\infty, \tag{2.10}$$

Remark. (2.9) and (2.10) mean the convergence in the Fréchet topology of the solution to the set of solitary waves (1.4) centered at the charge's position.

Note that $(E_{v(t)}(x), B_{v(t)}(x))$ is a *co-moving* soliton, and convergence to a certain fixed soliton was not yet proved. In the present paper we establish the convergence (except for the charge's position) to a fixed soliton. The main results of the paper are the following two theorems. The first step is to prove an orbital stability of solitons which extends the result [1] to the relativistic charge.

Theorem 2.4 *Let* (C) *hold. Fix a certain* $v \in V$. *Let* $Y(t) = (E(t), B(t), q(t), p(t)) \in C(\mathbb{R}, \mathcal{M})$ *be a solution to the system* (1.1) *with initial state* $Y(0) = Y^0 = (E^0, B^0, q^0, p^0) \in \mathcal{M}$ *and denote*

$$\delta = |E^0(\cdot) - E_v(\cdot - q^0)| + |B^0(\cdot) - B_v(\cdot - q^0)| + |p^0 - p_v|\,. \tag{2.11}$$

Then for every $\varepsilon > 0$ *there exists a* $\delta(\varepsilon) > 0$ *such that*

$$|E(q(t)+\cdot, t) - E_v(\cdot)| + |B(q(t)+\cdot, t) - B_v(\cdot)| + |p(t) - p_v| \le \varepsilon \text{ for all } t \in \mathbb{R} \tag{2.12}$$

provided $\delta \le \delta(\varepsilon)$.

The second step is to prove the convergence to a fixed soliton.

Theorem 2.5 *Let* (C) *and* (W) *hold. Let* $Y^0 = (E^0, B^0, q^0, p^0) \in \mathcal{M}^\sigma$ *with some* $\sigma > 1/2$. *Let* $Y(t) \in C(\mathbb{R}, \mathcal{M})$ *be the solution to* (1.1) *with* $Y(0) = Y^0$. *Then there exist the limits* (1.6) *and for every* $R > 0$

$$\lim_{t \to \pm\infty} \Big(|E(q(t) + \cdot\,,\, t) - E_{v_\pm}(\cdot)|_R + |B(q(t) + \cdot\,,\, t) - B_{v_\pm}(\cdot)|_R \Big) = 0\,. \tag{2.13}$$

Remark. Note that (2.13) follows obviously from (2.10) and (1.6). Hence, the crucial point of the proof is just the convergence (1.6).

3 Orbital stability of solitons

The main idea is to derive the orbital stability from the energy conservation developing the Lyapunov function method [1].

3.1 Hamiltonian variables and dynamics

We set the system (2.5) into a Hamiltonian form. Set

$$E_s(x,t) = E(x,t) + \nabla\varphi_\rho(x - q(t)), \text{ where } \nabla\varphi_\rho \in L^2(\mathbb{R}^3),\ \Delta\varphi_\rho(x) = -\rho(x), \tag{3.1}$$

φ_ρ is defined uniquely. Introduce a magnetic potential $A(x,t)$ which satisfies the Coulomb gauge,

$$B(x,t) = \nabla \wedge A(x,t),\ \nabla \cdot A(x,t) = 0. \tag{3.2}$$

Let us introduce the charge momentum in the magnetic field as

$$P(t) := p(t) + \int \rho(x - q(t)) A(x,t) d^3x. \tag{3.3}$$

Assume that the fields E, B are sufficiently smooth, vanish at infinity, and the equations (1.1) hold for (E, B, q, p). Then by a straightforward computation one obtains that (A, E_s, q, P) obeys the following constraints and equations,

$$\nabla \cdot E_s(x,t) = 0, \quad \nabla \cdot A(x,t) = 0, \tag{3.4}$$

$$\dot{E}_s(x,t) = -\Delta A(x,t) - \Pi_s(\rho(x-q(t))v(t)), \quad \dot{A}(x,t) = -E_s(x,t), \tag{3.5}$$

$$\dot{q}(t) = \frac{P(t) - \displaystyle\int \rho(x-q(t))A(x,t)d^3x}{\left[1 + \left(P(t) - \displaystyle\int \rho(x-q(t))A(x,t)d^3x\right)^2\right]^{1/2}} =: v(t), \tag{3.6}$$

$$\dot{P}(t) = \sum_{k=1}^{3} \int \rho(x-q(t))v_k(t) \cdot \nabla A_k(x,t)d^3x, \tag{3.7}$$

where Π_s is the projection to the divergence-free (solenoidal) fields. Consider the functional

$$H_s(E_s, A, q, P) = \frac{1}{2}\int \left(|E_s|^2 + |\nabla A|^2\right)d^3x + \left[1 + \left(P - \int \rho(x-q)A(x)d^3x\right)^2\right]^{1/2}. \tag{3.8}$$

The equations (3.5), (3.6), (3.7) are Hamiltonian with the Hamiltonian functional $H(E_s, A, q, P)$. Namely, the equations are equivalent respectively to

$$\begin{aligned} \dot{E}_s &= \frac{\delta H_s}{\delta A}, \quad \dot{A} = -\frac{\delta H_s}{\delta E_s}, \\ \dot{q} &= \frac{\partial H_s}{\partial P}, \quad \dot{P} = -\frac{\partial H_s}{\partial q}. \end{aligned} \tag{3.9}$$

Thus we call the variables E_s, A, q, P "Hamiltonian variables". The conserved total momentum (1.3) in Hamiltonian variables reads

$$\mathcal{P}(E_s, A, q, P) = P + \int E_s(x) \wedge (\nabla \wedge A(x))d^3x = P + \int E_s(x) \cdot \nabla A(x)d^3x, \tag{3.10}$$

where we denote $E \cdot \nabla A := \sum_{k=1}^{3} E_k \cdot \nabla A_k$; in the sequel we use the second expression for $\mathcal{P}$. It is easy to check that $\mathcal{P}(E_s, A, q, P) = \mathbf{P}(E, B, q, p)$ and

$$H_s(E_s, A, q, P) = H(E, B, q, p) - \frac{1}{2}\int |\nabla \varphi_\rho(x)|^2 dx, \tag{3.11}$$

where variables (E_s, A, q, P) and (E, B, q, p) are connected through (3.1), (3.2), (3.3).

We now introduce a phase space for the system (3.4) to (3.7) and state the existence of dynamics. Set $H^0 = L^2(\mathbb{R}^3, \mathbb{R}^3)$, $\dot{H}^1$ is the closure of $C_0^\infty(\mathbb{R}^3, \mathbb{R}^3)$ with respect to the norm $\|A\|_1 = |\nabla A| = \|\nabla A\|_{L^2(\mathbb{R}^3,\mathbb{R}^3)}$. Let H_s^0, $\dot{H}_s^1$ be the subspaces constituted by solenoidal vector fields, namely the closure in H^0, $\dot{H}^1$ respectively of C_0^∞ vector fields with vanishing divergence. Define the phase space

$$\mathcal{M}_0 = H_s^0 \oplus \dot{H}_s^1 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3,$$

where the norm of $Y_s = (E_s, A, q, P)$ is

$$\|Y_s\|_{\mathcal{M}_0} = |E_s| + \|A\|_1 + |q| + |P|.$$

Proposition 2.2 implies the following result.

Proposition 3.1 *Let* (C) *hold, let* $Y_s^0 = (E_s^0, A^0, q^0, P^0) \in \mathcal{M}_0$. *Then*

i) *There exists a unique solution* $Y_s(t) \in C(\mathbb{R}, \mathcal{M}_0)$ *to the system* (3.4) *to* (3.7) *with* $Y_s(0) = Y_s^0$.

ii) *The energy and the total momentum are conserved,*

$$H_s(Y_s(t)) = H_s(Y_s^0), \;\; \mathcal{P}(Y_s(t)) = \mathcal{P}(Y_s^0), \;\; t \in \mathbb{R}.$$

iii) *Consider the vector* $Y(t) = (E(t), B(t), q(t), p(t))$ *with*

$$E(t) = E_s(t) - \nabla\varphi_\rho(\cdot - q(t)), \;\; B(t) = \nabla \wedge A(t),$$

$$p(t) = P(t) - \int \rho(x - q(t))A(x,t)d^3x,$$

where $(E_s(t), A(t), q(t), P(t)) = Y_s(t)$ *is the solution to the system* (3.4) *to* (3.7) *with* $Y_s(0) = Y_s^0$. *Then* $Y(t)$ *is the unique solution in* $C(\mathbb{R}, \mathcal{M})$ *of the system* (1.1) *with the initial data*

$$E^0 = E_s^0 - \nabla\varphi_\rho(\cdot - q^0), \;\; B^0 = \nabla \wedge A^0,$$

$$q^0, \;\; p^0 = P^0 - \int \rho(x - q^0)A^0(x)d^3x.$$

3.2 Canonical transform and reduced system

The Hamiltonian (3.8) is invariant with respect to translations in the space $\mathbb{R}^3$. Hence, it is not an appropriate Lyapunov function. We exclude the translation degeneracy of the Hamiltonian reducing the system (3.4) to (3.7) by a canonical transform (cf. [1], [9]). Define the following transform of the space $\mathcal{M}_0$,

$$\mathcal{T}(E_s(x), A(x), q, P) = (\mathcal{E}(x), \mathcal{A}(x), \mathcal{Q}, \mathcal{P}),$$

where

$$\mathcal{E}(x) = E_s(x+q), \; \mathcal{A}(x) = A(x+q), \; \mathcal{Q} = q, \; \mathcal{P} = P + \int E_s(x) \cdot \nabla A(x) d^3x. \quad (3.12)$$

The transform $\mathcal{T} : \mathcal{M}_0 \to \mathcal{M}_0$ is continuous and Fréchet differentiable at points $(E_s(x), A(x), q, P)$ with sufficiently smooth functions $E_s(x), A(x)$, but not everywhere differentiable. Since

$$E_s(x) = \mathcal{E}(x - \mathcal{Q}), \; A(x) = \mathcal{A}(x - \mathcal{Q}), \; q = \mathcal{Q}, \; P = \mathcal{P} - \int \mathcal{E}(x) \cdot \nabla\mathcal{A}(x) d^3x,$$

the transform is invertible. Set $\mathcal{H}(\mathcal{E}, \mathcal{A}, \mathcal{Q}, \mathcal{P}) = H_s(\mathcal{T}^{-1}(\mathcal{E}, \mathcal{A}, \mathcal{Q}, \mathcal{P}))$, thus

$$\mathcal{H}(\mathcal{E}, \mathcal{A}, \mathcal{Q}, \mathcal{P})$$
$$= \frac{1}{2}\int (|\mathcal{E}|^2 + |\nabla\mathcal{A}|^2)d^3x + \left[1 + \Big(\mathcal{P} - \int \mathcal{E}\cdot\nabla\mathcal{A}\, d^3x - \int \rho\mathcal{A}\, d^3x\Big)^2\right]^{1/2}. \quad (3.13)$$

Lemma 3.2 *Let $Y_s(x,t) = (E_s(x,t), A(x,t), q(t), P(t))$ be a solution in $C(\mathbb{R}, \mathcal{M}_0)$ of the system* (3.4) *to* (3.7). *Consider*

$$(\mathcal{E}(x,t), \mathcal{A}(x,t), \mathcal{Q}(t), \mathcal{P}(t)) = \mathcal{T}Y_s$$
$$= (E_s(x + q(t), t), A(x + q(t), t), q(t), P(t) + \int E_s(x)\cdot\nabla A(x)d^3x)\,.$$

Then i) the constraints hold,

$$\nabla\cdot\mathcal{E}(x,t) = 0,\ \ \nabla\cdot\mathcal{A}(x,t) = 0; \quad (3.14)$$

ii) $(\mathcal{E}(x,t), \mathcal{A}(x,t), \mathcal{Q}(t), \mathcal{P}(t))$ is the solution in $C(\mathbb{R}, \mathcal{M}_0)$ of the following Hamiltonian system,

$$\begin{aligned}\dot{\mathcal{E}} &= \frac{\delta\mathcal{H}}{\delta\mathcal{A}},\ \ \dot{\mathcal{A}} = -\frac{\delta\mathcal{H}}{\delta\mathcal{E}},\\ \dot{\mathcal{Q}} &= \frac{\partial\mathcal{H}}{\partial\mathcal{P}},\ \ \dot{\mathcal{P}} = -\frac{\partial\mathcal{H}}{\partial\mathcal{Q}}.\end{aligned} \quad (3.15)$$

Proof. i) Follows from (3.4) by definition of $\mathcal{E}$, $\mathcal{A}$.
ii) We check, similar to [9], that the transform $\mathcal{T}$ is canonical, i.e., conserves the canonical form. This can be seen from the Lagrangian viewpoint. We focus on our particular case and do not develop any general theory of infinite-dimensional Hamiltonian systems which is beyond the scope of this paper.

By definition we have $\mathcal{H}(\mathcal{E}, \mathcal{A}, \mathcal{Q}, \mathcal{P}) = H_s(E_s, A, q, P)$ with $(\mathcal{E}, \mathcal{A}, \mathcal{Q}, \mathcal{P}) = \mathcal{T}(E_s, A, q, P)$. To each Hamiltonian we associate a Lagrangian through the Legendre transformation

$$\begin{aligned} L(E_s, \dot{E}_s, q, \dot{q}) &= \langle A, \dot{E}_s\rangle + P\cdot\dot{q} - H_s(E_s, A, q, P)\ , \qquad \dot{E}_s = \frac{\delta H}{\delta A}\ ,\ \dot{q} = \frac{\partial H}{\partial P}\ ,\\ \mathcal{L}(\mathcal{E}, \dot{\mathcal{E}}, \mathcal{Q}, \dot{\mathcal{Q}}) &= \langle \mathcal{A}, \dot{\mathcal{E}}\rangle + \mathcal{P}\cdot\dot{\mathcal{Q}} - \mathcal{H}(\mathcal{E}, \mathcal{A}, \mathcal{Q}, \mathcal{P})\ , \qquad \dot{\mathcal{E}} = \frac{\delta\mathcal{H}}{\delta\mathcal{A}}\ ,\ \dot{\mathcal{Q}} = \frac{\partial\mathcal{H}}{\partial\mathcal{P}}\,.\end{aligned}$$

These Legendre transforms are well defined because the Hamiltonian functionals are convex in the momenta. We claim the identity $\mathcal{L}(\mathcal{E}, \dot{\mathcal{E}}, \mathcal{Q}, \dot{\mathcal{Q}}) = L(E_s, \dot{E}_s, q, \dot{q})$. Clearly we have to check the invariance of the canonical 1-form,

$$\langle\mathcal{A}, \dot{\mathcal{E}}\rangle + \mathcal{P}\cdot\dot{\mathcal{Q}} = \langle A, \dot{E}_s\rangle + P\cdot\dot{q}\,. \quad (3.16)$$

For this purpose we substitute

$$\begin{aligned} \mathcal{A}(x) &= A(q+x)\,, & \dot{\mathcal{E}}(x) &= \dot{\mathcal{E}}(q+x) + \dot{q}\cdot\nabla E_s(q+x)\,, \\ \mathcal{P} &= P + \int E_s\cdot\nabla A\, d^3x\,, & \dot{\mathcal{Q}} &= \dot{q}\,. \end{aligned}$$

The left-hand side of (3.16) becomes then

$$\begin{aligned} \langle A(q+x), \dot{E}_s(q+x) + \dot{q}\cdot\nabla E_s(q+x)\rangle + (P + \langle E_s(x), \nabla A(x)\rangle)\cdot\dot{q} \\ = \langle A, \dot{E}_s\rangle + P\cdot\dot{q} \end{aligned}$$

after partial integration. Since $\mathcal{L}(\mathcal{E}, \dot{\mathcal{E}}, \mathcal{Q}, \dot{\mathcal{Q}}) = L(E_s, \dot{E}_s, q, \dot{q})$, the corresponding action functionals are identical when transformed by $\mathcal{T}$. The dynamical trajectories are stationary points of the corresponding action functionals. Therefore the two Hamiltonian systems (3.9) and (3.15) are equivalent. □

Remark. One can also check the equations (3.15) by a straightforward computation.

Since $\mathcal{H}$ does not depend on $\mathcal{Q}$, we may think of $\mathcal{P}$ as of a parameter and consider the reduced Hamiltonian

$$\mathcal{H}_{\mathcal{P}}(\mathcal{E}, \mathcal{A}) =$$

$$\frac{1}{2}\int(|\mathcal{E}|^2 + |\nabla\mathcal{A}|^2)\, d^3x + \left[1 + \Big(\mathcal{P} - \int \mathcal{E}\cdot\nabla\mathcal{A}\, d^3x - \int \rho\mathcal{A}\, d^3x\Big)^2\right]^{1/2}. \quad (3.17)$$

Then $\mathcal{E}$, $\mathcal{A}$ satisfy the reduced Hamiltonian system

$$\dot{\mathcal{E}} = \frac{\delta\mathcal{H}_{\mathcal{P}}}{\delta\mathcal{A}},\ \ \dot{\mathcal{A}} = -\frac{\delta\mathcal{H}_{\mathcal{P}}}{\delta\mathcal{E}}. \quad (3.18)$$

3.3 Soliton as global minimum of reduced Hamiltonian

The solitons in the Hamiltonian variables read

$$E_s(x,t) = E_{s,v}(x - vt),\ A(x,t) = A_{s,v}(x - vt),\ q(t) = vt,\ P_v = p_v + \int \rho A_v d^3x,$$

where $v \in V$, $E_{s,v} = E_v + \nabla\varphi_\rho$, $A_{s,v} = \Pi_s A_v$, and E_v, A_v, p_v are given by (1.5). The corresponding equations are

$$E_{s,v}(x) = v\cdot\nabla A_{s,v}(x), \quad (3.19)$$

$$v\cdot\nabla E_{s,v}(x) = \Delta A_{s,v}(x) + \Pi_s(\rho(x)v), \quad (3.20)$$

$$v = \frac{P_v - \displaystyle\int \rho(x) A_{v,s}(x)\, d^3x}{\left[1 + \Big(P_v - \displaystyle\int \rho(x) A_{s,v}(x)\, d^3x\Big)^2\right]^{1/2}}\,, \quad (3.21)$$

$$0 = \sum_{k=1}^{3} \int \rho(x) v \cdot \nabla (A_{s,v})_k(x) d^3x. \tag{3.22}$$

The map $v \to \mathcal{P}(v)$, where $\mathcal{P}(v)$ is the total momentum (3.10) of the soliton, is given, [10], by

$$\mathcal{P}(v) = \left[\frac{1}{\sqrt{1-v^2}} + \frac{1}{2} \int \frac{|\hat{\rho}(k)|^2}{k^2} \, d^3k \left(\frac{1+v^2}{2|v|^3} \log \frac{1+|v|}{1-|v|} - \frac{1}{v^2} \right) \right] v. \tag{3.23}$$

The map is differentiable and invertible, the inverse map is differentiable. Apply the canonical transform $\mathcal{T}$ to the soliton and obtain

$$\mathcal{E}_v := E_{s,v}(x), \ \mathcal{A}_v := A_{s,v}(x). \tag{3.24}$$

These are stationary fields and according to the reduced Hamiltonian system $(\mathcal{E}_v, \mathcal{A}_v)$ is a critical point of the reduced Hamiltonian $\mathcal{H}_{\mathcal{P}(v)}$, where $\mathcal{P}(v)$ is the total momentum of the soliton.

Lemma 3.3 *For $(\mathcal{E}, \mathcal{A}) \in H_s^0 \oplus \dot{H}_s^1$ the lower bound holds,*

$$\mathcal{H}_{\mathcal{P}(v)}(\mathcal{E}, \mathcal{A}) - \mathcal{H}_{\mathcal{P}(v)}(\mathcal{E}_v, \mathcal{A}_v) \geq \frac{1-|v|}{2} (|\mathcal{E} - \mathcal{E}_v|^2 + \|\mathcal{A} - \mathcal{A}_v\|_1^2), \tag{3.25}$$

where $\mathcal{E}_v$, $\mathcal{A}_v$ are defined by (3.24).

Proof. Set $\mathcal{E} = \mathcal{E}_v + e$, $\mathcal{A} = \mathcal{A}_v + a$, then $\nabla \cdot e = 0$, $\nabla \cdot a = 0$. We have

$$\mathcal{H}_{\mathcal{P}(v)}(\mathcal{E}_v + e, \mathcal{A}_v + a) - \mathcal{H}_{\mathcal{P}(v)}(\mathcal{E}_v, \mathcal{A}_v)$$

$$= \int (\mathcal{E}_v \cdot e + \nabla \mathcal{A}_v \cdot \nabla a) \, d^3x + \frac{1}{2} \int (|e|^2 + |\nabla a|^2) \, d^3x + [1 + (p_v + m)^2]^{1/2} - [1 + p_v^2]^{1/2},$$

where

$$m = -\int (\mathcal{E}_v \cdot \nabla a + e \cdot \nabla \mathcal{A}_v + e \cdot \nabla a) d^3x - \int \rho a \, d^3x.$$

Since the equations (3.19) and (3.20) hold, and $\nabla \cdot a = 0$, we obtain

$$\int (\mathcal{E}_v \cdot e + \nabla \mathcal{A}_v \cdot \nabla a) d^3x = \int (\mathcal{E}_v \cdot e - \Delta \mathcal{A}_v \cdot a) d^3x$$

$$= \int (\mathcal{E}_v \cdot e - (v \cdot \nabla \mathcal{E}_v - \Pi_s(\rho v)) \cdot a) d^3x = \int (v \cdot \nabla \mathcal{A}_v \cdot e - v \cdot \nabla \mathcal{E}_v \cdot a + \Pi_s(\rho v) \cdot a) d^3x$$

$$= \int (v \cdot \nabla \mathcal{A}_v \cdot e + v \cdot \mathcal{E}_v \cdot \nabla a + v \cdot \rho a - v \cdot e \cdot \nabla a + v \cdot e \cdot \nabla a) d^3x$$

$$= -v \cdot m - v \cdot \int e \cdot \nabla a \, d^3x.$$

Then

$$\mathcal{H}_{\mathcal{P}(v)}(\mathcal{E}_v + e, \mathcal{A}_v + a) - \mathcal{H}_{\mathcal{P}(v)}(\mathcal{E}_v, \mathcal{A}_v) = \frac{1}{2}\int (|e|^2 + |\nabla a|^2) d^3x - v \cdot \int e \cdot \nabla a \, d^3x$$

$$+[1 + (p_v + m)^2]^{1/2} - [1 + p_v^2]^{1/2} - v \cdot m.$$

The last line is non-negative, since the function $f(p) := (1 + p^2)^{1/2}$ is convex and $\nabla f(p_v) = v$. Hence, we obtain (3.25). □

Remark. The calculations are close to those in [9], since the algebraic structure of the Hamiltonian functional is similar.

3.4 Lyapunov function method: orbital stability

Let us finish the proof of Theorem 2.4. We denote by $\mathcal{P}^0$ the total momentum of the considered solution $Y_s(t)$. There exists a soliton-like solution $Y_{s,\tilde{v}} = (E_{s,\tilde{v}}, A_{s,\tilde{v}}, \tilde{v}t, P_{\tilde{v}})$ corresponding to some $\tilde{v} \in V$ and having the same total momentum $\mathcal{P}(\tilde{v}) = \mathcal{P}^0$. Then (2.11) implies $|\mathcal{P}^0 - \mathcal{P}(v)| = |\mathcal{P}(\tilde{v}) - \mathcal{P}(v)| = \mathcal{O}(\delta)$, hence also $|\tilde{v} - v| = \mathcal{O}(\delta)$ and

$$|E_s^0(x) - E_{s,\tilde{v}}(x - q^0)| + \|A^0(x) - A_{s,\tilde{v}}(x - q^0)\|_1 + |\mathcal{P}^0 - P_{\tilde{v}}| = \mathcal{O}(\delta)\,. \tag{3.26}$$

Therefore denoting $(\mathcal{E}^0, \mathcal{A}^0, q^0, \mathcal{P}^0) = \mathcal{T}Y_s^0$ we have

$$\mathcal{H}_{\mathcal{P}(\tilde{v})}(\mathcal{E}^0, \mathcal{A}^0) - \mathcal{H}_{\mathcal{P}(\tilde{v})}(E_{s,\tilde{v}}, A_{s,\tilde{v}}) = \mathcal{O}(\delta^2)\,. \tag{3.27}$$

Total momentum and energy conservation imply for $(\mathcal{E}(t), \mathcal{A}(t), q(t), \mathcal{P}^0) = \mathcal{T}Y_s(t)$

$$\mathcal{H}_{\mathcal{P}(\tilde{v})}(\mathcal{E}(t), \mathcal{A}(t)) = \mathcal{H}(\mathcal{T}Y_s(t)) = \mathcal{H}_{\mathcal{P}(\tilde{v})}(\mathcal{E}^0, \mathcal{A}^0) \ \text{ for } t \in \mathbb{R}\,.$$

Hence (3.27) and (3.25) with $\tilde{v}$ instead of v imply

$$|\mathcal{E}(t) - E_{s,\tilde{v}}| + \|\mathcal{A}(t) - A_{s,\tilde{v}}\|_1 = \mathcal{O}(\delta) \tag{3.28}$$

uniformly in $t \in \mathbb{R}$. On the other hand, total momentum conservation implies

$$\mathcal{P}(\tilde{v}) = P(t) + \langle \mathcal{E}(t), \nabla \mathcal{A}(t) \rangle \ \text{ for } t \in \mathbb{R}\,.$$

Therefore (3.28) leads to

$$|P(t) - P_{\tilde{v}}| = \mathcal{O}(\delta) \tag{3.29}$$

uniformly in $t \in \mathbb{R}$. Finally (3.28), (3.29) together imply the orbital stability for solutions $Y_s(t)$ in the space of Hamiltonian variables. By Proposition 3.1, (2.12) follows. □

4 Convergence of velocity

To prove Theorem 2.5 it suffices to prove the existence of the limits (1.6). We combine the orbital stability and the relaxation of the acceleration with a Hamiltonian formalism for the perturbed system (1.1). We prove (1.6) only for $t \to +\infty$ since the system is time-reversal. Introduce

$$\mathrm{osc}_{[T;+\infty)} v(t) := \sup_{t_1,t_2 \geq T} |v(t_1) - v(t_2)|.$$

The existence of the limits (1.6) follows from the following proposition.

Proposition 4.1 *Let the assumptions of Theorem* 2.5 *be fulfilled. Then*

$$\mathrm{osc}_{[T;+\infty)} v(t) \to 0 \quad \text{as} \quad T \to +\infty. \tag{4.1}$$

Proof. The idea of the proof is as follows. We modify the trajectory of the charge and the field part of the solution. The new trajectory and fields satisfy a new system of equations which is a small perturbation of the system (1.1) for large t.

Step 1. We introduce the above-mentioned modification of a solution. The following expansion holds, [8],

$$E(x,t) = E_{(r)}(x,t) + E_{(0)}(x,t), \; B(x,t) = B_{(r)}(x,t) + B_{(0)}(x,t). \tag{4.2}$$

Here

$$\begin{pmatrix} E_{(r)}(x,t) \\ B_{(r)}(x,t) \end{pmatrix} = \int_0^t ds\, g_{t-s}(x) * \begin{pmatrix} \rho(x - q(s)) \\ \rho(x - q(s))\dot{q}(s) \end{pmatrix}; \tag{4.3}$$

$$\begin{pmatrix} E_{(0)}(x,t) \\ B_{(0)}(x,t) \end{pmatrix} = m_t(x) * \begin{pmatrix} E^0 \\ B^0 \end{pmatrix}, \tag{4.4}$$

where m_t and g_t are respectively 6×6- and 6×4-matrix-valued distributions:

$$m_t = \begin{pmatrix} \dot{K}_t & \nabla \wedge K_t \\ -\nabla \wedge K_t & \dot{K}_t \end{pmatrix}, \quad g_t = \begin{pmatrix} -\nabla K_t & -\dot{K}_t \\ 0 & \nabla \wedge K_t \end{pmatrix},$$

and $K_t(x)$ denotes the Kirchhoff kernel

$$K_t(x) = \frac{1}{4\pi t}\delta(|x| - |t|).$$

It is important that the distributions m_t, g_t are concentrated on the sphere $\{|x| = |t|\}$, this means the strong Huygen's principle for the Maxwell-Lorentz system:

$$m_t(x) = 0 \;\text{ and }\; g_t(x) = 0 \;\text{ for }\; |x| \neq |t|. \tag{4.5}$$

Further, in [8] the following decay is established,

$$\begin{aligned}\left| m_t * \begin{pmatrix} E^0 \\ B^0 \end{pmatrix}(x) \right| &\le C_0|t|^{-2}\int_{|y-x|=|t|} d^3y\,(|E^0(y)|+|B^0(y)|)\\ &+ C_1|t|^{-1}\int_{|y-x|=|t|} d^3y\,(|\nabla E^0(y)|+|\nabla B^0(y)|). \quad (4.6)\end{aligned}$$

Recall that E^0 and B^0 satisfy the constraints

$$\nabla\cdot E^0(x)=\rho(x-q^0),\ \ \nabla\cdot B^0(x)=0, \qquad x\in\mathbb{R}^3 \tag{4.7}$$

providing that the fields defined through (4.2) to (4.4) satisfy the system (1.1). Further, by (2.9) for every $\varepsilon>0$ there exists t_ε such that

$$|\ddot q(t)|\le\varepsilon \ \text{ for } t\ge t_\varepsilon \ \text{ and } t_\varepsilon\to\infty \text{ as } \varepsilon\to 0\,. \tag{4.8}$$

Let us consider the points

$$t_{1,\varepsilon}=t_\varepsilon+1,\ \ t_{2,\varepsilon}=t_{1,\varepsilon}+\frac{R_\rho}{1-\overline{v}},\ \ t_{3,\varepsilon}=t_{2,\varepsilon}+\frac{R_\rho}{1-\overline{v}}. \tag{4.9}$$

Set

$$q_{3,\varepsilon}=q(t_{3,\varepsilon})\,,\ \ v_\varepsilon=\dot q(t_{3,\varepsilon}).$$

Then (4.8) implies that there exists $q_\varepsilon(t)\in C^2(\mathbb{R})$ such that

$$q_\varepsilon(t)=\begin{cases} q(t) & \text{for} \quad t\in[t_{1,\varepsilon},+\infty)\,,\\ l(t):=q_{3,\varepsilon}+v_\varepsilon(t-t_{3,\varepsilon}) & \text{for} \quad t\in(-\infty,t_\varepsilon]\,,\end{cases} \tag{4.10}$$

and

$$|\ddot q_\varepsilon(t)|\le C\varepsilon \ \text{ for all } t\in\mathbb{R} \tag{4.11}$$

with $C>0$ independent of $\varepsilon\in(0,1)$. Now set

$$\begin{pmatrix} E_\varepsilon(x,t)\\ B_\varepsilon(x,t)\end{pmatrix}=\int_{-\infty}^t ds\,g_{t-s}(x)*\begin{pmatrix}\rho(x-q_\varepsilon(s))\\ \rho(x-q_\varepsilon(s))\dot q_\varepsilon(s)\end{pmatrix},\ x\in\mathbb{R}^3,\ t>0. \tag{4.12}$$

Here the integrand, for a fixed s, is a convolution of two distributions of $\mathcal{S}'$, $\mathcal{S}'$ being the space of tempered distributions. One of the distributions, $g_{t-s}(\cdot)$, has a compact support by (4.5). Hence the integrand is as well a distribution of $\mathcal{S}'$, and this distribution depends continuously on s. Thus, the integral is understood as the Riemann integral of the continuous $\mathcal{S}'$-valued function on $\mathbb{R}$.

Step 2. We show that the modified fields satisfy the inhomogeneous Maxwell equations, coincide with soliton fields outside a certain light cone, and coincide with the retarded fields $(E_{(r)},B_{(r)})$ in a smaller light cone.

Lemma 4.2

i) *The fields E_ε, B_ε coincide with a soliton outside a light cone:*

$$E_\varepsilon(x,t) = E_{v_\varepsilon}(x-l(t)),\ B_\varepsilon(x,t) = B_{v_\varepsilon}(x-l(t)) \tag{4.13}$$

for

$$|x-q_\varepsilon(t_\varepsilon)| > t - t_\varepsilon + R_\rho. \tag{4.14}$$

ii) *E_ε, B_ε, q_ε satisfy the system*

$$\begin{aligned} \dot E_\varepsilon(x,t) &= \nabla\wedge B_\varepsilon(x,t) - \rho(x-q_\varepsilon(t))\dot q_\varepsilon(t), & \nabla\cdot E_\varepsilon(x,t) &= \rho(x-q_\varepsilon(t)),\\ \dot B_\varepsilon(x,t) &= -\nabla\wedge E_\varepsilon(x,t), & \nabla\cdot B_\varepsilon(x,t) &= 0, \end{aligned} \tag{4.15}$$

for $t\in\mathbb{R}$, $x\in\mathbb{R}^3$.

iii) *The fields E_ε, B_ε coincide with $E_{(r)}$, $B_{(r)}$ in the light cone $K = \{|x-q(t_{2,\varepsilon})| < t - t_{2,\varepsilon}\}$.*

Proof. i) Consider the soliton fields $(E_{v_\varepsilon}(x-l(t)), B_{v_\varepsilon}(x-l(t))$ as the solution of the Cauchy problem for system (4.15) with initial data at $-T$, $T>0$. These data equal $(E^{-T}, B^{-T}) = (E_{v_\varepsilon}(x-l(-T)), B_{v_\varepsilon}(x-l(-T))$. Let us apply the formulas of type (4.2) to (4.4) in the case, when Cauchy data are set at $-T$ instead of 0. Then we obtain

$$\begin{pmatrix} E_{v_\varepsilon}(x-l(t)) \\ B_{v_\varepsilon}(x-l(t)) \end{pmatrix} = \int_{-T}^{t} ds\, g_{t-s}(x) * \begin{pmatrix} \rho(x-l(s)) \\ \rho(x-l(s))v_\varepsilon \end{pmatrix} + m_{t+T} * \begin{pmatrix} E^{-T} \\ B^{-T} \end{pmatrix}, \tag{4.16}$$

since for (E^{-T}, B^{-T}) the constrains of type (4.7) are satisfied with $l(-T)$ instead of q^0. Here the last summand tends to zero in $L^2_{\rm loc}(\mathbb{R}^3)\oplus L^2_{\rm loc}(\mathbb{R}^3)$ (and hence in $\mathcal{S}'\oplus\mathcal{S}'$) as $T\to+\infty$. This follows by the bounds (4.6) using the formulas (1.5) and $|v_\varepsilon|<1$. Hence, proceeding to the limit as $T\to+\infty$ we obtain the identity of distributions,

$$\begin{pmatrix} E_{v_\varepsilon}(x-l(t)) \\ B_{v_\varepsilon}(x-l(t)) \end{pmatrix} = \int_{-\infty}^{t} ds\, g_{t-s}(x) * \begin{pmatrix} \rho(x-l(s)) \\ \rho(x-l(s))v_\varepsilon \end{pmatrix}. \tag{4.17}$$

Finally, in the region (4.14) the right-hand side of (4.17) coincides with (4.12) by (4.10) and (4.5).

ii) The strong Huygen's principle (4.5) implies that for $(x,t)\in K_0 = \{|x-q_\varepsilon(t_\varepsilon)| < t - t_\varepsilon + R_\rho\}$ and for sufficiently large t_ε

$$\begin{pmatrix} E_\varepsilon(x,t) \\ B_\varepsilon(x,t) \end{pmatrix} = \int_{-T}^{t} ds\, g_{t-s}(x) * \begin{pmatrix} \rho(x-q_\varepsilon(s)) \\ \rho(x-q_\varepsilon(s))\dot q_\varepsilon(s) \end{pmatrix}, \tag{4.18}$$

$x\in\mathbb{R}^3$, $t\in\mathbb{R}$ with a large T independent of $(x,t)\in K_0$. Introduce the fields

$$\begin{pmatrix} \tilde E_t \\ \tilde B_T \end{pmatrix} = \begin{pmatrix} E_\varepsilon \\ B_\varepsilon \end{pmatrix} + m_{t+T} * \begin{pmatrix} E^{-T} \\ B^{-T} \end{pmatrix}, \tag{4.19}$$

with (E^{-T}, B^{-T}) like in (4.18). Then the fields $(\tilde{E}, \tilde{B})$ satisfy all the equations of (4.15) for $t > -T$ by the same argument as in the proof of i) (the constraints of type (4.7) for initial data are satisfied). Finally, the second summand in the right-hand side of (4.19) tends to zero as $T \to +\infty$ like in (4.16). Hence, $(E_\varepsilon, B_\varepsilon)$ satisfy (4.15) for all $t \in \mathbb{R}$.

iii) follows from (4.3), (4.12), (4.10), and (4.5). □

Step 3. Now we show that at $t = t_{3,\varepsilon}$ in the ball, where the modified fields are not explicitly the soliton, they are sufficiently close to it. Indeed, we have

$$\begin{pmatrix} E_\varepsilon(x, t_{3,\varepsilon}) - E_{v_\varepsilon}(x - q_\varepsilon) \\ B_\varepsilon(x, t_{3,\varepsilon}) - B_{v_\varepsilon}(x - q_\varepsilon) \end{pmatrix}$$

$$= \int_{t_\varepsilon}^{t_{3,\varepsilon}} ds\, g_{t_{3,\varepsilon}-s}(x) \begin{pmatrix} \rho(x - q_\varepsilon(s)) - \rho(x - l(s)) \\ \rho(x - q_\varepsilon(t))\dot{q}_\varepsilon(s) - \rho(x - l(s))v_\varepsilon \end{pmatrix}.$$

Hence, by (4.10), (4.11) we obtain that

$$|E_\varepsilon(\cdot, t_{3,\varepsilon}) - E_{v_\varepsilon}(\cdot - q_\varepsilon)|_{L^2(B^\varepsilon)} + |B_\varepsilon(\cdot, t_{3,\varepsilon}) - B_{v_\varepsilon}(\cdot - q_\varepsilon)|_{L^2(B^\varepsilon)} = \mathcal{O}(\varepsilon), \quad (4.20)$$

where $B^\varepsilon = \{x : |x - q_\varepsilon(t_\varepsilon)| \le 2R_\rho/(1 - \overline{v}) + 1 + R_\rho\}$.

Step 4. We now express the Lorentz force equation for $t \ge T := t_{3,\varepsilon}$ in terms of the fields E_ε, B_ε. In this region $q_\varepsilon(t) = q(t)$. Thus, we can change $q_\varepsilon(t)$ by $q(t)$ in the equations (4.15) for E_ε, B_ε:

$$\dot{E}_\varepsilon(x, t) = \nabla \wedge B_\varepsilon(x, t) - \rho(x - q(t))\dot{q}(t), \ \ \nabla \cdot E_\varepsilon(x, t) = \rho(x - q(t)),$$

$$\dot{B}_\varepsilon(x, t) = -\nabla \wedge E_\varepsilon(x, t), \ \ \nabla \cdot B_\varepsilon(x, t) = 0 \quad (4.21)$$

for $t > T$. Further, one has $E_\varepsilon = E_{(r)}$ and $B_\varepsilon = B_{(r)}$ inside K by Lemma 4.2, iii). Thus, for $t > T$ in $\operatorname{supp} \rho(x - q(t))$ we have $E = E_\varepsilon + E_{(0)}$, $B = B_\varepsilon + B_{(0)}$ and hence

$$\dot{q}(t) = \frac{p(t)}{\sqrt{1 + p^2(t)}}, \ \ \dot{p}(t) = \int [E_\varepsilon(x, t) + \dot{q}(t) \wedge B_\varepsilon(x, t)]\rho(x - q(t))\, d^3x + f(t), \ \ t > T, \quad (4.22)$$

where

$$f(t) := \int [E_{(0)}(x, t) + \dot{q}(t) \wedge B_{(0)}(x, t)]\rho(x - q(t))\, d^3x.$$

Let us transfer to Hamiltonian variables $E_{s,\varepsilon}$, A_ε, q, P, where

$$E_{s,\varepsilon}(x, t) = \Pi_s E_\varepsilon(x, t), \ \ B_\varepsilon(x, t) = \nabla \wedge A_\varepsilon(x, t), \ \ \nabla \cdot A_\varepsilon(x, t) = 0,$$

$$P(t) := p(t) + \int \rho(x - q(t)) A_\varepsilon(x, t) d^3x.$$

Recall that the total momentum is $\mathcal{P} = p + \int E_s(x) \wedge (\nabla \wedge A(x)) d^3x$; for a soliton (E_v, B_v) the total momentum is $\mathcal{P}(v) = p_v + \int E_{s,v}(x) \wedge (\nabla \wedge A_{s,v}(x)) d^3x$.

Step 5. We establish a summable decay of $f(t)$ and prove that H_s and $\mathcal{P}$ are "almost conserved" along a trajectory solution $Y_\varepsilon(t)$, where $H_s(E_s, A, q, P)$ is defined by (3.8).

Lemma 4.3 *For $\sigma > 1/2$ introduced in Proposition* 2.3*:*

i) *The following asymptotics hold:*

$$|f(t)| = \mathcal{O}(t^{-1-\sigma}). \tag{4.23}$$

ii) *The oscillations of the Hamiltonian and the total momentum are small for large T:*

$$H_s(Y_\varepsilon(t)) = H_s(Y_\varepsilon(T)) + \mathcal{O}(T^{-\sigma}), \tag{4.24}$$

$$\mathcal{P}(Y_\varepsilon(t)) = \mathcal{P}(Y_\varepsilon(T)) + \mathcal{O}(T^{-\sigma}) \tag{4.25}$$

for $t > T$.

Proof. i) For $f(t)$ the asymptotics follow from the explicit formulas (4.4), the bounds (4.6), the estimate (2.8), and the decay (2.4) of the initial fields.

ii) By (3.11) it suffices to prove (4.24) for $H(Y_\varepsilon(t))$, where H is defined by (1.2). One has

$$\frac{d}{dt}\left(\sqrt{1+p^2} + \frac{1}{2}\int (E_\varepsilon^2 + B_\varepsilon^2) dx\right) = v \cdot \dot{p} + \langle E_s, \dot{E}_s \rangle + \langle B_s, \dot{B}_s \rangle =$$

$$v \cdot \left(\int (E_\varepsilon + \dot{q} \wedge B_\varepsilon)\rho(x-q)dx + f\right) + \langle E_\varepsilon, \nabla \wedge B_\varepsilon - \rho(x-q)\dot{q}\rangle + \langle B_\varepsilon, -\nabla \wedge E_\varepsilon \rangle = v \cdot f,$$

since $\langle E_\varepsilon, \nabla \wedge B_\varepsilon \rangle - \langle B_\varepsilon, \nabla \wedge E_\varepsilon \rangle = 0$ similar to the proof of Proposition A.5 of [8].

Similarly, one has

$$\frac{d}{dt}\mathcal{P}(Y_\varepsilon(t)) = f(t).$$

Then (4.24), (4.25) follow from (4.23). □

Step 6. Finally, we use the orbital stability estimate (3.25). For $t \geq T := t_{3,\varepsilon}$ one has $\mathcal{P}(Y_\varepsilon(t)) = \mathcal{P}(\tilde{v}(t))$, where $\mathcal{P}(\tilde{v}(t))$ is the total momentum of the soliton of velocity $\tilde{v}(t)$. From (4.25) and the differentiability of the map $\mathcal{P}(v) \mapsto v$, the inverse map to (3.23), it follows that

$$\operatorname{osc}_{[T,+\infty)} \tilde{v}(t) \to 0 \text{ as } T \to +\infty. \tag{4.26}$$

By the statement i) of Lemma 4.2 and (4.20) one has $\tilde{v}(t_{3,\varepsilon}) - v_\varepsilon = \mathcal{O}(\varepsilon)$. Together with (4.26) this implies the bound $|\tilde{v}(t)| \leq \overline{v}_1 < 1$ for $t \geq T$. Now apply the estimate (3.25) and get

$$\frac{1-|\tilde v(t)|}{2}\big(|E_{s,\varepsilon}(\cdot+q(t),t)-E_{s,\tilde v(t)}|^2+\|A_\varepsilon(\cdot+q(t),t)-A_{s,\tilde v(t)}\|^2\big)$$
$$\le \mathcal{H}_{\mathcal{P}(\tilde v(t))}(E_{s,\varepsilon}(\cdot+q(t),t),A_\varepsilon(\cdot+q(t),t))-\mathcal{H}_{\mathcal{P}(\tilde v(t))}(E_{s,\tilde v(t)},A_{s,\tilde v(t)}). \quad (4.27)$$

Lemma 4.4 *The right-hand side of* (4.27) *is arbitrary small uniformly in* $t\ge T$ *for sufficiently small* ε *and sufficiently large* T.

From this lemma it follows that

$$\mathrm{osc}_{[T,+\infty)}|E_{s,\varepsilon}(\cdot+q(t),t)|\to 0 \qquad\text{and}\qquad \mathrm{osc}_{[T,+\infty)}\|A_\varepsilon(\cdot+q(t),t)\|\to 0$$

as $T\to+\infty$. Indeed,

$$\begin{aligned}&E_{s,\varepsilon}(\cdot+q(t_2),t_2)-E_{s,\varepsilon}(\cdot+q(t_1),t_1)\\&\quad=(E_{s,\varepsilon}(\cdot+q(t_2),t_2)-E_{s,\tilde v(t_2)})\\&\qquad-(E_{s,\varepsilon}(\cdot+q(t_1),t_1)-E_{s,\tilde v(t_1)})+(E_{s,\tilde v(t_2)}-E_{s,\tilde v(t_1)}).\end{aligned}$$

For $t_1,t_2>T$ the first and the second summands are small by (4.27) and the lemma, the third is small by (4.26), since the soliton field E_v depends continuously on v in L^2. For the field A the argument is similar. Together with (4.25) this implies $\mathrm{osc}_{[T,+\infty)}p(t)\to 0$ as $T\to+\infty$ and hence (4.1) follows. Proposition 4.1 is proved. □

Proof of Lemma 4.4. Denote $\tilde{\mathcal{P}}(t)=\mathcal{P}(\tilde v(t))$, $\Phi(t)=(E_{s,\varepsilon}(\cdot+q(t),t),A_\varepsilon(\cdot+q(t),t))$, $\tilde\Phi(t)=(E_{s,\tilde v(t)},A_{s,\tilde v(t)})$. We claim that $\mathcal{H}_{\tilde{\mathcal{P}}(t)}(\Phi(t))-\mathcal{H}_{\tilde{\mathcal{P}}(t)}(\tilde\Phi(t))$ is close to $\mathcal{H}_{\tilde{\mathcal{P}}(T)}(\Phi(T))-\mathcal{H}_{\tilde{\mathcal{P}}(T)}(\tilde\Phi(T))$ and the last expression is small due to (4.13) and (4.20). Thus, it is sufficient to prove that $\mathcal{H}_{\tilde{\mathcal{P}}(t)}(\Phi(t))$ is close to $\mathcal{H}_{\tilde{\mathcal{P}}(T)}(\Phi(T))$ and $\mathcal{H}_{\tilde{\mathcal{P}}(t)}(\tilde\Phi(t))$ is close to $\mathcal{H}_{\tilde{\mathcal{P}}(T)}(\tilde\Phi(T))$. One has

$$\begin{aligned}&\mathcal{H}_{\tilde{\mathcal{P}}(t)}(\Phi(t))-\mathcal{H}_{\tilde{\mathcal{P}}(T)}(\Phi(T))\\&\quad=\mathcal{H}_{\tilde{\mathcal{P}}(t)}(\Phi(t))-\mathcal{H}_{\tilde{\mathcal{P}}(T)}(\Phi(t))+\mathcal{H}_{\tilde{\mathcal{P}}(T)}(\Phi(t))-\mathcal{H}_{\tilde{\mathcal{P}}(T)}(\Phi(T)),\end{aligned}$$

this is small due to (4.24), (4.25). For $\mathcal{H}_{\tilde{\mathcal{P}}(t)}(\tilde\Phi(t))-\mathcal{H}_{\tilde{\mathcal{P}}(T)}(\tilde\Phi(T))$ the argument is similar. □

Acknowledgments. V. Imaikin and A. Komech are supported partly by research grants of DFG (436 RUS 113/615/0-1(R)) and RFBR (01-01-04002).

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Communicated by Viencent Rivasseau
submitted 01/12/03, accepted 26/05/04