Book of Practical PDEs

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Chapter 1

Hyperbolic Equations. Method of Characteristics

1.1 Derivation of the d'Alembert equation

The d'Alembert equation

$$\frac{\partial^2 u}{\partial t^2}(x,t) = a^2 \frac{\partial^2 u}{\partial x^2} + f(x,t), \quad x \in [0,l], \quad t > 0. \tag{1.1.1}$$

describes small transversal oscillations of a stretched string or straight oscillations of a flexible rod. Let us give a brief derivation of this equation (for more rigorous derivation, see [Vla84, SD64, TS90]).

Transversal oscillations of a string

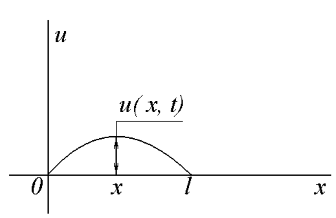


Figure 1.1:

We assume that a string of length l is stretched with the force T. We choose the direction of the axis Ox along the string in its equilibrium configuration. Let $x = 0$ corresponds to the left end of the string. Then the right end of the string is given by $x = l$. We choose the axis Ou normal to Ox, and only consider the transversal oscillations of the string, such that each point x moves only in the direction of the axis Ou. Such oscillations could be started by fixing the ends of the string, or by attaching the ends to the tiny rings which can move up and down along vertical rods as on Fig. 1.3.

We denote by $u(x,t)$ the displacement of the point x of the string at a moment t. We assume that the angles between the string and the axis $0x$ are small: $|\alpha|, |\beta| \ll 1$ (see Fig. 1.2). Let us prove that $u(x,t)$ satisfies the equation (1.1.1). To do so, we write Newton's Second Law for the piece of the string from x to $x+\Delta x$, and take its projection onto the axis $0u$:

$$a_u m = F_u. \tag{1.1.2}$$

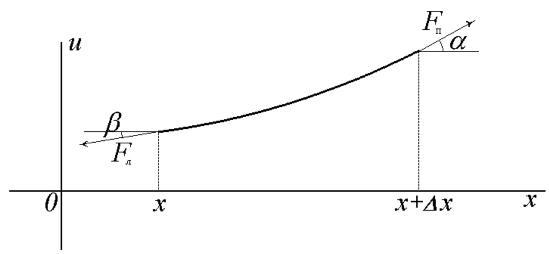


Figure 1.2:

Here $a_u \approx \frac{\partial^2 u}{\partial t^2}(x,t)$; $\quad m = \mu \cdot \Delta x$, where μ is the linear density of the string, that is, the mass of its unit length (we assume that the string is uniform), and

$$F_u \approx (F_l)_u + (F_r)_u + \tilde{f}(x,t)\Delta x. \tag{1.1.3}$$

By F_l (F_r) we denoted the force which acts on the region $[x, x+\Delta x]$ from the left (right) part of the string, and $(F_l)_u((F_r)_u)$ stands for the projections onto the axis Ou. $\tilde{f}(x,t)$ is the density of the transversal external forces. For example, in the gravitational field of the Earth, if the string is horizontal and the axis $0u$ is directed upward, then $\tilde{f}(x,t) = -g\mu$, where $g \approx 9{,}8\, m/s^2$.

Substituting a_u, m and F_u into (1.1.2), we obtain

$$\frac{\partial^2 u}{\partial t^2}\mu\Delta x \approx (F_l)_u + (F_r)_u + \tilde{f}(x,t)\Delta x. \tag{1.1.4}$$

Further, for an elastic string the force of tension T at each point is tangent to the string and has the same magnitude (see [Vla84]). Then

$$(F_l)_u = -T\sin\beta; \quad (F_r)_u = T\sin\alpha \tag{1.1.5}$$

and (1.1.4) takes the form

$$\frac{\partial^2 u}{\partial t^2}\mu\Delta x \approx -T\sin\beta + T\sin\alpha + \tilde{f}(x,t)\Delta x. \tag{1.1.6}$$

Since we consider the "small" oscillations, such that $|\alpha|$ and $|\beta| \ll 1$, with the precision up to higher powers of α and β

$$\sin\beta \approx \tan\beta = \frac{\partial u}{\partial x}(x,t); \quad \sin\alpha \approx \tan\alpha = \frac{\partial u}{\partial x}(x+\Delta x, t) \tag{1.1.7}$$

Substituting these expressions into (1.1.6), we have with the same precision

$$\frac{\partial^2 u}{\partial t^2}\mu\Delta x \approx T\Big(\frac{\partial u}{\partial x}(x+\Delta x,t) - \frac{\partial u}{\partial x}(x,t)\Big) + \tilde{f}(x,t)\Delta x. \tag{1.1.8}$$

Dividing this expression by Δx and sending $\Delta x \to 0$ we obtain with the specified precision the equation (1.1.1), where

$$a = \sqrt{\frac{T}{\mu}}; \quad f(x,t) = \frac{\tilde{f}(x,t)}{\mu}. \tag{1.1.9}$$

Remark 1.1.1. From our assumption about the tension we deduce that the projections of the forces F_l and F_r onto the axis $0x$ are equal to $-T\cos\beta$ and $T\cos\alpha$, respectively. Therefore their sum $(T\cos\alpha - T\cos\beta)$ is the quantity of magnitude $O(\alpha^2+\beta^2)$. The projection of the resulting force which acts on the piece of the string from x to $x+\Delta x$ is of the magnitude which is small in the approximation we use. Thus, under this assumption about the tension, the small oscillations of the string are transversal with the specified precision.

Remark 1.1.2. From (1.1.5) and (1.1.7) it follows that

$$T\frac{\partial u}{\partial x}(x,t) \tag{1.1.10}$$

is the vertical part of the tension of the string at a point x at a moment t.

Let us consider *the boundary conditions* for the string.

A. If the left end of the string, $x=0$, is fixed, then its displacement is equal to zero:

$$u(0,t) = 0, \quad t > 0. \tag{1.1.11}$$

B. Assume that the left end of the string is attached to a tiny ring of negligible mass, which can move without friction along a vertical rod (such an end of the string is called a free end). Then the vertical component of the force with which the rod acts on the left end of the string is equal to zero. Therefore, according to Newton's Third Law, the vertical component (1.1.10) of the force of tension of the string at $x=0$ is also equal to zero:

$$\frac{\partial u}{\partial x}(0,t) = 0, \quad t > 0. \tag{1.1.12}$$

C. In a more general case, when we attach a mass m to the left end of the string, there is the boundary condition

$$m\frac{\partial^2 u}{\partial t^2}(0,t) = T\frac{\partial u}{\partial x}(0,t), \quad t > 0 \tag{1.1.13}$$

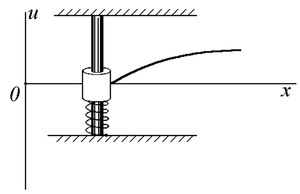


Figure 1.3:

If, besides, the mass m is attached to the spring (see Fig. 1.3) with the spring constant k, then in the right side of (1.1.13) we need to add the elasticity force $-ku(0,t)$. If the mass m experiences an additional friction force proportional to the velocity, then in the right hand side of (1.1.13) one needs to add a friction force $-\eta\frac{\partial u}{\partial t}(0,t)$. This way one obtains a physically reasonable linear boundary condition of the form

$$m\frac{\partial^2 u}{\partial t^2}(0,t) = T\frac{\partial u}{\partial x}(0,t) - ku(0,t) - \eta\frac{\partial u}{\partial t} + f(t). \tag{1.1.14}$$

Here $f(t)$ is an external force parallel to the axis Ou which is applied to the left end of the string.

Tangential oscillations of the elastic rod

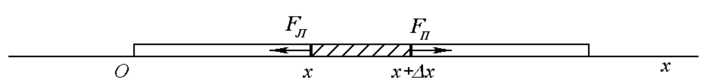


Figure 1.4:

Assume we have a uniform unstretched rod of length l. Choose the axis Ox along the rod, so that its left end is located at the point $x = 0$. Then $x = l$ is its right end. We will consider only tangential oscillations of the rod. Denote by $u(x,t)$ the displacement of the point x of the rod at the moment t, along the axis Ox.

Let us prove that $u(x,t)$ satisfies equation (1.1.1). For this, we write down the projection onto the axis Ox of Newton's Second Law for the piece of the rod from x to $x + \Delta x$:

$$a_x m = F_x; \quad a_x \approx \frac{\partial^2 u}{\partial t^2}(x,t); \quad m = \mu\Delta x. \tag{1.1.15}$$

The force F_x has the form

$$F_x = F_l + F_r + \tilde{f}(x,t)\Delta x. \tag{1.1.16}$$

where $F_l(F_r)$ is the force along the axis Ox, acting at the piece $[x, x+\Delta x]$ at the left (respectively, right) part of the rod, and $\tilde{f}(x,t)$ is the density of the external forces directed along the axis Ox. For example, if the rod is hanging vertically in the field of gravity of the Earth so that the axis Ox is directed downwards, then $\tilde{f}(x,t) = g\mu$.

Substituting F_x into (1.1.15), we get

$$\frac{\partial^2 u}{\partial t^2}(x,t)\mu\Delta x \approx F_l + F_r + \tilde{f}(x,t)\Delta x. \tag{1.1.17}$$

To find F_l and F_r, we use Hook's Law

$$\sigma(x,t) = E\varepsilon(x,t). \tag{1.1.18}$$

Here $\sigma(x,t)$ is a tension of the rod at the point x, that is, $\sigma(x,t) = T(x,t)/S$, where $T(x,t)$ is the tension force at the point x and S is the section area; E is Young's module of the material of the rod, and $\varepsilon(x,t)$ is the relative deformation at the point x. For the piece of the rod $[x, x+h]$, its initial length (when no force is applied) is equal to h, while under tension it is $h + u(x+h,t) - u(x,t)$. Therefore the absolute length increase is equal to $u(x+h,t) - u(x,t)$, while the relative length increase is

$$\frac{u(x+h,t) - u(x,t)}{h} \quad \longrightarrow \quad \frac{\partial u}{\partial x}(x,t), \quad h \to 0. \tag{1.1.19}$$

Thus,

$$\varepsilon(x,t) = \frac{\partial u}{\partial x}(x,t). \tag{1.1.20}$$

From here, by Hook's Law (1.1.18),

$$T(x,t) = S\sigma(x,t) = SE\varepsilon(x,t) = SE\frac{\partial u}{\partial x}(x,t) \tag{1.1.21}$$

Let us point out that Hook's Law (1.1.18) is a linear approximation for the dependence of $\sigma(x,t)$ of $\varepsilon(x,t)$, and is only applicable for small deformations, that is, small values of $\varepsilon(x,t)$.

Taking into account the direction of the forces F_l and F_r, we obtain

$$\begin{cases} F_l = -T(x,t) = -SE\frac{\partial u}{\partial x}(x,t), \\ F_r = -T(x+\Delta x,t) = -SE\frac{\partial u}{\partial x}(x+\Delta x,t). \end{cases} \tag{1.1.22}$$

Indeed, if, for example, $u(x,t)$ is monotonically increasing in x, then the rod is stretched out, hence $F_l \le 0$, while $F_r \ge 0$. At the same time $\frac{\partial u}{\partial x} \ge 0$. This means that the signs in (1.1.22) are correct.

Substituting (1.1.22) into (1.1.17), we get

$$\frac{\partial^2 u}{\partial t^2}(x,t)\mu\Delta x \approx SE\frac{\partial u}{\partial x}(x+\Delta x,t) - SE\frac{\partial u}{\partial x}(x,t) + \tilde{f}(x,t)\Delta x. \tag{1.1.23}$$

From here, dividing by Δx, at the limit $\Delta x \to 0$ we get (1.1.1) with

$$a = \sqrt{\frac{SE}{\mu}} = \sqrt{\frac{E}{\rho}}; \quad f(x,t) = \frac{\tilde{f}(x,t)}{\mu} \tag{1.1.24}$$

where $\rho = \mu/S$ is the density of the material of the rod.

Let us consider *boundary conditions* for the rod.

A. For the fixed end of the rod at $x = 0$ there is the boundary condition (1.1.11).

B. For the free end of the rod at $x = 0$ tension (1.1.21) is equal to zero. Therefore (1.1.12) holds.

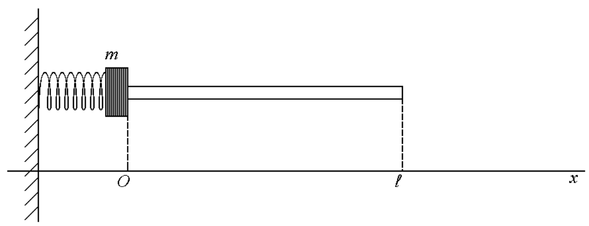


Figure 1.5:

C. In a more general case, assume that there is a mass m at the left end $x = 0$ of the rod, attached to the spring of stiffness $k > 0$, and that the equilibrium position of the spring corresponds to zero displacement of the left end of the rod. Assume that the mass moves with the viscous friction: $F_{fr} = -\eta v$, where $\eta > 0$, v is the speed of the mass. Then at $x = 0$ there is the boundary condition

$$m\frac{\partial^2 u}{\partial t^2}(0,t) = -ku(0,t) + SE\frac{\partial u}{\partial x}(0,t) - \eta\frac{\partial u}{\partial t}(0,t) + f(t), \tag{1.1.25}$$

where $f(t)$ is the external force, acting at the left end of the rod along the axis Ox.

1.2 Infinite string

The Cauchy problem for the d'Alembert equation

We consider the d'Alembert equation (1.1.1) in the real line:

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0. \tag{1.2.1}$$

This corresponds to the physical problem about a string of relatively large size. For simplicity we assume that $f(x,t) \equiv 0$, that is, that there are no external forces.

As we will see below, there are infinitely many solutions of (1.2.1). To be able to determine the movement of the string it suffices to prescribe initial position and velocity of all points of the string (as usually in mechanics):

$$u(x,0) = \varphi(x), \quad \frac{\partial u}{\partial t}(x,0) = \psi(x), \quad x \in \mathbb{R} \tag{1.2.2}$$

Here φ and ψ are prescribed functions, $\varphi(x)$ is initial displacement, and $\psi(x)$ is the initial velocity of a point x of the string.

The problem (1.2.1)–(1.2.2) is called the Cauchy problem (or initial value problem) for the d'Alembert equation (1.2.1). The relations (1.2.2) are called the boundary conditions, and the functions $\varphi(x)$, $\psi(x)$ are called initial data.

The d'Alembert method

The d'Alembert method is based on the fact that the general solution to (1.2.1) has the form

$$u(x,t) = f(x-at) + g(x+at), \tag{1.2.3}$$

where f and g are arbitrary functions of one variable.

Remark 1.2.1. If f and g belong to $C^2(\mathbb{R})$, then $u(x,t)$ also has two continuous derivatives. It turns out, though, that one can take f and g non-smooth and even non-continuous. Then $u(x,t)$ is also non-smooth or non-continuous, respectively.

As we will show in Section 4.6, such a non-continuous function satisfies the equation (1.2.1) in the sense of distributions.

To prove (1.2.3), let us change the variables in the differential equation (1.2.1)

$$\xi = x - at, \quad \eta = x + at \tag{1.2.4}$$

Change of variables in a differential equation

Let us express the function $u(x,t)$ in the new coordinates ξ, η:

$$u(x,t) = v(\xi,\eta), \tag{1.2.5}$$

where ξ,η are related to x,t by (1.2.4). For example,

$$u(x,t) = x \quad \Rightarrow \quad v(\xi,\eta) = \frac{1}{2}(\xi+\eta). \tag{1.2.6}$$

To make a change of variables in the differential equation (1.2.1) means to find a differential equation for the function $v(\xi,\eta)$, which would be equivalent to (1.2.1).

For this we need to express $\frac{\partial^2 u}{\partial t^2}$ and $\frac{\partial^2 u}{\partial x^2}$ via the derivatives of $v(\xi, \eta)$ with respect to ξ, η and to substitute the resulting expressions into (1.2.1). The necessary expressions are obtained with the aid of the chain rule applied to the identity

$$u(x,t) = v\big(\xi(x,t),\ \eta(x,t)\big) \tag{1.2.7}$$

Namely, differentiating (1.2.7) with respect to t and x, we obtain

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial v}{\partial \xi}\,\frac{\partial \xi}{\partial t} + \frac{\partial v}{\partial \eta}\,\frac{\partial \eta}{\partial t}, \\ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial \xi}\,\frac{\partial \xi}{\partial x} + \frac{\partial v}{\partial \eta}\,\frac{\partial \eta}{\partial x}. \end{cases} \tag{1.2.8}$$

In the same way one can express all other derivatives. Differentiating the first relation (1.2.8) with respect to t, we obtain:

$$\frac{\partial^2 u}{\partial t^2} = \Big(\frac{\partial}{\partial t}\,\frac{\partial v}{\partial \xi}\Big)\,\frac{\partial \xi}{\partial t} + \frac{\partial v}{\partial \xi}\,\frac{\partial^2 \xi}{\partial t^2} + \Big(\frac{\partial}{\partial t}\,\frac{\partial v}{\partial \eta}\Big)\,\frac{\partial \eta}{\partial t} + \frac{\partial v}{\partial \eta}\,\frac{\partial^2 \eta}{\partial t^2} \tag{1.2.9}$$

We express the operator $\frac{\partial}{\partial t}$ from the same relation (1.2.8):

$$\frac{\partial}{\partial t} = \frac{\partial \xi}{\partial t}\,\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t}\,\frac{\partial}{\partial \eta} \tag{1.2.10}$$

Substituting this expression in (1.2.9), we get

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} \quad = \quad & \Big(\frac{\partial \xi}{\partial t}\,\frac{\partial^2 v}{\partial \xi^2} + \frac{\partial \eta}{\partial t}\,\frac{\partial^2 v}{\partial \eta \partial \xi}\Big)\,\frac{\partial \xi}{\partial t} + \frac{\partial v}{\partial \xi}\,\frac{\partial^2 \xi}{\partial t^2} + \\ & +\Big(\frac{\partial \xi}{\partial t}\,\frac{\partial^2 v}{\partial \xi \partial \eta} + \frac{\partial \eta}{\partial t}\,\frac{\partial^2 v}{\partial \eta^2}\Big)\,\frac{\partial \eta}{\partial t} + \frac{\partial v}{\partial \eta}\,\frac{\partial^2 \eta}{\partial t^2} = \\ = \quad & \Big(\frac{\partial \xi}{\partial t}\Big)^2\,\frac{\partial^2 v}{\partial \xi^2} + 2\frac{\partial \xi}{\partial t}\,\frac{\partial \eta}{\partial t}\,\frac{\partial^2 v}{\partial \xi \partial \eta} + \Big(\frac{\partial \eta}{\partial t}\Big)^2\,\frac{\partial^2 v}{\partial \eta^2} \\ & +\frac{\partial v}{\partial \xi}\,\frac{\partial^2 \xi}{\partial t^2} + \frac{\partial v}{\partial \eta}\,\frac{\partial^2 \eta}{\partial t^2}. \end{aligned} \tag{1.2.11}$$

Here we used the identity

$$\frac{\partial^2 v}{\partial \eta \partial \xi} = \frac{\partial^2 v}{\partial \xi \partial \eta}. \tag{1.2.12}$$

In the same fashion (substituting in (1.2.11) t by x) one can obtain the formula

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} \quad = \quad & \Big(\frac{\partial \xi}{\partial x}\Big)^2\,\frac{\partial^2 v}{\partial \xi^2} + 2\frac{\partial \xi}{\partial x}\,\frac{\partial \eta}{\partial x}\,\frac{\partial^2 v}{\partial \xi \partial \eta} + \Big(\frac{\partial \eta}{\partial x}\Big)^2\,\frac{\partial^2 v}{\partial \eta^2} + \\ & + \frac{\partial v}{\partial \xi}\,\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial v}{\partial \eta}\,\frac{\partial^2 \eta}{\partial x^2}. \end{aligned} \tag{1.2.13}$$

Problem 1.2.1. Derive the formula

$$\begin{aligned}\frac{\partial^2 u}{\partial t\partial x} &= \frac{\partial\xi}{\partial t}\frac{\partial\xi}{\partial x}\frac{\partial^2 v}{\partial\xi^2}+\left(\frac{\partial\xi}{\partial t}\frac{\partial\eta}{\partial x}+\frac{\partial\eta}{\partial t}\frac{\partial\xi}{\partial x}\right)\frac{\partial^2 v}{\partial\xi\partial\eta}+\\ &+\frac{\partial\eta}{\partial t}\frac{\partial\eta}{\partial x}\frac{\partial^2 v}{\partial\eta^2}+\frac{\partial^2\xi}{\partial t\partial x}\frac{\partial v}{\partial\xi}+\frac{\partial^2\eta}{\partial t\partial x}\frac{\partial v}{\partial\eta}. \qquad (1.2.14)\end{aligned}$$

Remark 1.2.2. Usually the formulae (1.2.8) and (1.2.11)–(1.2.14) are written with u instead of v. For example, (1.2.8) is written as

$$\begin{cases}\frac{\partial u}{\partial t}=\frac{\partial u}{\partial\xi}\frac{\partial\xi}{\partial t}+\frac{\partial u}{\partial\eta}\frac{\partial\eta}{\partial t},\\ \frac{\partial u}{\partial x}=\frac{\partial u}{\partial\xi}\frac{\partial\xi}{\partial x}+\frac{\partial u}{\partial\eta}\frac{\partial\eta}{\partial x}.\end{cases} \qquad (1.2.15)$$

If so, the symbol $\frac{\partial u}{\partial\xi}$ (and $\frac{\partial u}{\partial\eta}$) in the right hand side is to be understood as the derivative along the line $\eta = const$ (or $\xi = const$):

$$\frac{\partial u}{\partial\xi}\equiv\frac{d}{d\xi}u\Big|_{\eta=const}, \qquad (1.2.16)$$

which is actually $\frac{\partial v}{\partial\xi}$ (or $\frac{\partial v}{\partial\eta}$), not as "partial derivative of $u(x,t)$ with respect to ξ (or η)"; the latter does not make sense until the other variable, η (or ξ), is chosen.

Indeed, from (1.2.16) one can see that $\frac{\partial u}{\partial\xi}$ depends not only on the choice of the variable ξ, but also on the variable η, although this is not reflected in the notation $\frac{\partial u}{\partial\xi}$. Thus, the usage of the notation u in the right hand side of (1.2.8), instead of v, as in (1.2.15), can lead to a confusion.

Problem 1.2.2. Find $\frac{\partial u}{\partial\xi}$, if $u(x,t)=t$, $\xi=x$, and $\eta=t+x$.

Solution. $t=\eta-x=\eta-\xi \quad\Longrightarrow \frac{\partial u}{\partial\xi}=-1.$

Problem 1.2.3. Find $\frac{\partial u}{\partial\xi}$, if $u(x,t)=t$, $\xi=x$ and $\eta=t-x$.

Solution. $t=\eta+x=\eta+\xi \quad\Longrightarrow \frac{\partial u}{\partial\xi}=1.$

Nevertheless, in the applied problems the formulae like (1.2.15) are often used, so that the introduction of new notations would not be needed. For example, the pressure is usually denoted by p, the current is denoted by j, the density is denoted by ρ, et cetera.

We will also use formulae like (1.2.15) everywhere below.

Proof of the d'Alembert representation (1.2.3)

From the generic formulae (1.2.15) for the change of variables (1.2.4) we derive

$$\begin{aligned}\frac{\partial}{\partial t} &= -a\frac{\partial}{\partial\xi}+a\frac{\partial}{\partial\eta};\\ \frac{\partial}{\partial x} &= \frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}; \qquad (1.2.17)\end{aligned}$$

From this we obtain:

$$\begin{cases} \frac{\partial^2}{\partial t^2} = a^2 \frac{\partial^2}{\partial \xi^2} - 2a^2 \frac{\partial^2}{\partial \xi \partial \eta} + a^2 \frac{\partial^2}{\partial \eta^2}, \\ \frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial \xi^2} + 2\frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2}, \end{cases} \tag{1.2.18}$$

Substituting (1.2.18) into (1.2.1), we get

$$\Big(a^2 \frac{\partial^2}{\partial \xi^2} - 2a^2 \frac{\partial^2}{\partial \xi \partial \eta} + a^2 \frac{\partial^2}{\partial \eta^2}\Big) u = a^2 \Big(\frac{\partial^2}{\partial \xi^2} + 2\frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2}\Big) u \tag{1.2.19}$$

After mutual cancellations we obtain

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0. \tag{1.2.20}$$

This is the canonical form of the d'Alembert equation (1.2.1), its simplest form that is, in which it could be easily solved. To solve (1.2.20), denote

$$\frac{\partial u}{\partial \eta}(\xi,\, \eta) = v(\xi,\, \eta) \tag{1.2.21}$$

Then (1.2.20) could be written as

$$\frac{\partial v}{\partial \xi} \equiv \frac{d}{d\xi} v \Big|_{\eta = const} = 0. \tag{1.2.22}$$

It then follows that $v\Big|_{\eta=const}$ does not depend on ξ, that is,

$$v(\xi,\, \eta) \equiv c(\eta), \tag{1.2.23}$$

or, taking into account (1.2.21),

$$\frac{d}{d\eta} u \Big|_{\xi = const} = c(\eta). \tag{1.2.24}$$

Integrating this ordinary differential equation, we obtain

$$u\Big|_{\xi=const} = \int c(\eta)\, d\eta + c_1(\xi). \tag{1.2.25}$$

Thus,

$$u = g(\eta) + f(\xi), \tag{1.2.26}$$

where g and f – are some functions of one variable. On the other hand, a function of the form (1.2.26) satisfies the equation (1.2.20) for any f and g. At last, changing the variables in (1.2.26) according to (1.2.4), we obtain the d'Alembert representation (1.2.3).

Remark 1.2.3. The graph of a function $f(x-at)$ in (1.2.4) is a wave moving along the direction of the axis Ox to the right, with the speed a, while $g(x+at)$ represents a wave moving with the same speed to the left. This means that the graph of the function $f(x-at)$ $\Big(g(x+at)\Big)$ for any $t>0$ as a function of x is obtained from the graph of the function $f(x)$ $\Big(g(x)\Big)$ with the aid of a parallel transform to the right (left) along the axis Ox by the distance at. Therefore, a form of the graph of the function $f(x-at)$ considered as a function of x with fixed t is the same. In Physics, such functions are called traveling waves. Thus, the d'Alembert decomposition (1.2.3) means that any solution of the d'Alembert equation is the sum (physicists also use words superposition and interference) of two traveling waves.

Solution of the Cauchy problem (1.2.1), (1.2.2) for the d'Alembert equation. The d'Alembert formula

We apply the d'Alembert method to the problem (1.2.1), (1.2.2). To do so, we substitute the equation (1.2.1) by its equivalent (1.2.3). Thus, we are left to take into account the initial conditions (1.2.2). It is from these conditions that we will determine the unknown functions f and g from the given φ ψ.

Namely, substitute (1.2.3) into (1.2.2):

$$\begin{cases} f(x)+g(x)=\varphi(x), \\ f'(x)(-a)+g'(x)a=\psi(x), \quad x\in\mathbb{R}. \end{cases} \tag{1.2.27}$$

Remark 1.2.4. In the second equation (1.2.27) we have used the chain rule:

$$\Big(\frac{\partial}{\partial t}f(x-at)\Big)\Big|_{t=0}=\Big(f'(x-at)\frac{\partial}{\partial t}(x-at)\Big)\Big|_{t=0}=f'(x)(-a) \tag{1.2.28}$$

Here $f'(x)$ is an ordinary derivative (not a partial one). This is the advantage of the d'Alembert method, which allows to reduce the equations (1.2.1), (1.2.2) with partial derivatives to the equations (1.2.27) with ordinary derivatives.

Then, integrating the second equation (1.2.27) and dividing it by a, we obtain:

$$-f(x)+g(x)=\frac{1}{a}\int_0^x\psi(s)\,ds+\frac{c}{a}. \tag{1.2.29}$$

Taking the sum of this equation with the first equation from (1.2.27) and dividing by 2, we obtain

$$g(x)=\frac{1}{2}\varphi(x)+\frac{1}{2a}\int_0^x\psi(s)\,ds+\frac{c}{2a}; \tag{1.2.30}$$

instead, taking the difference of these two equations, we obtain

$$f(x) = \frac{1}{2}\varphi(x) - \frac{1}{2a}\int_0^x \psi(s)\,ds - \frac{c}{2a} = \frac{1}{2}\varphi(x) + \frac{1}{2a}\int_x^0 \psi(s)\,ds - \frac{c}{2a}, \quad (1.2.31)$$

Substituting these expressions into the d'Alembert decomposition (1.2.3), we obtain the d'Alembert formula

$$u(x,t) = \frac{\varphi(x-at)+\varphi(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \psi(s)\,ds \quad (1.2.32)$$

Remark 1.2.5. One can see from (1.2.30)–(1.2.31) that the waves $f(x-at)$ and $g(x+at)$ are determined by the initial data φ and ψ not uniquely, but only up to an additive constant. At the same time, the solution $u(x,t)$ to the Cauchy problem is uniquely defined.

1.3 Analysis of the d'Alembert formula

Propagation of the waves

Problem 1.3.1. Take the following initial data in (1.2.2) (see Remark 1.2.1):

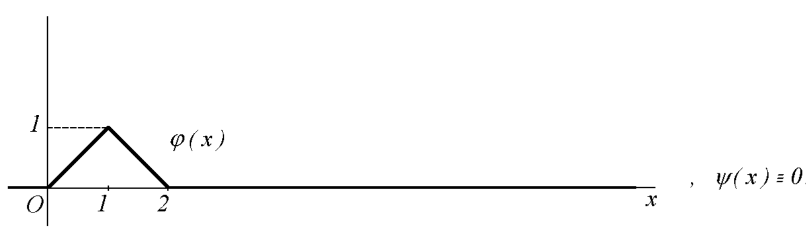


Figure 1.6:

Let us draw the shape of the string at $t = 1, 2, 3, 4, 5$, taking $a = 1$. (We may assume that $\varphi(x)$ is a piecewise-linear function. Then the solution is also going to be a piecewise-linear function, which is a solution of equation (1.2.1) in the sense of distributions (see Remark 1.2.1). Instead, one can think that the graph φ is slightly smoothed out at the corner points, so that $\varphi(x) \in C^2(\mathbb{R})$. Then the solution is also going to be of class C^2, and one should think that all the corners are slightly smoothed out at all the drawings below.)

Solution. According to the d'Alembert formula (1.2.32),

$$u(x,t) = \frac{1}{2}\varphi(x-t) + \frac{1}{2}\varphi(x+t). \quad (1.3.1)$$

This means that the graph $\varphi(x)$ should be compressed to the axis Ox by the factor of 2, shifted to the right by t, to the left by t, and the results added up (see Figure 1.7). Thereafter these humps of hight $\frac{1}{2}$ and width

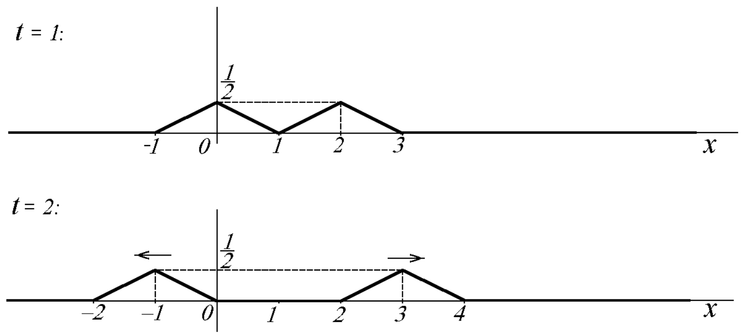


Figure 1.7:

2 propagate to the left and to the right, each with the speed 1.

Problem 1.3.2. In the settings of the previous problem, draw the shape of the string at $t = \frac{1}{4}, \frac{1}{2}$

Problem 1.3.3 (Hit at the string by a hammer). In (1.2.2), take the following initial data: Draw the shape of the string at $t = 1, 2, 3, 4, 5$, setting $a = 1$.

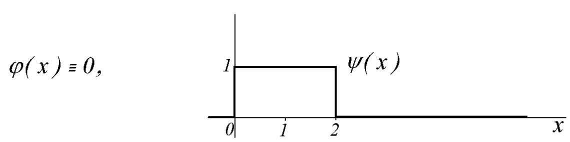


Figure 1.8:

Solution. According to the d'Alembert formula (1.2.32),

$$u(x,t) = \frac{1}{2}\int\limits_{x-t}^{x+t} \psi(s)\,ds = \phi(x+t) - \phi(x-t), \tag{1.3.2}$$

where

$$\phi(x) = \frac{1}{2}\int\limits_{0}^{x} \psi(s)\,ds. \tag{1.3.3}$$

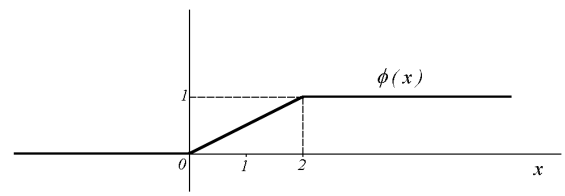


Figure 1.9:

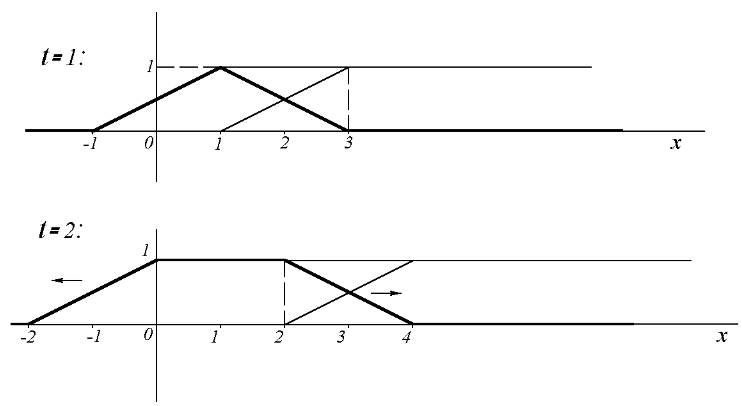


Figure 1.10:

See Figure 1.9. This formula means that the graph of the function $\phi(x)$ should be shifted to the left and to the right by t, and to subtract the results (see Figure 1.10). Thereafter this trapezoid spreads out to the left and to the right with the speed 1.

Problem 1.3.4. Under the settings of the previous problem, draw the shape of the string at $t = \frac{1}{4}$ and $t = \frac{1}{2}$.

Characteristics

When solving two previous problems, we have seen that the lines $x \pm t = const$ play a special role. For example, the corner points of the graphs of the solutions $u(x,t)$ lie on the lines $x \pm t = 0$ and $x \pm t = 2$.

For equation (1.1.1) with the coefficient a the similar role is played by the lines $x \pm at = const$. They are called the characteristics of equation (1.1.1). Thus, the characteristics of the d'Alembert equation are two families of lines (see Fig. 1.11). We will call the lines $x - at = const$ the characteristics moving to the right (with the speed a). Obviously, they are the level curves

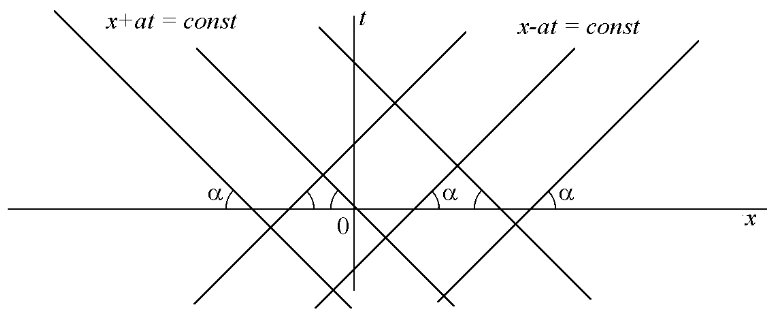


Figure 1.11:

of the wave $f(x-at)$. The greater the speed a, the smaller is the angle between the characteristic and the axis Ox (if the scale on the axes Ox and Ot is the same):

$$\tan\alpha = \frac{1}{a}. \tag{1.3.4}$$

Similarly, the lines $x+at = const$ are called characteristics, moving to the left. They are the level curves of the wave $g(x+at)$.

Discontinuities of the solution

Let us take $f(x)$ to be discontinuous:

$$f(x) = \begin{cases} 0, & x < 2, \\ 1, & x \geq 2. \end{cases} \tag{1.3.5}$$

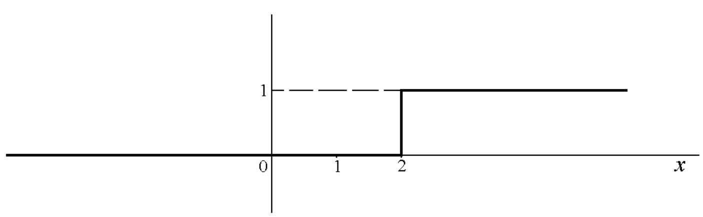


Figure 1.12:

Then the function

$$u(x,t) = f(x-at) \tag{1.3.6}$$

is discontinuous along the characteristic $x-at=2$. See Fig. 1.13. Function

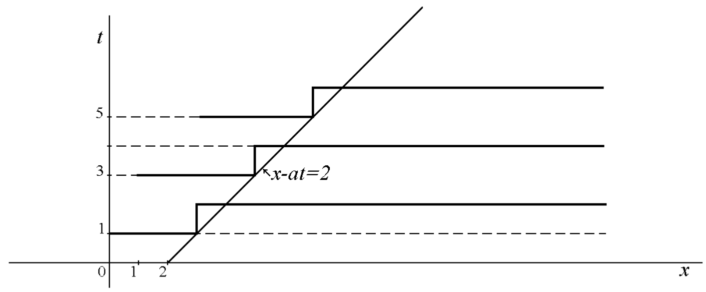


Figure 1.13: Profiles of the function $u(x,t)$ at $t = 1,\ 3,\ 5$.

(1.3.6) satisfies the d'Alembert equation (1.2.1) in the sense of distributions (see Remark 1.2.1).

Thus:

$$\left\| \begin{array}{l} \textit{Solutions of the d'Alembertequation could have discontinuities;} \\ \textit{Discontinuities propagate along characteristics.} \end{array} \right. \tag{1.3.7}$$

Remark 1.3.1. One can take a smooth function $f_\varepsilon(x)$, which changes from 0 to 1 on a small interval from $x = 2$ to $x = 2 + \varepsilon$, where $\varepsilon > 0$:

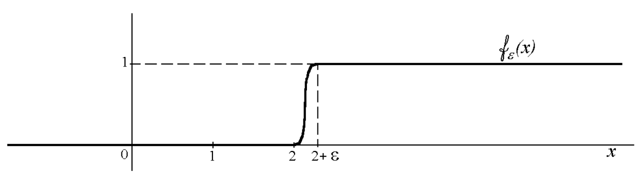


Figure 1.14:

Then the function $f_\varepsilon(x - at)$ will be a classical (smooth) solution of the d'Alembert equation, rapidly changing from 0 to 1 near the points of the characteristic $x - at = 2$. In the limit $\varepsilon \to 0+$ the solutions $f_\varepsilon(x - at)$ converge to a discontinuous function $f(x - at)$. It is in this sense it is natural to treat such a discontinuous function as a solution of d'Alembert equation in the sense of distributions (see also Remark 1.2.1).

Remark 1.3.2. Discontinuous solutions $u(x,t)$ to the d'Alembert equation for the string and for the rod do not make a physical sense. Still, the d'Alembert equation also describes the gas pressure $p(x,t)$ in a long narrow pipe (such as a flute or an organ). The function $p(x,t)$ can be discontinuous.

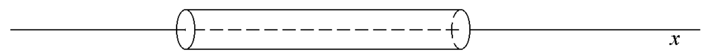


Figure 1.15:

Discontinuous solutions in the Dynamics of Gas are called the shock waves. When the plane travels with the supersonic speed (see Section 1.8) there is such a shock wave coming from the front edge of the wings, with the pressure being higher behind the front of this wave than ahead of it. We hear a bang when the wave front passes our ear (see Fig. 1.62).

Region of dependence and its graphical representation

Question. What do we need to know in order to compute the solution u of problem (1.2.1)–(1.2.2) at the point $(x_\circ, t_\circ)$?

Answer. From the d'Alembert formula (1.2.32) we see that one needs the initial displacements $\varphi(x)$ at two points: $x = x_\circ + at_\circ$ and $x = x_\circ - at_\circ$, and also the initial velocities $\psi(x)$ on the interval $[x_\circ - at_\circ, x_\circ + at_\circ]$ between these points. Knowing $\varphi(x)$ and $\psi(x)$ beyond the interval $[x_\circ - at_\circ, x_\circ + at_\circ]$ is not needed. Therefore the interval $[x_\circ - at_\circ, x_\circ + at_\circ]$ is called the region of dependence of the Cauchy problem (1.2.1)–(1.2.2) for the point $(x_\circ, t_\circ)$.

Remark 1.3.3. Now we can explain precisely when we can treat the string as infinite: When the point under consideration $x_\circ$ is located at a distance larger than $at_\circ$ from the endpoints of the string, where $t_\circ$ is the moment of time that we are interested at.

For the graphical representation of the region of dependence of the solution we draw from $(x_\circ, t_\circ)$ the two characteristics of the d'Alembert equation until their intersection with the axis Ox:

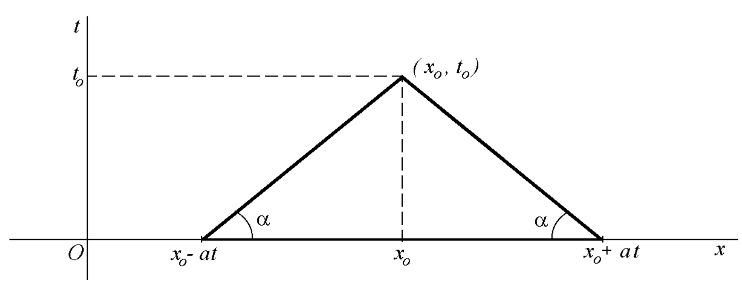


Figure 1.16:

The intersection of these characteristics with the axis Ox are the points $x_\circ - at_\circ$ and $x_\circ + at_\circ$, and the interval of the axis Ox between these points is the region of dependence of the solution u at the point $(x_\circ, t_\circ)$. Let us check this. The equations of the characteristics are

$$x - at = c_1; \quad x + at = c_2. \tag{1.3.8}$$

Since the point $(x_\circ, t_\circ)$ lies on these characteristics, $x_\circ - at_\circ = c_1$ and $x_\circ + at_\circ = c_2$. To find the intersection of the characteristics with the axis Ox, we need to set $t = 0$ in (1.3.8), getting

$$x = c_1 = x_\circ - at_\circ \qquad \text{and} \qquad \mathrm{x = c_2 = x_\circ + at_\circ}. \tag{1.3.9}$$

Propagation of waves

Problem 1.3.5. We know that $\varphi(x) = \psi(x) = 0$ for $x \notin [2, 5]$.

Find the region where the solution $u(x, t)$ to the problem (1.2.1)–(1.2.2) is equal to zero for $t > 0$.

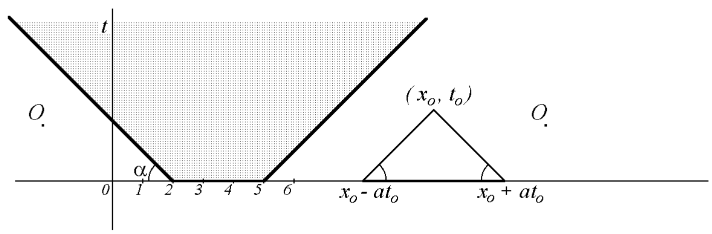


Figure 1.17:

Solution. From the points 2 and 5 of the Ox axis we draw the characteristics to the left and to the right, respectively. In the triangular region bounded by the characteristics the solution is equal to zero. Indeed, for the point $(x_\circ, t_\circ)$ in the region bounded by these characteristics the region of dependence does not intersect the interval $[2, 5]$. Therefore in this region of dependence $\varphi(x) \equiv \psi(x) \equiv 0$. Consequently, $u(x_\circ, t_\circ) = 0$.

1.4 The method of characteristics for hyperbolic equations of the second order with two independent variables. The Cauchy problem

Decomposition of the d'Alembert operator into factors

Let us bring the d'Alembert equation to a canonical form (1.2.20) using a new method. For this, we rewrite it as

$$\Box(u) \equiv \Big(\frac{\partial^2}{\partial t^2} - a^2\frac{\partial^2}{\partial x^2}\Big)u = 0. \tag{1.4.1}$$

We decompose this operator into factors:

$$\Box(u) \equiv \Big(\frac{\partial}{\partial t} - a\frac{\partial}{\partial x}\Big)\Big(\frac{\partial}{\partial t} + a\frac{\partial}{\partial x}\Big)u = 0 \tag{1.4.2}$$

It is known that

$$L_{(-a,1)} \equiv \frac{\partial}{\partial t} - a\frac{\partial}{\partial x}; \quad L_{(a,1)} \equiv \frac{\partial}{\partial t} + a\frac{\partial}{\partial x} \tag{1.4.3}$$

are the operators of differentiating along the vectors $(-a,1)$ and $(a,1)$, respectively. These vectors are directed along characteristics

$$x + at = const \quad \text{and} \quad x - at = const. \tag{1.4.4}$$

If we take the characteristic lines as the new coordinates, that is, to set

$$\xi = x - at; \quad \eta = x + at, \tag{1.4.5}$$

then due to (1.2.17) the d'Alembert operator takes the form

$$\Box = \frac{\partial^2}{\partial t^2} - a^2\frac{\partial^2}{\partial x^2} = .L_{(-a,1)} \cdot L_{(a,1)} = -2a\frac{\partial}{\partial \xi}2a\frac{\partial}{\partial \eta} = -4a^2\frac{\partial^2}{\partial \xi \partial \eta}. \tag{1.4.6}$$

Conclusion. The characteristics of equation (1.4.1) are the lines such that the operators of differentiating along them, $L_{(\mp a,1)}$, are the factors of the d'Alembert operator.

Remark 1.4.1. Since the operators of differentiating along the characteristics are the factors of the d'Alembert operator, this operator sends to zero any function that is constant along characteristics of one of the families (in particular, any such function that is discontinuous; see Remark 1.2.1). This explains why the solutions to the d'Alembert equation can have discontinuities along characteristics.

Hyperbolic equations of the second order with constant coefficients in the plane

We consider the equation of the form

$$Au \equiv a\frac{\partial^2 u}{\partial t^2} + 2b\frac{\partial^2 u}{\partial t \partial x} + c\frac{\partial^2 u}{\partial x^2} = 0, \quad x \in \mathbb{R}, \quad , t > 0. \tag{1.4.7}$$

In this section we assume that the coefficients a, b, and c are constants.

Let us try to apply the method of Section 1 to equation (1.4.7) instead of (1.4.1). To obtain the factorization like (1.4.2), we need to decompose into linear factors the "characteristic" quadratic form

$$a(\xi,\tau) \equiv a\tau^2 + 2b\tau\xi + c\xi^2 = \xi^2\Big(a\Big(\frac{\tau}{\xi}\Big)^2 + 2b\frac{\tau}{\xi} + c\Big). \tag{1.4.8}$$

To achieve this, we solve the characteristic equation

$$a\lambda^2 + 2b\lambda + c = 0 \tag{1.4.9}$$

Its roots

$$\lambda_{1,2} = \frac{b \pm \sqrt{b^2 - ac}}{a} \tag{1.4.10}$$

are real and different if the discriminant is positive:

$$D \equiv b^2 - ac > 0 \tag{1.4.11}$$

This is precisely the strict hyperbolicity condition for equation (1.4.7).

According to the Viett theorem,

$$a\lambda^2 + 2b\lambda + c = a(\lambda - \lambda_1)(\lambda - \lambda_2) \tag{1.4.12}$$

Therefore, the quadratic form turns into

$$A(\xi,\tau) = \xi^2 a\Big(\frac{\tau}{\xi} - \lambda_1\Big)\Big(\frac{\tau}{\xi} - \lambda_1\Big) = a(\tau - \lambda_1\xi)(\tau - \lambda_2\xi). \tag{1.4.13}$$

Accordingly, the differential equation takes the form

$$Au = \Big(\frac{\partial}{\partial t} - \lambda_1\frac{\partial}{\partial x}\Big)\Big(\frac{\partial}{\partial t} - \lambda_2\frac{\partial}{\partial x}\Big)u = 0 \tag{1.4.14}$$

Denote

$$L_{(-\lambda_1,1)} = \frac{\partial}{\partial t} - \lambda_1\frac{\partial}{\partial x} \quad \text{and} \quad L_{(-\lambda_2,1)} = \frac{\partial}{\partial t} - \lambda_2\frac{\partial}{\partial x}. \tag{1.4.15}$$

Analogously to (1.4.5), we set

$$\xi = x + \lambda_1 t; \quad \eta = x + \lambda_2 t. \tag{1.4.16}$$

Then

$$L_{(-\lambda_1,1)}\xi \equiv 0; \quad L_{(-\lambda_2,1)}\xi \equiv 0. \tag{1.4.17}$$

Here $L_{(-\lambda_1,1)}$ is the operator of differentiation along the lines $\xi = const$, while $L_{(-\lambda_2,1)}$ differentiates along the lines $\eta = const$. Hence,

$$L_{(-\lambda_1,1)} = c_1 \frac{\partial}{\partial \eta}\Big|_{\xi=const} ; \quad L_{(-\lambda_2,1)} = c_2 \frac{\partial}{\partial \xi}\Big|_{\eta=const} . \tag{1.4.18}$$

It follows that (1.4.14) is equivalent to the equation

$$\frac{\partial}{\partial \eta}\frac{\partial}{\partial \xi} u = 0. \tag{1.4.19}$$

Similarly to (1.2.26), the general solution to equation (1.4.7) is given by

$$u = f(\xi) + g(\eta) = f(x + \lambda_1 t) + g(x + \lambda_2 t) \tag{1.4.20}$$

The wave $f(x + \lambda_1 t)$ propagates along the axis x with the speed λ_1, while the wave $g(x + \lambda_2 t)$ propagates with the velocity λ_2 (to the left if $\lambda_1 > 0$, $\lambda_2 > 0$).

In particular, for the d'Alembert equation (1.4.1), characteristic equation (1.4.9) takes the form $\lambda^2 - a^2 = 0$, so that $\lambda_1 = -a$, $\lambda_2 = a$, and (1.4.16) turns into (1.2.4), while (1.4.20) – into (1.2.26).

With the aid of representation (1.4.20), all the conclusions of Section 1.3 about discontinuities of the solution, propagation of waves, and the regions of dependence are easily generalized for equation (1.4.7) (see Remark 1.2.1).

Solutions of equation (1.4.7) may have singularities along the characteristics that are defined by equations

$$\xi \equiv x + \lambda_1 t = const \quad \text{or} \quad \eta \equiv x + \lambda_2 t = const \tag{1.4.21}$$

This is seen from (1.4.20) when f or g are not smooth (see also Remark 1.4.1).

Cauchy problem (1.4.7) with initial data (1.2.2) has a solution

$$u(x,t) = \frac{\lambda_2 \varphi(x + \lambda_1 t) - \lambda_1 \varphi(x + \lambda_2 t)}{\lambda_2 - \lambda_1} + \frac{1}{\lambda_2 - \lambda_1} \int_{x+\lambda_1 t}^{x+\lambda_2 t} \psi(s)\, ds \tag{1.4.22}$$

Problem 1.4.1. Derive formula (1.4.22).

Let us point out that for the d'Alembert equation one has $\lambda_1 = -a$, $\lambda_2 = a$, so that (1.4.22) turns into the d'Alembert formula (1.2.32).

As seen from (1.4.22), the region of dependence for the solution u at a point $(x_\circ, t_\circ)$ is the interval $[x_\circ + \lambda_1 t_\circ, x_\circ + \lambda_2 t_\circ])$ of the axis Ox. Its ends are the intersection points of the axis Ox with characteristics (1.4.21) sent back in time from the point $(x_\circ, t_\circ)$:

Let us point out that the roots λ_1 and λ_2 could be of the same sign; then the waves $f(x + \lambda_1 t)$ and $g(x + \lambda_2 t)$ run into the same direction.

Example. For the equation

$$\Big(\frac{\partial^2}{\partial t^2} + 5\frac{\partial^2}{\partial t \partial x} + 6\frac{\partial^2}{\partial x^2}\Big) u = 0 \tag{1.4.23}$$

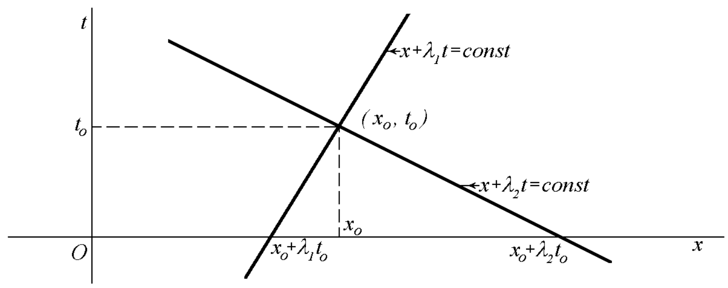


Figure 1.18:

the characteristic equation

$$\lambda^2 + 5\lambda + 6 = 0 \tag{1.4.24}$$

has the roots $\lambda_1 = -2$, $\lambda_2 = -3$, and the general solution

$$u = f(x - 2t) + g(x - 3t) \tag{1.4.25}$$

consists of two waves propagating to the right.

Let us find the differential equation of the characteristics of equation (1.4.7).

We note that according to (1.4.21) the tangent vector (dx, dt) to the characteristic satisfies the equation

$$dx + \lambda_1\, dt = 0, \quad \text{or} \quad dx + \lambda_2\, dt = 0 \tag{1.4.26}$$

Therefore, either $\frac{dx}{dt} = -\lambda_1$ or $\frac{dx}{dt} = -\lambda_2$, that is, $\lambda \equiv -\frac{dx}{dt}$ satisfies characteristic equation (1.4.9):

$$a\Big(\frac{dx}{dt}\Big)^2 - 2b\frac{dx}{dt} + c = 0. \tag{1.4.27}$$

This is the differential equation for the characteristics. It can be written in a symmetric form, as follows:

$$a\, dx^2 - 2b\, dx\, dt + c\, dt^2 = 0 \tag{1.4.28}$$

Hyperbolic equations of the second order with varying coefficients in the plane

Now let the coefficients a, b, and c in (1.4.7) be varying, that is, are functions of x and t:

$$Au(x,t) \equiv a(x,t)\frac{\partial^2 u}{\partial t^2} + 2b(x,t)\frac{\partial^2 u}{\partial t \partial x} + c(x,t)\frac{\partial^2 u}{\partial x^2} = 0; \quad x \in \mathbb{R}, \quad t > 0 \tag{1.4.29}$$

We will try to generalize the method of Section 2 in order to bring (1.4.29) to canonical form (1.4.19) or at least a form close to it (see [Smi66]).

In a small neighborhood of each point (x,t), we substitute equation (1.4.29) by equation (1.4.7) with constant coefficients, equal to the values of the coefficients of equation (1.4.29) at this particular point (x,t). This procedure is called "the freezing of the coefficients".

If we do so, the characteristics of the equation "frozen" at the point (x,t) will have directions that depend on (x,t). The vectors $(dx,\, dt)$ tangent to these characteristics will satisfy equation (1.4.28) (see Fig. 1.19)

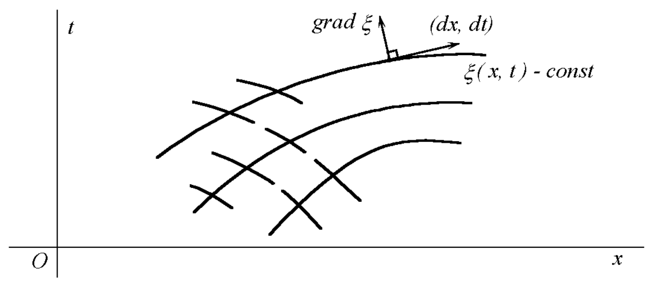


Figure 1.19:

Integral curves of equation (1.4.28) are called the characteristics of equation (1.4.29) (see [Smi66]). Thus, due to (1.4.28), the differential equation of the characteristics of equation (1.4.29) is given by

$$a(x,t)\,dx^2 - 2b(x,t)\,dx\,dt + c(x,t)\,dt^2 = 0. \tag{1.4.30}$$

Characteristic equation (1.4.30) is obtained by a formal substitution

$$\frac{\partial}{\partial t} \mapsto dx; \quad \frac{\partial}{\partial x} \mapsto -dt. \tag{1.4.31}$$

Assume that in the region of the (x,t)-plane where we are to solve equation (1.4.29), the strict hyperbolicity condition (1.4.11) is satisfied:

$$b^2(x,t) - a(x,t)\,c(x,t) > 0. \tag{1.4.32}$$

Then, dividing equation (1.4.30) by dt^2 and solving the resulting quadratic equation, we obtain two different differential equations:

$$\frac{dx}{dt} = \frac{b \pm \sqrt{b^2 - ac}}{a}. \tag{1.4.33}$$

If the functions a, b, and c are smooth, equation (1.4.33) have two corresponding families of the integral curves. We will denote these corresponding families of characteristics by the signs "+" "−", respectively. In the (x,t)-plane, we introduce new coordinates ξ, η so that $\xi = const$ on the characteristics of the family "+" while $\eta = const$ on the characteristics of the family "−". This means that the characteristics will be the new coordinate curves, and ξ, η are the first integrals of equations (1.4.33), respectively.

Let us mention that the change of variables $(x,t) \mapsto (\xi,\eta)$ is non-degenerate at each point where condition (1.4.32) is satisfied. Indeed, from (1.4.33) one can see that at each point the characteristics have different directions, and since $\operatorname{grad}\xi$ and $\operatorname{grad}\eta$ are orthogonal to the corresponding characteristics, they also have different directions. Incidentally, this means that the coordinates ξ, η may be defined in a sufficiently small open neighborhood of every point. These coordinates may not exist in the whole region under consideration.

Let us check that in the coordinates ξ, η equation (1.4.29) could be brought to canonical form (1.4.19) up to the terms that only contains derivatives of the first order. We first need to derive the differential equation for the functions $\xi(x,t), \eta(x,t)$, the so-called characteristic equation.

Since $\xi(x,t) = const$ on any characteristic from the family "+", that is, such a characteristic is the level curve of the function ξ, the vector $\operatorname{grad}\xi$ is orthogonal to this characteristic (see Fig. 1.19):

$$\operatorname{grad}\xi \perp (dx,\, dt). \tag{1.4.34}$$

Therefore, $\operatorname{grad}\xi \| (dt, -dx)$, so that $\frac{\partial\xi}{\partial x}\Big/\frac{\partial\xi}{\partial t} = -\frac{dt}{dx}$, or

$$dt = -k\,dx, \quad \text{where} \quad k = \frac{\partial\xi}{\partial x}\Big/\frac{\partial\xi}{\partial t}. \tag{1.4.35}$$

Substituting (1.4.35) into (1.4.30), we obtain the desired differential equation:

$$a(x,t)\Big(\frac{\partial\xi}{\partial t}\Big)^2 + 2b(x,t)\frac{\partial\xi}{\partial t}\frac{\partial\xi}{\partial x} + c(x,t)\Big(\frac{\partial\xi}{\partial x}\Big)^2 = 0 \tag{1.4.36}$$

In the same fashion one derives the differential equation for $\eta(x,t)$, and it coincides with (1.4.36):

$$a(x,t)\Big(\frac{\partial\eta}{\partial t}\Big)^2 + 2b(x,t)\frac{\partial\eta}{\partial t}\frac{\partial\eta}{\partial x} + c(x,t)\Big(\frac{\partial\eta}{\partial x}\Big)^2 = 0 \tag{1.4.37}$$

It is of no surprise, since (1.4.30) contains both equations from (1.4.33).

Now let us recall formulas (1.2.11)–(1.2.14) for the change of variables in a differential equation. Substituting expressions (1.2.11)–(1.2.14) into (1.4.29), we obtain the following differential equation for the function $v(\xi,\eta) = u(x,t)$:

$$\alpha(\xi,\eta)\frac{\partial^2 v}{\partial\xi^2} + 2\beta(\xi,\eta)\frac{\partial^2 v}{\partial\xi\partial\eta} + \gamma(\xi,\eta)\frac{\partial^2 v}{\partial\eta^2} + \cdots = 0, \tag{1.4.38}$$

where ... stands for the terms that only contain derivatives of the first order of v. The expressions for the coefficients α, β are γ are as follows:

$$\alpha = a\Big(\frac{\partial\xi}{\partial t}\Big)^2 + 2b\frac{\partial\xi}{\partial t}\frac{\partial\xi}{\partial x} + \Big(\frac{\partial\xi}{\partial x}\Big)^2; \tag{1.4.39}$$

$$\gamma = a\Big(\frac{\partial\eta}{\partial t}\Big)^2 + 2b\frac{\partial\eta}{\partial t}\frac{\partial\eta}{\partial x} + \Big(\frac{\partial\eta}{\partial x}\Big)^2; \tag{1.4.40}$$

$$\beta = a\frac{\partial\xi}{\partial t}\frac{\partial\eta}{\partial t} + 2b\Big(\frac{\partial\xi}{\partial t}\frac{\partial\eta}{\partial x} + \frac{\partial\eta}{\partial t}\frac{\partial\xi}{\partial x}\Big) + c\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}. \tag{1.4.41}$$

Denote by $\tilde{A}$ the so-called characteristic polynomial of the operator A from (1.4.29) that corresponds to the point (x,t):

$$\tilde{A}(\xi_1,\xi_2) = \tilde{A}(x,t;\xi_1,\xi_2) \equiv \alpha(x,t)\xi_2^2 + 2b(x,t)\xi_2\xi_1 + c(x,t)\xi_1^2. \tag{1.4.42}$$

Then (1.4.36) and (1.4.37) are equivalent with

$$\alpha = \tilde{A}(\operatorname{grad}\xi) = 0; \quad \gamma = \tilde{A}(\operatorname{grad}\eta) = 0. \tag{1.4.43}$$

Finally, (1.4.29) takes the form similar to (1.4.19):

$$2\beta(\xi,\eta)\frac{\partial^2 v}{\partial\xi\partial\eta} + \ldots = 0. \tag{1.4.44}$$

Problem 1.4.2. Prove that $\beta(\xi,\eta) \neq 0$ when $\xi = \xi(x,t)$, $\eta = \eta(x,t)$, if condition (1.4.32) holds. (Use (1.4.39)–(1.4.43).)

Equation (1.4.44) could be solved approximately. In a number of cases, when equation (1.4.44) is sufficiently simple, it is possible to find its general solution and thus to find the general solution to equation (1.4.29).

Problem 1.4.3 (18, [Smi66]). Find the general solution to the equation

$$\frac{\partial^2 u}{\partial x^2} - 2\sin x\frac{\partial^2 u}{\partial x\partial y} - \cos^2 x\frac{\partial^2 u}{\partial y^2} - \cos x\frac{\partial u}{\partial y} = 0. \tag{1.4.45}$$

Solution. Characteristic equation (1.4.30) is obtained from (1.4.45) by substituting $\frac{\partial}{\partial x} \mapsto dy$; $\frac{\partial}{\partial y} \mapsto -dx$ (see (1.4.31)):

$$dy^2 + 2\sin x\,dy\,dx - \cos^2 x\,dx^2 = 0 \tag{1.4.46}$$

or

$$\Big(\frac{dy}{dx}\Big)^2 + 2\sin x\frac{dy}{dx} - \cos^2 x = 0. \tag{1.4.47}$$

From here,

$$\frac{dy}{dx} = -\sin x \pm \sqrt{\sin^2 x + \cos^2 x} = -\sin x \pm 1. \tag{1.4.48}$$

Integrating, we get

$$y = \cos x \pm x = c. \tag{1.4.49}$$

Hence the functions

$$c(x,y) = y - \cos x \mp x \tag{1.4.50}$$

are constant along the integral curves, that is, it is them that are the first integrals of equations (1.4.48). Therefore,

$$\begin{cases} \xi = y - \cos x - x, \\ \eta = y - \cos x + x. \end{cases} \tag{1.4.51}$$

We already know that equation (1.4.45) in the variables ξ, η has the form (1.4.44). But we also need to know the form of the terms containing $\frac{\partial v}{\partial \xi}$, $\frac{\partial v}{\partial \eta}$, that are not written explicitly in (1.4.44). We could use the known formulas (1.2.11)–(1.2.14), but let us make the change of variables (1.4.51) in (1.4.45) directly. Instead of v, we will write u:

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial u}{\partial \xi}(\sin x - 1) + \frac{\partial u}{\partial \eta}(\sin x + 1), \\ \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}. \end{cases} \tag{1.4.52}$$

We then have

$$\frac{\partial^2 u}{\partial x^2} = \ldots + 2\frac{\partial^2 u}{\partial \xi \partial \eta}(\sin^2 x - 1) + \ldots + \frac{\partial u}{\partial \xi}\cos x + \frac{\partial u}{\partial \eta}\cos x. \tag{1.4.53}$$

Dots denote the terms containing $\frac{\partial^2 u}{\partial \xi^2}$ and $\frac{\partial^2 u}{\partial \eta^2}$, which, as we already know, (see (1.4.44)), cancel out in (1.4.45). Therefore, we do not have to write them out!

Analogously,

$$\frac{\partial^2 u}{\partial x \partial y} = \ldots + \frac{\partial^2 u}{\partial \xi \partial \eta}(\sin x - 1) + \frac{\partial^2 u}{\partial \eta \partial \xi}(\sin x + 1) + \ldots \tag{1.4.54}$$

Finally,

$$\frac{\partial^2 u}{\partial y^2} = \ldots + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \ldots \tag{1.4.55}$$

Substituting (1.4.52)–(1.4.55) into (1.4.45), we get

$$\frac{\partial^2 u}{\partial \xi \partial \eta}\Big(2(\sin^2 x - 1) - 2\sin x \cdot 2\sin x - 2\cos^2 x\Big) + \\ + \Big(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\Big)\cos x - \Big(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\Big)\cos x = 0.$$

After cancellations and collecting the terms, we obtain

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0 \quad \Longrightarrow \quad u = f(\xi) + g(\eta). \tag{1.4.56}$$

Answer. $u(x,y) = f(y - \cos x - x) + g(y - \cos x + x)$.

For better understanding of the material we recommend to solve the following problems:

Problems 1.4.1. Find the general solution for the following equations:

1. $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} - 3\frac{\partial^2 u}{\partial y^2} + 2\frac{\partial u}{\partial x} + 6\frac{\partial u}{\partial y} = 0.$
2. $x\frac{\partial^2 u}{\partial x^2} - y\frac{\partial^2 u}{\partial y^2} + \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right) = 0, \ x > 0, \quad y > 0.$
3. $x^2\frac{\partial^2 u}{\partial x^2} - y^2\frac{\partial^2 u}{\partial y^2} - 2y\frac{\partial u}{\partial y} = 0.$
4. $\frac{\partial}{\partial x}\left(x^2\frac{\partial u}{\partial x}\right) = x^2\frac{\partial^2 u}{\partial y^2}.$
5. $(x-y)\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0.$
6. $\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} + xyu = 0.$

Problems 1.4.2. Solve the following Cauchy problems:

1. $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} - 3\frac{\partial^2 u}{\partial y^2} = 0, \quad u\Big|_{y=0} = 3x^2, \quad \frac{\partial u}{\partial y}\Big|_{y=0} = 0.$
2. $4y^2\frac{\partial^2 u}{\partial x^2} + 2(1-y^2)\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y^2} - \frac{2y}{1+y^2}\left(2\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right) = 0, \ u\Big|_{y=0} = \varphi_\circ(x),$ $\frac{\partial u}{\partial y}\Big|_{y=0} = \varphi_1(x).$
3. $(1+x^2)\frac{\partial^2 u}{\partial x^2} - (1+y^2)\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = 0, \ u\Big|_{y=0} = \varphi_\circ(x), \ \frac{\partial u}{\partial y}\Big|_{y=0} = \varphi_1(x).$
4. $\frac{\partial^2 u}{\partial x^2} + 2\cos x\frac{\partial^2 u}{\partial x \partial y} - \sin^2 x\frac{\partial^2 u}{\partial y^2} - \sin x\frac{\partial u}{\partial y} = 0, \ u\Big|_{y=\sin x} = \varphi_\circ(x), \ \frac{\partial u}{\partial y}\Big|_{y=\sin x} = \varphi_1(x).$
5. $\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} - 5\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = 0, \ u\Big|_{y=0} = f(x), \ \frac{\partial u}{\partial y}\Big|_{y=0} = F(x).$
6. $x^2\frac{\partial^2 u}{\partial x^2} - 2xy\frac{\partial^2 u}{\partial x \partial y} - 3y^2\frac{\partial^2 u}{\partial y^2} = 0, \ u\Big|_{y=1} = \varphi_\circ(x), \ \frac{\partial u}{\partial y}\Big|_{y=1} = \varphi_1(x).$

Non-hyperbolic equations

Let us consider the case, when instead of the strict hyperbolicity condition (1.4.32) the opposite inequality holds:

$$b^2(x,t) - a(x,t)c(x,t) < 0. \tag{1.4.57}$$

In this case, equation (1.4.29) is called elliptic at the point (x,t). The right-hand side of equations (1.4.33) are complex conjugates, and the integration yields the "first integrals" ξ and $\eta = \bar{\xi}$ that are also complex conjugates. It turns out that if one takes $z_1 = \Re\xi = \frac{\xi+\eta}{2}$ and $z_2 = \Im\xi = \frac{\xi-\eta}{2i}$ as the new coordinates, then equation (1.4.29) takes the form

$$\frac{\partial^2 u}{\partial z_1^2} + \frac{\partial^2 u}{\partial z_2^2} + \ldots = 0, \tag{1.4.58}$$

that is, its principal part coincides with the Laplace operator (see problems N 9 and 12–17 from [Smi66]). This allows to solve such equations exactly or approximately.

Problem 1.4.4 (N 14, [Smi66]). Bring the equation

$$y\frac{\partial^2 u}{\partial x^2}+x\frac{\partial^2 u}{\partial y^2}=0,\quad y>0,\quad x>0 \tag{1.4.59}$$

to a canonical form.

Solution. The equation of the characteristics $y\,dy^2+x\,dx^2=0$ takes the form $\sqrt{y}\,dy=\pm i\sqrt{x}\,dx$, that is, the equation (1.4.59) is elliptic. Integrating, we get $y^{3/2}\mp ix^{3/2}=c$. Take the new coordinates $z_1=\Re c=y^{3/2}$, $z_2=\Im c=x^{3/2}$. Then

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial z_2}\frac{3}{2}x^{1/2};\quad \frac{\partial u}{\partial y}=\frac{\partial u}{\partial z_1}\frac{3}{2}y^{1/2} \tag{1.4.60}$$

Differentiating the above relations in x and y, respectively, we get (in accordance with (1.4.58), we do not write the terms with $\frac{\partial^2 u}{\partial z_1\partial z_2}$):

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial z_2^2}\frac{9}{4}x+\frac{\partial u}{\partial z_2}\frac{3}{4}x^{-1/2}+\dots,\quad \frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial z_1^2}\frac{9}{4}y+\frac{\partial u}{\partial z_1}\frac{3}{4}y^{-1/2}+\dots. \tag{1.4.61}$$

Substituting this into (1.4.59), we find

$$\Big(\frac{\partial^2 u}{\partial z_1^2}+\frac{\partial^2 u}{\partial z_2^2}\Big)\frac{9}{4}xy+\frac{3}{4}\frac{\partial u}{\partial z_1}xy^{-1/2}+\frac{3}{4}\frac{\partial u}{\partial z_2}yx^{-1/2}=0. \tag{1.4.62}$$

From here we get the canonical form:

$$\frac{\partial^2 u}{\partial z_1^2}+\frac{\partial^2 u}{\partial z_2^2}+\frac{1}{3z_1}\frac{\partial u}{\partial z_1}+\frac{1}{3z_2}\frac{\partial u}{\partial z_2}=0. \tag{1.4.63}$$

Now let us consider the case, when in (1.4.32) instead of "$>$" one has "$=$". Then equation (1.4.29) is called degenerate, or parabolic in the broad sense, at the point (x,t). If (1.4.29) is parabolic in a certain region, then equations (1.4.33) coincide and consequently there is only one independent first integral $\xi(x,t)$. In this case, for bringing (1.4.29) to the canonical form, one could choose as a second variable any function so that the change of variables $x,t\ \to\ \xi,\eta$ were non-degenerate. It turns out that (1.4.29) takes the form

$$\frac{\partial^2 u}{\partial \eta^2}+\dots=0. \tag{1.4.64}$$

Problem 1.4.5 (N 10, [Smi66]). Bring to the canonical form the equation

$$sin^2 x\frac{\partial^2 u}{\partial x^2}-2y\sin x\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=0,\quad 0<x<\pi,\quad y>0. \tag{1.4.65}$$

Solution. The equation of the characteristics, $\sin^2 x\,dy^2+2y\sin x\,dy\,dx+y^2\,dx^2=0$ takes the form $(\sin x\,dy+y\,dx)^2=0$, that is, equation (1.4.65) is parabolic. Separating the variables, we get $dx/\sin x=-dy/y$, hence

$\ln\tan(x/2) = -\ln y + c$, or $y\tan(x/2) = c_1$. We take $\xi = y\tan(x/2)$. Then, setting $\eta = y$, we find

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} y \frac{1}{2\cos^2\frac{x}{2}}; \qquad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\tan\frac{x}{2} + \frac{\partial u}{\partial \eta}. \tag{1.4.66}$$

Differentiating, we get (omitting the terms with $\frac{\partial^2 u}{\partial\xi\partial\eta}$ and $\frac{\partial^2 u}{\partial\xi^2}$, in accordance with (1.4.64)):

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial \xi} y \frac{\sin(x/2)}{2\cos^3(x/2)} + \ldots, \quad \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial \eta^2} + \ldots, \quad \frac{\partial^2 u}{\partial x\partial y} = \frac{\partial u}{\partial \xi}\frac{1}{2\cos^2(x/2)} + \ldots. \tag{1.4.67}$$

Substituting into (1.4.65), we find

$$\sin^2 x\Big(\frac{\partial u}{\partial \xi} y \frac{\sin(x/2)}{2\cos^3(x/2)}\Big) - 2y\sin x\frac{\partial u}{\partial \xi}\frac{1}{2\cos^2(x/2)} + y^2\frac{\partial^2 u}{\partial \eta^2} = 0, \tag{1.4.68}$$

from where we obtain the canonical form:

$$\frac{\partial^2 u}{\partial \eta^2} + \frac{\partial u}{\partial \xi}\Big(\frac{\xi\sin^2 x}{y^2 2\cos^2(x/2)} - \frac{\sin x}{y\cos^2(x/2)}\Big) = \frac{\partial^2 u}{\partial \eta^2} + \frac{\partial u}{\partial \xi}\Big(-\frac{2\xi}{\eta^2+\xi^2}\Big) = 0. \tag{1.4.69}$$

Problems 1.4.3. Bring to the canonical form the following equations:

1. $\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x\partial y} + \frac{\partial^2 u}{\partial y^2} + \alpha\frac{\partial u}{\partial x} + \beta\frac{\partial u}{\partial y} + cu = 0.$
2. $\tan^2 x\frac{\partial^2 u}{\partial x^2} - 2y\tan x\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + \tan^3 x\frac{\partial u}{\partial x} = 0.$
3. $\operatorname{cth}^2 x\frac{\partial^2 u}{\partial x^2} - 2y\operatorname{cth} x\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + 2y\frac{\partial u}{\partial y} = 0.$

1.5 Semi-infinite string

Mixed problem for the d'Alembert equation

Let us consider the d'Alembert equation (1.2.1) in the region $x > 0$. Physically, this corresponds to the string with one (left) end located at the origin and the other located far away from the origin (at a distance $\gg at$):

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}, \qquad x > 0, \qquad t > 0. \tag{1.5.1}$$

Initial conditions (1.2.2) are also required here:

$$u(x,0) = \varphi(x); \qquad \frac{\partial u}{\partial t}(x,0) = \psi(x), \qquad x > 0. \tag{1.5.2}$$

Besides, it is physically obvious that one needs the boundary condition at the left end of the string (at $x = 0$). For example, if this end is fixed, then its displacement is equal to zero:

$$u(0,t) = 0, \qquad t > 0. \tag{1.5.3}$$

Other physically sensible boundary conditions are also possible (see (1.1.14) and (1.1.25)).

Problem (1.5.1)–(1.5.3) is called a *mixed problem*, since it contains both the initial data (1.5.2) and the boundary conditions (1.5.3).

Solution of the mixed problem (1.5.1)–(1.5.3). Method of the incident and reflected waves

Let us use the d'Alembert method, that is, let us search for a solution in the form

$$u(x,t) = f(x - at) + g(x + at) \tag{1.5.4}$$

Substituting this decomposition into the initial data (1.5.2), we get, as in Section 1.2, equations (1.2.27)–(1.2.32), that is, the d'Alembert formula for $u(x,t)$.

Question. Why do we need the boundary condition (1.5.3), if we seem to have found the solution using only the initial data?

Answer. Equations (1.2.27)–(1.2.31) only make sense for $x > 0$, since the initial data (1.5.2), as opposed to (1.2.2), are only given for $x > 0$. Correspondingly, the d'Alembert formula (1.2.32) only holds for $x - at > 0$, and not for all $x > 0,\ t > 0$.

Conclusion. Solution of the mixed problem (1.5.1)–(1.5.3) is given by the d'Alembert formula (1.2.32) for $x - at > 0$.

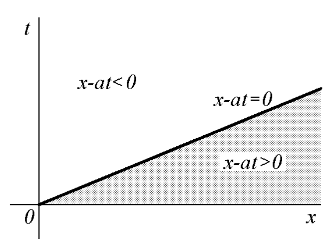


Figure 1.20:

This is the region below the "principal" characteristic $x - at = 0$. The characteristic $x - at = 0$ is called principal since it comes out of the special point (corner point) of the region $x > 0,\ t > 0$ where equation (1.5.1) is considered. Now let us find the solution above the principal characteristic (in the region $x - at < 0$). Decomposition (1.5.4) holds everywhere in the region $x > 0,\ t > 0$. The wave $g(x + at)$ is found from (1.2.30) for all $x > 0,\ t > 0$. On the other hand, the wave $f(x-at)$ is found from (1.2.31) only in the region $x - at > 0$, that is, below the principal characteristic. Thus, it remains to find $f(x - at)$ above the principal characteristic, for $x - at < 0$.

Let us find $f(x-at)$ for $x-at<0$. We use boundary condition (1.5.3):

$$f(-at)+g(at)=0, \qquad t>0. \tag{1.5.5}$$

It is this formula that relates the unknown values of the function f for the negative values of its argument with the values of the function g for the positive values of its argument, that are already known from (1.2.30).

Let us make the change of variables: We set

$$-at=z. \tag{1.5.6}$$

Then (1.5.5) takes the form

$$f(z)=-g(-z), \qquad z<0. \tag{1.5.7}$$

Due to (1.2.30), the above relation shows that for $x-at<0$

$$\begin{aligned} f(x-at) &= -g(at-x) = -\frac{\varphi(at-x)}{2} - \frac{1}{2a}\int\limits_0^{at-x} \psi(s)\,ds - \frac{c}{2a} = \\ &- \frac{\varphi(at-x)}{2} + \frac{1}{2a}\int\limits_{at-x}^{0} \psi(s)\,ds - \frac{c}{2a}. \end{aligned} \tag{1.5.8}$$

Substituting (1.5.8) and (1.2.30) into (1.5.4), we find: for $x>at$, we get the d'Alembert formula (1.2.32); for $0<x<at$, we get:

$$u(x,t)=\frac{-\varphi(at-x)+\varphi(x+at)}{2}+\frac{1}{2a}\int\limits_{at-x}^{x+at}\psi(s)\,ds. \tag{1.5.9}$$

Thus, the solution of the mixed problem (1.5.1)–(1.5.3) is given by two different formulas: The d'Alembert formula (1.2.32) for $x>at$ (below the principal characteristic) and (1.5.9) for $0<x<at$ (above the principal characteristic).

Definition 1.5.1. In the region $0<x<at$ the wave $g(x+at)$ is called an incident wave (onto the left end, $x=0$), while $f(x-at)$ is called a reflected wave.

Let us give *the graphical interpretation* of constructing the solution to problem (1.5.1)–(1.5.3).

Solution of this problem consists of two steps:

A. We substitute the d'Alembert decomposition (1.5.4) into the *initial data* (1.5.2), which are specified at $t=0$ at the points $x>0$ of the Ox axis. Solving system (1.2.27)

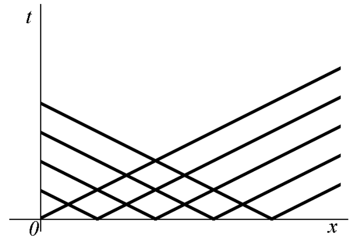


Figure 1.21:

for $x > 0$, we find the waves $f(x - at)$ and $g(x + at)$ at these same points $x > 0$, $t = 0$. Now $f(x - at)$ is known on all characteristics going to the right from these points (Fig. 1.21), since $f(x - at)$ is constant on all such characteristics. These characteristics fill the entire region $x - at > 0$. On the other hand, the wave $g(x + at)$ is known everywhere. Indeed, it is constant on the characteristics going to the left, while such characteristics, sent out of the points (x, t) with $x > 0$ and $t = 0$, fill the entire region $x > 0$, $t > 0$. Thus, the initial data allow to determine the solution in the region where on Fig. 1.21 that contains the characteristics of both families, that is, below the principal characteristic.

It is transparent (see Fig. 1.21) that above the principal characteristic the wave $f(x - at)$ (the reflected wave) is not known yet, while the incident wave $g(x + at)$ is already known.

B. We substitute the d'Alembert decomposition (1.5.4) into boundary condition (1.5.3), which is specified at the points of the time axis Ot ($t > 0$, $x = 0$). At these points the wave $g(x + at)$ is already determined from the initial data. Therefore boundary condition (1.5.5) relates the values of the wave $f(x - at)$ (unknown at these points) with the already known values of $g(x + at)$. This allows to determine the wave $f(x - at)$. But then $f(x - at)$ (and hence $u(x, t)$) is known on the characteristics going to the right from all these points (the dashed line on Fig. 1.22), that is, in the entire region $x < at$ above the principal characteristic.

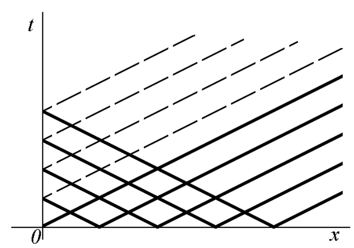


Figure 1.22:

Other boundary conditions

Instead of (1.5.3) one may consider the boundary condition (1.1.12):

$$\frac{\partial u}{\partial x}(0, t) = 0, \qquad t > 0. \tag{1.5.10}$$

Problem 1.5.1. Solve the mixed problem (1.5.1)–(1.5.2), (1.5.10).

Solution.

1. Below the principal characteristics, that is, for $x > at$, the d'Alembert formula (1.2.32) is valid, and formulas (1.2.30)–(1.2.31) hold for $x > 0$;

2. Above the principal characteristic, that is, for $x < at$, instead of (1.5.5) we substitute (1.5.4) into (1.5.10), obtaining

$$f'(-at) + g'(at) = 0, \qquad t > 0. \tag{1.5.11}$$

After the substitution $-at = z$, we have:

$$f'(z) + g'(-z) = 0, \qquad z < 0. \tag{1.5.12}$$

Integrating, we obtain

$$f(z) - g(-z) = c_1 = const, \qquad z < 0. \tag{1.5.13}$$

In view of (1.2.30), for $x < at$, we obtain:

$$f(x-at) = g(at-x)+c_1 = \frac{1}{2}\varphi(at-x)+\frac{1}{2a}\int\limits_0^{at-x} \psi(s)\,ds+\frac{c}{2a}+c_1. \tag{1.5.14}$$

Taking $g(x+at)$ from the same formula (1.2.30), for $x < at$ we obtain:

$$u(x,t) = \frac{\varphi(at-x)+\varphi(x+at)}{2} + \frac{1}{2a}\int\limits_0^{at-x} \psi(s)\,ds + \frac{1}{2a}\int\limits_0^{x+at} \psi(s)\,ds + c_2. \tag{1.5.15}$$

The constant c_2, as we will now show, could be determined from the condition that the solution $u(x,t)$ is continuous at the characteristic $x = at$, when problem (1.5.1)–(1.5.2), (1.5.10) describes a string or a rod.

Discontinuities of a solution along a principal characteristic. Continuity conditions

It follows that the solution to problem (1.5.1)–(1.5.2) is given by different expressions for $x - at > 0$ and $x - at < 0$, therefore it could be discontinuous along the line $x - at = 0$. It turns out that the discontinuity of any solution to (1.5.1) along the line $x - at = 0$ does not depend on time.

Indeed, this could be seen from (1.5.4):

1. The wave $g(x + at)$ is continuous when passing through the principal characteristic, since its level curves $x + at = const$ intersect the line $x = at$.

2. The wave $f(x - at)$ below the principal characteristic $x - at = 0$ has a limit, equal to $f(0+)$, since $x - at > 0$; analogously, its limit from above is equal to $f(0-)$. Thus,

$$u\Big|_{x-at=0-} - u\Big|_{x-at=0+} = f(0-) - f(0+). \tag{1.5.16}$$

Therefore the condition that the solution $u(x,t)$ is continuous on the principal characteristic has the form

$$f(0-) = f(0+). \tag{1.5.17}$$

Problem 1.5.2. Find the condition for the solution to problem (1.5.1)–(1.5.3) to be continuous at the principal characteristic.

Solution. As follows from (1.2.31),

$$f(0+) = \frac{\varphi(0)}{2} - \frac{c}{2a}, \tag{1.5.18}$$

while from (1.5.8) we have

$$f(0-) = -g(0) = -\frac{\varphi(0)}{2} - \frac{c}{2a}. \tag{1.5.19}$$

Therefore, condition (1.5.17) gives

$$-\frac{\varphi(0)}{2} = \frac{\varphi(0)}{2} \quad \Longleftrightarrow \quad \varphi(0) = 0. \tag{1.5.20}$$

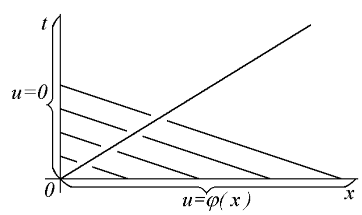


Figure 1.23:

Remark 1.5.1. Let us consider the region $x > 0$, $t > 0$, where problem (1.5.1)–(1.5.3) is being solved. On its boundary at the axis Ot the solution is equal to zero due to (1.5.3), while at the axis Ox the solution is equal to $\varphi(x)$. Therefore condition (1.5.20) is merely the continuity condition of the boundary values of $u(x,t)$ at the point $(0,0)$. As we have seen, this condition is necessary and sufficient for the continuity of the solution at all the points of the principal characteristic.

Problem 1.5.3. Find the continuity condition of the solution to problem (1.5.1)–(1.5.2), (1.5.10) at the principal characteristic.

Solution. Formula (1.5.18) is valid here, while instead of (1.5.19) we get from (1.5.14):

$$f(0-) = \frac{\varphi(0)}{2} + \frac{c}{2a} + c_1. \tag{1.5.21}$$

Therefore, (1.5.17) takes the form (see (1.5.15)):

$$\frac{\varphi(0)}{2} + \frac{c}{2a} + c_1 = \frac{\varphi(0)}{2} - \frac{c}{2a} \qquad \Longleftrightarrow \qquad c_1 + \frac{c}{a} = c_2 = 0. \tag{1.5.22}$$

Remark 1.5.2. A discontinuous solution to problem (1.5.1)–(1.5.2), (1.5.10) (when $c_2 \neq 0$) does not make a physical sense for a string or a rod, since it implies their breaking. Yet, in Acoustics and Gas Dynamics a discontinuous solution makes physical sense and is called a shock wave. In this case, the value of the discontinuity, represented by c_2, could not be found from equations (1.5.1)–(1.5.2), (1.5.10).

This value could be determined from additional physical or chemical information, and this allows to pinpoint a unique solution to the problem. For example, in the process of propagation of the detonation wave in the hasoline wapor the value of the pressure jump at the front of the shock wave depends on the type of the gasoline, pressure, temperature, presence of additional substances, etc.

Mixed problem (1.5.1)–(1.5.2) with more general boundary conditions (1.1.14) or (1.1.25) is solved similarly as in the case of boundary condition (1.5.10), but the equation of type (1.5.11) for the boundary condition for the reflected wave will be the differential equation of the second order, and its solution will contain two arbitrary constants. These constants are determined in each particular problem from the auxiliary conditions. For example, condition (1.5.43) below means that the mass at $t = 0$ is attached to the left end of the string and its speed is equal to 7.

Problem 1.5.4. Find a continuous solution to the problem

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} = 9\frac{\partial^2 u}{\partial x^2}, \quad x > 0,\ t > 0; \quad u(x,0) = e^{-x}; \\ &\frac{\partial u}{\partial t}(x,0) = \cos 5x; \quad \frac{\partial u}{\partial x}(0,t) = u(0,t) + t. \end{aligned}$$

Solution. At $x > 3t$ the d'Alembert formula holds:

$$u(x,t) = \frac{e^{-(x-3t)} + e^{-(x+3t)}}{2} + \frac{1}{6}\,\frac{\sin\Big(5(x+3t)\Big) - \sin\Big(5(x-3t)\Big)}{5}. \tag{1.5.23}$$

Therefore, for $x < 3t$, one needs to look for a solution in the form

$$u(x,t) = f(x-3t) + \frac{e^{-(x+3t)}}{2} + \frac{\sin\Big(5(x+3t)\Big)}{30}. \tag{1.5.24}$$

Substituting this expression into the boundary condition, we find:

$$f'(-3t) - \frac{e^{3t}}{2} + \frac{1}{6}\cos 15t = f(-3t) + \frac{e^{-3t}}{2} + \frac{\sin 15t}{30} + t, \quad t > 0. \tag{1.5.25}$$

Substituting $y=-3t$, we obtain:

$$f'(y)-\frac{e^y}{2}+\frac{1}{6}\cos 5y=f(y)+\frac{e^y}{2}-\frac{\sin 5y}{30}-\frac{y}{3},\quad y<0, \tag{1.5.26}$$

or

$$f'(y)-f(y)=e^y-\frac{1}{6}\cos 5y-\frac{\sin 5y}{30}-\frac{y}{3},\quad y<0. \tag{1.5.27}$$

It follows that

$$f(y)=Ce^y+ye^y+A\cos 5y+B\sin 5y+\frac{y}{3}+\frac{1}{3},\quad y<0. \tag{1.5.28}$$

We find the constants A and B substituting $f(y)$ into (1.5.27):

$$-5A\sin 5y-A\cos 5y+5B\cos 5y-B\sin 5y=-\frac{\cos 5y}{6}-\frac{\sin 5y}{30}. \tag{1.5.29}$$

Therefore $-5A-B=\frac{1}{30}$; $-A+5B=-\frac{1}{6}$, and thus $-26A=-\frac{1}{3} \quad\Rightarrow\quad A=\frac{1}{78}$; $B=-5A+\frac{1}{30}=-\frac{5}{78}+\frac{1}{30}$. Finally, C could be found from continuity condition (1.5.17): $C+A+\frac{1}{3}=\frac{1}{2} \quad\Rightarrow\quad C=\frac{1}{6}-A=\frac{1}{6}-\frac{1}{78}=\frac{2}{13}$.

Answer. For $x<3t$,

$$u(x,t)=\frac{2}{13}e^{x-3t}+(x-3t)e^{x-3t}+\frac{1}{78}\cos 5(x-3t)+$$
$$+\Big(\frac{1}{30}-\frac{5}{78}\Big)\sin 5(x-3t)+\frac{x-3t}{3}+\frac{1}{3}+\frac{e^{-(x+3t)}}{2}+\frac{\sin 5(x+3t)}{30}.$$

Propagation of waves

Problem 1.5.5. Stretched semi-infinite rope is initially at rest. Starting at $t=0$, its left end $x=0$ is moving up and down, with the displacement being equal to $\sin\pi t$. We assume $a=1$. Draw the shape of the rope at $t=1,\ 2,\ 3,\ldots$.

Solution. We need to solve the mixed problem (1.5.1)–(1.5.2), where $\varphi(x)\equiv\psi(x)\equiv 0$, with the boundary condition

$$u(0,t)=\sin\pi t,\quad t>0. \tag{1.5.30}$$

1) $x>t \quad\Rightarrow\quad u(x,t)=0$, since $\varphi(x)\equiv\psi(x)\equiv 0$ by the condition of the problem. In particular, $g(x+t)\equiv 0$.

2) $x<t$: since $g(x+t)\equiv 0$,

$$u(x,t)\equiv f(x-t). \tag{1.5.31}$$

Substituting (1.5.31) into (1.5.30), we get

$$f(-t)=\sin\pi t,\quad t>0. \tag{1.5.32}$$

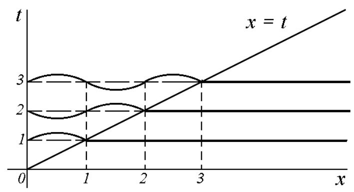


Figure 1.24:

Substituting $-t = z$,

$$f(z) = \sin \pi(-z), \quad z < 0. \tag{1.5.33}$$

Therefore

$$u(x,t) = f(x-t) = \sin \pi(t-x) = -\sin \pi(x-t), \quad x < t. \tag{1.5.34}$$

Answer. See Fig. 1.24.

Problem 1.5.6. The stretched rope is initially at rest. Starting at $t = 0$ its left end $x = 0$ is moved up and down with a given force $\sin \pi t$. Assume that $a = 1$ and $T = 1$. Draw the shape of the string at $t = 1, 2, 3, \ldots$.

Solution. We need to find the continuous solution to the mixed problem (1.5.1)–(1.5.2) with $\varphi(x) \equiv \psi(x) \equiv 0$ and with the boundary condition

$$\frac{\partial u}{\partial x}(0,t) = -\sin \pi t, \quad t > 0, \tag{1.5.35}$$

(see (1.1.14)).

1. $x > 0 \quad \Rightarrow \quad u(x,t) \equiv 0$; in particular, $g(x+t) \equiv 0$.
2. $x < t$:

$$u(x,t) = f(x-t) \tag{1.5.36}$$

Substituting (1.5.36) into (1.5.35), we get

$$f'(-t) = -\sin \pi t, \quad t > 0. \tag{1.5.37}$$

Substituting $-t = z$, we can write

$$f'(z) = \sin \pi z, \quad z < 0 \quad \Rightarrow \quad f(z) = -\frac{\cos \pi z}{\pi} + c, \quad z < 0. \tag{1.5.38}$$

Therefore

$$u(x,t) = f(x-t) = -\frac{\cos \pi(x-t)}{\pi} + c, \quad x < t. \tag{1.5.39}$$

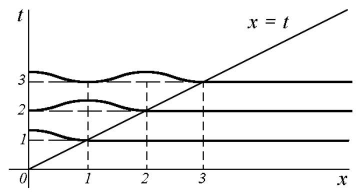


Figure 1.25:

The continuity condition at $x = t$ requires that

$$u(t,t) = 0 = -\frac{1}{\pi} + c \qquad \Longleftrightarrow \qquad c = \frac{1}{\pi}. \tag{1.5.40}$$

Finally:

$$u(x,t) = \frac{1}{\pi}\Big(-\cos\pi(x-t) + 1\Big), \quad x < t. \tag{1.5.41}$$

Answer. See Fig. 1.25.

Problem 1.5.7. The mass $m = 2$ moving with the speed $v = 7$ clings to to the end of the semi-infinite rod which was initially at rest. Find the displacement of the rod for $t > 0$, assuming that $a = 3$ in (1.5.1) and $SE = 5$ in (1.1.25).

Solution. The mathematical setup of the problem looks as follows:

$$\frac{\partial^2 u}{\partial t^2} = 9\frac{\partial^2 u}{\partial x^2}; \quad u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0; \quad 2\frac{\partial^2 u}{\partial t^2}(0,t) = 5\frac{\partial u}{\partial x}(0,t). \tag{1.5.42}$$

The clinging of the mass to the end of the rod gives the following conditions:

$$u(0,0+) = 0; \quad \frac{\partial u}{\partial t}(0,0+) = 7 \tag{1.5.43}$$

The last equality is due to the fact that the mass at the end of the rod is only due to the newly acquired mass m. For $x > 3t$ the d'Alembert formula holds, so that $u(x,t) = 0$, since the initial data are equal to zero. For $x < 3t$ the solution has the form $u(x,t) = f(x-3t)$, since $g(x+3t) \equiv 0$. Substituting u into the boundary conditions, we find

$$2\cdot 9f''(-3t) = 5f'(-3t), \quad t > 0; \quad f(0-) = 0; \quad -3f'(0-) = 7. \tag{1.5.44}$$

Hence

$$18f''(y) - 5f'(y) = 0; \quad y < 0 \quad \Rightarrow \quad f(y) = c_1 + c_2 e^{\frac{5}{18}y}; \quad c_1 + c_2 = 0;$$
$$-3c_2\frac{5}{18} = 7; \quad c_2 = -\frac{42}{5}.$$

Answer. $u = 0$ for $x > 3t$ and $u = \frac{42}{5}\Big(1 - e^{\frac{5}{18}(x-3t)}\Big)$ for $x < 3t$.

The reflection of waves

Besides the general method described above, the problem (1.5.1)–(1.5.2) with the boundary conditions (1.5.3) or (1.5.10) could be approached using the method of odd and even extension.

Let us first consider the method of odd extension.

The following problem describes the oscillations of a pitched string.

Problem 1.5.8. Solve the mixed problem (1.5.1)–(1.5.3) with $a = 1$ and the following initial data:

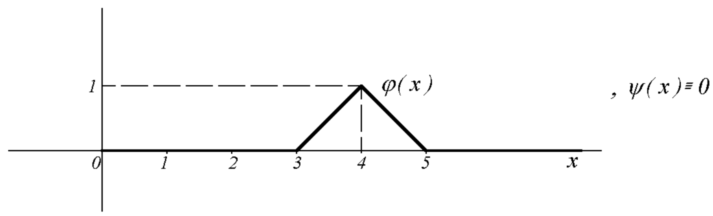


Figure 1.26:

Draw the shape of the string at $t = 1,\ 2,\ 3,\ 4,\ 5$.

Solution. Let us consider the solution $\hat{u}(x,t)$ to the Cauchy problem (1.2.1)–(1.2.2) on the entire axis, with $\frac{\partial}{\partial t}\hat{u}(x,0) = \hat{\psi}(x) \equiv 0$ and with $\hat{\varphi}$ being the odd extension of $\varphi(x)$ onto $\mathbb{R}$:

$$\hat{u}(0,x) = \hat{\varphi}(x) = \begin{cases} \varphi(x), & x \geq 0, \\ -\varphi(x), & x < 0. \end{cases} \tag{1.5.45}$$

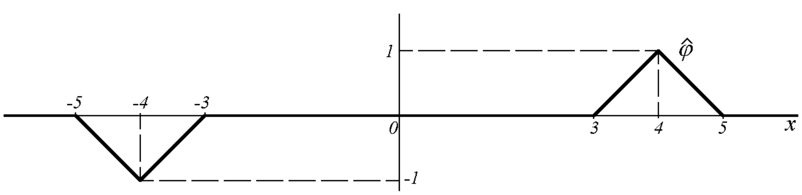


Figure 1.27:

Set

$$u(x,t) \equiv \hat{u}(x,t)\Big|_{x \geq 0}. \tag{1.5.46}$$

Obviously, u satisfies equation (1.5.1) and the initial data (1.5.2). Below, we will see that boundary condition (1.5.3) is also satisfied, since $\hat{u}(x,t)$ is odd in x. The region $x < 0$ will be called fictional, or non-physical.

Construction of $\hat{u}(x,t)$. According to the d'Alembert formula (1.2.32),

$$\hat{u}(x,t) = \frac{\hat{\varphi}(x-t)}{2} + \frac{\hat{\varphi}(x+t)}{2}, \tag{1.5.47}$$

that is, we need to divide $\hat{\varphi}(x)$ into two, shift by t to the right and to the left, and to add up the results.

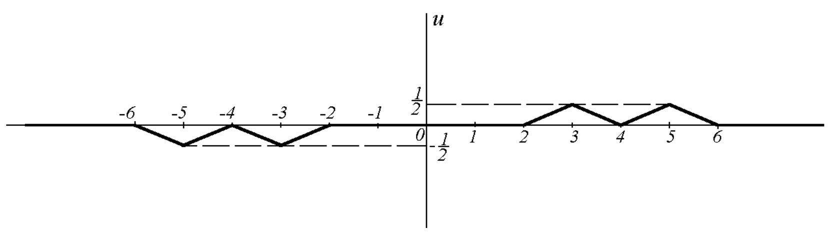


Figure 1.28: $t = 1$.

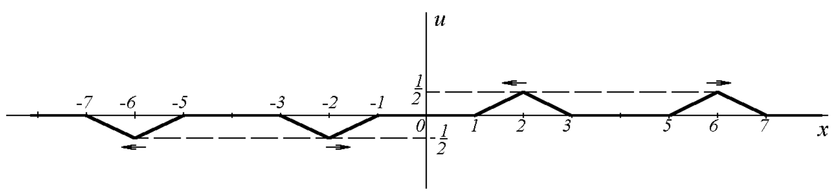


Figure 1.29: $t = 2$. Arrows indicate the direction of the motion of the humps.

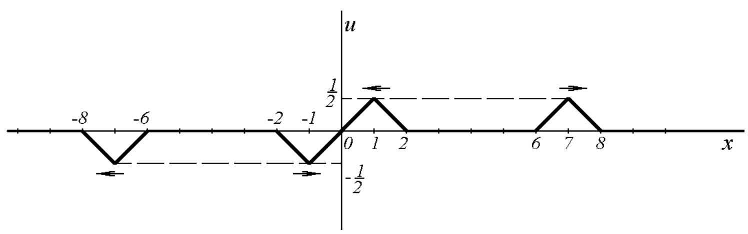


Figure 1.30: $t = 3$. The left hump in the physical region $x > 0$ approaches the nail at $x = 0$.

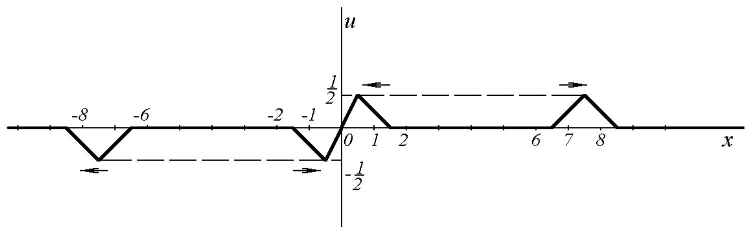


Figure 1.31: $t = 3.5$. The nail pulls the hump over.

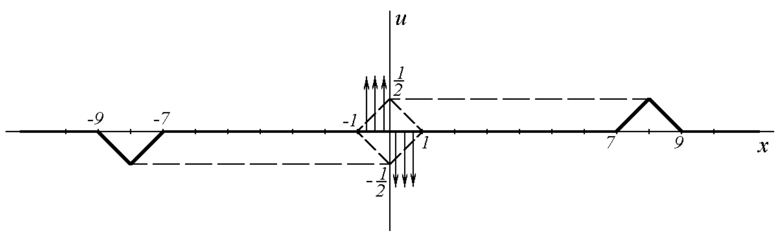


Figure 1.32: $t = 4$. The deviation for $x \in [-1, 1]$ is identically equals zero; arrows show the velocities of the points of the string.

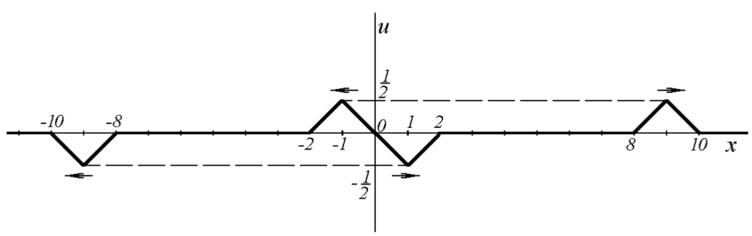


Figure 1.33: $t = 5$. The humps have parted (the arrows indicate the directions of motion of humps).

And so on: in the physical region $x > 0$ the two humps move to the right (while in the non-physical region $x < 0$ the two humps move to the left).

Remark 1.5.3. We see that boundary condition (1.5.3) at $x = 0$ holds for all $t > 0$ since $\hat{u}(x, t)$ is an odd function in x.

Problem 1.5.9. Draw the shape of the string at $t = 3.25$.

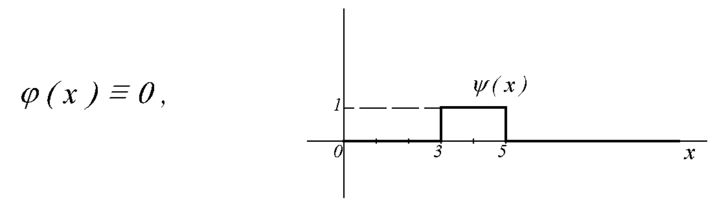


Figure 1.34:

Let us consider the oscillations of a piano string after being hit with a hammer.

Problem 1.5.10. Solve the problem (1.5.1)–(1.5.3) with $a = 1$ and the initial data as on Figure 1.34. Plot the string at $t = 1,\ 2,\ 3,\ 4,\ 5$, and 6.

Solution. Let us set $\hat{\varphi}(x) \equiv 0,\ x \in \mathbb{R}$, and let us extend $\psi(x)$ onto $\mathbb{R}$ so that it is odd:

$$\hat{\psi}(x) = \begin{cases} \psi(x), & x \geq 0, \\ -\psi(-x), & x < 0. \end{cases} \tag{1.5.48}$$

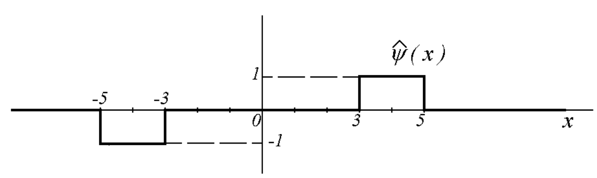


Figure 1.35:

Consider the solution $\hat{u}$ to the Cauchy problem (1.2.1)–(1.2.2) with the initial data $\hat{\varphi}$ and $\hat{\psi}$. As before,

$$\hat{u}(x,t) = \hat{\phi}(x+t) - \hat{\phi}(x-t), \quad \text{where} \quad \hat{\phi}(x) \equiv \frac{1}{2}\int\limits_{-\infty}^{x} \hat{\psi}(s)\,ds : \tag{1.5.49}$$

We set $u(x,t) \equiv \hat{u}(x,t)\Big|_{x>0}$. Obviously, $u(x,t)$ satisfies (1.5.1) and (1.5.2). As will be seen below, boundary condition (1.5.3) is also satisfied.

Construction of $\hat{u}(x,t)$ according to forumlas (1.5.49): Figs. 1.37–1.42.

Problem 1.5.11. Draw the shape of the string at $t = 3.5$ and $t = 4.5$.

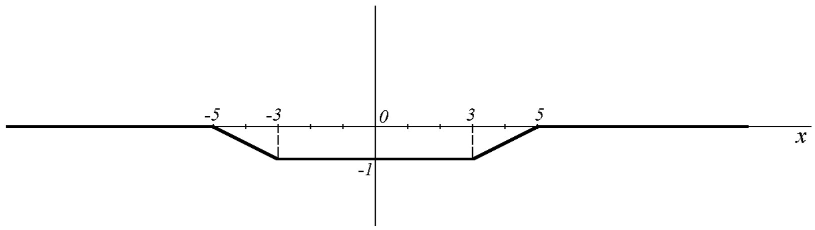


Figure 1.36:

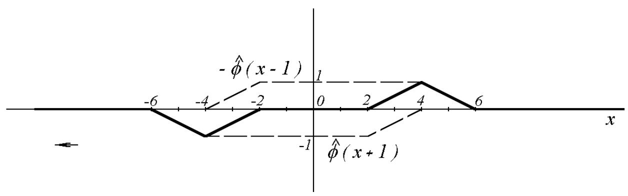


Figure 1.37: $t = 1$.

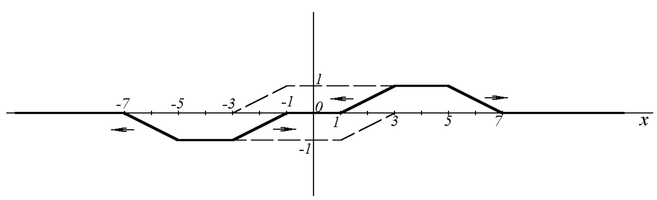


Figure 1.38: $t = 2$.

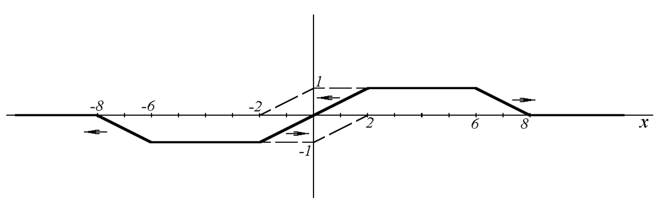


Figure 1.39: $t = 3$.

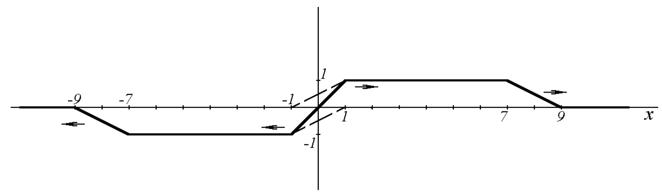


Figure 1.40: $t = 4$.

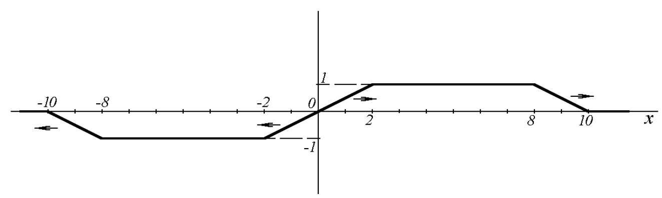


Figure 1.41: $t = 5$.

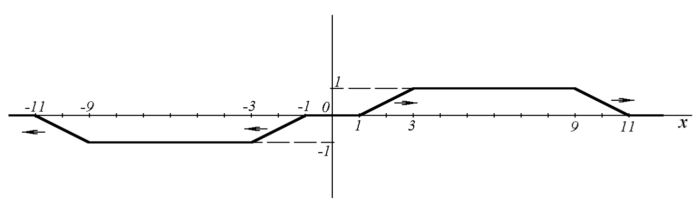


Figure 1.42: $t = 6$.

And so on: In the physical region $x > 0$ the trapoeoid is moving to the right (while the trapezoid in the unphysical region is moving to the left). Boundary condition (1.5.3) is obviously satisfied.

Let us now consider the *method of even extension.*

Problem 1.5.12. Solve the mixed problem (1.5.1)–(1.5.2), (1.5.10) with $a = 1$ and initial data (1.5.43). Draw the shape of the string at $t = 1;\ 2;\ 3;\ 3.5;\ 4;\ 4.5$.

Hint. Use the even extension for $\varphi(x)$ and $\psi(x)$. Then boundary condition (1.5.10) will be satisfied, since $\hat{u}(x,t)$ will be even in x.

Problem 1.5.13. Solve the mixed problem (1.5.1)–(1.5.2), (1.5.10) with $a = 1$ and initial data (1.5.49). Draw the shape of the string at $t = 1, 2, 3;\ 3,5;\ 4;\ 4,5;\ 6$.

Problem 1.5.14. For $t < 0$, there is a deformation wave propagating to the left along the elastic semi-infinite rod:

$$u(x,t) = \begin{cases} \sin(x+3t), & x > -3t, \\ 0, & 0 < x < -3t, \quad t < 0. \end{cases} \tag{1.5.50}$$

The left end of the rod at $x = 0$ is elastically attached (see (1.1.25)):

$$0 = -2u(0,t) + 3\frac{\partial u}{\partial x}(0,t), \quad t > 0. \tag{1.5.51}$$

Find $u(x,t)$ for $t > 0$.

Solution. As follows from the condition of the problem,

$$\frac{\partial^2 u}{\partial t^2} = 9\frac{\partial^2 u}{\partial x^2}, \quad x > 0,\ t > 0; \qquad u(x,0) = \sin x; \quad \frac{\partial u}{\partial t}(x,0) = 3\cos x, \quad x > 0. \tag{1.5.52}$$

From here, for $x > 3t$, the d'Alembert formula yields $u(x,t) = \sin(x+3t)$, as in (1.5.50). For $x < 3t$ we are looking for a solution in the form $u = f(x-3t) + \sin(x+3t)$. Substituting into boundary condition (1.5.51), we get

$$0 = -2f(-3t) - 2\sin 3t + 3f'(-3t) + 3\cos 3t. \tag{1.5.53}$$

The substitution $y = -3t$ gives

$$3f'(y) - 2f(y) = -2\sin y - 3\cos y. \tag{1.5.54}$$

Therefore $f(y) = Ce^{(2/3)y} + A\cos y + B\sin y$. The constants A and B are found by substituting $f(y)$ into (1.5.54):

$$-3A\sin y - 2A\cos y + 3B\cos y - 2A\sin y = -2\sin y - 3\cos y. \tag{1.5.55}$$

Therefore $-3A - 2B = -2;\quad -2A + 3B = -3$, so that

$$-9A - 4A = -12; \quad A = 12/13; \quad B = -1 + \frac{2}{3}A = -1 + \frac{8}{13} = -\frac{5}{13}. \tag{1.5.56}$$

Finally, C is found from continuity condition (1.5.17): $C + A = 0;\quad C = -\frac{12}{13}$.

Answer. For $x < 3t$

$$u(x,t) = -\frac{12}{13}e^{\frac{2}{3}(x-3t)} + \frac{12}{13}\cos(x-3t) - \frac{5}{13}\sin(x-3t) + \sin(x+3t). \tag{1.5.57}$$

1.6 Finite string

The d'Alembert method

Transversal oscillations of a string of length l in the absense of external forces are described by equation (1.5.1):

$$\frac{\partial^2 u(x,t)}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}; \quad 0 < x < l, \quad t > 0. \tag{1.6.1}$$

For the unique determination of the motion of the string we need the initial data

$$u(x,0) = \varphi(x); \qquad \dot{u}(x,0) = \psi(x), \quad 0 < x < l \tag{1.6.2}$$

and the boundary conditions at the ends. For example, if the ends are fixed, then

$$u(0,t) = 0; \quad u(l,t) = 0, \qquad t > 0. \tag{1.6.3}$$

Solution of the mixed problem (1.6.1)–(1.6.3) could be found by the d'Alembert method along the lines of Section 1.5, as follows:

1) Substituting (1.5.4) into initial data (1.6.2), which are given at the points $t = 0$, $0 < x < l$, we find by formulas (1.2.30)–(1.2.31) the waves $f(x - at)$ and $g(x + at)$ at these same points. This gives the solution $u(x,t)$ in region I (the triangle OAB):

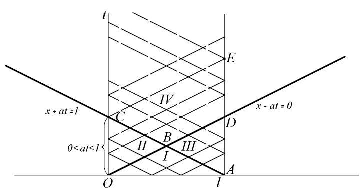


Figure 1.43:

2) Substituting (1.5.4) into boundary condition (1.6.3) at $x = 0$, we find the reflected wave $f(x - at)$ from knowing the incident wave $g(x + at)$ at the poinds of the interval OC. This gives the solution $u(x,t)$ in region II (the triangle OBC);

3) Substituting (1.5.4) into boundary condition (1.6.3) for $x = l$, we find the reflected wave $g(x + at)$ fron knowing the incident wave $f(x - at)$ at the points of the interval AE.

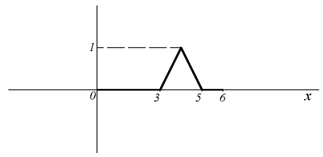
And so on. This allows to find the solution $u(x,t)$ in the entire semi-strip $0 < x < l$, $t > 0$, successively decomposing it into regions, bounded by characteristics similar to characteristics OD, AC, and CE. In the same fasion one can solve the mixed problem (1.6.1)–(1.6.2) with boundary conditions more difficult than (1.6.3).

Remark 1.6.1. The asymptotic properties of solutions to problem (1.6.1)–(1.6.3) as $t \to \infty$, and, in particular, the frequencies of oscillations, are easier to investigate using the Fourier method, which is described in Chapter 2.

Method of "even" and "odd" extension

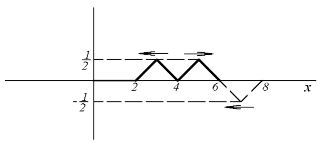
Problem 1.6.1. Solve problem (1.6.1)–(1.6.3) for $a = 1$, $l = 6$ and the initial data from Fig. 1.26. Plot the shape of the string for $t = 1, 2, \ldots$ and find the period T of the string oscillations.

$t = 0$:

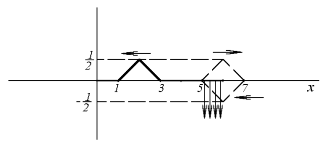


Solution.

$t = 1$:

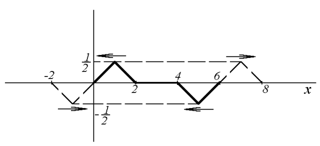


$t = 2$:

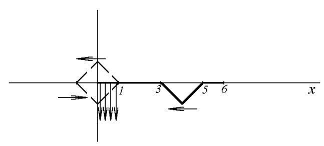


We send a virtual hump from the right.

$t = 3$:

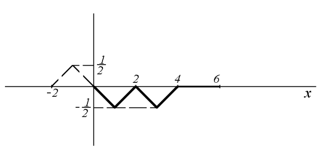


$t = 4$:

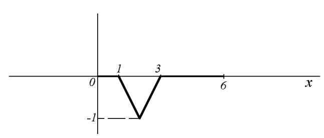


We send a virtual hump from the left.

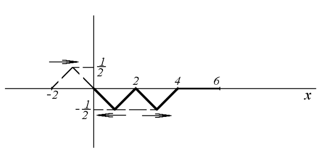
$t = 5$:



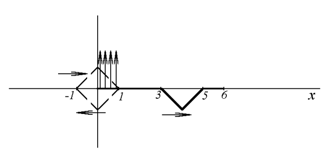
$t = 6$:



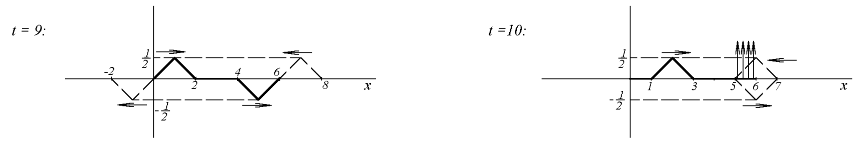
$t = 7$:



$t = 8$:



We send a virtual hump from the left.



We send a virtual hump from the right.

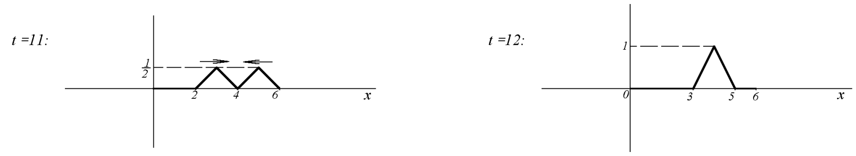


Figure 1.44:

Thus, the process is periodic, with the period being equal to $T = 12 = \frac{2l}{a}$.

Problem 1.6.2 (The piano string). Solve the problem (1.6.1)–(1.6.3) with $a = 1$ and the initial data from Fig. 1.34, $l = 6$. Plot the shape of the string at $t = 1, 2, \dots$ and find the period of oscillations.

Problem 1.6.3. Solve the problem (1.6.1)–(1.6.2) with the boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(l,t) = 0, \quad t > 0. \tag{1.6.4}$$

Hint. One should apply the method of "even" reflections, that is, send reflected virtual humps (see Fig. 1.44) with the same "polarization" as the incedent ones (not with the opposite).

1.7 The wave equation with many independent variables

Plane waves, characteristics, discontinuities

A multidimensional analog of the d'Alembert equation (1.1.1) is the wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \Delta u(x,t) \equiv a^2 \Big(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} \Big), \quad t > 0, \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3, \tag{1.7.1}$$

where $a > 0$. This equation describes the air pressure $p(x,t)$ (the sound wave in Acoustics), the potentials $\varphi(x,t)$ and $A(x,t)$ of the electromagnetic field in Electrodynamics, etc.

Let us try to find solutions to equation (1.7.1) in the form

$$u(x,t) = f(\xi_\circ t + \xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3) = f\Big(\xi_\circ t + \langle \vec{\xi} x\rangle\Big) \tag{1.7.2}$$

where $\vec{\xi} = (\xi_1, \xi_2, \xi_3) \neq 0$; $\quad \langle \vec{\xi}, x\rangle \equiv \xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3$.

Remark 1.7.1. Such a function is called a plane wave. This is related with the following:

A) At fixed $t = t_\circ$, the level surfaces $u(x, t_\circ) = const$ are represented by the planes

$$\xi_\circ t + \langle \vec{\xi}, x\rangle = c \tag{1.7.3}$$

orthogonal to the vector $\vec{\xi}$;

B) For different $t = t_\circ, t_1$, the function $u(t_1, x)$ differs from $u(t_\circ, x)$ by the shift by the vector

$$-\frac{\vec{\xi}}{|\vec{\xi}|^2}\xi_\circ(t_1 - t_\circ). \tag{1.7.4}$$

Indeed,

$$u\Big(x + \frac{\vec{\xi}}{|\vec{\xi}|^2}\xi_\circ(t_1 - t_\circ), t_\circ\Big) = f\Big(\xi_\circ t_\circ + \langle \vec{\xi}, x + \frac{\vec{\xi}}{|\vec{\xi}|^2}\xi_\circ(t_1 - t_\circ)\rangle\Big) =$$

$$= f\Big(\xi_\circ t_\circ + \langle \vec{\xi}, x\rangle + \frac{\langle \vec{\xi}, \vec{\xi}\rangle}{|\vec{\xi}|^2}\xi_\circ(t_1 - t_\circ)\Big) = f\Big(\xi_\circ t_1 + \langle \vec{\xi}, x\rangle\Big) = u(t_1, x). \tag{1.7.5}$$

Thus, (1.7.2) is a wave moving along the direction of the vector $-\vec{\xi}$ with the speed

$$v = \frac{\xi_\circ}{|\vec{\xi}|}. \tag{1.7.6}$$

We denote the unit vector in the direction $-\vec{\xi}$ by $\vec{\omega} = -\frac{\vec{\xi}}{|\vec{\xi}|}$. Then $\xi_\circ = v|\vec{\xi}|$; $\vec{\xi} = -\vec{\omega}|\vec{\xi}|$ and, therefore, (1.7.2) could be written as

$$u(x,t) = f\Big(v|\vec{\xi}|t - \langle \vec{\omega}, x\rangle|\vec{\xi}|\Big) = f\Big((vt - \langle \vec{\omega}, x\rangle)|\vec{\xi}|\Big) = g\Big(vt - \langle \vec{\omega}, x\rangle\Big), \tag{1.7.7}$$

where $g(z) \equiv f\Big(x|\vec{\xi}|\Big)$, $\quad |\vec{\omega}| = 1$.

After these preliminary remarks let us proceed to finding the solution to equation (1.7.1) in the form (1.7.2). We substitute (1.7.2) into (1.7.1), and, using the Chain Rule, we get:

$$f''(\xi_\circ t + \langle \vec{\xi}, x\rangle)\xi_\circ^2 = a^2 f''(\xi_\circ t + \langle \vec{\xi}, x\rangle)(\xi_1^2 + \xi_2^2 + \xi_3^2). \tag{1.7.8}$$

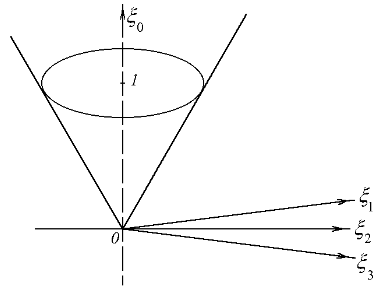


Figure 1.45:

Assuming that $f''(z) \not\equiv 0$, we get from here the characteristic equation

$$\xi_\circ^2 = a^2|\vec{\xi}|^2. \tag{1.7.9}$$

Solutions of this equation are vectors $\xi = (\xi_\circ, \vec{\xi}) \in \mathbb{R}^4$, lying on the (three-dimensional) cone Q in $\mathbb{R}^4$, whose base is a two-dimensional sphere $|\vec{\xi}| = \frac{1}{a}, \quad \xi_\circ = 1$ (Fig. 1.45).

Conversely, for any $\xi \in \mathbb{R}^4$ satisfying (1.7.9), the plane wave (1.7.2) with any function $f(z)$ is a solution to equation (1.7.1).

In particular, $f(z)$ could be taken discontinuous (or rapidly changing) at some point, for example, at $z = 2$ (see Fig. 1.14). Then the solution (1.7.2) will have the same discontinuity (or rapid change) along the entire hyperplane in $\mathbb{R}^4_{x,t}$ (if $\xi \neq 0$):

$$\xi_\circ t + \langle \vec{\xi}, x \rangle = 2. \tag{1.7.10}$$

For fixed t this discontinuity is located on the plane in $\mathbb{R}^3_x$ described by equation (1.7.10). As the time increases, this plane moves in the direction of its normal, represented by $-\vec{\xi}$, with the speed $v = \frac{|\xi_\circ|}{|\vec{\xi}|} = a$ (see (1.7.9)).

Definition 1.7.1. The vector $\xi = (\xi_\circ, \xi_1, \xi_2, \xi_3) \in \mathbb{R}^4, \quad \xi \neq 0$ satisfying (1.7.9) is called a characteristic normal of the wave equation (1.7.1).

The hyperplane $\xi^\perp = \Big\{(t,x) \in \mathbb{R}^4 : \xi_\circ t + \langle \vec{\xi}, x \rangle = const\Big\}$, orthogonal to a particular characteristic normal ξ, is called a characteristic hyperplane (or simply a characteristic) of the wave equation (1.7.1).

The hypersurface in $\mathbb{R}^4$ is called a characteristic hyperplane if the tangent hyperplane at each point is characteristic.

Remark 1.7.2. Due to characteristic equation (1.7.6) the speed of propagation of all the plane waves that satisfy the wave equation (1.7.1) is equal to a:

$$v^2 = \frac{\xi_\circ^2}{|\vec{\xi}|^2} = a^2. \tag{1.7.11}$$

Conclusion. Any characteristic hyperplane could be a surface of the discontinuity of solutions to equation (1.7.1) (see Remark 1.2.1).

All the plane waves satisfying equation (1.7.1) propagate with the speed a.

It is the formula (1.7.11) that the discovery of *the electromagnetic nature of light* and *the special theory of relativity* is connected with. From the equations of Electrodynamics Maxwell derived that the potentials of the electromagnetic field satisfy the wave equation (1.7.1) with the coefficient

$$a^2 = \frac{1}{\varepsilon_\circ \mu_\circ}. \tag{1.7.12}$$

Here $\varepsilon_\circ$ and $\mu_\circ$ are the electric and magnetic permeability of vacuum, respectively, are found experimentally from purely electromagnetic measurements. When Maxwell computed the speed of propagation of the electromagnetic waves, it turned out that this speed with the great accuracy coincided with the speed of light:

$$a = \frac{1}{\sqrt{\varepsilon_\circ \mu_\circ}} \approx 299976 \frac{km}{h}. \tag{1.7.13}$$

This led Maxwell to the conclusion that the light also has an electromagnetic nature!

Another great discovery related with furmulas (1.7.11), (1.7.12), was the special theory of relativity. The question naturally arises: In what reference frame the value of the speed of light is actually equal to $\frac{1}{\sqrt{\varepsilon_\circ \mu_\circ}}$? It is known that all the laws of Mechanics are the same in any inertial reference frame. Thus, it is natural to assume that the laws of Electrodynamics also hold in any inertial reference frame. But the, according to (1.7.12), the speed of light should also be the same in all such systems! Such a property of the velocity, though, contradicts Newton's Mechanics. It follows that either the Maxwell equations are only valid in a particular reference frame, related to the stationary "ether", or the Newton laws of Mechanics are not exact. It is for settling this question that Michelson and Morley built their famous experiment to justify that the speed of light is the same in different inertial reference frames, and, consequently, the absence of the statinary "ether" and inexactness of Newton's Mechanics (at high speeds). The necessary refinement of the Mechanics laws was later given by A. Einstein.

The region of dependence. The Kirchhoff formula

Let us try to find the region of dependence for equation (1.7.1) with the aid of characteristics, as in Section 1.4 (Fig. 1.18). That is, let us consider the Cauchy problem for equation (1.7.1) with initial conditions at $t = 0$:

$$u\Big|_{t=0} = \varphi(x), \quad \frac{\partial u}{\partial t}\Big|_{t=0} = \psi(x), \quad x \in \mathbb{R}^3. \tag{1.7.14}$$

Let us draw through a particular point $(x_\circ, t_\circ) \in \mathbb{R}^4$, $t_\circ > 0$ all the characteristics (*characteristic hyperplances*) of equation (1.7.1) (Fig. 1.46). On Fig. 1.46, $\xi_{\rm I}$ and $\xi_{\rm II}$ are the characteristic normals, while $\xi_{\rm I}^\perp$ and $\xi_{\rm II}^\perp$ are orthogonal to them characteristic hyperplanes that pass through $(x_\circ, t_\circ)$. These characteristics intersect the "initial" hyperplane $t = 0$ along the planes P_I and P_{II}.

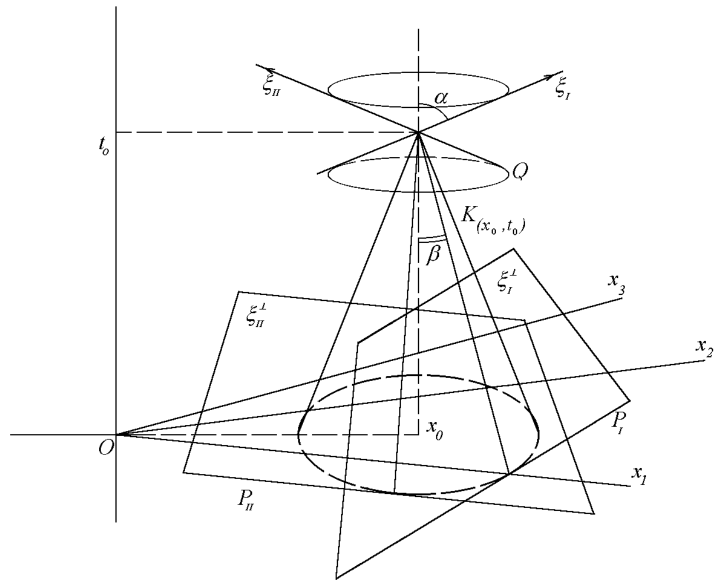


Figure 1.46:

Hypothesis. The region of dependence of the solution u at the point $(x_\circ, t_\circ)$ is the region of the hyperplane $t = 0$ bounded by all the hyperplanes P_I, $P_{II}, \dots$ (this is in the analorgy with Fig. 1.18).

This region is a ball of radius at centered at $x_\circ$. To see this, one should notice the following: The normals ξ_{I}, ξ_{II} belong to the cone Q described by equation (1.7.9), while all the hyperplanes $\xi_{\mathrm{I}}^\perp$, $\xi_{\mathrm{II}}^\perp$ are tangent to the cone $K_{(x_\circ,t_\circ)}$ orthogonal to the cone Q (see Fig. 1.46). Therefore the hyperplanes P_I, $P_{II}, \dots$ are tangent to the base of the cone $K_{(x_\circ,t_\circ)}$, that is, the sphere $S_{(x_\circ,t_\circ)}$, which is bounded by all these hyperplanes.

Remark 1.7.3. The cone $K_{(x_\circ,t_\circ)}$ is called the characteristic cone of equation (1.7.1) at the point $(x_\circ, t_\circ)$. It is the characteristic hypersurface.

As seen on Fig. 1.46, $\alpha + \beta = \frac{\pi}{2}$. As follows from (1.7.9),

$$\tan\alpha = \frac{|\vec{\xi}|}{|\xi_\circ|} = \frac{1}{a} \quad \Longrightarrow \quad \tan\beta = a. \tag{1.7.15}$$

Therefore the cone $K_{(x_\circ,t_\circ)}$ is given by

$$|x - x_\circ| = a|t - t_\circ|. \tag{1.7.16}$$

When $t = 0$, one gets the equation of a sphere:

$$S_{(x_\circ,t_\circ)} = \left\{x \in \mathbb{R}^3 \ : \ |x - x_\circ| = at_\circ\right\} \tag{1.7.17}$$

Thus, our hypothesis is that the region of dependence of u at the point $(x_\circ, t_\circ)$ is a ball of radius $at_\circ$ centered at $x_\circ$. This hypothesis is equivalent to saying that all the solutions of equation (1.7.1) propagate with the speed a. Let us point out that we already proved this for the plane waves.

Our hypothesis is correct indeed. Moreover, it turns out that the region of dependence is smaller than a ball: It only consists of the sphere $S_{(x_\circ, t_\circ)}$. Obviously, this follows from the Kirchhoff formula for the solution to the Cauchy problem (1.7.1), (1.7.14) (for the derivation of this formula, see [Pet91]):

$$u(x,t) = \frac{1}{4\pi a^2 t} \int\limits_{|y-x|=at} \psi(y)\, dS_y + \frac{\partial}{\partial t}\Big(\frac{1}{4\pi a^2 t} \int\limits_{|y-x|=at} \varphi(y)\, dS_y \Big). \qquad (1.7.18)$$

Distribution of waves. The Huygens principle

Problem 1.7.1. Given: $a = 1$, $\varphi(x) \equiv \psi(x) \equiv 0$ at $|x| > 1$; Find where (for certain) $u(x,t) \equiv 0$ at $t = 1, 2, 3, 4$.

Solution. First, assume that a is arbitrary. Then $u(x,t) = 0$ if the region

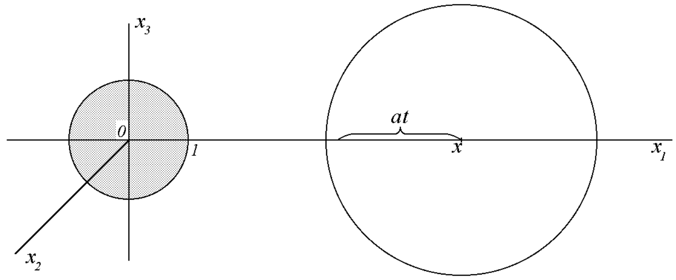


Figure 1.47:

of integration in (1.7.18), that is, the sphere $|y - x| = at$, does not intersect the region $|y| \leq 1$ where $\varphi(y)$ and $\psi(y)$ are supported. See Figure 1.47.

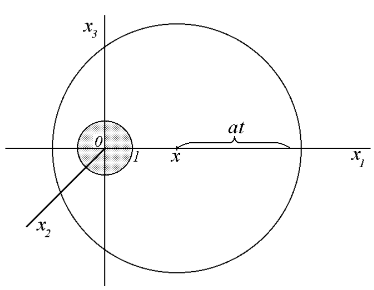


Figure 1.48:

Clearly, this condition is equivalent to the following (see Fig. 1.47):

$$1 + at < |x|, \qquad (1.7.19)$$

or, another possibility,

$$at > 1 = |x|. \qquad (1.7.20)$$

when the sphere $|y - x| = at$ contains the ball $|y| \leq 1$ (see Fig. 1.48).

Condition (1.7.19) at $a = 1$ yields the identity $u(x,t) \equiv 0$ in the following regions:

$$\begin{cases} t = 1 & \Rightarrow & |x| > 2; \\ t = 2 & \Rightarrow & |x| > 3; \\ t = 3 & \Rightarrow & |x| > 4; \\ t = 4 & \Rightarrow & |x| > 5. \end{cases} \tag{1.7.21}$$

Condition (1.7.20) at $a = 1$ yields the identity $u(x,t) \equiv 0$ in the following regions:

$$\begin{cases} t = 1 \quad \Rightarrow \quad x \in \emptyset; \quad t = 2 \quad \Rightarrow \quad |x| < 1; \\ t = 3 \quad \Rightarrow \quad |x| < 2; \quad t = 4 \quad \Rightarrow \quad |x| < 3. \end{cases} \tag{1.7.22}$$

Therefore, $u(x,t)$ has the form of the spherical wave contained in the spherical layer of thickness 2:

$$\begin{aligned} t = 1 \quad \Rightarrow \quad |x| \le 2; \quad t = 2 \quad \Rightarrow \quad 1 \le |x| \le 3; \\ t = 3 \quad \Rightarrow \quad 2 \le |x| \le 4; \quad t = 4 \quad \Rightarrow \quad 3 \le |x| < 5. \end{aligned} \tag{1.7.23}$$

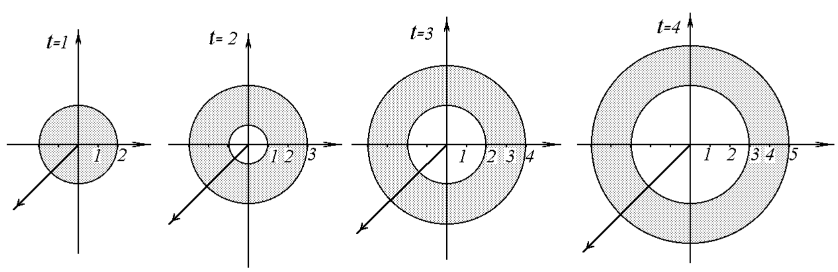


Figure 1.49:

Answer. $u(x,t)$ for certain is equal to zero outside the spherical layers (1.7.23) (although it could also be equal to zero somewhere inside these layers).

Conclusion. As seen from (1.7.23), the front of the spherical wave propagates with the speed 1. In the case of an arbitrary a, it is seen from (1.7.19) and (1.7.20) that the solution $u(x,t)$ could only be different from zero in a spherical layer

$$at - 1 \le |x| \le at + 1 \tag{1.7.24}$$

of thickness 2. This wave has two fronts: the forward front $|x| = at + 1$ and the rear front $|x| = at - 1$, both propagating with the speed a.

Problem 1.7.2. Given: $a = 1$, $\varphi(x) \equiv \psi(x) \equiv 0$ at $|x| < 2$ or $|x| > 4$ (as on Fig. 1.49 for $t = 3$). Where $u(x,t) \equiv 0$ for $t = 1, 2, 3, 4, 5$?

Solution. There are three possibilities, I, II, and III (see Fig. 1.50), of the location of the sphere $|y-x|=t$ so that we would have $u(x,t)\equiv 0$.

For location I, analogously to (1.7.19), in the case of a general value of a, $4+at<|x|$. For location II, analogously to (1.7.20), $at>4+|x|$.

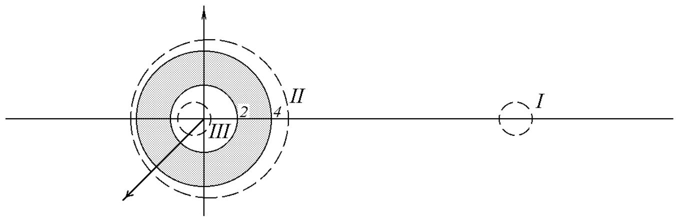


Figure 1.50:

Finally, for location III, $|x|+at<2$.

Since we are given $a=1$, we get the following: 1) At $t=1$ the sphere $|y-x|=t$ is of radius 1 and locations I and III are possible, while II is not. As a result, we see that $u(x,1)$ is supported in the layer $1\le|x|\le 5$ (see Fig. 1.51).

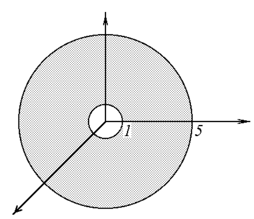


Figure 1.51:

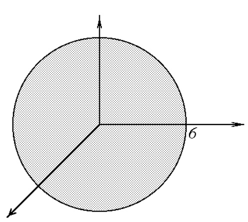


Figure 1.52:

Let us point out that this result seriously differs from Fig. 1.49 at $t=4$; 2) At $t=2$ the radius of the sphere of integration is equal to 2, therefore, only location I is possible. Therefore, the wave occupies the ball $|x|\le 6$;

3) At $t = 3$ it is also only location I that is possible (the sphere of integration is of radius 3), therefore the wave occupies the ball $|x| \leq 7$;

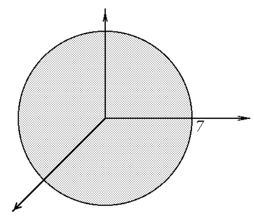


Figure 1.53:

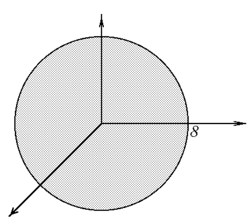


Figure 1.54:

4) The same happens for $t = 4$: the ball $|x| \leq 8$;

5) Finally, at $t = 5$, in addition to location I, location II also becomes possible (the sphere of integration is of radius 5), et cetera.
We now see that $u(x,t)$ for $t > 4$ is a spherical wave that occupies the spherical layer of thickness 8.

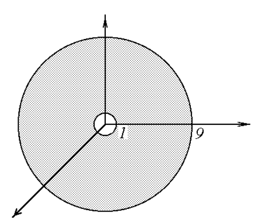


Figure 1.55:

The Huygens principle is the rule that allows to build the forward front F_t of the wave at the moment t if it is known at $t = 0$. This rule follows from the Kirchhoff formula (1.7.18) and consists of the following: Let $u\Big|_{t=0}$

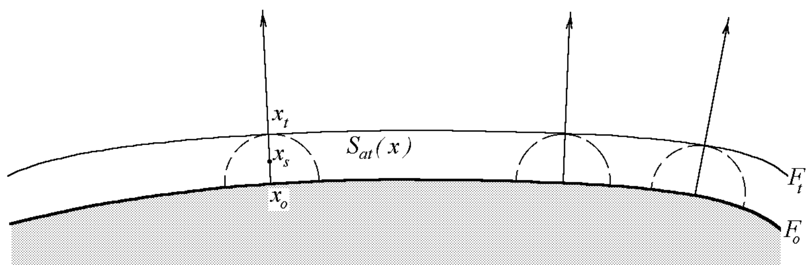


Figure 1.56:

and $\dot{u}\Big|_{t=0}$ be equal to zero outside of the dashed region on Fig. 1.56, with the smooth boundary $F_\circ$. Then $u(x,t) \equiv 0$ outside the region bounded by the surface F_t. The front F_t is constructed as follows: For each $x_\circ$, $x_\circ \in F_\circ$ we consider the sphere $S_{at}(x_\circ)$ of radius at centered at $x_\circ$; the sphere F_t is the envelope of all such spheres.

Let us assume that there is a unique point where the front F_t touches the sphere $S_{at}(x_\circ)$, and denote this point by x_t. It is easy to see that the interval $[x_\circ, x_t] \perp F_t$, if F_t is a smooth surface. One can also check that $[x_\circ, x_t] \perp F_\circ$ (problem). Consequently, the front F_t could also be constructed in the following way: From each point $x_\circ \in F_\circ$ we draw an interval $[x_\circ, x_t] \perp F_\circ$ of length at. The front F_t is then the set of all such points x_t. The intervals $[x_\circ, x_t]$ are called *the light rays.* Therefore, the Huygens principle means that the waves "propagate along the rays".

Diffusion of waves in two dimensions. The Poisson formula

The wave equation in the plane,

$$\frac{\partial^2 u}{\partial t^2}(x,t) = a^2 \Delta_2 u \equiv a^2 \Big(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_1^2} \Big), \quad x \in \mathbb{R}^2, \quad t > 0 \tag{1.7.25}$$

is obtained from (1.7.1) when $u(x_1, x_2, x_3, t)$ does not depend on x_3. This is the case when neither the initial data nor the external sources (such as the current or charges in Electrodynamics or the sound sources in Acoustics) depend on x_3. For example, the potentials of the magnetic field generated by the current in a straight wire and acoustic field of a long straight autobahn satisfy equation (1.7.25). The waves $u(x_1, x_2, t)$ that do not depend on x_3 are called cylindrical.

In this case, the initial data φ and ψ also do not depend on x_3:

$$u\Big|_{t=0} = \varphi(x); \quad \dot{u}\Big|_{t=0} = \psi(x); \quad x \in \mathbb{R}^2 \tag{1.7.26}$$

Solution to the problem (1.7.25)–(1.7.26) is given by the Poisson formula

$$u(x,t) = \frac{1}{2\pi a} \int\limits_{|y-x|<at} \frac{\psi(y)\,dy}{\sqrt{(at)^2 - |y-x|^2}} + \frac{1}{2\pi a}\frac{\partial}{\partial t}\Big(\int\limits_{|y-x|<at} \frac{\varphi(y)\,dy}{\sqrt{(at)^2 - |y-x|^2}} \Big). \tag{1.7.27}$$

These integrals are evaluated over the disc $|y-x| < at$ and not over its boundary; this is different from the Kirchhoff formula (1.7.18). Consequently, the propagation of the cylindrical waves (or simply "the plane waves") is different from that of the spherical waves.

Problem 1.7.3. Given: $a = 1$ and $\varphi(x) \equiv \psi(x) \equiv 0$ at $|x| > 1$, $x \in \mathbb{R}^2$. Where $u(x,t) \equiv 0$ for $t = 1, 2, 3, 4, 5$?

Answer. $t = 1 \quad \Rightarrow \quad |x| > 2; \quad t = 2 \quad \Rightarrow \quad |x| > 3; \quad t = 3 \quad \Rightarrow \quad |x| > 4; \quad t = 4 \quad \Rightarrow \quad |x| > 5$.

Remark 1.7.4. In this problem the cylindrical wave has the forward front but does not have the rear front, contrary to the spherical waves in two previous problems. This phenomenon is called the diffusion of waves. It turns out that for all odd $n \geq 3$ the wave equation with n spatial variables $x_1, \ldots, x_n$ has both the forward and rear fronts, while for all even $n \geq 2$ (and for $n = 1$ as well!) there is the forward front but no rear front.

Remark 1.7.5. If in the last problem the functions φ and ψ that enter (1.7.26) are bounded, then the solution converges to zero: $u(x,t) \to 0$ for $t \to \infty$, $\forall x \in \mathbb{R}^2$. This is seen from (1.7.27). (Prove this!)

Remark 1.7.6 ("The method of descent" from $n = 3$ to $n = 2$). One can obtain the Poisson formula (1.7.27) from the kirchhoff formula (1.7.18) using the independence of φ and ψ from x_3 (see [Pet91]):

1.8 General hyperbolic equations. Examples of nonhyperbolic equations

General hyperbolic equations with constant coefficients

Let us first consider the equation $Au = 0$ where A is *homogeneous* differential *operator*, that is an operator such that all the terms are the partial derivatives of the same total order m:

$$Au(x) \equiv \sum_{|\alpha|=m} a_\alpha \partial_x^\alpha u(x) = 0; \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n. \tag{1.8.1}$$

Here $\alpha = (\alpha_1, \ldots, \alpha_n)$; $|\alpha| = \alpha_1 + \cdots + \alpha_n$; $\alpha_k = 0, 1, 2, \ldots$,

$$\partial_x^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}. \tag{1.8.2}$$

Let us look for the solutions of the type of the plane waves:

$$u(x) = f(\langle \xi, x \rangle) = f(\xi_1 x_1 + \ldots + \xi_n x_n); \quad x \in \mathbb{R}^n, \tag{1.8.3}$$

where f is some function of one variable. Substituting (1.8.3) into (1.8.1) we get, analogously to (1.7.8),

$$\sum_{|\alpha|=m} a_\alpha \xi^\alpha f^{(m)}(\langle \xi, x \rangle) = 0; \quad \xi^\alpha \equiv \xi_1^{\alpha_1} \ldots \xi_n^{\alpha_n}. \tag{1.8.4}$$

From here, assuming that $f^{(m)}(z) \not\equiv 0$, we get, analogously to (1.7.9), the algebraic equation ot the characteristics (compare with (1.4.43)):

$$\tilde{A}(\xi) \equiv \sum_{|\alpha|=m} a_\alpha \xi^\alpha = 0. \tag{1.8.5}$$

This equatino defines the cone Q in $\mathbb{R}^n$, that is,

$$\xi \in Q \quad \Longrightarrow \quad t\xi \in Q; \qquad \forall t \in \mathbb{R}. \tag{1.8.6}$$

Thus, we see from (1.8.4) that the plane wave (1.8.3) for an arbitrary function f satisfies the differential equation (1.8.1) if and only if the "wave" vector ξ satisfies the algebraic equation (1.8.5).

Definition 1.8.1. 1. A vector $\xi \in \mathbb{R}^n$, $\xi \neq 0$ satisfying (1.8.5) is called a characteristic normal of the differential equation (1.8.1);

2. The hyperplane $\xi^\perp \equiv \{x \in \mathbb{R}^n : \langle \xi, x \rangle = const\}$ orthogonal to some characteristic normal is called a characteristic of the differential equation (1.8.1);

3. The hypersurface in $\mathbb{R}^n$ is called a characteristic hypersurface of equation (1.8.1) if at all its points the tangent hyperplanes are characteristics.

Definition 1.8.2. Equation (1.8.1) is called (strictly) hyperbolic in the direction of the axis Ox_1 if equation (1.8.5) on ξ_1 for any fixed

$$\xi' \equiv (\xi_2, \ldots, \xi_n) \in \mathbb{R}^{n-1} \setminus 0 \tag{1.8.7}$$

has exactly m different real roots $\xi_1^{(k)} = \lambda_k(\xi')$, $k = 1, \ldots, m$

$$\lambda_1(\xi') < \ldots < \lambda_m(\xi'). \tag{1.8.8}$$

Geometrically, condition (1.8.8) means that the cone Q has exactly m different compartments.

Example. For the wave equation (1.7.1) its order is $m = 2$ and equation (1.8.5), equivalent to (1.7.9), has 2 roots $\xi_\circ = \pm a|\xi|$; hence,

$$\lambda_1 = -a|\xi| < \lambda_2 = a|\xi|; \quad \xi \in \mathbb{R}^3 \setminus 0. \tag{1.8.9}$$

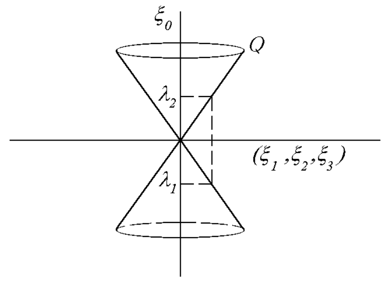


Figure 1.57:

Accordingly, the cone Q has two compartments. Therefore the wave equation is hyperbolic in the direction of the axis Ot.

Example. For the equation

$$\Big(\frac{\partial^2}{\partial t^2} - \Delta\Big)\Big(\frac{\partial^2}{\partial t^2} - 9\Delta\Big)u(x,t) = 0; \quad x \in \mathbb{R}^3, \quad t > 0 \tag{1.8.10}$$

the order is $m = 4$; the characteristic equation (1.8.5) has the form

$$(\xi_\circ^2 - |\xi|^2)(\xi_\circ^2 - 9|\xi|^2) = 0. \tag{1.8.11}$$

It has 4 roots: $\xi_\circ = \pm|\xi|$ and $\xi_\circ = \pm 3|\xi|$, and hence

$$\lambda_1 = -3|\xi| < \lambda_2 = -|\xi| < \lambda_3 = |\xi| < \lambda_4 = 3|\xi|, \quad \xi \in \mathbb{R}^3 \setminus 0. \tag{1.8.12}$$

Therefore the cone Q has four compartments (see Fig. 1.58).

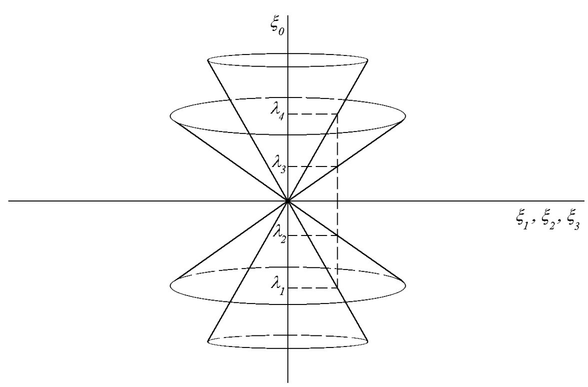


Figure 1.58:

Question. How is the strict hyperbolicity condition related with condition (1.4.11)?

Answer. For the second order equations with two independent variables they are equivalent. Indeed, in the case of equation (1.4.7), equation (1.8.5) has the form

$$A(\xi_\circ, \xi_1) \equiv a\xi_\circ^2 + 2b\xi_\circ\xi_1 + c\xi_1^2 = 0 \tag{1.8.13}$$

Under condition (1.4.11), its roots

$$\xi_\circ = \frac{b \pm \sqrt{D}}{a}\xi_1 \tag{1.8.14}$$

are real and different.

Taking in (1.8.3) a discontinuous function $f(z)$, we see that the solution to equation (1.8.1) could have a discontinuity along any given characteristic hyperplane (see Remark 1.2.1).

Remark 1.8.1. Let us take the direction of the characteristic normal ξ as a new coordinate axis, so that the plane $y_1 = 0$ coincides with $\xi^\perp$, while other coordinate axes $y_2, \dots, y_n$ are chosen arbitrarily, as long as it is a linear nondegenerate change of variables. Then, as turns out (problem!), equation (1.8.1) in the new coordinates contains the term $b_{(m,0,\dots,0)} \frac{\partial^m u}{\partial y_1^m}$ with the following coefficient (compare with (1.4.39)–(1.4.40), (1.4.42):

$$b_{(m,0,\dots,0)} = \tilde{A}(\operatorname{grad} y_1) = C\tilde{A}(\xi) \tag{1.8.15}$$

But in view of (1.8.5) this coefficient is equal to zero. Therefore equation (1.8.1) takes the form

$$\sum_{|\alpha|=m, \alpha_1 \le m-1} b_\alpha \partial_y^\alpha u(y) = 0. \tag{1.8.16}$$

This property of the vector ξ is usually taken as the definition of the characteristic normal (see [Vla79, Ole76, Pet91, TS90]). It is transparent from (1.8.16) why solutions to equation (1.8.1) could have discontinuities along the hyperplane $\xi^\perp$. This is because each term in equation (1.8.16) contains at least one derivative with respect to $y_2, \dots, y_n$. Consequently, any function of y_1 satisfies equations (1.8.16) and (1.8.1); in particular, any discontinuous function of y_1 (compare with Remark 1.4.1).

Now let us consider the equation $Au = 0$ where A is a general *nonhomogeneous* operator:

$$\sum_{|\alpha| \le m} a_\alpha \partial_x^\alpha u(x) = 0; \quad x \in \mathbb{R}^n. \tag{1.8.17}$$

We no longer know the solutions to this equation in the form of the plane waves. But, by the definition, it is accepted that the characteristic equation for (1.8.17) is (1.8.5), that is, we omit the lower order terms.

We know that solutions to equation (1.8.1) could have discontinuities along any given characteristic hyperplane. It turns out that this is also the case for equation (1.8.17) if it is strictly hyperbolic. The following example shows that, if the hyperbolicity condition is not satisfied, this may no longer be the case!

Examples of nonhyperbolic equations

Degenerate or parabolic heat equation (see Appendix):

$$\frac{\partial u}{\partial t} = a^2 \Delta u(x,t); \quad x \in \mathbb{R}^3, \quad t > 0. \tag{1.8.18}$$

For this equation the characteristic equation (1.8.5) has the form

$$0 = a^2|\xi|^2 \quad \Longleftrightarrow \quad \xi = 0. \tag{1.8.19}$$

It does not have the roots $\xi_\circ(\xi)$ for $\xi \neq 0$, hence, the heat equation is not hyperbolic in t (it is called parabplic instead). The cone Q consists of vectors parallel to the axis Ot:

$$Q = \{(\xi_\circ, 0, 0, 0)\}, \tag{1.8.20}$$

where $\xi_\circ$ is arbitrary.

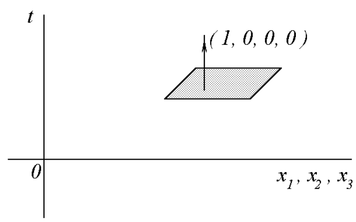


Figure 1.59:

The characteristic hyperplanes are given by equations $t = const$ and are orthogonal to the axis Ot (Fig. 1.59).

Question. Is it true that equation (1.8.18) has solutions with discontinuities along the planes $t = const$?

Answer. No, it is not true. This is because equation (1.8.18) is not hyperbolic and because we neglected the term $\frac{\partial u}{\partial t}$ when writing the characteristic equation (1.8.19).

As the matter of fact, all solutions to the heat equation are smooth. On the other hand, it has solutions that are smooth on the characteristic planes $t = const$ but not analytic.

Example. The function

$$E(x,t) = \begin{cases} \frac{1}{(2\pi t)^{3/2}} e^{-\frac{|x|^2}{2t}}, & t > 0, \\ 0, \quad t \leq 0 & x \in \mathbb{R}^3 \end{cases} \tag{1.8.21}$$

1. satisfies the heat equation (1.8.18) everywhere in $\mathbb{R}^4$, except for the point $t = 0,\ x = 0$;
2. For $t \neq 0$ or $x \neq 0$ it is smooth;
3. At $t = 0,\ x \neq 0$ it is not analytic.

Problem 1.8.1. Prove the statements 1), 2), 3) formulated above.

Let us point out that if we remove the term $\frac{\partial u}{\partial t}$ from equation (1.8.18), the resulting equation $0 = \Delta u$, obviously, has solutions discontinuous on any given characteristic hyperplane $t = const$; for example, we could take functions of the form $u(x,t) \equiv f(t)$, where $f(t)$ is piecewise continuous. Therefore, contrary to the case of nondegenerate equations, the properties of solutions to degenerate equations strongly depend on lower order terms. *Question.* Is it possible to find the region of dependence for a general equation (1.8.17) with the aid of characteristics, as in Section 1.4? In other words, is the hypothesis from Section 1.7 holds for this equation?

Answer. This hypothesis is true indeed for a strictly hyperbolic equation (just as for the wave equation from Section 1.7). See [BJS79].

Remark 1.8.2. In a certain sense, this hypothesis is also true for the heat equation (1.8.18). Namely, let us consider the Cauchy problem for equation (1.8.18) with the initial data

$$u\Big|_{t=0} = \varphi(x). \tag{1.8.22}$$

For any point $(x_\circ, t_\circ)$, $x_\circ \in \mathbb{R}^3$, $t_\circ > 0$ the characteristic hyperplane passing through it as unique and given by $t = t_\circ$. It does not intersect the hyperplane $t = 0$ at all, or instead one can think that they intersect at infinity. The region contained "inside" the intersections of the characteristics with plane $t = 0$ is the entire hyperplane $t = 0$. Indeed, this is precisely the region of dependence for the heat equation. This can be seen from the Poisson formula for the solution to the Cauchy problem (1.8.18), (1.8.22) (see [Vla84, Pet91, TS90]):

$$u(x,t) = \frac{1}{(2\pi a t)^{3/2}} \int\limits_{\mathbb{R}^3} e^{-\frac{|x-y|^2}{2at}} \varphi(y)\, dy. \tag{1.8.23}$$

This means that the speed of propagation of perturbations for the heat equation is equal to infinity.

Example. The Laplace equation (it is elliptic, see Appendix):

$$\Delta u(x) \equiv \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} = 0; \quad x \in \mathbb{R}^3 \tag{1.8.24}$$

It is obtained from the wave equation (1.7.1) and from the heat equation (1.8.18) when u does not depend on t. These are so-called stationary solutions. Physically, they describe the stationary states of (1.7.1) or the limiting temperature distributions $t \to +\infty$ for solutions of equation (1.8.18) and are of particular interest in applications.

For (1.8.24), let us find the plane wave solutions:

$$u(x) = f(\langle \xi, x \rangle) = f(\xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3), \quad x \in \mathbb{R}^3. \tag{1.8.25}$$

Substituting into (1.8.24), we obtain, as above,

$$f''(\langle \xi, x\rangle)\xi_1^2 + f''(\langle \xi, x\rangle)\xi_2^2 + f''(\langle \xi, x\rangle)\xi_3^2 = 0, \tag{1.8.26}$$

getting the characteristic equation

$$\xi_1^2 + \xi_2^2 + \xi_3^2 = 0. \tag{1.8.27}$$

It follows that

$$\xi_1 = \xi_2 = \xi_3 = 0. \tag{1.8.28}$$

Conclusion. Equation (1.8.24) is not hyperbolic (in either variable).

Question. Does this mean that the Laplace equation has no solutions similar to the plane waves?

Answer. No, it does not. Let us consider the complex solutions to (1.8.27), for example,

$$\xi_1 = i\sqrt{\xi_2^2 + \xi_3^2}; \quad (\xi_2, \xi_3) \in \mathbb{R}^2. \tag{1.8.29}$$

But then the function $f(z)$ in (1.8.25) should be determined for complex values of z. Moreover, in the first term in (1.8.26), $f''(\langle \xi, x\rangle)$ is the derivative of f in the direction of the imaginary axis, while in the second and the third – in the direction of the real axis! Therefore, to cancel f'' out of (1.8.26) and to get (1.8.27), we need $f(z)$ to have the same values of the derivatives in the directions of the real and imaginary axes at each point. But, as known from the theory of functions of complex variable, this means that $f(z)$ is analytic! Consequently, $u(x) = f(\langle \xi, x\rangle)$ is also an analytic function of real variables x_1, x_2, x_3 and can not be discontinuous. For example, $u(x) = \langle \xi, x\rangle^3 = (x_1 i\sqrt{\xi_2^2 + \xi_3^2} + \xi_2 x_2 + \xi_3 x_3)^3$.

Corollary 1.8.1. *All solutions to the Laplace equation (1.8.24) that are similar to the plane waves are analytic and, consequently, are smooth. Let us point out, though, that these are complex-valued solutions and their level surfaces are complex hypersurfaces in* C^3.

It turns out that all the solutions to the Laplace equation are analytic [Ole76, Pet91].

The shock waves and the Vavilov-Cherenkov radiation

Let us consider the electromagnetic field of a charge that is moving steadily in a certain substance. If its velocity is equal to v and it moves in the positive direction of the axis Ox_1, then its electromagnetic field is described by four potentials, each of them having the form

$$\varphi(x,t) = u(x_1 - vt, x_2, x_3) \tag{1.8.30}$$

and satisfies the wave equation (1.7.1) everywhere away from the point $(x_1 - vt, 0, 0) \equiv 0$ where the charge is located. The value of a in (1.7.1) is given by $a = c_b$, where c_b is the speed of light in the substance. Let us point out that $c_b < c$, where c is the speed of light in the vacuum, while v could be greater or smaller than c_b (but less than c).

Substituting (1.8.30) into (1.7.1), we get the equation

$$v^2 \frac{\partial^2 u}{\partial x_1^2}(x_1 - vt, x_2, x_3) = c_b^2 \Big(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} \Big), \quad x \neq x(t), \tag{1.8.31}$$

from where, denoting $x_1 - vt = y_1$, we get the following equation for $u(y_1, x_2, x_3)$:

$$(c_b^2 - v^2) \frac{\partial^2 u}{\partial y_1^2} + c_b^2 \Big(\frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} \Big) = 0, \quad (y_1, x_2, x_3) \neq 0. \tag{1.8.32}$$

Characteristic equation (1.8.5) that corresponds to (1.8.32) is given by

$$(c_b^2 - v_b^2)\xi_1^2 + c_b^2(\xi_2^2 + \xi_3^2) = 0. \tag{1.8.33}$$

From here, we see that 1) when $v < c_b$, equation (1.8.32) does not have (real) characteristics (the same is true for the Laplace equation). It is of the elliptic type (see Appendix). It turns out that all its solutions are smooth, that is, the electromagnetic field does not have singularities for $x \neq x(t)$; 2) when $v > c_b$, equation (1.8.32) is hyperbolic in y_1, and, consequently, has discontinuous solutions similar to the plane waves. For the characteristic cone Q represented by equation (1.8.33), as we know from Section 1.7, there is a corresponding "orthogonal" characteristic cone K described by the equation

$$c_b^2 y_1^2 + (c_b^2 - v^2)(x_2^2 + x_3^2) = 0 \tag{1.8.34}$$

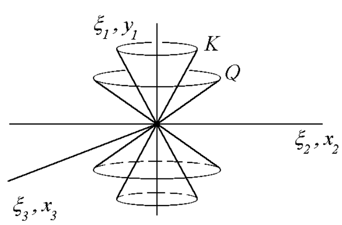


Figure 1.60:

It turns out that the considered solution u will be infinite in the part of a cone (1.8.34) where $y_1 < 0$. From (1.8.34) we get the equation of the surface of singularities of potential (1.8.30):

$$c_b^2(x_1 - vt)^2 = (v^2 - c_b^2)(x_2^2 + x_3^2), \\ x_1 - vt < 0. \tag{1.8.35}$$

For each fixed t this surface in $\mathbb{R}^3$ is a cone with a vertex at the point $x(t)$ where the charge is located (see Fig. 1.61). Along this surface the potentials and the field intensity are infinite, and the molecules of the matter at the points of the cone become exited and emit the light. This is the selebrated Vavilov-Cherenkov radiation.

The same situation arises when one is to find the sound generated by the body moving through the air: there are no pressure jumps for $v < c_{sound}$, but the jumps appear for $v > c_{sound}$. This is why behind the supersonic plane there is the shock wave located on the cone (1.8.35), that is, the pressure is discontinuous at the points of the cone (Fig. 1.62).

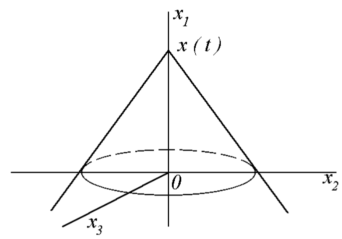


Figure 1.61:

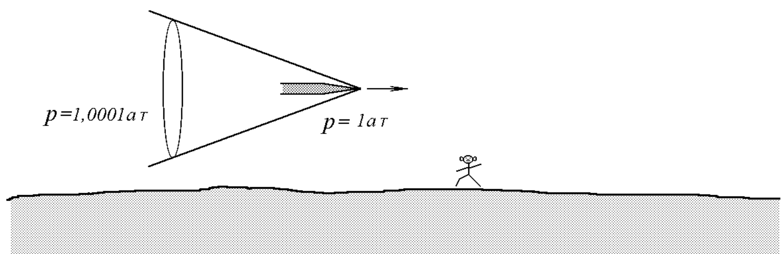


Figure 1.62:

We hear a bang when the pressure front passes our ear (see Fig. 1.62). The conic front of this shock wave is called the Mach cone.

Chapter 2

The Fourier Method

2.1 Derivation of the heat equation

We consider a straight homogeneous metal rod of length l. We choose the axis x along the rod, and let $x = 0$ be the left end of the rod, so that $x = l$ is its right end. Denote by $u(x, t)$ the temperature of the rod at a point x at the moment $t > 0$. It turns out that $u(x, t)$ satisfies the differential equation called the heat equation,

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}(x, t) + bf(x, t) \tag{2.1.1}$$

where $f(x, t)$ is the density of the external heat source at the point x at the moment t. This means that the piece $[x, x + \Delta x]$ of the rod during the time interval from t until $t + \Delta t$ receives from the outside the amout of heat equal to

$$Q_{external} = f(x, t)\Delta x \Delta t. \tag{2.1.2}$$

Let us derive (2.1.1). For this, we write down the equation of the heat balance for the piece of the rod $[x, x + \Delta x]$ during the time interval from t until $t + \Delta t$:

$$cm\Delta T = Q \tag{2.1.3}$$

Here c is the specific heat capacity of the material of the rod,

$$m(\text{mass}) = \mu \Delta x, \quad \Delta T \approx u(x, t + \Delta t) - u(x, t), \tag{2.1.4}$$

$$Q = Q_{external} + Q_l + Q_r, \tag{2.1.5}$$

where Q is the amount of heat received by the piece under consideration from the external sources, Q_l is the amount of heat received from the left (that is, through the section of the rod at the point x), while Q_r is the amount of heat received from the right (that is, through the section of the rod at the point $x + \Delta x$). See Fig. 2.1.

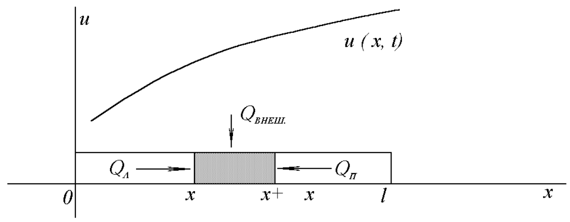


Figure 2.1:

According to the Fourier law of heating,

$$Q_l = -\lambda S \frac{\partial u}{\partial x}(x,t)\Delta t; \quad Q_r = \lambda S \frac{\partial u}{\partial x}(x+\Delta x,t)\Delta t \tag{2.1.6}$$

where λ is the heat transfer coefficient and S is the cross-section area of the rod. Roughly speaking, law (2.1.4) means that the rate of the heat transfer through the cross-section of the rod at the point x is proportional to the rate of change of the temperature, $\frac{\partial u}{\partial x}(x,t)$. Signs in (2.1.6) are chosen so that the heat would be transferred from warmer bodies to cooler ones (the Second Law of Thermodynamics). For example, for $u(x,t)$ on Fig. 2.1, $Q_l \le 0$; $Q_r \ge 0$, while $\frac{\partial u}{\partial x} \ge 0$ everywhere, hence the signs in the left- and right-hand sides of (2.1.6) coincide. For other cases (other choices of $u(x,t)$) the signs in (2.1.6) are checked in the same fasion.

Substituting (2.1.6) and (2.1.2) into (2.1.5), and then (2.1.5) and (2.1.4) into (2.1.3), we get

$$c\mu\Delta x\Big(u(x,t+\Delta t) - u(x,t)\Big) \approx$$
$$\approx f(x,t)\Delta x\Delta t + \lambda S\Big(\frac{\partial u}{\partial x}(x+\Delta x,t) - \frac{\partial u}{\partial x}(x,t)\Big)\Delta t. \tag{2.1.7}$$

From here, dividing by $\Delta x\Delta t$ and considering the limit $\Delta x \to 0$ and $\Delta t \to 0$, we get

$$c\mu\frac{\partial u}{\partial t} = \lambda S\frac{\partial^2 u}{\partial x^2} + f(x,t). \tag{2.1.8}$$

Then (2.1.1) follows.

2.2 The mixed problem for the heat equation. The operator form of the problem. The idea of the Fourier method

For the unique determination of the temperature of the rod, besides equation (2.1.1), one needs to specify the initial temperature

$$u(x,0) = \varphi(x), \quad 0 < x < l \tag{2.2.1}$$

and the boundary conditions. For example, if the ends of the rod are submerged into the melting ice, then their temperature will be equal to zero (0° C):

$$u(0,t)=0, \quad u(l,t)=0, \quad t>0. \tag{2.2.2}$$

The problem (2.1.1), (2.2.1)–(2.2.2) is called the mixed problem for the heat equation.

Let us write it in the operator form:

$$\begin{cases} \frac{d\hat{u}}{dt}=a^2A\hat{u}(t)+\hat{f}(t), \quad t>0, \\ \hat{u}(0)=\hat{\varphi}. \end{cases} \tag{2.2.3}$$

Here $A=\frac{d^2}{dx^2}$; $\hat{f}(t)\equiv f(x,t)$; $\hat{u}(t)\equiv u(x,t)$; $\hat{\varphi}=\varphi(x)$. As follows from the boundary conditions (2.2.2), $\hat{u}(t)\in C^2_\circ[0,l]$ for all $t>0$, where

$$C^2_\circ[0,l]\equiv\{u(x)\in C^2[0,l] \ : u(0)=u(l)=0\}. \tag{2.2.4}$$

Thus, the operator A is equal to $-\frac{d^2}{dx^2}$, with the domain $D(A)=c^2_\circ[0,l]$.

The idea of the Fourier method consists of the following. For $f\equiv 0$, one tries to find the solution to problem (2.2.3) in the form of the sum of particular solutions to the first equation of this problem that have the form $T(t)\cdot X(x)$.

Let us illustrate this idea on an example of the system of n ordinary differental equations with n unknown functions, also written in the vector form (2.2.3) (with $f\equiv 0$):

$$\begin{cases} \frac{d\hat{u}(t)}{dt}=A\hat{u}(t), \quad \hat{u}(t)=\Big(\hat{u}_1(t),\dots,\hat{u}_n(t)\Big)\in\mathbb{R}^n, \quad t>0, \\ \hat{u}(0)=\hat{\varphi}=(\hat{\varphi}_1,\dots,\hat{\varphi}_n)\in\mathbb{R}^n, \end{cases} \tag{2.2.5}$$

where A is a matrix of size $n\times n$.

Assume that there is a basis of the eigenvectors $e_1,\dots,e_n$ of the matrix A:

$$Ae_k=\lambda_k e_k, \quad k=1,\dots,n. \tag{2.2.6}$$

Then the solution $\hat{u}(t)$ we are looking for, as well as the initial vector $\hat{\varphi}$, could be represented as

$$\hat{u}(t)=\sum_1^n T_k(t)e_k, \quad \hat{\varphi}=\sum\varphi_k e_k. \tag{2.2.7}$$

Substituting into (2.2.5) we get

$$\sum_1^n \frac{dT_k(t)}{dt}e_k=\sum_1^n\lambda_k T_k(t)e_k, \quad \sum_1^n T_k(0)e_k=\sum_1^n\varphi_k e_k, \tag{2.2.8}$$

hence

$$\frac{dT_k(t)}{dt}=\lambda_k T_k(t), \quad t>0, \quad T_k(0)=\varphi_k. \tag{2.2.9}$$

We see that $T_k(t) = \varphi_k e^{\lambda_k t}$, and, therefore,

$$\hat{u}(t) = \sum_1^n \varphi_k e^{\lambda_k t} \cdot e_k. \tag{2.2.10}$$

In what follows we will obtain the analogs of formulas (2.2.6)–(2.2.10) for the operator $A = -\frac{d^2}{dx^2}$.

2.3 The Sturm-Liouville problem and its solution

The Sturm-Liouville problem

Let us find in $D(A) = C_0^2[0,l]$ the eigenvectors $X_1(x), \ldots, X_k(x), \ldots$ of the operator A:

$$\begin{cases} AX_k = \lambda_k X_k; \\ X_k \in D(A), \quad X_k \neq 0. \end{cases} \tag{2.3.1}$$

Relation (2.3.1) means that

$$\begin{cases} X_k''(x) = \lambda_k X_k(x), \quad 0 < x < l, \\ X_k(0) = X_k(l) = 0, \quad X_k(x) \not\equiv 0. \end{cases} \tag{2.3.2}$$

Remark 2.3.1. We will show below in Section 2.5 that in the basis $X_1, \ldots, X_k, \ldots$ of the eigenvectors of the operator A the solution to problem (2.2.3) with $f(x,t) \equiv 0$ has the form analogous to (2.2.10):

$$u(x,t) = \sum_1^\infty e^{a^2 \lambda_k t} \varphi_k X_k(x), \tag{2.3.3}$$

where φ_k are the coordinates $\hat{\varphi}$ in the basis $\{X_k\}$.

Let us point out that in view of (2.3.1) each term in series (2.3.3) satisfies the operator equation (2.2.3). Therefore any finite (partial) sum of this series also satisfies (2.2.3). The entire series (2.3.3) satisfies equation (2.2.3) if it allows termwise differentiation: once in t and twice in x, that is, if the series converges sufficiently fast.

We introduce the notation

$$\langle u, v \rangle = \int_0^l u(x)\, v(x)\, dx \quad \text{for} \quad \forall u,\, v \in L_2[0,l]. \tag{2.3.4}$$

Lemma 2.3.1. *The operator $\frac{d^2}{dx^2}$ with the domain $D(A) = C_0^2[0,l]$ is symmetric and negative, that is,*

$$\left\langle \frac{d^2u}{dx^2}, v \right\rangle = \left\langle u, \frac{d^2v}{dx^2} \right\rangle, \quad \forall u, v \in D(A), \tag{2.3.5}$$

$$\left\langle \frac{d^2u}{dx^2}, u \right\rangle u < 0, \quad \forall u \in D(A), \quad u(x) \not\equiv 0. \tag{2.3.6}$$

Proof. 1) Equality (2.3.5) means that

$$\int_0^l u''(x)\, v(x)\, dx = \int_0^l u(x)\, v''(x)\, dx. \tag{2.3.7}$$

To prove it, we integrate by parts:

$$\int_0^l u''(x)\, v(x)\, dx = u'\, v\Big|_0^l - \int_0^l u'(x)\, v'(x)\, dx, \tag{2.3.8}$$

$$\int_0^l u(x)\, v''(x)\, dx = u\, v'\Big|_0^l - \int_0^l u'(x)\, v'(x)\, dx. \tag{2.3.9}$$

The substitution into (2.3.8) yields zero since $v(0) = v(l) = 0$, while the substitution into (2.3.9) gives zero since $u(0) = u(l) = 0$. Thus relation (2.3.7) is proved;

2) When $u = v$, it follows from (2.3.8) that

$$\left\langle \frac{d^2u}{dx^2}, u \right\rangle = \int_0^l u''(x)\, u(x)\, dx = -\int_0^l \big(u'(x)\big)^2\, dx \le 0. \tag{2.3.10}$$

This proves (2.3.6). Indeed, if

$$\int_0^l \big(u'(x)\big)^2\, dx = 0, \tag{2.3.11}$$

then $u'(x) \equiv 0 \quad \Rightarrow \quad u(x) \equiv \text{const}$. But

$$u(0) = u(l) = 0 \quad \Rightarrow \quad u(x) \equiv 0, \tag{2.3.12}$$

contradicting the condition $u(x) \not\equiv 0$ in (2.3.6). □

Corollary 2.3.1. *1. All the eigenvalues of the operator $A = d^2/dx^2$ are negative. Indeed, as follows from (2.3.6),*

$$0 > \left\langle \frac{d^2X_k}{dx^2}, X_k \right\rangle = \lambda_k \langle X_k, X_k \rangle. \tag{2.3.13}$$

2. The eigenvectors X_k, X_n with different eigenvalues $\lambda_k \ne \lambda_n$ are orthogonal:

$$\int_0^l X_k(x)\, X_n(x)\, dx = 0. \tag{2.3.14}$$

Indeed, it follows from (2.3.5) that

$$\begin{aligned} &\langle AX_k, X_n \rangle = \langle X_k, AX_n \rangle \quad \Rightarrow \quad \lambda_k \langle X_k, X_n \rangle = \lambda_n \langle X_k, X_n \rangle \quad \Rightarrow \\ &(\lambda_k - \lambda_n)\, \langle X_k, X_n \rangle = 0 \quad \Rightarrow \quad \langle X_k, X_n \rangle = 0. \end{aligned}$$

Solution the Sturm-Liouville problem (2.3.1)

From equation (2.3.2) we get

$$X_k(x) = A_k\, e^{\sqrt{\lambda_k}x} + B_k e^{-\sqrt{\lambda_k}x}. \qquad (2.3.15)$$

Substituting this into the boundary conditions (2.3.2), we get

$$\begin{cases} A_k + B_k = 0, \\ A_k\, e^{\sqrt{\lambda_k}l} + B_k e^{-\sqrt{\lambda_k}l} = 0. \end{cases} \qquad (2.3.16)$$

The matrix of this system should be degenerate, or else $A_k = B_k = 0$ and $X_k(x) \equiv 0$, contradicting (2.3.2). Thus, λ_k satisfy the so-called characteristic equation

$$\det \left[\begin{cases} 1 & 1 \\ e^{\sqrt{\lambda_k}l} & e^{-\sqrt{\lambda_k}l} \end{cases} \right] = e^{-\sqrt{\lambda_k}l} - e^{\sqrt{\lambda_k}l} = 0. \qquad (2.3.17)$$

It then follows that

$$e^{-\sqrt{\lambda_k}l} = e^{\sqrt{\lambda_k}l} \quad \Rightarrow \quad e^{2\sqrt{\lambda_k}l} = 1. \qquad (2.3.18)$$

Therefore, $2\sqrt{\lambda_k}l = 2k\pi i, \quad k \in \mathbf{Z} \quad \Rightarrow$

$$\sqrt{\lambda_k} = \frac{k\pi i}{l} \quad \Rightarrow \quad \lambda_k = -\Big(\frac{k\pi}{l}\Big)^2; \qquad (2.3.19)$$

Here we may assume that $k \geq 0$. As one might have expected, $\lambda_k \leq 0$.

Thus, the eigenvalues λ_k are found. Now let us find the eigenfunctions $X_k(x)$. For this, we take into account that system (2.3.16) is degenerate. Therefore, its equations are proportional to one another, and it suffices to consider only the first one: $B_k = -A_k$. Hence, from (2.3.15) and in view of (2.3.19), we get:

$$X_k(x) = A_k(e^{\frac{k\pi i}{l}x} - e^{-\frac{k\pi i}{l}x}) = A_k 2i \sin\frac{k\pi x}{l}. \qquad (2.3.20)$$

Here we applied the Euler formula

$$e^{i\varphi} - e^{-i\varphi} = 2i\sin\varphi. \qquad (2.3.21)$$

Since the eigenfunctions X_k are defined up to a factor, we can finally set

$$X_k(x) = \sin\frac{k\pi x}{l}; \quad k = 1, 2, \dots. \qquad (2.3.22)$$

Here we can assume that $k > 0$, since for $k = 0$ we have $X_0(x) \equiv 0$.

Answer.

$$\lambda_k = -(\frac{k\pi}{l}^2), \quad X_k(x) = \sin\frac{k\pi x}{l}, \quad k = 1, 2, \dots. \qquad (2.3.23)$$

Properties of solutions to the Sturm-Liouville problem

1. $X_k(x)$ form a complete system in $L_2(0,l)$ (this property is known from the theory of the Fourier series).
2. Orthogonality:

$$\langle X_k, X_n \rangle = \int_0^l X_k(x) X_n(x)\, dx = 0 \quad \text{for} \quad k \neq n. \tag{2.3.24}$$

3. Asymptotics: $\lambda_k \sim -k^2$ for $k \to \infty$. That is, there exists a limit

$$\lim_{k\to\infty} \frac{\lambda_k}{-k^2} > 0. \tag{2.3.25}$$

Problem 2.3.1. Check directly the orthogonality property (2.3.24) for X_k.
Solution. Since $k \neq n$,

$$\int_0^l \sin\frac{k\pi x}{l} \sin\frac{n\pi x}{l}\, dx =$$

$$= \frac{1}{2}\int_0^l \left[\cos\left(\frac{k\pi x}{l} - \frac{n\pi x}{l}\right) - \cos\left(\frac{k\pi x}{l} + \frac{n\pi x}{l}\right)\right] dx =$$

$$= \frac{1}{2}\left[\frac{\sin\frac{(k-n)\pi x}{l}}{\frac{(k-n)\pi}{l}}\Bigg|_0^l - \frac{\sin\frac{(k+n)\pi x}{l}}{\frac{(k+n)\pi}{l}}\Bigg|_0^l\right] = 0.$$

Problem 2.3.2. Find the norm of X_k in $L_2(0,l)$.
Solution.

$$\begin{aligned} ||X_k||^2 \equiv \int_0^l X_k^2(x)\, dx &= \int_0^l \sin^2\frac{k\pi x}{l}\, dx = \\ = \int_0^l \frac{1-\cos\frac{2k\pi x}{l}}{2}\, dx &= \int_0^l \frac{1}{2}\, dx - \frac{\sin\frac{2k\pi x}{l}}{2\frac{2k\pi}{l}}\Bigg|_0^l = \frac{l}{2}. \end{aligned} \tag{2.3.26}$$

Problem 2.3.3. Plot the graph of $X_k(x)$.
Solution.

Problem 2.3.4. Solve the Sturm-Liouville problem, that is, find the eigenfunctions of the operator $A \equiv \frac{d^2}{dx^2}$ on the interval $[0,l]$ but at the different boundary conditions:

$$X_k(0) = X_k'(l) = 0, \tag{2.3.27}$$

$$X_k'(0) = X_k(l) = 0, \tag{2.3.28}$$

$$X_k'(0) = X_k'(l) = 0. \tag{2.3.29}$$

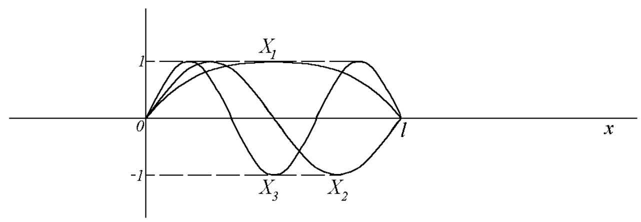


Figure 2.2:

Problem 2.3.5. For each of the boundary conditions (2.3.27)-(2.3.29) repeat the last three exercises.

Answer.

For (2.3.27), see Fig. 2.3.

$$\lambda_k = -\Big(\frac{(k+\frac{1}{2})\pi}{l}\Big)^2,$$
$$X_k(x) = \sin\frac{(k+\frac{1}{2})\pi x}{l},$$
$$k = 0, 1, 2, \ldots.$$

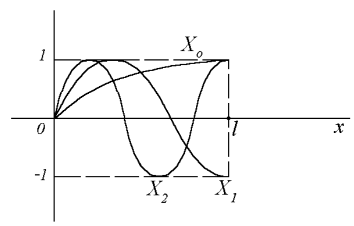


Figure 2.3:

For (2.3.28), see Fig. 2.4.

$$\lambda_k = -\Big(\frac{(k+\frac{1}{2})\pi}{l}\Big)^2,$$
$$X_k(x) = \cos\frac{(k+\frac{1}{2})\pi x}{l},$$
$$k = 0, 1, 2, \ldots.$$

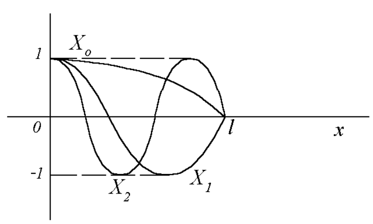


Figure 2.4:

For (2.3.29), see Fig. 2.5:

$$\lambda_k = -(\frac{k\pi}{l})^2,$$
$$X_k(x) = \cos\frac{k\pi x}{l},$$
$$k = 0, 1, 2, \ldots.$$

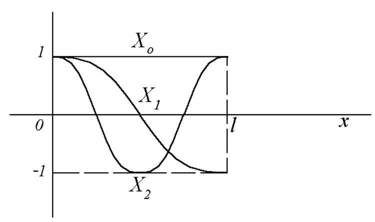


Figure 2.5:

One can also consider arbitrary boundary conditions of the form

$$\begin{aligned} \alpha_0 X_k'(0) + \beta_0 X_k(0) = 0; \\ \alpha_1 X_k'(l) + \beta_1 X_k(l) = 0, \end{aligned} \tag{2.3.30}$$

where $\alpha_{0,1}$ and $\beta_{0,1}$ are real numbers such that $\alpha_0^2+\beta_0^2 \neq 0$ and $\alpha_1^2+\beta_1^2 \neq 0$.

Problem 2.3.6. Prove that the operator $\frac{d^2}{dx^2}$ with the boundary conditions (2.3.30) is symmetric.

Remark 2.3.2. The eigenfunctions and the eigenvalues of the problems (2.3.27)-(2.3.29) possess all the properties 1, 2, 3 of problem (2.3.1) (completeness, orthogonality, the asymptotics of the eigenvalues). See [Vla84, Mik78, Pet91, SD64, TS90].

Multidimentional problem on the eigenvalues

Let us consider an arbitrary bounded region $\Omega \subset \mathbf{R}$ with a smooth boundary $\partial\Omega$ and the problem of finding the eigenfunctions of the Laplace operator in Ω with the Dirichlet boundary conditions:

$$\triangle X_k(x) = \lambda_k X_k(x), \quad x \in \Omega, \tag{2.3.31}$$

$$X_k \Big|_{\partial\Omega} = 0 \tag{2.3.32}$$

It turns out that its eigenfunctions that correspond to different λ_k are also orthogonal in $L_2(\Omega)$, while its eigenvalues λ_k are negative.

Problem 2.3.7. Prove that:

1. The Laplace operator with the boundary conditions (2.3.32) is symmetric and negative, and
2. If instead of (2.3.32) one uses the Newmann boundary conditions,

$$\frac{\partial X_k}{\partial n}\Big|_{\partial\Omega} = 0 \tag{2.3.33}$$

   (here n is the normal to $\partial\Omega$), then the Laplace operator is symmetric and non-positive; $\lambda = 0$ is the eigenvalue corresponding to the eigenfunction $X_0(x) \equiv 1$.

2.4 Decomposition over the eigenvalues of the Sturm-Liouville problem

As we already pointed out, the eigenfunctions $\sin\frac{k\pi x}{l}$, $\quad k = 1, 2, \ldots$ form a complete system in $L_2(0, l)$. Therefore they make up an orthogonal basis in $L_2(0, l)$ and, consequently, any function $\varphi(x) \in L_2(0, l)$ could be decomposed over this basis:

$$\varphi(x) = \sum_1^\infty \varphi_k X_k(x). \tag{2.4.1}$$

Let us find the formula for the coefficients φ_k. This is accomplished with the aid of the orthogonality conditions (2.3.24): we multiply (2.4.1) by $X_k(x)$ and integrate from 0 to l. Then we get

$$\int_0^l \varphi(x)X_n(x)\,dx = \sum_{k=1}^{\infty}\varphi_k \int_0^l X_k(x)X_n(x)\,dx = \varphi_n \int_0^l X_n^2(x)\,dx, \tag{2.4.2}$$

since all the terms in the summation in (2.4.2) with numbers $k \neq n$ are equal to zero! Termwise integration of the series in (2.4.2) is justified since the series in (2.4.1) converges in $L_2(0,l)$, while the scalar product in $L_2(0,l)$ is continuous in each of the two arguments.

Finally, let us take into account (2.3.26). Then from (2.4.2) we get the desired expression:

$$\varphi_n = \frac{\int_0^l \varphi(x)X_n(x)\,dx}{\int_0^l X_n^2(x)\,dx} = \frac{2}{l}\int_0^l \varphi(x)X_n(x)\,dx. \tag{2.4.3}$$

Problem 2.4.1. Find the conditions on the function $\varphi(x)$ so that the following is true:

1. Series (2.4.1) converges uniformly on the interval $[0,l]$;
2. Series (2.4.1) is termwise differentiable two times.

Solution. 1) It is sufficient (but not necessary) that

$$\sum_1^{\infty} |\varphi_k| < \infty. \tag{2.4.4}$$

For this inequality to hold, it suffices to require that

$$\varphi(x) \in C^1[0,l]; \quad \varphi(0) = \varphi(l) = 0. \tag{2.4.5}$$

Let us derive (2.4.4) from (2.4.5). Integrating by parts, we get:

$$\varphi_k = \frac{2}{l}\int_0^l \varphi(x)\sin\frac{k\pi x}{l}\,dx = \frac{2}{l}\int_0^l \varphi(x)\frac{(-\cos\frac{k\pi x}{l})'}{\frac{k\pi}{l}}\,dx = \tag{2.4.6}$$

$$\frac{2}{k\pi}\left[-\varphi(x)\cos\frac{k\pi x}{l}\Big|_0^l + \int_0^l \varphi'(x)\cos\frac{k\pi x}{l}\,dx\right]. \tag{2.4.7}$$

Above, the boundary term is equal to zero due to the boundary conditions in (2.4.5). Therefore $\varphi_k = \frac{2}{k\pi}\varphi_k'$, where $\varphi_k' = \int_0^l \varphi'(x)\cos\frac{k\pi x}{l}\,dx$. But $\{\cos\frac{k\pi x}{l}\}$

is the orthogonal system in $L_2(0,l)$, and $\int\limits_0^l \cos^2 \frac{k\pi x}{l}\,dx = \frac{l}{2}$, hence, due to the Bessel inequality,

$$\sum_1^\infty |\varphi_k'|^2 \le \frac{2}{l}\int\limits_0^l |\varphi'(x)|^2\,dx < \infty. \tag{2.4.8}$$

Therefore, from the Cauchy-Bunyakovsky inequality, we get:

$$\sum_1^\infty |\varphi_k| \le \Big(\sum_1^\infty \left|\frac{2}{k\pi}\right|^2\Big)^{\frac{1}{2}} \Big(\sum_1^\infty |\varphi_k'|^2\Big)^{\frac{1}{2}} < \infty; \tag{2.4.9}$$

2) For series (2.4.1) to be twice differentiable, it suffices that the series for $\varphi''(x)$ to be convergent uniformy in x. This, in turn, is satisfied if

$$\sum_1^\infty k^2|\varphi_k| < \infty. \tag{2.4.10}$$

For this, we require that in addition to (2.4.5) we also have

$$\varphi(x) \in C^3[0,l] \quad \text{and} \quad \varphi''(0) = \varphi''(l) = 0. \tag{2.4.11}$$

Let us derive (2.4.10) from (2.4.11), (2.4.5). For this, we remart that, due to (2.4.5), $(4.5')$

$$\begin{aligned}\varphi_k &= \frac{2l}{(k\pi)^2}\int\limits_0^l \varphi''(x)\frac{(\cos\frac{k\pi x}{l})'}{\frac{k\pi}{l}}\,dx = \\ &= \frac{2l^2}{(k\pi)^3}\left[\varphi''(x)\cos\frac{k\pi x}{l}\Big|_0^l - \int\limits_0^l \varphi'''(x)\cos\frac{k\pi x}{l}\,dx\right].\end{aligned} \tag{2.4.12}$$

The boundary term vanishes due to the boundary conditions (2.4.11). Therefore $\varphi_k = \frac{-2l^2}{(k\pi)^3}\varphi_k'''$, where $\varphi_k''' = \int\limits_0^l \varphi'''(x)\cos\frac{k\pi x}{l}\,dx$ But $\varphi''' \in L^2(0,l)$, thus (see (2.4.8)):

$$\sum_1^\infty |\varphi_k'''|^2 \le \frac{2}{l}\int\limits_0^l |\varphi'''(x)|^2\,dx < \infty, \tag{2.4.13}$$

and, similarly to $(4.6')$,

$$\sum_1^\infty k^2|\varphi_k| \le \frac{2l^2}{\pi^3}\sum_1^\infty \frac{1}{|k|}|\varphi_k'''| < \infty \tag{2.4.14}$$

Problem 2.4.2. For a function $\varphi(x) \in C^{(N)}[0,l]$ the estimates

$$|\varphi_k| \leq \frac{C}{|k|^N}, \quad k = 1, 2, \ldots \tag{2.4.15}$$

are satisfied if and only if

$$\varphi(0) = \varphi(l) = 0; \quad \varphi''(0) = \varphi''(l) = 0, \ldots; \varphi^{2n}(0) = \varphi^{2n}(l) = 0 \tag{2.4.16}$$

for all $2n \leq N - 2, \quad n = 0, 1, 2, \ldots$.

Let us point out that the boundary conditions $(4.10')$ are satisfied, in particular, for all the eigenfunctions $\sin \frac{k\pi x}{l}$. On the other hand, under condition (2.4.15) series (2.4.1) is convergent on the interval $[0,l]$ uniformly together with its derivatives up to the order $N-2$. Therefore, since the homogeneous boundary conditions $(4.10')$ are satisfied for the eigenfunctions $\sin \frac{k\pi x}{l}$, it follows that the same boundary conditions are also satisfied for the sum of series (2.4.1). This proves the necessity of conditions (2.4.16) for (2.4.15).

Remark 2.4.1. Similarly, let us consider the decomposition of the function $\varphi(x)$ over a system of eigenfunctions $X_k(x)$ that corresponds to different boundary conditions ((2.3.27)–(2.3.29)). For estimate (2.4.15) for the Fourier coefficients φ_k of this decomposition to be true, it is necessary that $\varphi(x)$ satisfies the same homogeneous boundary conditions as the eigenfunctions $X_k(x)$ and their derivatives up to the order $N-2$. When $\varphi \in C^{(N)}[0,l]$, it is easy to check that these conditions are not only necessary but also sufficient for (2.4.15).

Problem 2.4.3. Solve the previous problem for the decomposition over the eigenfunctions of the Sturm-Liouville problem with the boundary conditions (2.3.27)–(2.3.29).

Problems 2.4.1. Decompose over the system $\sin \frac{k\pi x}{l}, \quad k = 1, 2, \ldots$ the following functions:

1. $\varphi(x) \equiv 1, \quad 0 < x < l.$

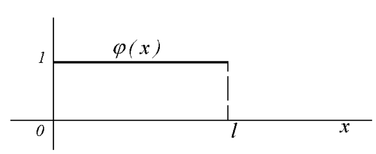


Figure 2.6:

Solution.

$$\varphi_k = \frac{2}{l} \int_0^l \sin \frac{k\pi x}{l}\, dx = -\frac{2}{l} \frac{\cos \frac{k\pi x}{l}}{\frac{k\pi}{l}} \Bigg|_0^l$$

$$= \frac{2}{k\pi}\left[1-(-1)^k\right]. \tag{2.4.17}$$

Let us point out that now condition (2.4.4) is not satisfied. This is because $\varphi(x) \equiv 1$ is not equal to zero at the ends of the interval (Fig. 2.6).

2. $\varphi(x) \equiv x, \quad 0 < x < l.$

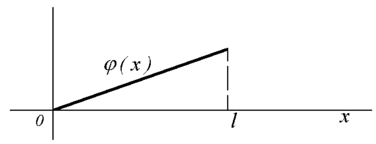


Figure 2.7:

Solution.

$$\varphi_k = \frac{2}{l}\int_0^l x \sin\frac{k\pi x}{l}\,dx = \frac{2}{l}\int_0^l x\frac{\left(-\cos\frac{k\pi x}{l}\right)'}{\frac{k\pi}{l}}\,dx = \ldots = -\frac{2}{k\pi}l(-1)^k. \tag{2.4.18}$$

Here $|\varphi_k| \sim \frac{1}{k}$ because $\varphi(l) \neq 0$ (see (2.4.15)–(4.10′) and Fig. 2.7).

3. $\varphi(x) = x(l-x)$. Is it true that $\varphi_k = O(\frac{1}{k})$, or $O(\frac{1}{k^2})$, or $O(\frac{1}{k^3}), \ldots$?

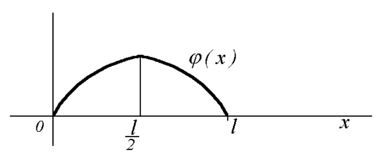


Figure 2.8:

Problem 2.4.4. Decompose the functions $\varphi(x) = 1,\, x,\, x^2,\, x(l-x)$ over the eigenfunctions of the Sturm-Liouville problem (2.3.27)–(2.3.29). In each of these cases, find the asymptotics:

$$\varphi_k = O\Big(\frac{1}{k}\Big), \quad O\Big(\frac{1}{k^2}\Big), \quad \ldots? \tag{2.4.19}$$

Hint. Use Remark 2.4.1.

2.5 The Fourier method for solving mixed problem (2.2.3) for the heat equation

So, let us solve problem (2.2.3). For simplicity, let us first assume that $f(x,t) \equiv 0$. Then the problem takes the form

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = u(l,t) = 0, \quad t > 0, \tag{2.5.1}$$

$$u(x,0) = \varphi(x), \quad 0 < x < l. \tag{2.5.2}$$

The general case $f(x,t) \not\equiv 0$ is considered in Section 2.7 below.

Let us look for the solution to problem (2.5.1)–(2.5.2) in the form of the series

$$u(x,t) = \sum_1^\infty T_k(t) X_k(x); \quad X_k(x) = \sin \frac{k\pi x}{l}. \tag{2.5.3}$$

One can write in this form any function $u(x,t)$ as long as $u(x,t) \in L_2(0,l)$ for each fixed t. The completeness property of the eigenfunctions $\sin \frac{k\pi x}{l}$ in $L_2(0,l)$ is important here. The choice of the basis $\{\sin \frac{k\pi x}{l}\}$ is determined by boundary conditions that enter (2.5.1). Namely, each term of series (2.5.3) satisfies these boundary conditions since $\sin \frac{k\pi x}{l}$ satisfy the boundary conditions in (2.3.2).

To find the solution $u(x,t)$, it remains to determine so-called "temporal" functions $T_k(t)$ (while $\sin \frac{k\pi x}{l}$ are called the spatial functions). $T_k(t)$ are found substituting series (2.5.3) into equations (2.5.1) and (2.5.2).

Remark 2.5.1. Equalities (2.5.1) for the function $u(x,t)$ from (2.5.3) are formally satisfied since they are satisfied for each term of series (2.5.3). Writing the solution in the form as in (2.5.3) explains the name of the method of "separation of variables" (and also "eigenfunctions").

Determining the temporal functions

A. We substitute series (2.5.3) into equation (2.5.1): For $t > 0$,

$$\sum_1^\infty T_k'(t) \sin \frac{k\pi x}{l} = -a^2 \sum_1^\infty T_k(t) \Big(\frac{k\pi}{l}\Big)^2 \sin \frac{k\pi x}{l}, \quad 0 < x < l. \tag{2.5.4}$$

Here we interchanged the operators of differentiation, $\frac{\partial}{\partial t}$ and $\frac{\partial^2}{\partial x^2}$ with the summation of the series. We will return below to discussing why this interchange is allowed. It is the validity of this interchange that serves the justification of the Fourier method.

In (2.5.4) we also used the identity

$$\frac{\partial^2}{\partial x^2} \sin \frac{k\pi x}{l} = -\Big(\frac{k\pi}{l}\Big)^2 \sin \frac{k\pi x}{l} \tag{2.5.5}$$

for the eigenfunctions the Sturm-Liouville problem (2.3.1)–(2.3.2). Let us point out that the boundary conditions for the Sturm-Liouville problem have already been used (see Remark 2.5.1).

Further, if the series in (2.5.4) converge in $L_2(0,l)$, then due to the orthogonality of the basis $\{\sin \frac{k\pi x}{l}\}$ we get the equality of the coefficients of these series:

$$T_k'(t) = -a^2\Big(\frac{k\pi}{l}\Big)^2 T_k(t) = -\Big(\frac{ak\pi}{l}\Big)^2 T_k(t), \quad t>0, \quad k=1,2,\dots. \tag{2.5.6}$$

This is a homogeneous linear differential equation with constant coefficients. Let us write its characteristic equation:

$$\lambda = -(\frac{ak\pi}{l})^2 \tag{2.5.7}$$

then the general solution is

$$T_k(t) = C_k e^{-(\frac{ak\pi}{l})^2 t} \tag{2.5.8}$$

Substituting into (2.5.3), we get

$$u(x,t) = \sum_1^\infty C_k e^{-(\frac{ak\pi}{l})^2 t} \sin\frac{k\pi x}{l} \tag{2.5.9}$$

B. The unknown constants C_k in (2.5.9) are found from the initial conditions (2.5.2). Namely, substituting series (2.5.3) into (2.5.2), we find:

$$\sum_1^\infty T_k(0)\sin\frac{k\pi x}{l} = \varphi(x), \quad 0<x<l. \tag{2.5.10}$$

Hence, $T_k(0)$ coincide with the Fourier coefficients of the decomposition of the function $\varphi(x)$ over the system $\sin\frac{k\pi x}{l}$ (see (2.4.3)):

$$T_k(0) = \varphi_k \equiv \frac{2}{l}\int_0^l \varphi(x)\sin\frac{k\pi x}{l}\,dx \tag{2.5.11}$$

Substituting here (2.5.8), we find

$$C_k = \varphi_k. \tag{2.5.12}$$

Thus, from (2.5.9) we get

$$u(x,t) = \sum_1^\infty \varphi_k e^{-(\frac{ak\pi}{l})^2 t}\sin\frac{k\pi x}{l}. \tag{2.5.13}$$

Checking validity of solution (2.5.13)

Is series (2.5.13) indeed the to the problem (2.5.1)–(2.5.2)?

A. For $t > 0$, series (2.5.13) converge for each $x \in [0, l]$. For example, let

$$\varphi(x) \in L_2(0, l). \tag{2.5.14}$$

Then series (2.5.10) converge in the same space $L_2(0, l)$. Indeed, from the Cauchy-Bunyakovsky inequality,

$$|\varphi_k| \le \frac{2}{l}\int_0^l |\varphi(x)|\,dx \le \frac{2}{l}\Big(\int_0^l dx\Big)^{\frac{1}{2}}\Big(\int_0^l \varphi^2(x)\,dx\Big)^{\frac{1}{2}} \le \text{const}. \tag{2.5.15}$$

Therefore, series (2.5.13) for each fixed $t > 0$ is dominated by the series

$$\text{const}\cdot\sum_1^\infty e^{-(\frac{ak\pi}{l})^2 t} = \text{const}\cdot\sum_1^\infty e^{-\varepsilon k^2}, \tag{2.5.16}$$

where $\varepsilon = (\frac{a\pi}{l})^2 t > 0$, which converge fast. Hence, according to the Weierstrass theorem, functional series (2.5.13) converges uniformly on $[0, l]$ for $\forall t > 0$ to a function that is continuous in x.

Corollary 2.5.1. *Series (*2.5.*13) satisfies the boundary conditions (*2.2.*2).*

B. Series (2.5.13) is a differentiable function in $x \in [0, l]$ for $\forall t > 0$. Indeed, according to the theorem about the termwise differentiation of a series, for $\forall t > 0$,

$$\frac{\partial u}{\partial x}(x,t) = \sum_1^\infty \varphi_k e^{-(\frac{ak\pi}{l})^2 t}\Big(-\cos\frac{k\pi x}{l}\Big)\frac{k\pi}{l}, \tag{2.5.17}$$

if the series in the right-hand side converges uniformly in x on $[0, l]$. But the last condition is satisfied since series (2.5.17) is dominated by the convergent series

$$\text{const}\cdot\frac{\pi}{l}\sum_1^\infty k e^{-\varepsilon k^2} < \infty. \tag{2.5.18}$$

C. Series (2.5.13) has derivatives in x and in t of all orders for $t > 0$ This is proved similarly to B.

Corollary 2.5.2. *All termwise differentiations of series in (*2.5.*4) are justified, hence series (*2.5.*13) satisfies the heat equation (*2.1.*1).*

Finally, for $t = 0$ series (2.5.13) satisfies the initial condition (2.2.1) in view of (2.5.10), (2.5.11) in the following sense (prove this!):

$$||u(t,x) - \varphi(x)||_{L_2(0,l)} \to 0 \quad \text{for} \quad t \to 0+. \tag{2.5.19}$$

Remark 2.5.2. Condition (2.5.14) allows that the function $\varphi(x)$ has discontinuities: For example, let $\varphi(x) \equiv 0$ for $x < \frac{l}{2}$, $u(x) \equiv 1$ for $x < \frac{l}{2}$. Then the function $u(x,0) = \varphi(x)$ will be discontinuous. At the same time, the solution $u(x,t)$ for $\forall t > 0$ will be a smooth function on $[0,l]$! It is said that, the heat equation (2.1.1) "smoothens" the initial data.

Problem 2.5.1. Find the solution to the mixed problem

$$\begin{cases} \frac{\partial u}{\partial t} = 9\frac{\partial^2 u}{\partial x^2}(x,t), \quad 0 < x < 5, \quad t > 0, \\ u(0,t) = u(5,t) = 0, \\ u(x,0) = 1. \end{cases}$$

Solution. According to (2.5.13),

$$u(x,t) = \sum_1^\infty \varphi_k e^{-(\frac{3k\pi}{5})^2 t} \sin\frac{k\pi x}{5}, \tag{2.5.20}$$

where φ_k are found using (2.5.11):

$$\varphi_k = \frac{2}{5}\int_0^5 \sin\frac{k\pi x}{5}\,dx = \frac{2}{k\pi}\big[1 - (-1)^k\big]. \tag{2.5.21}$$

Problem 2.5.2. Find the limit of solution (2.5.20) for $t \to \infty$.

Solution.

$$\begin{aligned} \lim_{t\to+\infty} u(x,t) &= \lim_{t\to+\infty}\sum_1^\infty \varphi_k e^{-(\frac{3k\pi}{5})^2 t}\sin\frac{k\pi x}{5} = \\ &= \sum_1^\infty \varphi_k \lim_{t\to+\infty} e^{-(\frac{3k\pi}{5})^2 t}\sin\frac{k\pi x}{5} = \sum_1^\infty 0 = 0. \end{aligned} \tag{2.5.22}$$

Problem 2.5.3. Justify the interchange of taking the limit with the summation in (2.5.22).

Problem 2.5.4. Find the solution to the mixed problem

$$\begin{cases} u_t(x,t) = 4u_{xx}(x,t), \quad 0 < x < 3, \quad t > 0, \\ u(0,t) = 0, \quad u_x(3,t) = 0, \\ u(x,0) = x. \end{cases} \tag{2.5.23}$$

Solution. Here the solution should be decomposed over the eigenfunctions of the Sturm-Liouville problem (2.3.27) (see Fig. 2.3):

$$u(x,t) = \sum_0^\infty T_k(t)\sin\frac{(k+\frac{1}{2})\pi x}{3}. \tag{2.5.24}$$

Substituting this series into (2.5.23), we obtain

$$\sum_0^\infty T_k'(t)\sin\frac{(k+\frac12)\pi x}{3} = 4\sum_0^\infty -\Big(\frac{(k+\frac12)\pi}{3}\Big)^2 T_k(t)\sin\frac{(k+\frac12)\pi x}{3}. \quad (2.5.25)$$

From this relation, for $\forall k = 0, 1, 2, \ldots$,

$$T_k'(t) = -\Big(\frac{2(k+\frac12)\pi}{3}\Big)^2 T_k(t) \quad \Rightarrow \quad T_k(t) = C_k e^{-\left(\frac{2(k+\frac12)\pi}{3}\right)^2 t}. \quad (2.5.26)$$

Substituting (2.5.24) into the initial condition of problem (2.5.23), we get

$$\sum_0^\infty T_k(0)\sin\frac{(k+\frac12)\pi x}{3} = x \quad \Longrightarrow$$

$$T_k(0) = \frac23\int_0^3 x\sin\frac{(k+\frac12)\pi x}{3}\,dx =$$

$$= \frac23\, x\,\frac{-\cos\frac{(k+\frac12)\pi x}{3}}{\frac{(k+\frac12)\pi}{3}}\Bigg|_0^3 + \frac23\int_0^3 \frac{\cos\frac{(k+\frac12)\pi x}{3}}{\frac{(k+\frac12)\pi}{3}}\,dx = 0 + \frac23\,\frac{\sin\frac{(k+\frac12)\pi x}{3}}{\left(\frac{(k+\frac12)\pi}{3}\right)^2}\Bigg|_0^3 =$$

$$= \frac{\frac23\sin(k+\frac12)\pi}{\left(\frac{(k+\frac12)\pi}{3}\right)^2} = \frac23\,\frac{(-1)^k\,9}{(k+\frac12)^2\pi^2} = \frac{6(-1)^k}{(k+\frac12)^2\pi^2}.$$

Since $C_k = T_k(0)$, in (2.5.26), substituting $T_k(t)$ into (2.5.24), we find

$$u(x,t) = \sum_0^\infty \frac{6(-1)^k}{(k+\frac12)^2\pi^2}\quad e^{-\frac{4\pi^2(k+\frac12)^2 t}{9}}\sin\frac{(k+\frac12)\pi x}{3}. \quad (2.5.27)$$

Problem 2.5.5. Find the solution to the mixed problem

$$\begin{cases} u_t(x,t) = 16u_{xx}(x,t), \quad 0 < x < 3, \quad t > 0,\\ u_x(0,t) = u_x(3,t) = 0,\\ u(x,0) = x.\end{cases}$$

Problem 2.5.6. Find the limit $t \to \infty$ of the solution of the previous problem.

Answer. $\lim\limits_{t\to\infty} u = \varphi_0 \equiv \frac13\int_0^3 x\,dx = \frac13\frac92 = \frac32.$

2.6 Mixed problem for the d'Alembert equation

Let us solve the mixed problem

$$u_{tt}(x,t) = a^2u_{xx}(x,t), \quad 0 < x < l, \quad t > 0, \quad (2.6.1)$$

$$u(0,t) = 0, \quad u(l,t) = 0, \quad (2.6.2)$$

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x). \quad (2.6.3)$$

Similarly to (2.2.3), it is written in the operator form as

$$\begin{cases} \frac{\partial^2 \hat{u}}{\partial t^2}(t) = a^2 A\hat{u}(t), \quad t > 0, \\ \hat{u}(0) = \varphi, \quad \frac{\partial \hat{u}}{\partial t}(0) = \psi. \end{cases} \tag{2.6.4}$$

Solution to the problem (2.6.1)–(2.6.3)

We will look for the solution in the form of series (2.5.3):

$$u(x,t) = \sum_1^\infty T_k(t) \sin \frac{k\pi x}{l}. \tag{2.6.5}$$

A. Substituting (2.6.5) into (2.6.1), we formally get

$$\sum_1^\infty T_k''(t) \sin \frac{k\pi x}{l} = a^2 \sum_1^\infty -\left(\frac{k\pi}{l}\right)^2 T_k(t) \sin \frac{k\pi x}{l}. \tag{2.6.6}$$

From here, if these series converge in $L_2(0,l)$, we find the equations for the temporal functions (compare with (2.5.6)):

$$T_k''(t) = -\left(\frac{ak\pi}{l}\right)^2 T_k(t), \quad \forall k = 1, 2, \ldots. \tag{2.6.7}$$

The general solution is (compare with (2.5.8)):

$$T_k(t) = A_k \cos \frac{ak\pi}{l} t + B_k \sin \frac{ak\pi}{l} t. \tag{2.6.8}$$

B. The unknown constants A_k and B_k are found from the initial conditions (2.6.3):

$$\begin{cases} u(x,0) = \sum_1^\infty T_k(0) \sin \frac{k\pi x}{l} = \varphi(x) \quad \Rightarrow \quad T_k(0) = \varphi_k \quad (\text{see}(2.5.11)), \\ u_t(x,0) = \sum_1^\infty T_k'(0) \sin \frac{k\pi x}{l} = \psi(x) \quad \Rightarrow \quad T_k'(0) = \psi_k \equiv \frac{2}{l} \int_0^l \psi_k(x) \sin \frac{k\pi x}{l}\, dx. \end{cases}$$

Substituting (2.6.8) in the above relation, we find:

$$\begin{aligned} T_k(0) &= A_k = \varphi_k, \\ T_k'(0) &= B_k \frac{ak\pi}{l} = \psi_k \quad \Rightarrow \quad B_k = \frac{\psi_k}{\left(\frac{ak\pi}{l}\right)}. \end{aligned} \tag{2.6.9}$$

Therefore, according to (2.6.8),

$$T_k(t) = \varphi_k \cos \frac{ak\pi}{l} t + \frac{\psi_k}{\left(\frac{ak\pi}{l}\right)} \sin \frac{ak\pi}{l} t. \tag{2.6.10}$$

Finally, substituting (2.6.9) into (2.6.5), we get

$$u(x,t) = \sum_1^\infty \left(\varphi_k \cos \frac{ak\pi}{l} t + \frac{\psi_k}{\left(\frac{ak\pi}{l}\right)} \sin \frac{ak\pi}{l} t\right) \sin \frac{k\pi x}{l} t. \tag{2.6.11}$$

Question. While deriving (2.6.7) we again interchanged differentiation in x and t with the summation. Is this justified?

Checking the validity of solution (2.6.11)

A. Does series (2.6.11) converge? It is dominated by the series

$$\text{const} \cdot \sum_{1}^{\infty} \Big(|\varphi_k| + \frac{|\psi_k|}{k} \Big). \tag{2.6.12}$$

For the convergence of this series, it suffices that

$$\begin{cases} \varphi(x) \in C^1[0,l], \quad \varphi(0) = \varphi(l) = 0; \\ \psi(x) \in C[0,l]. \end{cases} \tag{2.6.13}$$

This is proved similarly to the derivation of (2.4.4) from (2.4.5).

B. We need that series (2.6.5) could be differentiated twice in x and once in t. For this, the convergence of the following series suffices:

$$\sum_{1}^{\infty} \Big(k^2 |\psi_k| + k|\varphi_k| \Big) < \infty. \tag{2.6.14}$$

For the convergence of this series, it is sufficient that

$$\begin{cases} \varphi(x) \in C^3[0,l], \quad \varphi(0) = \varphi(l) = 0, \quad \varphi''(0) = \varphi''(l) = 0; \\ \psi(x) \in C^2[0,l], \quad \psi(0) = \psi(l) = 0. \end{cases} \tag{2.6.15}$$

This is proved analogously to the derivation of (2.4.10) from (2.4.11).

Conclusion. Series (2.6.11) is a solution of problem (2.6.1)–(2.6.3), if the functions φ and ψ satisfy conditions (2.6.15).

Remark 2.6.1. More precise (less restrictive) conditions on φ, ψ are given in terms of the Sobolev spaces (see [Mik78, Ole76], and also in Section 2.7).

Problem 2.6.1. Find the solution of the mixed problem

$$u_t = 9u_{xx}(x,t), \quad 0 < x < 4, \quad t > 0, \tag{2.6.16}$$

$$u_x(0,t) = 0, \quad u(4,t) = 0, \tag{2.6.17}$$

$$u(x,0) = 0, \quad u_t(x,0) = 16 - x^2. \tag{2.6.18}$$

Solution. One needs to decompose the solution over the eigenfunctions of the Sturm-Liouville problem (2.3.28) (see Fig. 2.4):

$$u(x,t) = \sum_{0}^{\infty} T_k(t) \cos \frac{(k+\frac{1}{2})\pi x}{4}. \tag{2.6.19}$$

Substitution into (2.6.16) gives, similarly to (2.6.7),

$$T_k''(t) = -9 \Big(\frac{(k+\frac{1}{2})\pi}{4} \Big)^2 T_k(t). \tag{2.6.20}$$

The initial conditions (2.6.18) give

$$T_k(0) = \varphi_k = 0,$$
$$T_k'(0) = \psi_k \equiv \frac{2}{4}\int_0^4 (16 - x^2)\cos\frac{(k+\frac{1}{2})\pi x}{4}\,dx = \frac{4^3(-1)^k}{(k+\frac{1}{2})^3\pi^3}. \quad (2.6.21)$$

Let us point out that here $\varphi_k \equiv 0$, while $\psi(x)$ satisfies conditions similar to (2.6.15): $\psi(x) \equiv 16 - x^2 \in C^2[0,4]$; $\psi'(0) = \psi(4) = 0$, that is, $\psi(x)$ satisfies the same homogeneous boundary conditions as the eigenfunctions $X_k(x) = \cos\frac{(k+\frac{1}{2})\pi x}{4}$ do, and $\psi_k \le C/k^3$, due to Remark 2.4.1. Therefore estimate (2.6.14) takes place.

Therefore from (2.6.20)–(2.6.21) we find, similarly to (2.6.8)–(2.6.9), that

$$T_k(t) = \frac{\psi_k \sin\frac{3(k+\frac{1}{2})\pi t}{4}}{\frac{3(k+\frac{1}{2})\pi}{4}}. \quad (2.6.22)$$

Answer.

$$u(x,t) = \sum_1^\infty \frac{256(-1)^k}{3\big((k+\frac{1}{2})\pi\big)^4}\sin\frac{3(k+\frac{1}{2})\pi t}{4}\cos\frac{(k+\frac{1}{2})\pi x}{4}. \quad (2.6.23)$$

2.7 Generalization of the Fourier method for nonhomogeneous equations

The heat equation

A. Let us consider the mixed problem for the nonhomogeneous heat equation with the homogeneous boundary conditions (nonhomogeneous boundary conditions will be the next step in developing the Fourier method):

$$\begin{cases} \frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2} + f(x,t), \quad 0 < x < l, \\ u(0,t) = 0, \quad u(l,t) = 0, \\ u(x,0) = \varphi(x). \end{cases} \quad (2.7.1)$$

Again, we look for solution of this problem in the form (2.5.3), (2.6.5):

$$u(x,t) = \sum_1^\infty T_k(t)\sin\frac{k\pi x}{l}. \quad (2.7.2)$$

The new step will be the decomposition of $f(x,t)$ over the eigenfunctions of the Sturm-Liouville problem:

$$f(x,t) = \sum_1^\infty f_k(t)\sin\frac{k\pi x}{l}; \quad f_k(t) = \frac{2}{l}\int_0^l f(x,t)\sin\frac{k\pi x}{l}\,dx. \quad (2.7.3)$$

This decomposition is possible doe to the completeness of the family of eigenfunctions $\sin\frac{k\pi x}{l}$ in the space $L_2(0,l)$ as long as $f(x,t)\in L_2(0,l)$ for each fixed $t>0$.

B. For finding the temporal functions $T_k(t)$ we substitute decompositions (2.7.2), (2.7.3) into (2.7.1):

$$\sum_1^\infty T_k'(t)\sin\frac{k\pi x}{l}=-a^2\sum_1^\infty\Big(\frac{k\pi}{l}\Big)^2T_k(t)\sin\frac{k\pi x}{l}+\sum_1^\infty f_k(t)\sin\frac{k\pi x}{l}. \quad (2.7.4)$$

From here, due to the orthogonality of the family of eigenfunctions, we get

$$T_k'(t)=-\Big(\frac{ak\pi}{l}\Big)^2T_k(t)+f_k(t),\quad t>0,\quad k=1,2,\dots. \quad (2.7.5)$$

Thus, the differential equation for the temporal functions is obtained. To determine these functions uniquely, one needs to take into account the initial condition from (2.7.1):

$$\sum_1^\infty T_k(0)\sin\frac{k\pi x}{l}=\varphi(x)\quad\Rightarrow\quad T_k(0)=\frac{2}{l}\int_0^l\varphi(x)\sin\frac{k\pi x}{l}\,dx. \quad (2.7.6)$$

Let us point out that the boundary conditions in (2.7.1) are automatically satisfied due to decomposition (2.7.2) (since they are satisfied for the eigenfunctions $\sin\frac{k\pi x}{l}$) if $T_k(t)=O\Big(\frac{1}{k^2}\Big)$.

C. Let us apply this scheme for solving problems.

Problem 2.7.1. Solve the mixed problem

$$\begin{cases} u_t=16u_{xx}+2, & 0<x<7,\quad t>0,\\ u_x(0,t)=u(7,t)=0,\\ u(x,0)=0.\end{cases} \quad (2.7.7)$$

Solution. As follows from the boundary conditions, the solution should be decomposed over the eigenfunctions of the Sturm-Liouville problem (2.3.28) (see Fig. 2.4):

$$u(x,t)=\sum_0^\infty T_k(t)\cos\frac{(k+\frac12)\pi x}{7}. \quad (2.7.8)$$

Substituting this series into (2.7.7), we get the equation similar to (2.7.5):

$$T_k'(t)=-\Big(\frac{4(k+\frac12)\pi}{7}\Big)^2T_k+f_k,\quad t>0,\quad k=1,2,\dots. \quad (2.7.9)$$

where

$$f_k\equiv\frac27\int_0^7 2\cos\frac{(k+\frac12)\pi x}{7}\,dx=\frac47\,\frac{\sin\frac{(k+\frac12)\pi x}{7}}{\big(\frac{(k+\frac12)\pi}{7}\big)}\Bigg|_0^7=4\frac{(-1)^k}{(k+\frac12)\pi}. \quad (2.7.10)$$

As follows from the initial condition of the problem,

$$T_k(0) = 0. \tag{2.7.11}$$

Let us solve the problem (2.7.9), (2.7.11). The general solution to equation (2.7.9) has the following form:

$$T_k(t) = T_k^0(t) + T_k^p(t), \tag{2.7.12}$$

where T_k^0 is the general solution to the homogeneous equation:

$$T_k^0(t) = C_k e^{-\left(\frac{4(k+\frac{1}{2})\pi}{7}\right)^2 t} \tag{2.7.13}$$

The particular solution is a constant: $T_k^p(t) = A_k$. Substituting into (2.7.9), we get

$$\begin{aligned} 0 &= -\Big(\frac{4(k+\frac{1}{2})\pi}{7}\Big)^2 A_k + f_k \qquad \Longrightarrow \\ A_k &= \frac{49 f_k}{16\big((k+\frac{1}{2})\pi\big)^2} = \frac{49(-1)^k}{4\big((k+\frac{1}{2})\pi\big)^3}. \end{aligned} \tag{2.7.14}$$

Substituting (2.7.13) and (2.7.14) into (2.7.12), we get

$$T_k(t) = C_k e^{-\left(\frac{4(k+\frac{1}{2})\pi}{7}\right)^2 t} + \frac{49(-1)^k}{4\big((k+\frac{1}{2})\pi\big)^3}. \tag{2.7.15}$$

Now we need to take into account (2.7.11):

$$0 = C_k + \frac{49(-1)^k}{4\big((k+\frac{1}{2})\pi\big)^3} \quad \Rightarrow \quad C_k = -\frac{49(-1)^k}{4\big((k+\frac{1}{2})\pi\big)^3}. \tag{2.7.16}$$

Finally, substituting (2.7.15) into (2.7.8), we get

$$u(x,t) = \sum_0^\infty (-1)^k \frac{49}{4\big((k+\frac{1}{2})\pi\big)^3} \left[-e^{-(\frac{4(k+\frac{1}{2}\pi}{7})^2 t} + 1 \right] \cos\frac{(k+\frac{1}{2})\pi x}{7}. \tag{2.7.17}$$

Problem 2.7.2. Find the limit of the solution to problem (2.7.7) as $t \to +\infty$.

Solution. Taking the limit $t \to \infty$ in each term in series (2.7.17), we get (justify!)

$$u_\infty(x) \equiv \lim_{t\to+\infty} u(x,t) = \sum_0^\infty \frac{49(-1)^k}{4\big((k+\frac{1}{2})\pi\big)^3} \cos\frac{(k+\frac{1}{2})\pi x}{7}. \tag{2.7.18}$$

Let us compute the sum of this series. For this, we notice that

$$u'_\infty(x) = -\sum_0^\infty \frac{7}{4}\frac{(-1)^k}{\left((k+\frac{1}{2})\pi\right)^2}\sin\frac{(k+\frac{1}{2})\pi x}{7}, \tag{2.7.19}$$

$$u''_\infty(x) = -\sum_0^\infty \frac{(-1)^k}{4(k+\frac{1}{2})\pi}\cos\frac{(k+\frac{1}{2})\pi x}{7} = -\frac{1}{8} \tag{2.7.20}$$

where the last equality follows from decomposition (see (2.7.10))

$$2 = \sum_0^\infty \frac{4(-1)^k}{(k+\frac{1}{2})\pi}\cos\frac{(k+\frac{1}{2})\pi x}{7}. \tag{2.7.21}$$

Integrating twice identity (2.7.20), we get

$$u_\infty(x) = \frac{1}{16}(-x^2 + C_1 x + C_2). \tag{2.7.22}$$

To find C_1 and C_2, we notice that due to (2.7.18) and (2.7.19)

$$u_\infty(7) = 0, \quad u'_\infty(0) = 0. \tag{2.7.23}$$

Substituting here (2.7.22), we find $C_1 = 0, \quad C_2 = 49$:

$$u_\infty(x) = \frac{1}{16}(49 - x^2). \tag{2.7.24}$$

Remark 2.7.1. We could obtain u_∞ directly from (2.7.7), without using the nonstationary solution (2.7.17), substituting u_t by 0 and solving the problem

$$\begin{cases} 0 = 16u''_\infty(x) + 2, \quad 0 < x < 7, \\ u'_\infty(0) = 0, \quad u_\infty(7) = 0. \end{cases} \tag{2.7.25}$$

Remark 2.7.2. The fundamental property of the heat equation is that under stationary external conditions (that is, when the nonhomogeneous terms of the equation and the boundary conditions do not depend explicitly on t), the solution $u(x,t)$ stabilizes as $t \to +\infty$:

$$u(x,t) \to u_\infty(x), \quad t \to +\infty. \tag{2.7.26}$$

The limit function $u_\infty(x)$ is the solution to the corresponding stationary problem.

Problem 2.7.3. Find the limit as $t \to +\infty$ of the solution to the mixed problem

$$\begin{cases} u_t = 25u_{xx}(x,t) + 3x^2, \quad 0 < x < 6, \\ u(0,t) = 0, \quad u'(6,t) = 1, \\ u(x,0) = \sin x. \end{cases} \tag{2.7.27}$$

Solution. As we said above, we get from (2.7.27), (7.25′) the boundary value problem for $u_\infty(x) = \lim_{t\to\infty} u(x,t)$

$$\begin{cases} 0 = 25u''_\infty(x) + 3x^2, \quad 0 < x < 6, \\ u_\infty(0) = 0; \quad u'_\infty(6) = 1. \end{cases}$$

Integrating this equation, we get $u_\infty(x) = -\frac{x^4}{100} + C_1 x + C_2$. From the boundary conditions we get $C_2 = 0, \quad -\frac{6^3}{25} + C_1 = 1$.

Answer. $u_\infty(x) = -\frac{x^4}{100} + \frac{241}{25}x$.

The wave equation

Let us consider the nonhomogeneous wave equation.

Problem 2.7.4. Solve the mixed problem ($\omega > 0$):

$$\begin{cases} u_{tt}(x,t) = 25u_{xx} + \sin(\omega t)x(3-x), \quad 0 < x < 3, \quad t > 0, \\ u(0,t) = u(3,t) = 0, \\ u(x,0) = 0, \quad u_t(x,0) = 0. \end{cases} \tag{2.7.28}$$

Solution. A. In view of the boundary conditions in (2.7.28), we are looking for the solution u in form of the decomposition over the eigenfunctions of the Sturm-Liouville problem (2.3.1):

$$u(x,t) = \sum_1^\infty T_k(t) \sin\frac{k\pi x}{3}. \tag{2.7.29}$$

For this, the function $\sin(\omega t)x(3-x)$ in equation (2.7.28) is also decomposed in the series over the system $\sin\frac{k\pi x}{3}$:

$$\sin(\omega t)x(3-x) = \sin(\omega t)\sum_1^\infty g_k \sin\frac{k\pi x}{3}, \tag{2.7.30}$$

where $g_k = \frac{2}{3} \int\limits_0^3 x(3-x)\sin\frac{k\pi x}{3} \quad dx = \frac{36}{(k\pi)^3}\big(1-(-1)^k\big)$.

B. Finding the temporal functions $T_k(t)$. Substituting decomposition (2.7.29) and (2.7.30) into equation (2.7.28) and using the orthogonality of the family $\sin\frac{k\pi x}{3}$, we get, similarly to (2.6.7),

$$T''_k(t) = -\Big(\frac{5k\pi}{3}\Big)^2 T_k(t) + g_k \sin(\omega t). \tag{2.7.31}$$

Substituting series (2.7.29) into the initial conditions (2.7.28), we get, clearly,

$$T_k(0) = 0, \quad T'_k(0) = 0. \tag{2.7.32}$$

The Cauchy problem (2.7.31)–(2.7.32) uniquely determines the temporal functions $T_k(t)$.

It is known that the general solution to equation (2.7.31) has the form

$$T_k(t) = T_k^0(t) + T_k^p(t), \tag{2.7.33}$$

where $T_k^0(t)$ is the general solution of the corresponding homogeneous equation

$$T_k^0(t) = A_k \cos\big(\frac{5k\pi}{3}t\big) + B_k \sin\big(\frac{5k\pi}{3}t\big), \tag{2.7.34}$$

while $T_k^p(t)$ is a particular solution to the nonhomogeneous equation (2.7.31).

When finding the particular solution, one needs to distinguish two cases; namely, the resonant and non-resonant cases.

1. Non-resonant case: For all $k \in N$,

$$\omega \neq \frac{5k\pi}{3}. \tag{2.7.35}$$

Then $T_k^p(t)$ are to be looked for in the form

$$T_k^p(t) = A \sin(\omega t). \tag{2.7.36}$$

Substitution into (2.7.31) gives

$$-\omega^2 A \sin(\omega t) = -\Big(\frac{5k\pi}{3}\Big)^2 A \sin(\omega t) + g_k \sin(\omega t), \tag{2.7.37}$$

from where, in view of (2.7.35),

$$A = \frac{g_k}{\big(\frac{5k\pi}{3}\big)^2 - \omega^2}. \tag{2.7.38}$$

Then (2.7.33) takes the form

$$T_k(t) = A_k \cos\big(\frac{5k\pi}{3}t\big) + B_k \sin\big(\frac{5k\pi}{3}t\big) + \frac{g_k \sin(\omega t)}{\big(\frac{5k\pi}{3}\big)^2 - \omega^2}. \tag{2.7.39}$$

Finally, the initial conditions (2.7.32) yield

$$A_k = 0, \quad B_k \frac{5k\pi}{3} + \frac{g_k \omega}{(\frac{5k\pi}{3})^2 - \omega^2} = 0 \quad \Rightarrow \quad B_k = -\frac{g_k \omega}{\frac{5k\pi}{3}\big((\frac{5k\pi}{3})^2 - \omega^2\big)}. \tag{2.7.40}$$

Thus, in the case when (2.7.35) is satisfied for all $k = 1, 2, \dots,$ we have

$$u(x,t) = \sum_1^\infty \frac{g_k}{\big(\frac{5k\pi}{3}^2\big) - \omega^2}\Big(-\frac{\omega}{\big(\frac{5k\pi}{3}\big)} \sin\big(\frac{5k\pi}{3}t\big) + \sin(\omega t)\Big) \sin\frac{k\pi x}{3}. \tag{2.7.41}$$

2. Resonant case: For some $m \in \mathbb{N}$,

$$\omega = \frac{5m\pi}{3}. \tag{2.7.42}$$

In this case,

$$T_m^p(t) = t(A\cos\omega t + B\sin\omega t). \tag{2.7.43}$$

Taking $k = m$ and substituting into (2.7.31), we get

$$\begin{aligned}
&2\big(-A\omega\sin(\omega t) + B\omega\cos(\omega t)\big) + t\big(-A\omega^2\cos(\omega t) - B\omega^2\sin(\omega t)\big) = \\
&= -\big(\frac{5m\pi}{3}\big)^2 t\big(A\cos(\omega t) + B\sin(\omega t)\big) + g_m\sin(\omega t).
\end{aligned} \tag{2.7.44}$$

Here in the left-hand side we used the Leibniz formula for computing

$$\frac{d^2}{dt^2}\Big[t\big(A\cos(\omega t) + B\sin(\omega t)\big)\Big]. \tag{2.7.45}$$

Taking into account (2.7.42) and collecting the terms in (2.7.44), we get

$$2\big(-A\omega\sin(\omega t) + B\omega\cos(\omega t)\big) + g_m\sin(\omega t). \tag{2.7.46}$$

We compare the coefficients at $\cos(\omega t)$ and $\sin(\omega t)$ on the left and on the right:

$$2B\omega = 0, \qquad -2A\omega = g_m. \tag{2.7.47}$$

Since $\omega > 0$,

$$B = 0, \qquad A = -\frac{g_m}{2\omega}. \tag{2.7.48}$$

Thus,

$$T_m^p(t) = -t\frac{g_m}{2\omega}\cos(\omega t). \tag{2.7.49}$$

Therefore

$$T_m(t) = A_m\cos\big(\frac{5k\pi}{3}t\big) + B_m\sin\big(\frac{5k\pi}{3}t\big) - t\frac{g_m}{2\omega}\cos(\omega t). \tag{2.7.50}$$

Substituting into the initial conditions (2.7.32), we get

$$A_m = 0; \quad B_m\frac{5m\pi}{3} - \frac{g_m}{2\omega} = 0 \Longrightarrow B_m = \frac{3g_m}{10m\pi\omega}. \tag{2.7.51}$$

Therefore,

$$T_m^p(t) = \frac{3g_m}{10m\pi\omega}\sin\big(\frac{5k\pi}{3}t\big) - t\frac{g_m}{3}\cos(\omega t). \tag{2.7.52}$$

Thus, if for some $m \in \mathbb{N}$ condition (2.7.42) is satisfied, we get (compare with (2.7.41)):

$$\begin{aligned}
u(x,t) = &\sum_{\substack{k=1\\k\neq m}}^{\infty} \frac{g_k}{(\frac{5k\pi}{3})^2 - \omega^2}\Big(-\frac{\omega}{(\frac{5k\pi}{3})}\sin\big(\frac{5k\pi}{3}t\big) + \sin(\omega t)\Big)\sin\frac{k\pi x}{3} + \\
&+\Big(\frac{3g_m}{10m\pi\omega}\sin\big(\frac{5m\pi}{3}t\big) - t\frac{g_m}{2\omega}\cos(\omega t)\Big)\sin\frac{m\pi x}{3}.
\end{aligned} \tag{2.7.53}$$

Remark 2.7.3. In the non-resonant case, all the terms in series (2.7.41) are bounded functions of x, t, while in the resonant case (2.7.42) one of the terms in (2.7.53) is unbounded when $t \to +\infty$. Therefore, for large t, the solution will be represented mainly by the last term in (2.7.53). For very large t, the solution will become considerably large. If it were a string, it will get torn. As the matter of fact, when the solution becomes large, it is no longer described by the linear wave equation, and formula (2.7.53) is no longer valid.

Problem 2.7.5. Find the solution to the mixed problem

$$\begin{cases} u_{tt}(x,t) = 16u_{xx} + \sin \frac{7\pi x}{10}, & 0 < x < 5, \quad t > 0, \\ u(0,t) = 0, \quad u_x(5,t) = 0, \\ u(0,x) = 0, \quad u_t(0,x) = 0. \end{cases}$$

2.8 Generalization of the Fourier method to the case of non-homogeneous boundary conditions

Up to now, we were using the Fourier method only for problems with homogeneous boundary conditions It turns out that the problem with non-homogeneous boundary conditions is easily reduced to a problem with homogeneous boundary conditions.

The heat equation

Problem 2.8.1. Find the solution to the mixed problem

$$\begin{cases} u_t = 9u_{xx}, & 0 < x < 4, \quad t > 0, \\ u(0,t) = f(t), \quad u(4,t) = g(t), \\ u(x,0) = 0. \end{cases} \tag{2.8.1}$$

Solution. Let us find an auxiliary function $v(x,t)$ that satisfies the given boundary conditions:

$$v(0,t) = f(t), \quad v(4,t) = g(t), \quad t > 0. \tag{2.8.2}$$

Such a function can easily be found, for example, using a linear interpolation

$$v(x,t) = \frac{x}{4} g(t) + \frac{4-x}{4} f(t). \tag{2.8.3}$$

Denote $w = u - w$. Then w satisfies the homogeneous boundary conditions

$$w(0,t) = 0, \quad w(4,t) = 0, \quad t > 0. \tag{2.8.4}$$

Question. What equation and boundary conditions does the function w satisfy?

Answer. We substitute $u = w + v$ into (2.8.1), then

$$\begin{cases} w_t + v_t = 9(w_{xx} + v_{xx}), \\ w(x,0) + v(x,0) = 0. \end{cases} \tag{2.8.5}$$

Then

$$\begin{cases} w_t = 9w_{xx} + 9(v_{xx} - v_t), \\ w(x,0) = -v(x,0). \end{cases} \tag{2.8.6}$$

Thus, unlike u, w satisfies the nonhomogeneous heat equation! But the boundary conditions (2.8.4) are now homogeneous, hence w cound be found using the method of Section 2.7; then $u = w + v$ is the solution to problem (2.8.1). Thus, we sent the nonhomogeneity from the boundary conditions into the differential equation (2.8.1) and into the initial condition.

The wave equation

Problem 2.8.2. Solve the mixed problem

$$\begin{cases} u_{tt} = 16u_{xx}, \quad 0 < x < 5, \quad t > 0, \\ u(0,t) = 0, \quad u_x(5,t) = \sin(\omega t), \\ u(x,0) = 0 \quad u_t(x,0) = 0. \end{cases} \tag{2.8.7}$$

Solution. A. The auxiliary function

$$v(x,t) = x\sin(\omega t) \tag{2.8.8}$$

satisfies the required boundary conditions. For $w \equiv u - v$ we have

$$\begin{cases} u_{tt} = 16u_{xx} + \omega^2 x\sin(\omega t), \quad 0 < x < 5, \quad t > 0, \\ w(0,t) = 0, \quad w_x(5,t) = 0, \\ w(x,0) = -v(x,0) = 0 \quad w_t(x,0) = -v_t(x,0) = -x\omega. \end{cases} \tag{2.8.9}$$

B. Following the method of Section 2.7, we are looking for w in the form

$$w(x,t) = \sum_0^\infty T_k(t) \sin\frac{(k+\frac{1}{2})\pi x}{5}. \tag{2.8.10}$$

For this, we expand the right-hand side of equation (2.8.9):

$$\omega^2 x\sin(\omega t) = \omega^2 \sin(\omega t) \cdot \sum_0^\infty x_k \sin\frac{(k+\frac{1}{2})\pi x}{5}, \tag{2.8.11}$$

where

$$x_k \;=\; \frac{2}{5}\int_0^5 x\sin\frac{(k+\frac{1}{2})\pi x}{5}\,dx = -\frac{2}{5}\frac{5}{(k+\frac{1}{2})\pi}\int_0 5x\,d\cos\frac{(k+\frac{1}{2})\pi x}{5} =$$

$$= -\frac{5}{(k+\frac{1}{2})\pi}\Big[x\cos\frac{(k+\frac{1}{2})\pi x}{5}\Big|_0^5 - \int_0^5 \cos\frac{(k+\frac{1}{2})\pi x}{5}\,dx\Big] =$$

$$= \frac{2\cdot 5}{\big((k+\frac{1}{2})\pi\big)^2}\sin\frac{(k+\frac{1}{2})\pi x}{5}\Big|_0^5 = \frac{10}{\big((k+\frac{1}{2})\pi\big)^2}\cdot(-1)^k. \tag{2.8.12}$$

B. Substituting (2.8.10)–(2.8.12) в equation (2.8.9), we find the equations for the temporal functions $T_k(t)$:

$$T_k''(t) = -16\Big(\frac{(k+\frac{1}{2})\pi}{5}\Big)^2 T_k(t) + \omega 2\sin(\omega t)\cdot x_k, \quad k = 0,1,2,\ldots. \tag{2.8.13}$$

From initial conditions (2.8.9) we find $T_k(0) = 0$, and, taking into account (2.8.12), we get:

$$T_k'(0) = -\omega\frac{2}{5}\int_0^5 x\sin\frac{(k+\frac{1}{2})\pi x}{5}\,dx = -\omega\frac{10\cdot(-1)^k}{\big((k+\frac{1}{2})\pi\big)^2}. \tag{2.8.14}$$

The problem (2.8.13)–(2.8.14) could be solved in the same way as in Section 2.7, Again, two cases are possible: resonant and non-resonant.

Complete the solution of problem (2.8.1) and write the answer.

Remark 2.8.1. For problems (2.8.9) the condition analogous to (2.6.15) is not satisfied. Still, the new function $w(x,t)$ satisfies the initial and boundary conditions in the usual sense. It is only the first equation (2.8.9) that is satisfied in the sense of the theory of distributions (see Chapter 3).

Problem 2.8.3. Find the resonance condition in problem (2.8.7).

Answer. For some $m = 0,1,2,\ldots$,

$$\omega = \frac{4(m+\frac{1}{2})\pi x}{5} \tag{2.8.15}$$

2.9 The Fourier method for the Laplace equation

Boundary value problems in a rectangle

A. Let us consider the boundary value problem in the rectangle $\Omega = [0,a]\times[0,b]$:

$$\begin{cases} \triangle u(x,y) \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad 0 < x < a, \quad 0 < y < b; \\ u(0,y) = 0, \quad u(a,y) = 0; \\ u(x,0) = f(x), \quad u(x,b) = g(x). \end{cases} \tag{2.9.1}$$

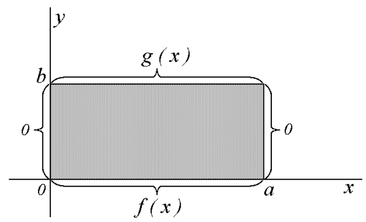


Figure 2.9:

This is the boundary value problem, or *the Dirichlet problem*, when the function u is given at the boundary of the considered region.

Solution. Problem (2.9.1) can be solved by the method of Section 2.7, where the role of the variable t is now played by the variable y, as could be seen from comparing problems (2.9.1) and (2.7.1). We are looking for the solution in the following form:

$$u(x,y)=\sum_{1}^{\infty} Y_k(y)\sin\frac{k\pi x}{a}. \tag{2.9.2}$$

Then the boundary conditions at $x=0$ and $x=a$ in (2.9.1) are automatically satisfied. We substitute (2.9.2) into equation (2.9.1). This gives equations for $Y_k(y)$:

$$-\left(\frac{k\pi}{a}\right)^2 Y_k(y)+Y_k''(y)=0, \quad 0<y<b. \tag{2.9.3}$$

Substitution into boundary conditions (2.9.1) at $y=0$ and $y=b$ yields

$$\left\{\begin{array}{l} Y_k(0)=f_k\equiv\frac{2}{a}\int\limits_0^a f(x)\sin\frac{k\pi x}{a}\,dx,\\ Y_k(b)=g_k\equiv\frac{2}{a}\int\limits_0^a g(x)\sin\frac{k\pi x}{a}\,dx. \end{array}\right. \tag{2.9.4}$$

The general solution to equation (2.9.3) has the form

$$Y_k(y)=A_k e^{\frac{k\pi}{a}y}+B_k e^{-\frac{k\pi}{a}y}. \tag{2.9.5}$$

The constants A_k and B_k are found from boundary conditions (2.9.4):

$$A_k+B_k=f_k, \quad A_k e^{\frac{k\pi}{a}b}+B_k e^{-\frac{k\pi}{a}b}=g_k. \tag{2.9.6}$$

Solving this system, we find

$$\left\{\begin{array}{l} A_k=\frac{1}{e^{\frac{k\pi}{a}b}-e^{-\frac{k\pi}{a}b}}\left(g_k-f_k e^{-\frac{k\pi}{a}b}\right),\\ B_k=\frac{1}{e^{\frac{k\pi}{a}b}-e^{-\frac{k\pi}{a}b}}\left(f_k e^{\frac{k\pi}{a}b}-g_k\right). \end{array}\right. \tag{2.9.7}$$

Thus, the solution of problem (2.9.1) is given by (2.9.2), (2.9.5), (2.9.7).

Let us check the validity of solution (2.9.2). We need to justify the possibility of the termwise differentiation of series (2.9.2). If $f(x)$ and $g(x)$ are summable functions, then $f(x)$ and $g(x)$ are bounded:

$$|f_k|\le\frac{2}{a}\int\limits_0^a|f(x)|\,dx, \quad |g_k|\le\frac{2}{a}\int\limits_0^a|g(x)|\,dx.$$

But then from (2.9.7) we see that

$$|A_k| \le \frac{c}{e^{\frac{k\pi}{a}b}}, \quad |B_k| \le \text{const.}$$

Therefore, it follows from (2.9.5) that

$$|Y_k(y)| \le ce^{-\frac{k\pi}{a}(b-y)} + ce^{-\frac{k\pi}{a}y}.$$

As a consequence, for $0 < \varepsilon < y < b - \varepsilon$

$$|Y_k(y)| \le ce^{-\frac{k\pi}{a}\varepsilon},$$

and series (2.9.2) for these values of y is dominated by the convergent series

$$\sum_1^\infty ce^{-\frac{k\pi}{a}\varepsilon}.$$

It is easy to see that the derivatives of the second order in x and in y of series (2.9.2) are dominated by the series

$$\sum_1^\infty ck^2 e^{-\frac{k\pi}{a}\varepsilon}, \tag{2.9.8}$$

which is also convergent. In the same way one proceeds with the derivatives of any order in x and y.

Conclusion. Solution of the Dirichlet problem (2.9.1) is a smooth function inside the rectangle Ω. Let us assume that, as in (2.4.5), $f(x), g(x) \in C_0^2[0, a]$. Then, analogously to (2.4.4), $f_k, g_k = O(\frac{1}{k^2})$ and, consequently, $|Y_k(y)| \le \frac{c}{k^2}, \quad y \in [0, b]$. Therefore, series (2.9.2) converges uniformly in the rectangle $\Omega = [0, a] \times [0, b]$, and its sum is a function that is continuous in this rectangle and satisfies boundary conditions in (2.9.1).

B. More general boundary value problem of the Dirichlet type in the rectangle

$$\begin{cases} \triangle u(x, y) = 0, \quad 0 < x < a, \quad 0 < y < b; \\ u(0, y) = \varphi(y), \quad u(a, y) = \psi(y); \\ u(x, 0) = f(x), \quad u(x, b) = g(x). \end{cases} \tag{2.9.9}$$

could be solved by decomposing the solution u into two terms:

$$u = u_1 + u_2. \tag{2.9.10}$$

Here u_1 solves problem (2.9.1), while u_2 solves the problem

$$\begin{cases} \triangle u_2 = 0, \quad 0 < x < a, \quad 0 < y < b; \\ u_2(0, y) = \varphi(y), \quad u_2(a, y) = \psi(y); \\ u_2(x, 0) = 0, \quad u_2(x, b) = 0. \end{cases} \tag{2.9.11}$$

This problem takes the same form as (2.9.1) if one interchanges x and y. Therefore u_2 should be tried in the form (compare with (2.9.2)):

$$u_2(x,y) = \sum_1^\infty X_k(x) \sin \frac{k\pi y}{b}. \tag{2.9.12}$$

If $f, \quad g \in C_0^2[0,a]$, while $\varphi, \quad \psi \in C_0^2[0,b]$, then, according to what we said above, u_1 and u_2, and, consequently, u are continuous functions in Ω and satisfies the required boundary conditions.

In the general case, for the continuity of $u(x,y)$ in Ω, the following compatibility conditions are obviously required:

$$f(0) = \varphi(0), \quad \varphi(b) = g(0), \quad g(a) = \psi(b), \quad \psi(0) = f(a). \tag{2.9.13}$$

Problem 2.9.1. Prove that problem (2.9.9) has a solution that is continuous in Ω if $f, \quad g \in C^2[0,a]$, $\varphi, \quad \psi \in C^2[0,b]$, and the compatibility condition (2.9.13) is satisfied.

Hint. Try to find the solution to equation $\triangle v = 0$ in Ω that coincide with the boundary values given by functions f, g, φ and ψ at the corner points of the region Ω. Then the difference $u - v$ could be found using decomposition (2.9.10) described above.

C. Consider the nonhomogeneous Laplace equation (the Poisson equation).

Problem 2.9.2. Solve the boundary value problem

$$\begin{cases} \triangle u(x,y) = x^2 y, \quad 0 < x < a, \quad 0 < y < b; \\ u(0,y) = 0, \quad u(a,y) = 0; \\ u(x,0) = 0, \quad \frac{\partial u}{\partial y}(x,b) = 0. \end{cases} \tag{2.9.14}$$

Let us point out that here at $x = 0, \quad x = a$ and $y = 0$ one has the boundary value of the Dirichlet type, while at $y = b$ one has the boundary value of the Neumann type (that is, one is given the derivative of the solution in the normal direction).

Solution. Homogeneous boundary conditions at $x = 0$, and $x = a$ allow to write the solution in the form of the series over the eigenfunctions of the corresponding Sturm-Liouville problem:

$$u(x,y) = \sum_1^\infty Y_k(x) \sin \frac{k\pi y}{a}. \tag{2.9.15}$$

We also decompose over these functions the right-hand side:

$$x^2 y = y \cdot \sum_1^\infty g_k \sin \frac{k\pi y}{a}, \quad g_k = \frac{2}{a} \int_0^a x^2 \sin \frac{k\pi y}{a}\, dx. \tag{2.9.16}$$

Substituting these decompositions into (2.9.14), we get for $\forall k = 1, 2, \ldots$

$$-\left(\frac{k\pi}{a}\right)^2 Y_k(y) + Y_k''(y) = yg_k, \quad 0 < y < b; \quad Y_k(0) = Y_k'(b) = 0. \tag{2.9.17}$$

Then

$$Y_k(y) = A_k e^{\frac{k\pi x}{a}} + B_k e^{-\frac{k\pi x}{a}} + \frac{yg_k}{-\left(\frac{k\pi}{a}\right)^2}. \tag{2.9.18}$$

The constants A_k and B_k could be found after substituting this solution into the boundary conditions in (2.9.17):

$$\begin{cases} A_k + B_k = 0, \\ A_k \frac{k\pi}{a} e^{\frac{k\pi}{a}b} + B_k(-\frac{k\pi}{a})e^{-\frac{k\pi}{a}b} + \frac{g_k}{-(\frac{k\pi}{a})^2} = 0. \end{cases} \tag{2.9.19}$$

Solving this algebraic system, we find A_k and B_k.

Answer. solution is given by formulas (2.9.15), (2.9.18).

Boundary value problems in the annulus and in the disc

A. Let us solve the boundary value problem of the Dirichlet type in the annulus between the circles of radii r_1 and r_2:

$$\begin{cases} \triangle u(x,y) = 0, \quad r_1^2 < x^2 + y^2 < r_2^2; \\ u|_{x^2+y^2=r_1^2} = f_1(\varphi), \quad 0 < \varphi < 2\pi; \\ u|_{x^2+y^2=r_2^2} = f_2(\varphi), \quad 0 < \varphi < 2\pi. \end{cases} \tag{2.9.20}$$

Here f_1 and f_2 are given funcions of the angular variable φ.

Solution. Let us convert to polar coordinates r, φ:

$$r = \sqrt{x^2 + y^2}; \quad \tan\varphi = y/x. \tag{2.9.21}$$

Problem 2.9.3. Prove that in these coordinates problem (2.9.20) takes the form

$$\begin{cases} \triangle u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} = 0, \quad r_1 < r < r_2; \\ u|_{r=r_1} = f_1(\varphi), \quad 0 \le \varphi \le 2\pi; \\ u|_{r=r_2} = f_2(\varphi). \end{cases} \tag{2.9.22}$$

This is a problem in a rectangle $[0, 2\pi] \times [r_1, r_2]$ (Fig. 2.10). The boundary conditions are given in the lower and the upper sides of the rectangle.

Question. Are there the boundary conditions at the left and right sides of the rectangle?

Answer. Yes, it is the periodicity condition in the variable φ

$$\begin{cases} u(0, r) = u(2\pi, r), \\ \frac{\partial u}{\partial \varphi}(0, r) = \frac{\partial u}{\partial \varphi}(2\pi, r). \end{cases} \tag{2.9.23}$$

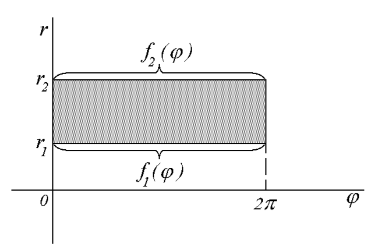


Figure 2.10:

that follows from representing the points of the (x, y)-plane in the polar coordinates $(0, r)$ and $(2\pi, r)$. Analogous periodicity conditions in φ also hold for $\frac{\partial u}{\partial r}, \frac{\partial^2 u}{\partial r^2}, \frac{\partial^3 u}{\partial r^3}, \dots$ — for all the derivatives of u in r and φ.

Problem 2.9.4. Show that conditions (2.9.23) together with the equation (2.9.22) guarantee the periodicity in φ of all the derivatives of u in r and φ, if $u(\varphi, r)$ is a smooth function in the rectangle $[0, 2\pi] \times [r_1, r_2]$.

The Sturm-Liouville problem that corresponds to homogeneous boundary conditions (2.9.23) has the form

$$\begin{cases} \frac{\partial^2 \Phi(\varphi)}{\partial \varphi^2} = \lambda \Phi(\varphi), \quad 0 < \varphi < 2\pi, \\ \Phi(0) = \Phi(2\pi), \quad \Phi'(0) = \Phi'(2\pi). \end{cases} \tag{2.9.24}$$

Solving this problem, we find:

$$\lambda_k = -k^2, \quad k = 0, 1, 2, \dots \quad \Phi_k(\varphi) = A_k \cos(k\varphi) + B_k \sin(k\varphi). \tag{2.9.25}$$

Therefore, for each $k \neq 0$ there are two linearly independent eigenfunctions: $\cos(k\varphi)$ and $\sin(k\varphi)$, while for $k = 0$ there is only one eigenfunction: $\Phi_0(\varphi) \equiv 1$.

As is known from the Fourier series theory, these eigenfunctions form a complete system in $L_2(0, 2\pi)$, and are mutually orthogonal:

$$\begin{cases} \int\limits_0^{2\pi} \Phi_0^2(\varphi)\, d\varphi = \int\limits_0^{2\pi} d\varphi = 2\pi, \\ \int\limits_0^{2\pi} \cos^2(k\varphi)\, d\varphi = \int\limits_0^{2\pi} \sin^2(k\varphi)\, d\varphi = \pi, \quad k \neq 0. \end{cases} \tag{2.9.26}$$

The Fourier method for problem (2.9.22) in the annulus consists in finding the solution in the form of a series over the eigenfunctions of problem (2.9.24):

$$u(\varphi, r) = \sum_0^\infty R_k(r) \cos(k\varphi) + \sum_1^\infty S_k(r) \sin^2(k\varphi). \tag{2.9.27}$$

Substituting this series into equation (2.9.22), we get the following equations for the "radial" functions $R_k(r)$:

$$R_k'' + \frac{1}{r} R_k' + \frac{1}{r^2} R_k(-k^2) = 0, \quad r_1 < r < r_2, \quad k = 0, 1, 2, \dots \tag{2.9.28}$$

and the same equations for S_k:

$$S_k'' + \frac{1}{r} S_k' + \frac{1}{r^2} S_k(-k^2) = 0, \quad r_1 < r < r_2, \quad k = 0, 1, 2, \dots \tag{2.9.29}$$

Let us solve the e*radial* equations (2.9.28)—(2.9.29). These are the Euler equations (see [Phi79]). Substituting into (2.9.28) $R_k = r^2$, we get

$$\lambda(\lambda - 1) r^{\lambda - 2} + \lambda r^{\lambda - 2} - k^2 r^{\lambda - 2} = 0, \tag{2.9.30}$$

and the following characteristic equation follows:

$$\lambda^2 - k^2 = 0 \Leftrightarrow \lambda = \pm k. \tag{2.9.31}$$

If $k \neq 0$, then the roots are simple, and the general solution (2.9.28) has the form:

$$R_k = A_k r^k + B_k r^{-k}, \quad k = 1, 2, 3, \dots \tag{2.9.32}$$

Analogously, for (2.9.29):

$$S_k = C_k r^k + D_k r^{-k}, \quad k = 1, 2, 3, \dots \tag{2.9.33}$$

For $k = 0$, the root of the equation $\lambda = 0$ has multiplicity 2, hence

$$\Rightarrow \quad R_0 = A_0 + B_0 \ln r. \tag{2.9.34}$$

Substituting (2.9.32)—(2.9.34) into (2.9.27), we get the general solution of a homogeneous Laplace equation in the annulus:

$$\begin{aligned} u(\varphi, r) = A_0 \quad &+ \quad B_0 \ln r + \sum_1^\infty (A_k r^k + B_k r^{-k}) \cos(k\varphi) \\ &+ \quad \sum_1^\infty (C_k r^k + D_k r^{-k}) \sin(k\varphi). \end{aligned} \tag{2.9.35}$$

Remark 2.9.1. This is a general form of a harmonic function in the annulus.

Arbitrary constants in (2.9.35) are found from boundary conditions (2.9.22):

$$\left\{ \begin{array}{l} A_0 + B_0 \ln r_1 + \sum_1^\infty (A_k r_1^k + B_k r_1^{-k}) \cos(k\varphi) + \\ + \sum_1^\infty (C_k r_1^k + D_k r_1^{-k}) \sin(k\varphi) = f_1(\varphi), 0 \le \varphi \le 2\pi; \\ A_0 + B_0 \ln r_2 + \sum_1^\infty (A_k r_2^k + B_k r_2^{-k}) \cos(k\varphi) + \\ + \sum_1^\infty (C_k r_2^k + D_k r_2^{-k}) \sin(k\varphi) = f_2(\varphi), 0 \le \varphi \le 2\pi. \end{array} \right. \tag{2.9.36}$$

Taking now into account the orthogonality of the eigenfunctions of the Sturm-Liouville problem (2.9.24) and relations (2.9.26), we get

$$\left\{ \begin{array}{l} A_0 + B_0 \ln r_1 = \frac{1}{2\pi} \int\limits_0^{2\pi} f_1(\varphi)\, d\varphi, \\ A_0 + B_0 \ln r_2 = \frac{1}{2\pi} \int\limits_0^{2\pi} f_2(\varphi)\, d\varphi, \end{array} \right. \tag{2.9.37}$$

and, similarly, for $\forall k = 1, 2, 3, \dots$,

$$\left\{ \begin{array}{l} A_k r_1^k + B_k r_1^{-k} = \frac{1}{\pi} \int\limits_0^{2\pi} f_1(\varphi) \cos(k\varphi)\, d\varphi, \\ A_k r_2^k + B_k r_2^{-k} = \frac{1}{\pi} \int\limits_0^{2\pi} f_2(\varphi) \cos(k\varphi)\, d\varphi; \end{array} \right. \tag{2.9.38}$$

$$\begin{cases} C_k r_1^k + D_k r_1^{-k} = \frac{1}{\pi} \int\limits_0^{2\pi} f_1(\varphi) \sin(k\varphi)\, d\varphi, \\ C_k r_2^k + D_k r_2^{-k} = \frac{1}{\pi} \int\limits_0^{2\pi} f_2(\varphi) \sin(k\varphi)\, d\varphi. \end{cases} \tag{2.9.39}$$

We find A and B from system (2.9.37) and A_k, B_k from (2.9.38). C_k and D_k are found from (2.9.39). Problem (2.9.20) is solved.

Problem 2.9.5. Prove that solution (2.9.35) of problem (2.9.20) is infinitely differentiable in the interior of the annulus.

Problem 2.9.6. Solve the Dirichlet problem in the annulus

$$\begin{cases} \triangle u(x, y) = 0, \quad 4 < x^2 + y^2 < 9; \\ u|_{x^2+y^2=4} = x, \quad u|_{x^2+y^2=9} = y. \end{cases} \tag{2.9.40}$$

Solution. Here $r_1 = 2, \quad r_2 = 3$, so that

$$f_1(\varphi) = 2\cos\varphi, f_2(\varphi) = 3\sin\varphi. \tag{2.9.41}$$

Therefore the right-hand sides in (2.9.37) are equal to zero and $A_0 = B_0 = 0$. Analogously, the right-hand sides of systems (2.9.38) and (2.9.39) are equal to zero for all $k \neq 1$. Therefore,

$$A_k = B_k = 0, \quad C_k = D_k = 0 \quad \text{for} \quad k \neq 1. \tag{2.9.42}$$

Hence, series (2.9.35) contains only two terms:

$$u = (A_1 r + B_1 r^{-1}) \cos\varphi + (C_1 r + D_1 r^{-1}) \sin\varphi. \tag{2.9.43}$$

The remaining coefficients are obtained from the systems of equations

$$\begin{cases} A_1 2 + B_1 \frac{1}{2} = 2, \\ A_1 3 + B_1 \frac{1}{3} = 0, \end{cases} \qquad \begin{cases} C_1 2 + D_1 \frac{1}{2} = 0, \\ C_1 3 + D_1 \frac{1}{3} = 3, \end{cases} \tag{2.9.44}$$

that are derived directly from (2.9.41). Namely, (2.9.44) is obtained by substituting (2.9.41) into (2.9.36) and comparing the Fourier coefficients in both sides of the relations, instead of evaluating integrals in (2.9.38)–(2.9.39). From (2.9.44) we find

$$\begin{cases} B_1 = \frac{36}{5}, \\ A_1 = -\frac{4}{5}, \end{cases} \qquad \begin{cases} D_1 = -\frac{36}{5}, \\ C_1 = \frac{9}{5}. \end{cases} \tag{2.9.45}$$

Finally, from (2.9.43) and (2.9.45) we find the answer:

$$u = \Big(-\frac{4}{5} r + \frac{36}{5} r^{-1}\Big) \cos\varphi + \Big(\frac{9}{5} r - \frac{36}{5} r^{-1}\Big) \sin\varphi. \tag{2.9.46}$$

B. Now let us consider the Dirichlet problem in the disc of radius R^2:

$$\begin{cases} \triangle u(x, y) = 0, \quad x^2 + y^2 < R^2; \\ u|_{x^2+y^2=R^2} = f(\varphi), \quad 0 < \varphi < 2\pi. \end{cases} \tag{2.9.47}$$

Solution of this problem also has the form (2.9.35), since the disc $x^2+y^2<R^2$ contains the (degenerate) annulus $0<x^2+y^2<R^2$. But the disc also contains the point $(0,0)$, where the solution has to be finite:

$$|u(0,0)|<\infty. \tag{2.9.48}$$

It can be shown [TS90] that (2.9.48) holds if and only if all the terms that have the singularity at $(0,0)$ of the form $\ln r$ and r^{-k} are absent from (2.9.35). This means that $B_0=B_k=D_k=0, \quad k=1,2,3,\ldots$. Thus, (2.9.35) takes the form

$$u(x,y)=A_0+\sum_1^\infty r^k(A_k\cos(k\varphi)+C_k\sin(k\varphi)). \tag{2.9.49}$$

This is the analog of the Taylor series for a harmonic function in a disc.

The coefficients of series (2.9.49) are found from the boundary condition of problem (2.9.47).

Problem 2.9.7. Let us solve the Dirichlet problem in the disc

$$\begin{cases} \triangle u(x,y)=0, \quad x^2+y^2<4; \\ u|_{x^2+y^2=4}=x^2. \end{cases} \tag{2.9.50}$$

Solution. We are looking for the solution u in the form as in (2.9.49). The substitution of this series into the boundary condition gives:

$$A_0+\sum_1^\infty 2^k(A_k\cos(k\varphi)+C_k\sin(k\varphi))=2+2\cos(2\varphi), \tag{2.9.51}$$

since

$$x^2\big|_{r=2}=(2\cos\varphi)^2=4\cos^2\varphi=4\frac{1+\cos(2\varphi)}{2}=2+2\cos(2\varphi). \tag{2.9.52}$$

Comparing the Fourier coefficients in the left- and right-hand sides of (2.9.51), we see that all A_k and C_k with $k\neq 0$ and $k\neq 2$ are equal to zero, and

$$\begin{cases} A_0=2 \\ 4A_2=2, \quad C_2=0. \end{cases} \tag{2.9.53}$$

This gives $A_2=\frac{1}{2}$, and formula (2.9.49) yields the answer:

$$u=2+r^2\frac{1}{2}\cos(2\varphi)=2+\frac{r^2}{2}(\cos^2\varphi-\sin^2\varphi)=2+\frac{x^2-y^2}{2}. \tag{2.9.54}$$

Problem 2.9.8. Solve the Dirichlet problem in the annulus

$$\begin{cases} \triangle u(x,y)=x^2, \quad 9<x^2+y^2<16; \\ u|_{x^2+y^2=9}=0, \quad u|_{x^2+y^2=16}=0. \end{cases} \tag{2.9.55}$$

Hint. Here the solution that we are looking for and the right-hand side of the equation are to be decomposed into the series of form (2.9.27). Equations for the radial functions R_k and S_k will be the nonhomogeneous Euler equations.

Problem 2.9.9. Solve the Neumann problem in the disc

$$\begin{cases} \triangle u(x,y) = 0, \quad x^2 + y^2 < 9; \\ \frac{\partial u}{\partial n}|_{x^2+y^2=9} = y. \end{cases} \tag{2.9.56}$$

Hint. Solution is to be looked for in the form of series (2.9.49); moreover, in the polar coordinates one has $\frac{\partial u}{\partial n} = \frac{\partial u}{\partial r}$.

Conclusion. The heat equation, the wave equation, and the Laplace equation possess different properties. As follows from the results of Chapter 2, solutions of the homogeneous Laplace equation and the heat equation are smooth inside the regions where they are considered, even if the boundary values are discontinuous. At the same time, solutions of the homogeneous wave equation could be discontinuous if, for example, the initial data are discontinuous functions.

Chapter 3

Distributions on the line. Green's functions for boundary problems on an interval

3.1 Different ways of defining a function

Continuous functions $u(x) \in C(\mathbb{R})$ could be defined using the following three ways.

1. The continuous function could be uniquely defined by its values

$$\{u(x)\}, \quad x \in \mathbb{R}. \tag{3.1.1}$$

2. It could be defined using its Fourier coefficients (if it is 2π-periodic):

$$u(x) = \sum_{k \in Z} u_k e^{ikx}. \tag{3.1.2}$$

Here

$$u_k = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikx}\, u(x)\, dx. \tag{3.1.3}$$

The sequence

$$\{u_k \quad k \in \mathbb{Z}\}, \tag{3.1.4}$$

uniquely defines a continuous (periodic) function by formula (3.1.2).

3. Let us introduce the space of so-called test functions. Let $C_0^\infty(\mathbb{R})$ be the space of smooth functions with the compact support, that is,

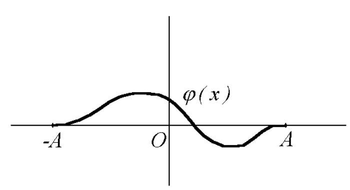


Figure 3.1:

1. $\varphi(x) \in C^\infty(\mathbb{R})$;
2. $\varphi(x) \equiv 0$ for $|x| \geq A,$, where $A \geq 0$ depends on φ (Fig. 3.1).

For any continuous function $u(x)$ define the scalar product with $\varphi \in C^\infty(\mathbb{R})$:

$$\langle u(x), \varphi(x) \rangle \equiv \int\limits_{-\infty}^{\infty} u(x)\varphi(x)\,dx. \tag{3.1.5}$$

This integral converges, since $\varphi(x) \equiv 0$ for $|x| \geq A$:

$$\langle u(x), \varphi(x) \rangle = \int\limits_{-A}^{A} u(x)\varphi(x)\,dx. \tag{3.1.6}$$

For a particular continuous function $u(x)$ consider the set of values

$$\{\langle u, \varphi \rangle, \varphi \in C_0^\infty(\mathbb{R})\}. \tag{3.1.7}$$

Question. Is the function $u(x)$ uniquely defined by this set of values?

Answer. Yes. (Prove this!)

Question. Can the formula be written for restoring the continuous function $u(x)$ from the set of values (3.1.7)?

Answer. Yes:

$$u(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int\limits_{-\infty}^{\infty} \varphi(\frac{x-y}{\varepsilon})u(y)\,dx = \lim_{\varepsilon \to 0} \langle \varphi_\varepsilon^x(y), u(y) \rangle. \tag{3.1.8}$$

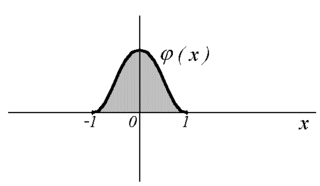


Figure 3.2:

Here $\varphi_\varepsilon^x(y) = \frac{1}{\varepsilon}\varphi(\frac{x-y}{\varepsilon}) \in C_0^\infty(\mathbb{R})$. The function $\varphi \in C_0^\infty(\mathbb{R})$ satisfies the following conditions:

1. $\varphi(y) \equiv 0$ for $|y| \geq 1$,
2. $\int\limits_{-1}^{1} \varphi(y)\,dy = 1$ (Fig. 3.2).

Let us prove (3.1.8). We change the variable of integration: $\frac{x-y}{\varepsilon} = z$. Then (3.1.8) takes the form

$$u(x) = \lim_{\varepsilon \to 0} \int\limits_{-1}^{1} \varphi(z)u(x - \varepsilon z)\,dz. \tag{3.1.9}$$

In this form, this formula follows from the continuity of u at the point x:

$$\lim_{\varepsilon\to 0}\int_{-1}^{1}\varphi(z)u(x-\varepsilon z)\,dz=\int_{-1}^{1}\varphi(z)\lim_{\varepsilon\to 0}u(x-\varepsilon z)\,dz=$$
$$=\int_{-1}^{1}\varphi(z)u(x)\,dz=u(x)\int_{-1}^{1}\varphi(z)\,dz=u(x).$$

Question. What is the essential difference of the three ways of defining a function $u(x)$ which we described above?

Answer.

1. The set of numbers $\{u(x), x\in\mathbb{R}\}$ could be more or less arbitrary: at any finite set of points $x_k\in\mathbb{R}$ the values $u(x_k)$ could be arbitrary.

2. The set of numbers $\{u_k, k\in Z\}$ could be arbitrary, as long as $|u_k|$ decay for $|k|\to\infty$ so that, for example,

$$\sum_{-\infty}^{\infty}|u_k|\infty. \tag{3.1.10}$$

3. The values $\{\langle u,\varphi\rangle, \varphi\in C_0^\infty(\mathbb{R})\}$ are not arbitrary: As could be seen from (3.1.5), they are connected by algebraic relations

$$\langle u,\varphi_1+\varphi_2\rangle=\langle u,\varphi_1\rangle+\langle u,\varphi_2\rangle, \tag{3.1.11}$$

for all $\varphi_1,\varphi_2\in C_0^\infty(\mathbb{R})$

Conclusion. For the abstract set of numbers $\{l_\varphi\varphi\in C_0^\infty(\mathbb{R})\}$, to correspond to some function $u(x)\in C(\mathbb{R})$ such that

$$l_\varphi=\langle u,\varphi\rangle, \forall\varphi\in C_0^\infty(\mathbb{R}), \tag{3.1.12}$$

it is necessary that this set satisfies the compatibility conditions (3.1.11):

$$l_{\varphi_1+\varphi_2}=l_{\varphi_1}+l_{\varphi_2},\quad \forall\varphi_1,\varphi_2\in C_0^\infty(\mathbb{R}) \tag{3.1.13}$$

Definition 3.1.1. The convergence $\varphi_n\xrightarrow[n\to\infty]{C_0^\infty}\varphi$ means the following:

1. $\varphi_n(x)$ converges to $\varphi(x)$ uniformly in $x\in\mathbb{R}$, and the same is true for derivatives of any order: $\forall k=0,1,2,\dots$

$$\varphi_n^{(k)}(x)\overset{\longrightarrow}{\longrightarrow}\varphi^{(k)}(x),\quad x\in\mathbb{R}\quad\text{as}\quad n\to\infty. \tag{3.1.14}$$

2. All φ_n "have the common support" $[-A, A]$: $\exists\, A \geq 0, \quad \forall n = 1, 2, 3, \ldots$

$$\varphi_n(x) \equiv 0 \quad \text{for} \quad |x| \geq A. \tag{3.1.15}$$

Question. Are the compatibility conditions (3.1.13) sufficient for the existence of a function $u(x) \in C(\mathbb{R})$ that correspond to the set $\{l_\varphi\}$ in the sense of identity (3.1.12)?

Answer. No, they are not. One also needs the continuity conditions:

$$\langle u, \varphi_n \rangle \underset{n\to\infty}{\longrightarrow} \langle u, \varphi \rangle, \quad \text{if} \quad \varphi_n \xrightarrow{C_0^\infty} \varphi \quad \text{for} \quad n \to \infty. \tag{3.1.16}$$

Under the conditions (3.1.14)—(3.1.15), convergence (3.1.16) follows from the theorem about the interchanging the integration and taking the limit:

$$\langle u, \varphi_n \rangle \equiv \int_{-A}^{A} u(x)\varphi_n(x)\, dx \underset{n\to\infty}{\longrightarrow} \int_{-A}^{A} u(x)\varphi_(x)\, dx = \langle u, \varphi \rangle, \tag{3.1.17}$$

since $u(x)\varphi_n(x) \overset{\longrightarrow}{\longrightarrow} u(x)\varphi(x)$ as $x \in [-A, A], \quad n \to \infty$.

Thus, for the existence of a function $u(x) \in C(\mathbb{R})$ that corresponds to the set $\{l_\varphi\}$ in the sense of (3.1.12), the following condition is necessary (besides (3.1.13):

$$l_{\varphi_n} \to l_\varphi \quad \text{as} \quad \varphi_n \xrightarrow{C_0^\infty} \varphi \tag{3.1.18}$$

Question. Are conditions (3.1.13) and (3.1.18) suffice for the existence of $u(x) \in C(\mathbb{R})$ that gives representation (3.1.12)?

Answer. No.

Problem 3.1.1. Give the example of the set $\{l_\varphi\}$ that satisfies conditions (3.1.13) and (3.1.18), but such that there is no corresponding function $u(x) \in C(\mathbb{R})$ (see [Vla81, p. 97]).

Answer.

$$l_\varphi = \varphi(0), \quad \forall \varphi \in C_0^\infty(\mathbb{R}) \tag{3.1.19}$$

Conclusion. The set of values $\{l_\varphi\}$ defines a function $u(x) \in C(\mathbb{R})$ satisfying identity (3.1.12) uniquely only if such function $u(x)$ exists, although it may not exist for the set $\{l_\varphi\}$. Conditions (3.1.13), (3.1.18) are necessary for the existence of a continuous function $U(x)$, but not sufficient.

3.2 Distributions

Definition 3.2.1. A distribution is a set $l = \{l_\varphi, \quad \varphi \in C_0^\infty(\mathbb{R})\}$ that satisfies conditions (3.1.13) and (3.1.18).

For brevity, we denote

$$D = D(\mathbb{R}) = C_0^\infty(\mathbb{R}). \tag{3.2.1}$$

Remark 3.2.1. From the Functional Analysis point of view, The set $\{l_\varphi\}$ satisfying conditions (3.1.13), (3.1.18) (that is, a distribution) is a continuous linear functional on $D(\mathbb{R})$, that is, the element of the dual space $D'(\mathbb{R})$:

$$l = \{l_\varphi\} \in D'(\mathbb{R}); \quad l(\varphi) \equiv l_\varphi, \quad \forall \varphi \in D(\mathbb{R}) \tag{3.2.2}$$

Thus, $D'(\mathbb{R})$ is the space of distributions.

Notation 3.2.1. For a distribution $\{l_\varphi\}$ the value $\{l_\varphi$ of the distribution l on a test function φ will be denoted by both $l(\varphi)$ and $\langle l(x), \varphi(x)\rangle$, and will be called the scalar product of the distribution $l(x)$ with the test function $\varphi(x)$:

$$l_\varphi = l(\varphi) = \langle l, \varphi\rangle = \langle l(x), \varphi(x)\rangle \tag{3.2.3}$$

Let us point out that $l(x)$ is not the value of the function l at the point x, but merely a symbol.

Example. Distribution (3.1.19) is called the Dirac δ-function:

$$\delta_\varphi = \delta(\varphi) = \langle \delta(x), \varphi(x)\rangle = \varphi(0), \quad \forall \varphi \in D(\mathbb{R}). \tag{3.2.4}$$

Remark 3.2.2. Formula (3.1.5) assigns a distribution to each continuous function $u(x) \in C(\mathbb{R})$:

$$u(x) \mapsto \{\langle u(x), \varphi(x)\rangle, \quad \varphi \in D(\mathbb{R})\}. \tag{3.2.5}$$

According to (3.1.8), this mapping is injective:

$$C(\mathbb{R}) \subset D'(\mathbb{R}). \tag{3.2.6}$$

But not every distribution could be represented by (3.1.5) using some continuous function (for example, $\delta(x)$ could not be represented in this way).

Let us consider examples of distributions.

$$1.\ \langle \delta_k(x), \varphi(x)\rangle = \varphi^{(k)}(0), \quad \forall \varphi \in D(\mathbb{R}) \tag{3.2.7}$$

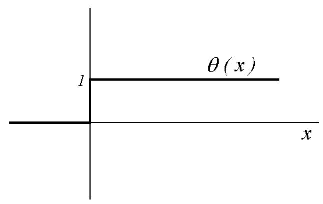


Figure 3.3:

Check that $\delta_k(x) \in D'(\mathbb{R})$ (that is, check conditions (3.1.13), (3.1.16)). 2. The Heaviside function (Fig. 3.3).

$$\Theta(x) = \begin{cases} 1, \ x > 0, \\ 0, \ x < 0 : \end{cases} \tag{3.2.8}$$

$$\langle \Theta(x), \varphi(x) \rangle \equiv \int_{-\infty}^{\infty} \Theta(x), \varphi(x)\, dx = \int_{0}^{\infty} \varphi(x)\, dx. \tag{3.2.9}$$

Check that $\Theta(x) \in D'(\mathbb{R})$.

3.3 Operations on distributions

Addition of distributions

Let us first notice that for the continuous functions $u_1(x)$, $u_2(x)$ and their sum $u_1(x) + u_2(x)$ we have, due to (3.1.15),

$$\langle u_1(x) + u_2(x), \varphi(x) \rangle = \langle u_1, \varphi \rangle + \langle u_2, \varphi \rangle, \quad \forall \varphi \in D(\mathbb{R}) \tag{3.3.1}$$

Definition 3.3.1. For $l^1, l^2 \in D'(\mathbb{R})$ we set

$$(l^1 + l^2)_\varphi = l^1_\varphi + l^2_\varphi, \quad \forall \varphi \in D(\mathbb{R}). \tag{3.3.2}$$

Remark 3.3.1. Under such definition, the addition of continuous functions $u_1(x)$, $u_2(x)$ coincides with the addition of the corresponding distributions. This is seen from (3.3.1) and (3.3.2).

Problem 3.3.1. Check that $l^1 + l^2 \in D'(\mathbb{R})$.

Multiplication of a distribution by a scalar

Let us notice that for $u(x) \in C(\mathbb{R})$ and $\alpha \in \mathbb{R}$, due to (3.1.5), we have

$$\langle \alpha u(x), \varphi \rangle = \alpha \langle u, \varphi \rangle, \quad \forall \varphi \in D(\mathbb{R}) \tag{3.3.3}$$

Definition 3.3.2. For $l(x) \in D'(\mathbb{R})$ and $\alpha \in \mathbb{R}$ we set

$$\langle \alpha l(x), \varphi \rangle = \alpha \langle l, \varphi \rangle, \quad \forall \varphi \in D(\mathbb{R}). \tag{3.3.4}$$

Multiplication of a distribution by a smooth function

Take $g(x) \in C^\infty(\mathbb{R})$. If $u(x) \in C(\mathbb{R})$, then, as seen from (3.1.5),

$$\langle g(x)u(x), \varphi\rangle = \langle u\, g(x)\varphi(x)\rangle, \quad \forall\varphi \in D(\mathbb{R}) \tag{3.3.5}$$

Definition 3.3.3. For $l(x) \in D'(\mathbb{R})$, we set

$$\langle g(x)l(x), \varphi\rangle = \langle l, g(x)\varphi(x)\rangle, \quad \forall\varphi \in D(\mathbb{R}) \tag{3.3.6}$$

Remark 3.3.2. The right-hand side of (3.3.6) makes sense, since

$$g(x)\varphi(x) \in D(\mathbb{R})! \tag{3.3.7}$$

Problems 3.3.1. 1. Verify that $g(x)l(x) \in D'(\mathbb{R})$

2. Compute $x\delta(x)$.

3. Prove that

$$g(x)\delta(x) = g(0)\delta(x). \tag{3.3.8}$$

"Shift" of the distributions

For $u(x) \in C(\mathbb{R})$ and $a \in (\mathbb{R})$,

$$\int u(x-a)\varphi(x)\,dx = \int u(y)\varphi(y+a)\,dy, \quad \forall\varphi \in D(\mathbb{R}). \tag{3.3.9}$$

Definition 3.3.4. For $l(x) \in D'(\mathbb{R})$, we set

$$\langle l(x-a), \varphi(x)\rangle = \langle l(y), \varphi(y+a)\rangle. \tag{3.3.10}$$

Example.

$$\langle \delta(x-a), \varphi(x)\rangle = \langle \delta(y), \varphi(y+a)\rangle = \varphi(a). \tag{3.3.11}$$

"Change of scale" in the argument of the distributions

For $u(x) \in C(\mathbb{R})$ and $k \neq 0$

$$\int_{-\infty}^{+\infty} u(kx)\varphi(x)\,dx = \frac{1}{|k|}\int_{-\infty}^{+\infty} u(y)\varphi\big(\frac{y}{k}\big)\,dy. \tag{3.3.12}$$

Definition 3.3.5. For $f(x) \in D(\mathbb{R})$ we set for $k \neq 0$

$$\langle f(kx), \varphi(x)\rangle = \frac{1}{|k|}\Big\langle f(y), \varphi\big(\frac{y}{k}\big)\Big\rangle. \tag{3.3.13}$$

Problem 3.3.2. Prove that

$$\delta(kx) = \frac{1}{|k|}\delta(x), \quad k \neq 0 \tag{3.3.14}$$

In particular, prove that δ is even:

$$\delta(-x) = \delta(x). \tag{3.3.15}$$

Remark 3.3.3. From definition (3.3.11) we get:

$$\langle \delta(y-x), \varphi(y) \rangle = \varphi(x). \tag{3.3.16}$$

This means that $\delta(y-x)$ is the integral kernel of the unit operator $I\varphi = \varphi$. In the linear algebra the matrix of the unit operator is the Kronecker δ-symbol, δ_{ij}. It is due to this analogue that Dirac called functional (3.2.4) the δ-function.

Problem 3.3.3. Write the formula of a general change of variable $x = g(y)$ in a distribution, where $g : \mathbb{R} \to \mathbb{R}$ is a smooth diffeomorphism.

Convergence of distributions

Definition 3.3.6. Distributions $u_n(x) \in D'(\mathbb{R})$ converge (weakly) to $u(x) \in D'(\mathbb{R})$ when $n \to \infty$ if for $\forall \varphi(x) \in C_0^\infty(\mathbb{R}) \equiv D(\mathbb{R})$

$$\langle u_n, \varphi \rangle \to \langle u, \varphi \rangle \quad \text{when} \quad n \to \infty \tag{3.3.17}$$

There is the following notation: $u_n(x) \overset{D'(\mathbb{R})}{\to} u(x)$ as $n \to \infty$.

Examples of convergent series of distributions:

1. If $u_n(x) \in C(\mathbb{R})$ and $U_n(x) \overset{\longrightarrow}{\longrightarrow} u(x)$ as $n \to \infty$, then $u_n \overset{D'(\mathbb{R})}{\to} u$ as $n \to \infty$. (Prove this!)

2. If $u_n(x) \in L_2(\mathbb{R})$ and $u_n \to u$ in $L_2(\mathbb{R})$ as $n \to \infty$, then $u_n \overset{D'(\mathbb{R})}{\to} u$. (Prove!)

3. $\sin kx \overset{D'(\mathbb{R})}{\to} 0$ as $k \to \infty$. Indeed, integrating by parts, we obtain:

$$\langle \sin kx, \varphi(x) \rangle = \int \frac{\cos kx}{k} \varphi'(x)\, dx \longrightarrow 0 \quad \text{as } k \to \infty \tag{3.3.18}$$

4. Analogously, $k^2 \sin kx \overset{D'(\mathbb{R})}{\to} 0$ as $k \to \infty$.

   Remark 3.3.4. Sequences of functions $\sin kx$ and $k^2 \sin kx$ do not converge neither in the space $C(\mathbb{R})$, nor in the space $L_2(\mathbb{R})$, but they do converge in $D'(\mathbb{R})$.

5. "δ-like" sequences. Consider Steklov step-functions

$$u_n(x) = \begin{cases} n, & x \in [0, \frac{1}{n}], \\ 0, & x \notin [0, \frac{1}{n}], \ n \in \mathbb{N}. \end{cases} \tag{3.3.19}$$

Obviously, $\int\limits_{-\infty}^{+\infty} u_n(x)\,dx = 1, \quad \forall n = 1, 2, 3, \dots.$

Problem 3.3.4. Prove that

$$u_n(x) \overset{D'(\mathbb{R})}{\to} \delta(x) \quad \text{as} \quad n \to \infty \tag{3.3.20}$$

Hint. Apply the mean value theorem to the integral $\int\limits_{-\infty}^{+\infty} u_n(x)\varphi(x)\,dx$.

Analogously, the Gauss distributions

$$\frac{e^{-\frac{x^2}{2\sigma}}}{\sqrt{2\pi\sigma}} \overset{D'(\mathbb{R})}{\to} \delta(x) \quad \text{as} \quad \sigma \to 0+ \tag{3.3.21}$$

converge weakly. (Prove this!)

Differentiation of distributions

For $u(x) \in C^1(\mathbb{R})$, integrating by parts, we get ($\varphi(x) \equiv 0$ for $|x| \geq A$):

$$\int\limits_{-\infty}^{+\infty} u'(x)\varphi(x)\,dx = u\varphi\Big|_{-A}^{A} - \int\limits_{-A}^{A} u(x)\varphi'(x)\,dx = -\int\limits_{-\infty}^{+\infty} u(x)\varphi'(x)\,dx, \tag{3.3.22}$$

since $\varphi(A) = \varphi(-A) = 0$, so that the boundary term is equal to zero.

Definition 3.3.7. For $u(x) \in D'(\mathbb{R})$ we set

$$\langle u'(x), \varphi(x)\rangle = -\langle u(x), \varphi'(x)\rangle, \quad \forall \varphi \in D(\mathbb{R}). \tag{3.3.23}$$

Problem 3.3.5. Prove that $u'(x) \in D'(\mathbb{R})$.

Thus, any distribution has a derivative (which is also a distribution), and hence derivatives of all orders!

Let us consider examples of derivatives of distributions.

1. $(\sin x)' = \cos x$.
2. Let us find $\Theta'(x)$ (see (3.2.8)—(3.2.9)). According to definition (3.3.23),

$$\langle \Theta(x), \varphi(x)\rangle = -\langle \Theta(x), \varphi'(x)\rangle = \int\limits_{0}^{+\infty} \varphi'(x)\,dx =$$
$$= -\varphi(x)\Big|_0^{+\infty} = \varphi(0) = \langle \delta(x), \varphi(x)\rangle. \tag{3.3.24}$$

From here, we see that $\Theta'(x) = \delta(x)$.

Continuity of the differentiation with respect to convergence of distributions

Lemma 3.3.1. *The operator* $\frac{d}{dx} : D'(\mathbb{R}) \to D'(\mathbb{R})$ *is continuous.*

Proof. Let $u_n(x) \overset{D'(\mathbb{R})}{\to} u(x)$. Then for $\forall \varphi \in D(\mathbb{R})$

$$\langle u_n'(x), \varphi(x)\rangle \equiv -\langle u_n(x), \varphi'(x)\rangle \underset{n\to\infty}{\longrightarrow} -\langle u(x), \varphi'(x)\rangle \equiv -\langle u'(x), \varphi(x)\rangle \quad (3.3.25)$$

Consequently, $u_n'(x) \overset{D'(\mathbb{R})}{\to} u'(x)$ according to definition (3.3.17). □

Problem 3.3.6. Prove that for $\forall u(x) \in D'(\mathbb{R})$

$$\frac{u(x+\varepsilon) - u(x)}{\varepsilon} \overset{D'(\mathbb{R})}{\to} u'(x) \quad \text{as} \quad \varepsilon \to 0. \quad (3.3.26)$$

3.4 Differentiation of piecewise smooth functions and the product rule

Differentiation of piecewise smooth functions

Lemma 3.4.1. *Let* $u(x) \in C^1$ *for* $x < a$ *and for* $x > a$, *while at the point* $x = a$ *it has the discontinuity of type one that is, there are one-sided limits* $u(a \pm 0)$ *(for simplicity, we assume that* $u'(a \pm 0)$ *also exist). See Fig. 3.4.*

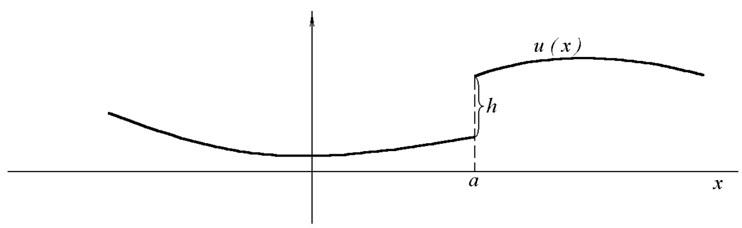


Figure 3.4:

Then the following formula is valid:

$$u'(x) = \{u'(x)\} + h \cdot \delta(x-a); \quad h = u(a+0) - u(a-0). \quad (3.4.1)$$

The function $u'(x)$ in the left-hand side of (3.4.1) is a generalized derivative of the distribution $u(x)$, while $\{u'(x)\}$ in the right-hand side is a function continuous for $x \neq a$, which is equal to a derivative of the function $u(x)$ at the points where this derivative exists. The distribution given for $\{u'(x)\}$ by formula (3.1.5) is called the regular part of the generalized derivative $u'(x)$. *Example.* For $u(x) = \Theta(x)$, we have: $a = 0, \{\Theta'(x)\} \equiv 0$, since $\Theta'(x) = 0$ for $x \neq 0$, $\quad h = \Theta(0+) - \Theta(0-) = 1$. Therefore, formula (3.4.1) gives

$$\Theta'(x) = \delta(x), \quad (3.4.2)$$

in agreement with (3.3.24).

Problem 3.4.1. Compute $|x|''$.

Solution. According to formula (3.4.1),

$$|x|' = \{|x|'\} + 0 \cdot \delta(x) = \operatorname{sgn} x \equiv \begin{cases} 1, & x > 0, \\ -1, & x < 0. \end{cases} \tag{3.4.3}$$

Again, using the same formula,

$$|x|'' = (\operatorname{sgn} x)' = \{\operatorname{sgn}' x\} + 2 \cdot \delta(x) = 2\delta(x). \tag{3.4.4}$$

Answer. $|x|'' = 2\delta(x)$.

Problem 3.4.2. Prove formula (3.4.1).

Solution. For $\varphi \in C_0^\infty(\mathbb{R})$,

$$\begin{aligned} \langle u', \varphi \rangle \quad &= \quad -\langle u, \varphi' \rangle = -\int\limits_{-\infty}^{a} u\varphi'\, dx - \int\limits_{a}^{+\infty} u\varphi'\, dx = -u\varphi\big|_{-\infty}^{a-0} - u\varphi\big|_{a+0}^{\infty} + \\ &+ \quad \int_{x \neq a} u'\varphi\, dx = -u(a)\varphi(a-0) + u(a)\varphi(a+0) + \langle \{u'\}, \varphi \rangle, \end{aligned}$$

which is equivalent to (3.4.1).

Product rule

For $g(x) \in C^\infty(\mathbb{R})$ and $u(x) \in D'(\mathbb{R})$, the product $g(x)u(x)$ is defined (see definition (3.3.6)). It turns out that the following common formula is valid:

$$\big(g(x)u(x)\big)' = g'(x)u(x) + g(x)u'(x). \tag{3.4.5}$$

Problem 3.4.3. Prove formula (3.4.5).

Problem 3.4.4. Using formula (3.4.5), compute the following:

$$\Big(\frac{d}{dx} + \lambda\Big)\big(\Theta(x)e^{-\lambda x}\big). \tag{3.4.6}$$

Solution. According to formulas (3.4.5) and (3.3.8),

$$\begin{aligned} \frac{d}{dx}(e^{-\lambda x}\Theta(x)) &= -\lambda e^{-\lambda x}\Theta(x) + e^{-\lambda x}\Theta'(x) = \\ &= -\lambda e^{-\lambda x}\Theta(x) + \delta(x). \end{aligned} \tag{3.4.7}$$

Adding $\lambda\Theta(x)e^{-\lambda x}$ to both sides, we see that (3.4.6) is equal to $\delta(x)$:

$$\Big(\frac{d}{dx} + \lambda\Big)\big(\Theta(x)e^{-\lambda x}\big) = \delta(x). \tag{3.4.8}$$

Problem 3.4.5. Compute (for $\omega \neq 0$):

$$\left(\frac{d^2}{dx^2} + \omega^2\right)\left(\Theta(x)\frac{\sin \omega x}{\omega}\right) = ? \tag{3.4.9}$$

Solution. According to formulae (3.4.5) and (3.3.8),

$$\frac{d}{dx}\left(\frac{\sin \omega x}{\omega} \cdot \Theta(x)\right) = \cos \omega x \cdot \Theta(x) + \frac{\sin \omega x}{\omega} \cdot \Theta'(x) = \cos \omega x \cdot \Theta(x). \tag{3.4.10}$$

Using the same formulae, we get

$$\frac{d^2}{dx^2}\left(\frac{\sin \omega x}{\omega} \cdot \Theta(x)\right) = \frac{d}{dx}(\cos \omega x \cdot \Theta(x)) = -\omega \sin \omega x \cdot \Theta(x) + \\ + \cos \omega x \cdot \Theta'(x) = -\omega \sin \omega x \cdot \Theta(x) + \delta(x). \tag{3.4.11}$$

Substituting $\omega^2 \Theta(x) \frac{\sin \omega x}{\omega}$, we get $\delta(x)$:

$$\left(\frac{d^2}{dx^2} + \omega^2\right)\left(\Theta(x)\frac{\sin \omega x}{\omega}\right) = \delta(x). \tag{3.4.12}$$

Problem 3.4.6. Prove (3.4.8) and (3.4.12) using formula (3.4.1) instead of (3.4.5).

Solution.

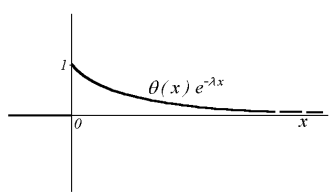


Figure 3.5:

Let us prove (3.4.8). We plot $\Theta(x)e^{-\lambda x}$ (Fig. 3.5). According to formula (3.4.1) ($a = 0$ and $h = 1$),

$$\frac{d}{dx}(\Theta(x)e^{-\lambda x}) = \Theta(x)(-\lambda)e^{-\lambda x} + \delta(x). \tag{3.4.13}$$

Adding $\lambda \Theta(x) e^{-\lambda x}$, we get (3.4.8).

Let us prove (3.4.12). The plot $\Theta(x)\frac{\sin \omega x}{\omega}$ is on Fig. 3.6.
According to formula (3.4.1) ($a = 0$ and $h = 0$),

$$\frac{d}{dx}\left(\Theta(x)\frac{\sin \omega x}{\omega}\right) = \Theta(x) \cos \omega x. \tag{3.4.14}$$

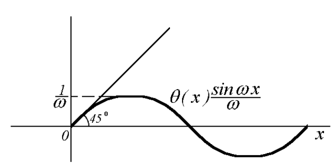


Figure 3.6:

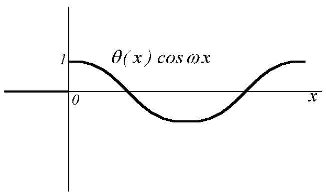


Figure 3.7:

The graph of $\Theta(x)\cos\omega x$ is plotted on Fig. 3.7. We use formula (3.4.1) ($a = 0$ and $h = 1$):

$$\frac{d^2}{dx^2}(\Theta(x)\frac{\sin\omega x}{\omega}) = \frac{d}{dx}(\Theta(x)\cos\omega x) = \\ = \Theta(x)(\sin\omega x)(-\omega) + \delta(x). \qquad (3.4.15)$$

Adding $\omega^2\Theta(x)\frac{\sin\omega x}{\omega}$, we get (3.4.12).

Remark 3.4.1. We have equality (3.4.8) since the function $\Theta(x)e^{-\lambda x}$ for $x \neq 0$ satisfies the homogeneous equation

$$\Big(\frac{d}{dx} + \lambda\Big)\big(\Theta(x)e^{-\lambda x}\big) = 0 \quad \text{for} \quad x \neq 0, \qquad (3.4.16)$$

while its jump is $h = 1$. Analogously, we have equality (3.4.12) since the function $y(x) = \Theta(x)\frac{\sin\omega x}{\omega}$ for $x \neq 0$ satisfies the homogeneous equation

$$\Big(\frac{d^2}{dx^2} + \omega^2\Big)\Big(\Theta(x)\frac{\sin\omega x}{\omega}\Big) = 0 \quad \text{for} \quad x \neq 0. \qquad (3.4.17)$$

Besides, the function $y(x)$ is continuous at $x = 0$, while its first derivative $y'(x) = \Theta(x)\cos\omega x$ has a jump equal to 1:

$$\begin{cases} y(0-) = y(0+), \\ y'(0+) = y'(0-) + 1. \end{cases} \qquad (3.4.18)$$

Thus, the regular parts in (3.4.8) and (3.4.12) cancel out due to identities (3.4.16) and (3.4.17), respectively.

3.5 Fundamental solutions of ordinary differential equations

Fundamental solutions of ordinary equations

Let us consider a linear differential operator of order m with constant coefficients:

$$A = A\big(\frac{d}{dx}\big) = \sum_{k=0}^{m} a_k\frac{d^k}{dx^k}, \quad a_m \neq 0. \qquad (3.5.1)$$

Using the chain rule, we get:

$$\frac{d^k}{dx^k}(u(x-y)) = u^{(k)}(x-y), \quad x \in \mathbb{R}. \qquad (3.5.2)$$

Therefore,

$$A\big(\frac{d}{dx}\big)(u(x-y)) = (Au)(x-y), \quad x \in \mathbb{R}. \qquad (3.5.3)$$

Definition 3.5.1. The fundamental solution of the operator A is a function (distribution) $\varepsilon(x) \quad x \in D'(\mathbb{R})$ such that

$$A(\frac{d}{dx})\, e(x) = \delta(x), \quad x \in \mathbb{R}, \tag{3.5.4}$$

where the derivatives are understood in the sense of distributions.

Remark 3.5.1. As follows from (3.5.3),

$$A(\frac{d}{dx})\, e(x-y) = \delta(x-y), \quad x \in \mathbb{R}. \tag{3.5.5}$$

Examples.

1. $A = \frac{d}{dx}: \quad e(x) = \Theta(x) \quad (see\ (3.4.2)).$
2. $A = \frac{d^2}{dx^2}: \quad e(x) = \frac{1}{2}|x| \quad (see\ (3.4.4)).$
3. $A = \frac{d}{dx} + \lambda: \quad e(x) = \Theta(x)e^{-\lambda x} \quad (see\ (3.4.8)).$
4. $A = \frac{d^2}{dx^2} + \omega^2: \quad e(x) = \Theta(x)\frac{\sin \omega x}{\omega} \quad (see\ (3.4.12)).$

Let us point out that for a fixed operator A there could be infinitely many fundamental solutions.
Question. Why does one need fundamental solutions?
Answer. To solve nonhomogeneous equations

$$A(\frac{d}{dx})u(x) = f(x), \quad x \in \mathbb{R}. \tag{3.5.6}$$

A particular solution could be found using the formula

$$u(x) = \int_{-\infty}^{+\infty} e(x-y)f(y)\,dy \equiv (e * f)(x) = \int_{-\infty}^{+\infty} e(y)f(x-y)\,dy, \tag{3.5.7}$$

if $f(x) = 0$ for $|x| \geq \text{const}$, and $f(x) \in C(\mathbb{R})$ (the operation $*$ in (3.5.7) is called a convolution of e with f).
Verification. for the case $f(x) \in C^m(\mathbb{R})$: For function (3.5.7) we get from (3.5.4), for $x \in \mathbb{R}$:

$$A(\frac{d}{dx})u(x) = \int_{-\infty}^{+\infty} e(y)A(\frac{d}{dx})f(x-y)\,dy = \Big\langle e(y), A(-\frac{d}{dy})f(x-y)\Big\rangle =$$

$$= \Big\langle A(\frac{d}{dy})\,e(y), f(x-y)\Big\rangle = \langle \delta(y), f(x-y)\rangle = f(x). \tag{3.5.8}$$

Examples.

1. For the equation
$$\frac{d}{dx}u(x) = f(x), \quad x \in \mathbb{R} \tag{3.5.9}$$
formula (3.5.7) gives a particular solution
$$u(x) = \int_{-\infty}^{+\infty} \Theta(x-y)f(y)\,dy = \int_{-\infty}^{x} f(y)\,dy, \quad x \in \mathbb{R} \tag{3.5.10}$$
which is well-known from Calculus.

2. For the equation
$$\frac{d^2}{dx^2}u(x) = f(x), \quad x \in \mathbb{R} \tag{3.5.11}$$
formula (3.5.7) gives a particular solution
$$u(x) = \int_{-\infty}^{+\infty} \frac{1}{2}|x-y|f(y)\,dy, \quad x \in \mathbb{R} \tag{3.5.12}$$
which is analogous to the Cauchy formula for double integrals.

The method of construction of the fundamental solutions for an arbitrary operator $A(\frac{d}{dx})$ of form (3.5.1)

Let us consider the function $u_0(x)$ for $x \geq 0$, which is a solution to the Cauchy problem

$$\begin{cases} A(\frac{d}{dx})u_0(x) = 0, \quad x > 0, \\ u_0(0) = 0, \\ \dots \\ u_0^{(m-2)}(0) = 0, \\ u_0^{(m-1)}(0) = \frac{1}{a_m}. \end{cases} \tag{3.5.13}$$

Then the function

$$e = \begin{cases} u_0(x), \quad x > 0, \\ 0, \quad x < 0. \end{cases} \tag{3.5.14}$$

is the fundamental solution of the operator A.

Problem 3.5.1. Prove (3.5.4) for function (3.5.14) using formula (3.4.1).

Problem 3.5.2. Find the solution to the equation

$$3u''(x) - u'(x) = \delta(x), \quad x \in \mathbb{R} \tag{3.5.15}$$

Solution. $3\lambda^2 - \lambda = 0 \iff \lambda_1 = 0, \lambda_2 = \frac{1}{3} \quad \Rightarrow$

$$u_0(x) = c_1 + c_2 e^{\frac{x}{3}}. \tag{3.5.16}$$

The initial conditions (3.5.13) yield

$$\begin{cases} c_1 + c_2 = 0 \\ \frac{1}{3}c_2 = \frac{1}{3} \end{cases} \quad \Rightarrow \quad \begin{cases} c_2 = 1 \\ c_1 = -1 \end{cases} \tag{3.5.17}$$

Answer.

$$u(x) = \Theta(x)(e^{\frac{x}{3}} - 1). \tag{3.5.18}$$

Problem 3.5.3. Find the formula for the particular solution to the equation

$$u''(x) - 3u'(x) + 2u(x) = f(x), \quad x \in \mathbb{R} \tag{3.5.19}$$

where $f(x) \in C(\mathbb{R}), \quad f(x) = 0$ for $|x| = \text{const}$.

Solution. Let us find the fundamental solution:

$$e''(x) - 3e'(x) + 2e(x) = \delta(x). \tag{3.5.20}$$

For this, we find the roots of the characteristic equation

$$\lambda^2 - 2\lambda + 2 = 0 \iff \lambda_1 = 1, \quad \lambda_2 = 2 \quad \Rightarrow \tag{3.5.21}$$

$$e(x) = \Theta(x)(c_1 e^x + c_2 e^{2x}). \tag{3.5.22}$$

Initial conditions (3.5.13) yield

$$\begin{cases} c_1 + c_2 = 0, \\ c_1 + 2c_2 = 1, \end{cases} \quad \Rightarrow \quad \begin{cases} c_2 = 1, \\ c_1 = -1, \end{cases} \tag{3.5.23}$$

Answer. According to formula (3.5.7),

$$\begin{aligned} u(x) \quad &= \quad \varepsilon * f(x) = \int_{-\infty}^{+\infty} \Theta(x-y)(e^{2(x-y)} - e^{x-y}) f(y)\, dy = \\ &= \int_{-\infty}^{x} (e^{2(x-y)} - e^{x-y}) f(y)\, dy. \end{aligned} \tag{3.5.24}$$

3.6 The Green function for the boundary value problems on an interval

The Green function

Let us find the solution to the boundary value problem ($\omega \neq 0$)

$$\begin{cases} u''(x) - \omega^2 u(x) = f(x), \quad 0 < x < l, \\ u(0) = u(l) = 0. \end{cases} \tag{3.6.1}$$

The Green function of this boundary value problem is a function $G(x,y)$ on $[0,l] \times [0,l]$, smooth for $x \neq y$ and satisfying the equations

$$\begin{cases} (\frac{d^2}{dx^2} - \omega^2)G(x,y) = \delta(x-y), \quad 0 < x < l, \\ G(0,y) = G(l,y) = 0. \end{cases} \tag{3.6.2}$$

Here y plays the role of a parameter, $y \in (0,l)$. We could say that the Green function is the fundamental solution that satisfies the boundary conditions.

Having the Green function, one can find the solution to the boundary value problem (3.6.1) using the formula

$$u(x) = \int_0^l G(x,y)f(y)\,dy. \tag{3.6.3}$$

Verification. The boundary conditions (3.6.1) follow from boundary conditions (3.6.2): At $x = 0$,

$$u(0) = \int_0^l G(0,l)f(y)\,dy = 0 \tag{3.6.4}$$

and similarly at $x = l$. Equation (3.6.1) could be checked formally:

$$\Big(\frac{d^2}{dx^2} - \omega^2\Big)u(x) = \int_0^l \Big(\frac{d^2}{dx^2} - \omega^2\Big)G(x,y)f(y)\,dy = \int_0^l \delta(x-y)f(y)\,dy = f(x). \tag{3.6.5}$$

Remark 3.6.1. Formula (3.6.3) means that the Green function $G(x,y)$ is the integral kernel of the operator G that is inverse to the operator $A = \frac{d^2}{dx^2} - \omega^2$ of the boundary value problem (3.6.1):

$$A = \frac{d^2}{dx^2} - \omega^2 : \quad C_0^2[0,l] \to C[0,l] \tag{3.6.6}$$

Here $C_0^2[0,l]$ is the space of functions $u(x) \in C^2[0,l]$ that satisfy the boundary conditions $u(0) = u(l) = 0$.

Remark 3.6.2. Operator (3.6.6) is symmetric in $L_2(0,l)$, as is shown in (2.3.7). Hence, the operator $G = A^{-1}$ is also symmetric in $L_2(0,l)$. It is here that the important symmetry property of the Green function is coming from:

$$G(x,y) = G(y,x), \quad \forall x, y \in [0,l]. \tag{3.6.7}$$

The method of constructing the Green function for the boundary value problems on an interval

The differential equation (3.6.2) is homogeneous for $x \neq y$, since $\delta(x-y) = 0$ for $x - y \neq 0$. Therefore, analogously to (3.4.17),

$$\Big(\frac{d^2}{dx^2} - \omega^2\Big)G(x,y) = 0 \quad \text{for} x \neq y. \tag{3.6.8}$$

Therefore,

$$G(x,y) = \begin{cases} Ae^{\omega x} + Be^{-\omega x}, & x < y, \\ Ce^{\omega x} + De^{-\omega x}, & x > y. \end{cases} \tag{3.6.9}$$

For determining the constants A, B, C, and D, we have two boundary conditions (3.6.2) and two matching conditions at $x = y$ (see Fig. 3.8). Similarly to (3.4.18),

$$\begin{cases} G(y-0,y) = G(y+0,y), \\ G'_x(y+0,y) = G'_x(y-0,y) + 1. \end{cases} \tag{3.6.10}$$

These four equations determine A, B, C, and D uniquely.

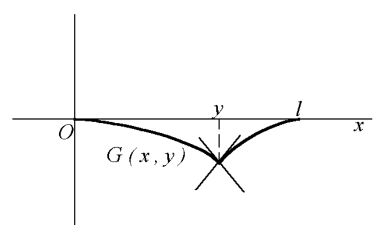


Figure 3.8:

Problem 3.6.1. Derive (3.6.2) from (3.6.8) and (3.6.10).

Hint. Apply formula (3.4.1) (twice) for computing $\frac{d^2}{dx^2}G(x,y)$.

We point out that one could take into account the boundary conditions (3.6.2) looking for the Green function in the form

$$G(x,y) = \begin{cases} A \operatorname{sh} \omega x, & x < y, \\ B \operatorname{sh} \omega(x-l), & x > y. \end{cases} \tag{3.6.11}$$

Then (3.6.9) is also satisfied. It remains to take into account matching conditions (3.6.10):

$$\begin{cases} A \operatorname{sh} \omega y = B \operatorname{sh} \omega(y-l), \\ B\omega \operatorname{ch} \omega(y-l) = A\omega \operatorname{ch} \omega y + 1. \end{cases} \tag{3.6.12}$$

Solving this system, we find

$$\begin{cases} A = \frac{\operatorname{sh} \omega(y-l)}{\omega \operatorname{sh} \omega l}, \\ B = \frac{\operatorname{sh} \omega y}{\omega \operatorname{sh} \omega l}. \end{cases} \tag{3.6.13}$$

Finally, from (3.6.11) we find the Green function for problem (3.6.1):

$$G(x,y)=\begin{cases}\frac{\operatorname{sh}\omega(y-l)\operatorname{sh}\omega x}{\omega\operatorname{sh}\omega l}, & x<y,\\ \frac{\operatorname{sh}\omega y\operatorname{sh}\omega(x-l)}{\omega\operatorname{sh}\omega l}, & x>y.\end{cases}\tag{3.6.14}$$

Substituting into (3.6.3), we find the solution of the boundary value problem (3.6.1):

$$u(x)=\int_0^x\frac{\operatorname{sh}\omega y\operatorname{sh}\omega(x-l)}{\omega\operatorname{sh}\omega l}f(y)\,dy+\int_x^l\frac{\operatorname{sh}\omega(y-l)\operatorname{sh}\omega x}{\omega\operatorname{sh}\omega l}f(y)\,dy.\tag{3.6.15}$$

The Green function (3.6.14) is symmetric in complete agreement with Remark 3.6.2

Problem 3.6.2. Let $\omega\neq 0$. Find the solution to the boundary value problem

$$\begin{cases}u''(x)+\omega^2u(x)=f(x), & 0<x<l,\\ u(0)=u(l)=0.\end{cases}\tag{3.6.16}$$

Solution. We find the Green function $G(x,y)$ for $\forall y\in[0,l]$:

$$\begin{cases}(\frac{d^2}{dx^2}+\omega^2)G(x,y)=\delta(x-y), & 0<x<l,\\ G(0,y)=G(l,y)=0.\end{cases}\tag{3.6.17}$$

Substituting in (3.6.8) the sign "$-$" by "$+$" we get, analogously to (3.6.11), the following:

$$G(x,y)=\begin{cases}A\sin\omega x, & x<y,\\ B\sin\omega(x-l), & x>y.\end{cases}\tag{3.6.18}$$

Substituting G into the matching conditions (3.6.10), we get, analogously to (3.6.13), that for $\sin\omega l\neq 0$

$$\begin{cases}A=\frac{\sin\omega(y-l)}{\omega\sin\omega l},\\ B=\frac{\sin\omega y}{\omega\sin\omega l}.\end{cases}\tag{3.6.19}$$

From here, similarly to (3.6.14),

$$G(x,y)=\begin{cases}\frac{\sin\omega(y-l)\sin\omega x}{\omega\sin\omega l}, & x<y,\\ \frac{\sin\omega y\sin\omega(x-l)}{\omega\sin\omega l}, & x>y.\end{cases}\tag{3.6.20}$$

Finally, we get the solution to problem (3.6.16):

$$u(x)=\int_0^x\frac{\sin\omega y\sin\omega(x-l)}{\omega\sin\omega l}f(y)\,dy+\int_x^l\frac{\sin\omega(y-l)\sin\omega x}{\omega\sin\omega l}f(y)\,dy.\tag{3.6.21}$$

Let us point out that Green's function (3.6.20) is also symmetric.

Problem 3.6.3. Find the solution to

$$\begin{cases} u''(x) - \omega^2 u(x) = f(x), & 0 < x < l, \\ u(x) = u'(l) = 0. \end{cases} \tag{3.6.22}$$

Problem 3.6.4. Find the solution to

$$\begin{cases} u''(x) + \omega^2 u(x) = f(x), & 0 < x < l, \\ u'(x) = u(l) = 0. \end{cases} \tag{3.6.23}$$

Problems 3.6.1. Construct the Green functions and write the solution formulas for the following boundary value problems:

1. $u''(x) = f(x),\ 0 < x < 1; \quad u'(0) = u(0), \quad u'(1) = -u(1).$
2. $u''(x) + u(x) = f(x),\ 0 < x < 1; \quad u'(0) = u(0), \quad u'(1) = 3u(1).$
3. $x^2 u''(x) + 2xu'(x) = f(x),\ 1 < x < 2; \quad u'(1) = 0, \quad u(2) + 5u'(2) = 0.$
4. $(3 + x^2)u''(x) + 2xu'(x) = f(x),\ 0 < x < 1; \quad u'(0) = u(0), \quad u(1) = 0.$

3.7 Well-posedness of the boundary value problems

Let us point out that formula (3.6.21) for the solution of problem (3.6.16) and the Green function (3.6.20) do not make sense for $\omega l = k\pi, \quad k \in Z$, since then $\sin \omega l = 0$.

Question. Could we forsee this without solving problem (3.6.16)?

Answer. Yes. The thing is that when $\omega l = k\pi$, problem (3.6.16) has a nonzero solution $u_0(x)$ for $f(x) \equiv 0$: $u_0(x) = \sin \frac{k\pi x}{l}$,

$$\begin{cases} (\frac{d^2}{dx^2} + \omega^2) u_0(x) = 0, & 0 < x < l, \\ u_0(0) = u_0(l) = 0. \end{cases} \tag{3.7.1}$$

Therefore, the operator $A \equiv \frac{d^2}{dx^2} + \omega^2 : C_0^2[0, l] \to C[0, l]$ is not invertible! Therefore the (left) inverse G does not exist, and neither does its integral kernel $G(x, y)$.

We point out, though, that the absense of the inverse operator to A does not mean that problem (3.6.16) does not have solutions for a single $f(x)$!

Question. Under which conditions on $f(x)$ does problem (3.6.16) have solution $u(x)$, and how could this solution be found?

To answer this question, we need to take a detour into the Linear Algebra. The thing is, the similar question arises when solving the system

$$Au = f, \tag{3.7.2}$$

where A is an $n \times n$ matrix and $f \in \mathbb{R}^n$. System (3.7.2) has a (unique) solution $u = A^{-1} f$ if $\det A \neq 0$. If instead $\det A = 0$, then system (3.7.2) may not have solutions.

The necessary and sufficient condition on f so that system (3.7.2) has a solution is the following orthogonality condition (see [CH53]):

$$f \perp \operatorname{Ker} A^* \tag{3.7.3}$$

Here $\operatorname{Ker} A^*$ is the subspace in в $\mathbb{R}^n$ that consists of solutions to the adjoint homogeneous system:

$$h \in \operatorname{Ker} A^* \iff A^* h = 0 \tag{3.7.4}$$

Thus, (3.7.3) means that

$$\langle f, h\rangle = 0, \quad \forall h \in \operatorname{Ker} A^* \tag{3.7.5}$$

Let us prove the necessity of conditions (3.7.3), (3.7.5). If for a given vector $f \in \mathbb{R}^n$ there is a solution u to system (3.7.2), then for each vector $h \in \operatorname{Ker} A^*$

$$\langle f, h\rangle = \langle Au, h\rangle = \langle u, A^* h\rangle = \langle u, 0\rangle = 0. \tag{3.7.6}$$

Problem 3.7.1. Prove the sufficiency of conditions (3.7.3), (3.7.5) for the well-posedness of system (3.7.2).

Due to the form of condition (3.7.3) we say that system (3.7.2) is "normally" well-posed.

It turns out that for the well-posedness of problem (3.6.16) the necessary and sufficient condition is the one of form (3.7.3)! We will prove this below.

Let us find $\operatorname{Ker} A^*$ for the operator of problem (3.6.16). Since $A^* = A$ (see Lemma 2.3.1) from Section 2.3),

$$\operatorname{Ker} A^* = \operatorname{Ker} A. \tag{3.7.7}$$

This means that $\operatorname{Ker} A^*$ coincides with the space of solutions of problem (3.7.1), that is,

$$\operatorname{Ker} A^* = \{C \sin \frac{k\pi x}{l}, \quad C \in \mathbb{R}\}. \tag{3.7.8}$$

Thus, the orthogonality condition (3.7.6) takes the following form:

$$\langle f(x), \sin \frac{k\pi x}{l}\rangle = \int_0^l f(x) \sin \frac{k\pi x}{l}\, dx = 0. \tag{3.7.9}$$

Problem 3.7.2. Prove the necessity of condition (3.7.9) for the well-posedness of problem (3.6.16) when $\omega = \frac{k\pi}{l}$.

Solution. Analogously to (3.7.6) and in view of (2.3.7), we have:

$$\begin{aligned}\langle f(x), \sin \frac{k\pi x}{l}\rangle &= \Big\langle \big(\frac{d^2}{dx^2} + \omega^2\big) u(x), \sin \frac{k\pi x}{l}\Big\rangle = \\ &= \Big\langle u(x), \big(\frac{d^2}{dx^2} + \omega^2\big) \sin \frac{k\pi x}{l}\Big\rangle = \langle u(x), 0\rangle = 0. \end{aligned} \tag{3.7.10}$$

Problem 3.7.3. Prove the sufficiency of condition (3.7.9).

Solution. Let us take $\omega \to \frac{k\pi}{l}$, but $\omega \neq \frac{k\pi}{l}$. Then problem (3.6.16) has solution (3.6.21). It turns out that, firstly, function (3.6.21) under condition (3.7.9) has a limit as $\omega \to \frac{k\pi}{l}$, and, secondly, this limit is the solution to problem (3.6.16).

Let us prove the first statement. By formula (3.6.21),

$$u(x) = \frac{1}{\omega \sin \omega l}\Big[\int_0^x \sin \omega y \sin \omega(x-l) f(y)\,dy + \int_x^l \sin \omega(y-l) \sin \omega x f(y)\,dy\Big]. \tag{3.7.11}$$

When $\omega = \frac{k\pi}{l}$, the integrands in both integrals have the same form. We use the identities

$$\begin{cases} \sin \omega y \cdot \sin \omega(x-l) = \sin \omega y \cdot \sin(\omega x - k\pi) = (-1)^k \sin \omega y \cdot \sin \omega x, \\ \sin \omega(y-l) \cdot \sin \omega x = \sin(\omega y - k\pi) \cdot \sin \omega x = (-1)^k \sin \omega y \cdot \sin \omega x. \end{cases} \tag{3.7.12}$$

Then the expression in the square brackets in (3.7.11) takes the form

$$(-1)^k \sin \omega x \int_0^l \sin \omega y f(y)\,dy. \tag{3.7.13}$$

But when $\omega = \frac{k\pi}{l}$ integral (3.7.13) is equal to zero due to the orthogonality condition (3.7.9)! Therefore, when $\omega \to \frac{k\pi}{l}$, both the numerator and the denominator of expression (3.7.11) tend to 0, and we obtain an indefinite expression of the form $\frac{0}{0}$.

Problem 3.7.4. Find the limit of expression (3.7.11) as $\omega \to \frac{k\pi}{l}$. (Apply the l'Hospital Rule.)

Answer.

$$\begin{aligned} u(x) \;=\; & \frac{1}{\omega l \sin k\pi}\Big\{ \\ & \int_0^x \big[y \cos \omega y \,\sin \omega(x-l) + \sin \omega y\,(x-l) \cos \omega(x-l)\big] f(y)\,dy + \\ & + \int_x^l [(y-l) \cos \omega(y-l) \,\sin \omega x + \sin \omega(y-l)\, x \cos \omega x] f(y)\,dy\Big\}. \end{aligned} \tag{3.7.14}$$

Problem 3.7.5. Prove that function (3.7.14) is a solution to problem (3.6.16) (for $\omega = \frac{k\pi}{l}$ and under condition (3.7.9)!).

Question. The solution to problem (3.6.16) for $\omega = \frac{k\pi}{l}$ is not defined uniquely, since one could add to it $C \cdot \sin \frac{k\pi x}{l}$ with any value of C. How does the solution given by formula (3.7.14) stand out of all other solutions?

Answer. Formula (3.7.14) gives a solution to problem (3.6.16) that satisfies the condition

$$\left\langle u(x), \sin \frac{k\pi x}{l} \right\rangle = 0. \tag{3.7.15}$$

Problem 3.7.6. Find the well-posedness condition and the solution formula for problem (3.6.23) with $\omega = \frac{(k+\frac{1}{2})\pi}{l}$, $k = 0, 1, 2, \ldots$.

3.8 Sobolev functional spaces

Let Ω be some region in $\mathbb{R}^n$, and $s = 0, 1, 2, \ldots$.

Definition 3.8.1. The space $H_s(\Omega)$ consists of all functions $u(x) \in L_2(\Omega)$, that satisfy

$$\partial_x^\alpha u(x) \in L_2(\Omega), \quad \text{for} \quad |\alpha| \le s, \tag{3.8.1}$$

where the derivatives are understood in the sense of distributions.

The Sobolev norm $\|u\|_s$ in the space $H_s(\Omega)$ is defined by

$$\|u\|_s^2 \equiv \sum_{|\alpha|\le s} \|\partial_x^\alpha u(x)\|_{L_2(\Omega)}^2 = \sum_{|\alpha|\le s} \int_\Omega |\partial_x^\alpha u(x)|^2 \, dx \tag{3.8.2}$$

Remark 3.8.1. $H_0(\Omega) \equiv L_2(\Omega)$, and, obviously, $C_0^\infty(\Omega) \subset H_s(\Omega)$.

Definition 3.8.2. $H_s^0(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the space $H_s(\Omega)$.

Let us list the most important properties of the Sobolev spaces.

I. $H_s(\Omega)$ is the complete Hilbert space. Later we will always assume that Ω is a bounded region in $\mathbb{R}^n$ (with a compact closure) and the smooth boundary $\partial\Omega$. Let us formulate the most significant *Sobolev embedding theorems.*

II. $H_s(\Omega) \subset C(\bar{\Omega})$ for $s > \frac{n}{2}$.

III. For $s_1 > s_2$, the inclusion $H_{s_1}(\Omega) \subset H_{s_2}(\Omega)$ is a compact mapping.

Proofs of properties I–III are in [Mik78, Ole76, Pet91, SD64]. Let us consider examples of the Sobolev spaces. Let $n = 1$ and $\Omega = (0, l)$, where $l > 0$. Then

1. $H_0((0,l)) = L_2(0,l)$, and, decomposing $u(x) \in L_2(0,l)$ into the Fourier series $u(x) = \sum_1^\infty u_k \sin \frac{k\pi x}{l}$ and applying the Bessel identity, we get:

$$\|u\|_0^2 = \frac{l}{2} \sum_1^\infty |u_k|^2 \tag{3.8.3}$$

2. The space $H_1(0,l)$ consists of functions $u(x) \in L_2(0,l)$ that satisfy

$$\|u\|_1^2 \equiv \int_0^l u^2(x)\,dx + \int_0^l |u'(x)|^2\,dx < \infty \tag{3.8.4}$$

3. The space $H_1^0((0,l))$ consists of functions $u(x) \in C[0,l]$ such that norm (3.8.4) is finite, and, moreover, $u(0) = u(l) = 0$. (Prove this!)

Problem 3.8.1. Prove that, analogously to (3.8.3), for $u \in H_1^0((0,l))$

$$\|u\|_1^2 = \frac{l}{2}\sum_1^\infty |u_k|^2 + \frac{\pi^2}{2l}\sum_1^\infty k^2|u_k|^2. \tag{3.8.5}$$

Hint. First prove (3.8.5) for $u(x) \in C_0^\infty(0,l)$.

Corollary 3.8.1. *For $u \in H_1^0((0,l))$ the norm $\|u\|_1^2$ is equivalent with*

$$|||u|||_1^2 \equiv \sum_1^\infty k^2 |u_k|_1^2 \tag{3.8.6}$$

Problem 3.8.2. Prove that (3.8.5) is not valid for $u(x) \in H_1((0,l)) \setminus H_1^0((0,l))$.

Problem 3.8.3. Prove the Friedrichs inequality: For $u(x) \in H_1^0((0,l))$,

$$\int_0^l u^2(x)\,dx \le C \int_0^l |u'(x)|^2\,dx, \tag{3.8.7}$$

where $c > 0$ does not depend on u.

Hint. Express the integrals in (3.8.7) via the Fourier coefficients u_k with respect to the basis $\{\sin\frac{k\pi x}{l}\}$.

Problem 3.8.4. Prove that there is no constant $C > 0$ such that inequality (3.8.7) holds for all $u \in H_1((0,l))$.

Problem 3.8.5. Prove that $H_1^0((0,l)) \subset C[0,l]$, using th finiteness of norm (3.8.5) and decomposition of $u(x)$ into the Fourier series.

Problem 3.8.6. Prove the compactness of the embedding $H_1^0(0,l) \subset H_0((0,l))$, using (3.8.5) and (3.8.3).

Problem 3.8.7. Let Ω be a ball $|x| < 1$ в $\mathbb{R}^n$. For which $\alpha \in \mathbb{R}$

$$|x|^\alpha \in H_1(\Omega)? \quad (\sin|x|)^\alpha \in H_1(\Omega)? \quad (\ln|x|)^\alpha \in H_1(\Omega)? \tag{3.8.8}$$

3.9 Well-posedness of the mixed problem for the wave equation in Sobolev spaces

Consider the problem (2.6.1)–(2.6.3) from Chapter 2 and formula (2.6.11) for its solution. Assume that

$$\varphi(x) \in H_1^0((0,l)) \quad \text{and} \quad \psi(x) \in H_0((0,l)) \equiv L_2(0,l) \tag{3.9.1}$$

Let us verify that formula (2.6.11) gives a solution to the problem (2.6.1)–(2.6.3) under conditions (3.9.1).

Problems 3.9.1. 1. Prove that equation (2.6.1) is satisfied in the sense of distributions, $D'((0,l)\times\mathbb{R})$.

2. Prove that for $\forall t \in \mathbb{R}$

$$u(x,t) \in H_1^0 \equiv H_1^0((0,l)) \quad \text{and} \quad \dot{u}(x,t) \in H^0 \equiv H_0((0,l)) \tag{3.9.2}$$

3. Prove that the mapping $t \mapsto u(x,t)$ is continuous from $\mathbb{R}$ into $H_1^0((0,l))$, while $t \mapsto \dot{u}(x,t)$ is continuous from $\mathbb{R}$ to $H_0((0,l))$, and, moreover, that

$$u(\cdot,t) \xrightarrow{H_1^0} \varphi(\cdot) \quad \text{and} \quad \dot{u}(\cdot,t) \xrightarrow{H_0} \psi(\cdot) \quad \text{as} \quad t \to 0. \tag{3.9.3}$$

4. Prove the uniqueness of the solution to the problem (2.6.1)–(2.6.3) in the class of functions $u(x,t)$ that possess properties formulated in Problems 1–3.

   For $t \in \mathbb{R}$, denote by S_t the mapping

$$\varphi, \psi \mapsto (u(\cdot,t), \dot{u}(\cdot,t)) \tag{3.9.4}$$

   according to formula (2.6.11). According to the statement of Problem 2, the mapping S_t maps the space $E \equiv H_1^0((0,l)) \times H_0((0,l))$ into itself.

5. Prove that for all $t \in \mathbb{R}$ the mapping $S_t : E \to E$ is continuous.

6. Prove that the mappings S_t defined above form a group: $S_t S_\tau = S_{t+\tau}, \quad \forall t, \tau \in \mathbb{R}$.

7. Prove the energy conservation for solution (2.6.11):

$$H \equiv \int [|\dot{u}(x,t)|^2 + |u'(x,t)|^2]\, dx = \text{const}, \quad t \in \mathbb{R}.$$

Remark 3.9.1. The solutions written in (2.6.11) under assumptions (3.9.1) are precisely the finite energy solutions. It is for justification of the well-posedness of problems of the form (2.6.1)–(2.6.3) in the class of the finite energy functions that S.L. Sobolev introduced the functional spaces H_s.

Chapter 4

Fundamental solutions and the Green function for the partial differential equations

4.1 Fundamental solutions of the Laplace operator in $\mathbb{R}^n$

Distributions of several variables $x_1, \dots, x_n$ and operations with them are defined similarly to the case $n = 1$ (see Sections 3.2, 3.3). For example,

$$\langle \delta_{(n)}(x), \varphi(x) \rangle \equiv \varphi(0), \quad \forall \varphi \in C_0^\infty(\mathbb{R}^n); \tag{4.1.1}$$

for each distribution $u(x) \in D'(\mathbb{R}^n)$

$$\langle \frac{\partial u}{\partial x_2}, \varphi(x) \rangle \equiv -\langle u, \frac{\partial \varphi}{\partial x_2} \rangle, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^n). \tag{4.1.2}$$

Denote

$$\triangle_n = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2} \tag{4.1.3}$$

Usually $\triangle_n$ is simply denoted by $\triangle$.

Problem 4.1.1. Find the fundamental solution of the operator $\triangle_3$, that is, the function $e(x) \in D'(\mathbb{R}^3)$ such that

$$\triangle_3 e(x) = \delta_{(3)}(x), \quad x \in \mathbb{R}^3 \tag{4.1.4}$$

For this, we approximate $\delta_{(3)}$-function by the step-functions $\delta_\rho(x)$, analogous to the Steklov step-functions (3.3.19):

$$\delta_\rho(x) = \begin{cases} \frac{1}{\Omega_\rho} & \text{for} \quad |x| < \rho, \\ 0 & \text{for} \quad |x| > \rho, \end{cases} \tag{4.1.5}$$

where $\Omega_\rho = \frac{4}{3}\pi\rho^3$ is the volume of a ball of radius $\rho > 0$.

Problem 4.1.2. Prove that (analogously to (3.3.20))

$$\delta_\rho(x) \stackrel{D'(\mathbb{R}^3)}{\longrightarrow} \delta_{(3)}(x) \quad \text{as} \quad \rho \to 0+\,. \tag{4.1.6}$$

We find the solution to equation (4.1.4) as the limit as $\rho \to 0+$ of solutions e_ρ to the equation

$$\triangle_3 e_\rho(x) = \delta_\rho(x), \quad x \in \mathbb{R}^3. \tag{4.1.7}$$

It is natural to look for the solution to this equation in the form $e_\rho(x) \equiv E_\rho(r)$, where $r = |x|$. To accomplish this, we introduce the spherical coordinates in (4.1.7).

Problem 4.1.3. Prove that

$$\triangle_3 E_\rho(r) = \frac{\partial^2 E_\rho}{\partial r^2} + \frac{2}{r}\frac{\partial E_\rho}{\partial r} = \frac{1}{r}\frac{\partial^2}{\partial r^2}(E_\rho \cdot r). \tag{4.1.8}$$

Corollary 4.1.1. *As follows from equation (4.1.7),*

$$\frac{1}{r}(E_\rho r)''_{rr} = \delta_\rho(r) \quad \text{for} \quad r > 0, \tag{4.1.9}$$

where

$$\delta_{(\rho)}(r) \equiv \begin{cases} \frac{1}{|\Omega_\rho|} & \text{for} \quad 0 < r < \rho, \\ 0 \quad \text{for} \quad r > \rho. \end{cases}$$

This equation is readily solved by integrating twice:

$$E_\rho \cdot r = \begin{cases} \frac{r^3}{6}\frac{1}{\Omega_\rho} + C_1 + C_2 r & \text{for} \quad 0 < r < \rho \\ C_3 + C_4 r \quad \text{for} \quad r > \rho. \end{cases} \tag{4.1.10}$$

It remains to find constants $C_1, \dots, C_4$.

We take into account that, according to (4.1.9), $r \cdot E_\rho$ and $(r \cdot E_\rho)'$ are continuous at $\rho = r$ (this follows from formula (3.4.1)):

$$\begin{cases} \frac{\rho^3}{6}\,\frac{1}{\Omega_\rho} + C_1 + C_2\rho = C_3 + C_4\rho, \\ \frac{\rho^2}{2}\,\frac{1}{\Omega_\rho} + C_2 = C_4. \end{cases} \tag{4.1.11}$$

The constant C_4 is arbitrary, since one can add an arbitrary constant to a solution of equation (4.1.7) (and (4.1.4)). So let us set $C_4 = 0$.

Then we obtain from (4.1.11) that $C_2 = -\frac{\rho^2}{2\Omega_\rho} = -\frac{3}{8\pi\rho}$, and

$$C_3 = \frac{1}{8\pi} + C_1 - \frac{3}{8\pi} = C_1 - \frac{1}{4\pi}. \tag{4.1.12}$$

We choose C_1 so that $E_\rho(0) < \infty$, that is, $C_1 = 0$. Then $C_3 = -\frac{1}{4\pi}$, and according to formulas (4.1.11) we get

$$E_\rho(r) = \begin{cases} \frac{r^2}{8\rho^3} - \frac{3}{8\rho} & \text{for} \quad 0 < r < \rho, \\ -\frac{1}{4\pi r} \quad \text{for} \quad r > \rho. \end{cases} \tag{4.1.13}$$

Remark 4.1.1. The condition $E_\rho(0)<\infty$ led to $e_\rho(x)\equiv E_\rho(|x|)$ being a smooth function in the ball $|x|<\rho$:

$$e_\rho(x)=\begin{cases}\frac{x_1^2+x_2^2+x_3^2}{8\rho^3}-\frac{3}{8\rho} & \text{for} \quad |x|<\rho,\\ -\frac{1}{4\pi|x|} & \text{for} \quad |x|>\rho.\end{cases} \tag{4.1.14}$$

Thus, it satisfies equation (4.1.7) not only for $x\in\mathbb{R}^3\setminus 0$, as follows from (4.1.9), but also in an open neighborhood of the point $x=0$, and hence for all $x\in\mathbb{R}^3$.

Problem 4.1.4. Prove that, in the sense of (weak) convergence of distributions in $D'(\mathbb{R}^3)$,

$$\varepsilon_\rho(x)\overset{D'(\mathbb{R}^3)}{\longrightarrow}-\frac{1}{4\pi|x|} \quad \text{as} \quad \rho\to 0+\,. \tag{4.1.15}$$

See Fig. 4.1.

Corollary 4.1.2. *As follows from (4.1.15),*

$$\triangle(-\frac{1}{4\pi|x|})=\delta_{(3)}(x), \quad x\in\mathbb{R}^3. \tag{4.1.16}$$

Indeed, from (4.1.15), due to the continuity of the operator $\triangle$ в $D'(\mathbb{R}^3)$ (see Lemma 3.3.1),

$$\triangle e_\rho(x)\overset{D'(\mathbb{R}^3)}{\to}\triangle(-\frac{1}{4\pi|x|}) \quad \text{as} \quad \rho\to 0+\,. \tag{4.1.17}$$

But, on the other hand, from (4.1.7) and (4.1.6) we also see that

$$\triangle e_\rho(x)=\delta_\rho(x)\overset{D'(\mathbb{R}^3)}{\to}\delta_{(3)}(x) \quad \text{as} \quad \rho\to 0+. \tag{4.1.18}$$

Relation (4.1.16) follows from (4.1.17) and (4.1.18).

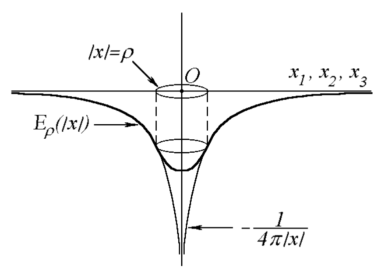


Figure 4.1:

Thus, the fundamental solution for $\triangle_3$ is a so-called "Coulomb" (or "Newton") potential

$$e(x)=\frac{1}{4\pi|x|}. \tag{4.1.19}$$

Answer.

$$e(x)=\frac{1}{2\pi}\ln|x|, \quad x\in\mathbb{R}^2. \tag{4.1.20}$$

Problem 4.1.5. Prove that if $C_1\neq 0$ in (4.1.10), then the function $E_\rho(|x|)$ is not a solution to equation (4.1.7).

Hint. Use (4.1.16).

Problem 4.1.6. Find the fundamental solutions for the operator $\triangle_2$.

Problem 4.1.7. Find the fundamental solutions for the operators $\triangle_n$, $n>3$; for $\triangle_3\pm k^2$, where $k>0$.

4.2 Potentials and their properties

Volume potentials

Once one knows the fundamental solutions to the Laplace equation (4.1.19) and (4.1.20), one can also find solutions to the nonhomogeneous Laplace equation in $\mathbb{R}^n$ for $n = 2$ and $n = 3$. For example, the solution to the equation

$$\triangle_2 u(x) = f(x), \quad x \in \mathbb{R}^2 \tag{4.2.1}$$

is the function

$$u(x) = \frac{1}{2\pi} \int\limits_{\mathbb{R}^2} \ln|x-y| f(y)\,dy, \tag{4.2.2}$$

if $f(x) \in C(\mathbb{R}^2)$, $\;f(x) = 0$ for $|x| > \text{const}$. In the same fasion, a solution to the equation

$$\triangle_3 u(x) = f(x), \quad x \in \mathbb{R}^3 \tag{4.2.3}$$

is given by

$$u(x) = -\frac{1}{4\pi} \int\limits_{\mathbb{R}^3} \frac{1}{|x-y|} f(y)\,dy. \tag{4.2.4}$$

Remark 4.2.1. Integrals of form (4.2.4) are called the *Coulomb* (or *Newton*) *volume* potentials. As the matter of fact, in Electrostatics, integral (4.2.4) up to a scalar factor (that depends on the choice of units) and up to the sign represents the potential of the electric field of the charge with the volume density $f(x)$. Equation (4.2.3) for the electric potential is called the Poisson equation. It takes form (4.2.1) in a particular case when the charge distribution $f(x)$ does not depend on the coordinate x_3. For example, the potential of a uniformly charged straight infinite wire satisfies (4.2.1).

It follows that the fundamental solution $-\frac{1}{4\pi}\frac{1}{|x|}$ from (4.1.19) is the potential of a point charge $+1$ located at the point $x = 0$, since in this case the charge distribution is given by the Dirac delta-function, $f(x) = \delta(x)$.

Problem 4.2.1. Compute the potential of a uniform distribution of charges with the density ρ in a spherical layer $R_1 < |x| < R_2$.

Solution. The potential we are looking for could be converted to a form

$$u(x) = -\frac{1}{4\pi} \int\limits_{R_1<|y|<R_2} \frac{\rho\,dy}{|x-y|} = \int\limits_{R_1}^{R_2} u_r(x)\,dr. \tag{4.2.5}$$

Here $u_r(x)$ is the potential of the same form as in (4.2.13), obtained according to formula (4.2.15) (see selow):

$$u_r(x) = -\frac{1}{4\pi} \int\limits_{|y|=r} \frac{\rho\,dS(y)}{|x-y|} = \begin{cases} -\rho r, & |x| < r, \\ -\frac{\rho r^2}{|x|}, & |x| > r. \end{cases} \tag{4.2.6}$$

We consider the three cases:

$$1)\quad |x| < R_1 \Longrightarrow u(x) = \int_{R_1}^{R_2} (-\rho r)\, dr = -\rho\Big(\frac{R_2^2}{2} - \frac{R_1^2}{2}\Big); \tag{4.2.7}$$

$$2)\quad R_1 < |x| < R_2 \Longrightarrow u(x) = \int_{R_1}^{|x|} \Big(-\frac{\rho r^2}{|x|}\Big)\, dr + \int_{|x|}^{R_2} (-\rho r)\, dr = \tag{4.2.8}$$

$$= -\frac{\rho}{|x|}\Big(\frac{|x|^3}{3} - \frac{R_1^3}{3}\Big) - \rho\Big(\frac{R_2^2}{2} - \frac{|x|^2}{2}\Big); \tag{4.2.9}$$

$$3)\quad |x| > R_2 \Longrightarrow u(x) = \int_{R_1}^{R_2} \Big(-\frac{\rho r^2}{|x|}\Big)\, dr = -\frac{\rho}{|x|}\Big(\frac{R_2^3}{3} - \frac{R_1^3}{3}\Big). \tag{4.2.10}$$

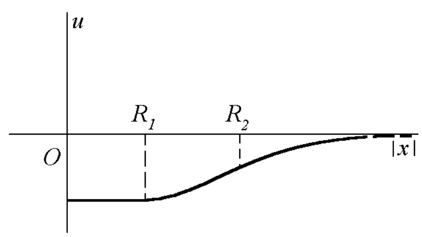


Figure 4.2:

The graph of potential (4.2.5) is plotted on Fig. 4.2.

Remark 4.2.2. For $|x| > R_2$ potential (4.2.10) is equal to the Coulomb potential of a point charge, of the value equal to the total charge of the spherical layer:

$$u(x) = -\frac{1}{4\pi} \frac{\frac{4}{3}\pi R_2^3 - \frac{4}{3}\pi R_1^3}{|x|} \tag{4.2.11}$$

The surface potentials

A. The *simple-layer potential* is a potential of the charge distributed over a surface:

$$u(x) = -\frac{1}{4\pi} \int_S \frac{1}{|x-y|} \sigma(y)\, dS(y). \tag{4.2.12}$$

Here S is a smooth compact surface in $\mathbb{R}^3$ and $\sigma(y)$ is the surface charge density.

Problem 4.2.2. Compute the potential of the uniform distribution of charge on a sphere $|x| = R$, with the surface density σ.

Solution.

$$u(x) = -\frac{1}{4\pi} \int\limits_{|y|=R} \frac{\sigma\, dS}{|x-y|} = -\frac{1}{4\pi} \int\limits_0^\pi \int\limits_0^{2\pi} \frac{\sigma R^2 \sin\Theta d\Theta d\varphi}{\sqrt{R^2 + |x|^2 - 2R|x|\cos\Theta}}. \tag{4.2.13}$$

Above, Θ, φ are the spherical coordinates of the point y, counted from the vector x, with φ being the longtitude and Θ being the altitude. By the Cosine theorem,

$$\begin{aligned} |x-y|^2 &= |x|^2 + |y|^2 - 2|x| \cdot |y| \cdot \cos\Theta = \\ &= |x|^2 + R^2 - 2|x| R \cos\Theta. \end{aligned}$$

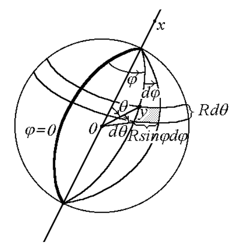


Figure 4.3:

Integral (4.2.13) is readily evaluated:

$$\begin{aligned} u(x) &= -\frac{\sigma R^2}{4\pi} 2\pi \int\limits_0^\pi \frac{-d\cos\Theta}{\sqrt{R^2 + |x|^2 - 2R|x|\cos\varphi}} = \\ &= -\frac{\sigma R^2}{2} \int\limits_1^{-1} \frac{-dt}{\sqrt{R^2 + |x|^2 - 2R|x|t}} = \frac{\sigma R^2}{2} 2 \frac{\sqrt{R^2 + |x|^2 - 2R|x|t}}{-2R|x|}\Bigg|_1^{-1} = \\ &= -\frac{\sigma R}{2|x|} \left(\sqrt{R^2 + |x|^2 + 2R|x|} - \sqrt{R^2 + |x|^2 - 2R|x|}\right) = \\ &= -\frac{\sigma R}{2|x|} \left(|R + |x|| - |R - |x||\right) \end{aligned}$$

$$= \begin{cases} -\sigma R, & |x| \le R, \\ -\frac{\sigma R^2}{|x|} & |x| > R. \end{cases} \tag{4.2.14}$$

Let us point out that for $|x| > R$ potential (4.2.15) coincides with the Coulomb potential u_Q of the point charge of magnitude $Q = 4\pi R^2 \sigma$ equal to the charge of the sphere:

$$u_Q(x) = -\frac{1}{4\pi} \frac{Q}{|x|} = -\frac{-1}{4\pi} \frac{4\pi R^2 \sigma}{2|x|} = -\frac{\sigma R}{2|x|}. \tag{4.2.16}$$

Remark 4.2.3. The simple-layer potential (4.2.15) is continuous on the sphere $|x| = R$, while its normal derivative is discontinuous, with

$$\frac{\partial u}{\partial n}\bigg|_{|x|=R+0} - \frac{\partial u}{\partial n}\bigg|_{|x|=R-0} = \frac{\sigma R^2}{|x|^2}\bigg|_{|x|=R} = \sigma. \tag{4.2.17}$$

Answer.

$$u(x) = \begin{cases} -\sigma R, & |x| \le R, \\ -\frac{\sigma R^2}{|x|}, & |x| > R. \end{cases} \tag{4.2.15}$$

See the plot on Fig. 4.4.

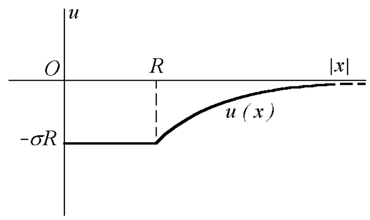


Figure 4.4:

Besides, one can easily see from (4.2.15) that

$$\triangle u(x) = 0 \quad \text{for} \quad |x| \neq R. \tag{4.2.18}$$

It turns out that these properties of the simple-layer potential are common for integrals of form (4.2.12).

Properties of the *simple-layer* potential:

1) If $\sigma(y)$ is a continuous function, then so is $u(x)$ for all $x \in \mathbb{R}^3$, including $x \in S$;

2) If $\sigma(y)$ has a continuous derivative, then

$$\frac{\partial u}{\partial n}(x + 0 \cdot n) - \frac{\partial u}{\partial n}(x - 0 \cdot n) = \sigma(x), \tag{4.2.19}$$

where n is the direction of the normal to S at the point $x \in S$;

3) For $x \notin S$, the potential is a harmonic function:

$$\triangle_3 u(x) = 0 \quad \text{for} \quad x \in \mathbb{R}^3 \setminus S \tag{4.2.20}$$

B. The *double-layer* potential is a potential of the surface distribution of *dipoles*. For starters, let us compute the potential of one dipole. A dipole in Electrostatics is a pair of point charges (di-pole) $+\frac{p}{\varepsilon}$ and $-\frac{p}{\varepsilon}$ at an "infinitely small" distance ε from one another.

The quantity p (the vector $p\vec{e}$, see Fig. 4.5) is called *the dipole moment.*

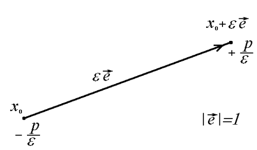


Figure 4.5:

The dipole potential is equal to

$$\begin{aligned} u(x) = \lim_{\varepsilon\to 0}\Big(-\frac{1}{4\pi}\Big(\frac{-\frac{p}{\varepsilon}}{|x - x_0|} + \frac{\frac{p}{\varepsilon}}{|x - x_0 - \varepsilon\vec{e}|}\Big)\Big) = \\ = -\frac{p}{4\pi}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big(\frac{1}{|x - x_0 - \varepsilon\vec{e}|} - \frac{1}{|x - x_0|}\Big) = \\ = -\frac{p}{4\pi}\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\frac{1}{|x - x_0 - \varepsilon\vec{e}|} = -\frac{p}{4\pi}\frac{1}{|x - x_0|^2}\cos(\widehat{x - x_0, \vec{e}}). \end{aligned} \tag{4.2.21}$$

So,

$$u(x) = -\frac{p}{4\pi}\frac{1}{|x-x_0|^2}\cos(\widehat{x-x_0,\vec{e}}). \tag{4.2.22}$$

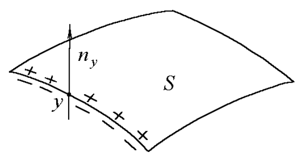


Figure 4.6:

The sign of expression (4.2.21) is determined by considering the case when the directions of the vectors $x - x_0$ and $\vec{e}$ coincide.

Now let us find the double-layer potential on the surface S in $\mathbb{R}^3$ with the dipole density $p(y)$, with the moments in the direction of the normal n_y to the surface at every $y \in S$:

$$u(x) = -\frac{1}{4\pi}\int\limits_S \frac{p(y)\cos(\widehat{x-y,n_y})\,dS(y)}{|x-y|^2}. \tag{4.2.23}$$

Remark 4.2.4. Potential (4.2.23) could be represented as the derivative in ε at $\varepsilon = 0$ of the simple-layer potential on the surface S_ε with the density $p(x)\frac{dS(x)}{dS_\varepsilon(x)}$. Here S_ε is the surface S, "shifted" by ε along the field of the field of normals (see Fig. 4.7).

Problem 4.2.3. Compute the potential of the double-layer potential for a sphere with the constant dipole density p.

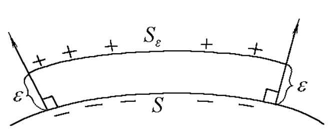


Figure 4.7:

Solution. We consider the simple-layer potential for the spheres of radii $R+\varepsilon$ and R with the charge density $\frac{p_\varepsilon}{\varepsilon}$ and $-\frac{p}{\varepsilon}$, respectively (Fig. 4.8).

The density p_ε could be determined from the fact that the total charge of the spheres is equal to zero:

$$\frac{p_\varepsilon}{\varepsilon}\cdot 4\pi(R+\varepsilon)^2 - \frac{p}{\varepsilon}4\pi R^2 = 0, \tag{4.2.24}$$

since the sum of charges in each dipole equals zero! Hence,

$$p_\varepsilon = p\frac{R^2}{(R+\varepsilon)^2}. \tag{4.2.25}$$

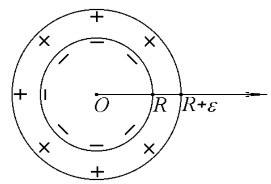


Figure 4.8:

Using formula (4.2.15), we obtain the desired double-layer potential:

$$u(x) \approx \begin{cases} -\frac{p_\varepsilon}{\varepsilon}(R+\varepsilon) + \frac{p}{\varepsilon}R, & |x| < R, \\ -\frac{p_\varepsilon(R+\varepsilon)^2}{\varepsilon|x|} + \frac{pR^2}{\varepsilon|x|}, & |x| > R+\varepsilon. \end{cases} \tag{4.2.26}$$

For $\varepsilon \to 0$ we obtain the exact formula

$$u(x) = \begin{cases} -\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} (R+\varepsilon)p_\varepsilon, & |x| < R, \\ -\frac{1}{|x|}\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} p_\varepsilon(R+\varepsilon)^2, & |x| > R. \end{cases} \tag{4.2.27}$$

Answer.

$$u(x) = \begin{cases} p, & |x| < R, \\ 0, & |x| > R. \end{cases} \tag{4.2.28}$$

Properties of the *double-layer* potential:

1. The double-layer potential (4.2.23) is a function that is discontinuous at the points of the surface S:

$$u(x + 0 \cdot n_x) - u(x - 0 \cdot n_x) = -p(x), \quad x \in S \tag{4.2.29}$$

   (if the function $u(x)$ is differentiable at the point x);

2. Beyond the surface S, the potential $u(x)$ is a harmonic function:

$$\triangle_3 u(x) = 0, \quad \text{for} \quad x \in \mathbb{R}^3 \setminus S. \tag{4.2.30}$$

Example. Potential (4.2.28) agrees with (4.2.29), (4.2.30).

4.3 Computing the potentials with the aid of the Ostrogradsky-Gauss formula

Since $\triangle u = \operatorname{div}\operatorname{grad} u$, the Poisson equation (4.2.3) could be written in the form

$$\operatorname{div}\operatorname{grad} u(x) = f(x). \tag{4.3.1}$$

Integrating this relation over an arbitrary domain $\Omega \subset \mathbb{R}^3$, and using the Ostrogradsky-Gauss theorem, we obtain:

$$\int\limits_{\partial\Omega} \operatorname{grad} u(x) \cdot n_x \, dS(x) = \int\limits_{\Omega} \operatorname{div}\operatorname{grad} u(x) \, dx = \int\limits_{\Omega} f(x) \, dx. \tag{4.3.2}$$

In Electrostatics, $\operatorname{grad} u(x) = E(x)$ (up to a sign) is the intensity vector of the electric field at the point x, while $Q(\Omega) = \int\limits_{\Omega} f(x)\, dx$ is the total charge of the region Ω. Hence, (4.3.2) could be rewritten in the form

$$\int\limits_{\partial\Omega} E(x) \cdot n_x \, dS(x) = Q(\Omega). \tag{4.3.3}$$

This identity is valid for any region $\Omega \subset \mathbb{R}^3$.

Let us compute potential (4.2.13) by the Ostrogradsky-Gauss method. The charge density in (4.2.13) is spherically symmetric, hence the potential $u(x)$ also possesses this property. Therefore,

$$u(x) = u_1(|x|). \tag{4.3.4}$$

Thus, the field $E(x) = \operatorname{grad} u(x)$ is radial:

$$E(x) = \frac{x}{|x|} u_1'(|x|). \tag{4.3.5}$$

Applying to this field identity (4.3.3) for the ball $\{|x| < r\} = \Omega$, we get:

$$|E(x)| \cdot 4\pi |x|^2 = \begin{cases} 0, & |x| < R, \\ 4\pi R^2 \sigma, & |x| > R. \end{cases} \tag{4.3.6}$$

According to (4.3.5), $|E(x)| = |u_1'(|x|)|$, and we get from (4.3.6):

$$u_1'(r) \cdot 4\pi r^2 = \begin{cases} 0, & r < R, \\ 4\pi R^2 \sigma, & r > R. \end{cases} \tag{4.3.7}$$

Hence,

$$u_1'(r) = \begin{cases} 0, & r < R, \\ \frac{R^2\sigma}{r^2}, & r > R. \end{cases} \tag{4.3.8}$$

Integrating, we get:

$$u_1(r) = \begin{cases} C_1, & r < R, \\ -\frac{R^2\sigma}{r} + C_2, & r > R. \end{cases} \tag{4.3.9}$$

Problem 4.3.1. Derive from (4.3.9) formula (4.2.15).

Hint. Constants C_1 and C_2 are determined from the continuity condition at $r = R$ and equality

$$\lim_{|x|\to\infty} u(x) = 0, \tag{4.3.10}$$

which obviously follows from (4.2.13).

Problem 4.3.2. Using the Ostrogradsky-Gauss method, compute potential (4.2.5).

4.4 Solution of the boundary value problems for the Laplace equation in three-dimensional domains. Constructing the Green functions by the method of reflections

The Dirichlet problem with zero boundary conditions is solved by the method of odd reflections, while the Neumann problem is solved by the method of even reflections. This is analogous to the situation with the wave equation (see Chapter I).

Solution of the Dirichlet problem in the half-space $\mathbb{R}^3_+ = \{x \in \mathbb{R}^3 : x_3 > 0\}$

$$\begin{cases} \triangle_3 u(x) = f(x_1, x_2, x_3), \quad x_3 > 0, \\ u(x_1, x_2, 0) = 0, \quad -\infty < x_1, \quad x_2 < +\infty; \quad u(x) \underset{|x|\to\infty}{\longrightarrow} 0. \end{cases} \tag{4.4.1}$$

Here $f(x)$ is a given function in $\mathbb{R}^3_+$, $f(x) \in C(\bar{\mathbb{R}}^3_+)$, $f(x) \equiv 0$ for $|x| > \text{const}$.

Let us find the Green function $G(x, y)$ for problem (4.4.1). By definition (compare with (3.6.2)), G is a solution to the problem

$$\begin{cases} \triangle_x G(x, y) = \delta(x - y), \quad x_3 > 0; \\ G((x_1, x_2, 0), y) = 0; \quad G(x, y) \to 0 \quad \text{as} |x| \to \infty, \end{cases} \tag{4.4.2}$$

smooth for $x \neq y$.

Here y is an arbitrary fixed point from $\mathbb{R}^3_+$. Denote by $\bar{y} = (y_1, y_2, -y_3)$ the point symmetric to y with respect to the boundary $x_3 = 0$ of the half-space $\mathbb{R}^3_+$:

Then the solution to problem (4.4.2) is given by the function

$$G(x, y) = -\frac{1}{4\pi}\frac{1}{|x - y|} + \frac{1}{4\pi}\frac{1}{|x - \bar{y}|}. \tag{4.4.3}$$

According to (4.1.16), $\triangle_x G(x, y) = \delta(x - y) - \delta(x - \bar{y})$. This yields the first equation in (4.4.2), since

$$\delta(x - \bar{y}) = 0 \quad \text{for} \quad x_3 > 0. \tag{4.4.4}$$

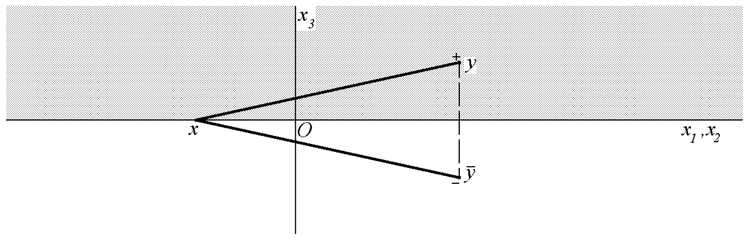


Figure 4.9:

Indeed, $\delta(x-\bar{y})$ is a distribution located at the point $\bar{y}$ of the lower half-space!

Let us verify the boundary condition in (4.4.2): If $x_3=0$, then the distances $|x-y|$ and $|(x-\bar{y})|$ are equal, as one sees from Fig. 4.9. Therefore from (4.4.3) one sees that $G(x,y)=0$. Finally, it is obvious that $G(x,y) \underset{|x|\to\infty}{\longrightarrow} 0$.

Solution (in the sense of distributions) to the boundary value problem (4.4.1) is given by the integral

$$u(x) = \int\limits_{\mathbb{R}^3_+} G(x,y)f(y)\,dy = -\frac{1}{4\pi}\int\limits_{y_3>0}\left(\frac{1}{|x-y|}-\frac{1}{|x-\bar{y}|}\right)f(y)\,dy. \tag{4.4.5}$$

Indeed, formally,

$$\triangle_x u(x) = \int\limits_{\mathbb{R}^3_+} \triangle_x G(x,y)f(y)\,dy = \int\limits_{\mathbb{R}^3_+} \delta(x-y)f(y)\,dy = f(x). \tag{4.4.6}$$

The boundary condition is readily verified:

$$u\Big|_{x_3=0} = \int\limits_{\mathbb{R}^3_+} G(x,y)\Big|_{x_3=0} f(y)\,dy = 0 \quad \text{and} \quad u(x) \underset{|x|\to\infty}{\longrightarrow} 0. \tag{4.4.7}$$

Electrostatic interpretation of problems (4.4.1), (4.4.2). The method of reflected charges

In the Electrostatics, the solution $u(x)$ to the boundary value problem (4.4.1), up to a sign and a factor that depends on the metric system, is the potential of the electrostatic field generated by the charge density $f(x)$ in the upper half-plane $\mathbb{R}^3_+$, located above the conducting surface $x_3=0$ (this could be, for example, the surface of the Earth or the flat tin roof). Electrostatically, the Green function $G(x,y)$ from (4.4.2) could be viewed as the potential of the point charge of magnitude $+1$, located at the point y above the conducting plane $x_3=0$. The field of the point charge redistributes the charges in the

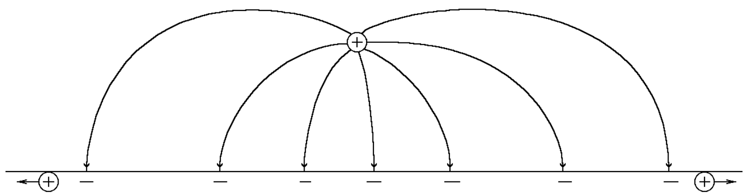


Figure 4.10:

plane $x_3 = 0$: It attracts negative charges while detracting positive charges (and they go to infinity):

It is known that, thereafter, the field lines (the integral curves) $E(x) = \operatorname{grad} u(x)$ (see Fig. 4.10) are orthogonal to the conducting surface (the Earth), or else the free charges in the conductor would start moving along the surface. It follows that the surface of a conductor is the level surface of the potential $u(x)$ (equipotential surface in Electrostatics).

It is this property of the field lines that allows to find the field $E(x)$. For this, let us recall the plot of the field curves of the field of two point charges of the same magnitude and opposite sign (see Fig. 4.11). As follows from the symmetry of the field curves with respect to the plane of symmetry of the charges, the field curves are orthogonal to this plane. Therefore the field above the plane of symmetry coincides with the field we are looking for. This yields formula (4.4.3).

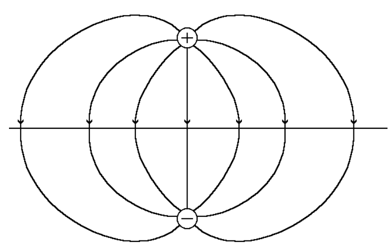


Figure 4.11:

The Dirichlet problem in the quarter-space

The quarter of the space $\mathbb{R}^3_{++}$ is the angular domain with the angle of magnitude 90° (see Fig. 4.12). Let $f(x) \in C(\mathbb{R}^3_{++})$, $f(x) \equiv 0$ for $x >$ const. Consider the Dirichlet problem in $\mathbb{R}^3_{++}$

$$\begin{cases} \triangle u(x_1, x_2, x_3) = f(x_1, x_2, x_3), \\ \qquad x_1 > 0, \quad x_2 > 0, \quad x_3 \in \mathbb{R}; \\ u|_{x_1=0} = 0, \quad u|_{x_2=0} = 0; \quad u(x) \to 0 \quad \text{as} \quad |x| \to \infty. \end{cases} \tag{4.4.8}$$

The Green function $G(x, y)$ of problem (4.4.8) by definition is a solution of the boundary value problem

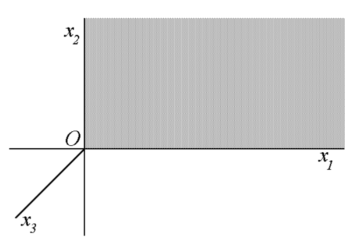


Figure 4.12:

$$\begin{cases} \triangle_x G(x,y) = \delta(x-y), \quad x \in \mathbb{R}^3_{++}; \\ G|_{x_1=0} = 0, \quad G|_{x_2=0} = 0; \\ G(x,y) \to 0 \quad \text{as} \quad |x| \to \infty. \end{cases} \tag{4.4.9}$$

Here $y \in \mathbb{R}^3_{++}$ is a parameter.

This Green function is also found by the method of (odd) reflections (Fig. 4.13).

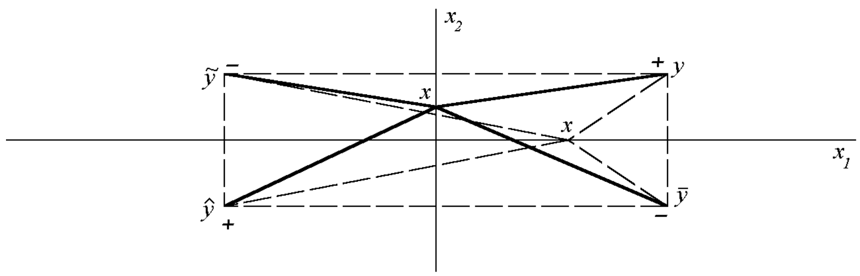


Figure 4.13:

Let $\bar{y} = (y_1, -y_2, y_3)$ be the reflection of the point y in the plane $x_2 = 0$, $\tilde{y} = (-y_1, y_2, y_3)$ be te reflection in the plane $x_1 = 0$, and $\hat{y} = (-y_1, -y_2, y_3)$ be the composition of these maps. We put the charges of magnitude $+1$ into the points y and $\hat{y}$, and the charges of magnitude -1 into the points $\bar{y}$ and $\tilde{y}$. Then their electrostatic field is represented by the potential

$$G(x,y) = -\frac{1}{4\pi}\frac{1}{|x-y|} + \frac{1}{4\pi}\frac{1}{|x-\bar{y}|} + \frac{1}{4\pi}\frac{1}{|x-\tilde{y}|} - \frac{1}{4\pi}\frac{1}{|x-\hat{y}|} \tag{4.4.10}$$

Let us verify that this function indeed satisfies equation (4.4.9). First of all, for $x \in \mathbb{R}^3_{++}$,

$$\triangle_x G(x,y) = \delta(x-y) - \delta(x-\bar{y}) - \delta(x-\tilde{y}) + \delta(x-\hat{y}) = \delta(x-y), \tag{4.4.11}$$

since $\delta(x-\bar{y})$, $\delta(x-\tilde{y})$ and $\delta(x-\hat{y})$ are equal to zero for $x \in \mathbb{R}^3_{++}$! Therefore the first equation in (4.4.9) is satisfied.

Further, let us verify the boundary conditions from (4.4.9): A) When $x_1 = 0$, the point x is equidistant from y and $\tilde{y}$, and also from $\bar{y}$ and $\hat{y}$ (see Fig. 4.13). Therefore in the right-hand side of (4.4.10) the first and the third term cancel out, and so do the second one with the fourth one; B) When $x_2 = 0$, the point x is equidistant from y and $\bar{y}$, and also from $\tilde{y}$ and $\hat{y}$ (see Fig. 4.13). Therefore, in the right-hand side of (4.4.10) the first and the second terms cancel out, and so do the third and the fourth ones. It is also obvious that $G(x,y) \to 0$ as $|x| \to \infty$.

Thus, the boundary conditions in (4.4.9) are also satisfied.

Therefore, $G(x,y)$ from (4.4.10) is the Green function of the Dirichlet problem (4.4.8). Hence the solution of the latter could be written in the form

$$u(x) = \int\limits_{\mathbb{R}^3_{++}} G(x,y)f(y)\,dy =$$
$$= -\frac{1}{4\pi}\int\limits_{\mathbb{R}^3_{++}} \Big(\frac{1}{|x-y|} - \frac{1}{|x-\bar{y}|} - \frac{1}{|x-\tilde{y}|} + \frac{1}{|x-\hat{y}|}\Big)f(y)\,dy. \tag{4.4.12}$$

Problem 4.4.1. Find the Green function and write the formula for the solution to the boundary value problem in a quadrant of the space (under the same conditions on $f(x)$ as in (4.4.8)):

$$\begin{cases} \triangle u(x) = f(x), \quad x_1 > 0, \quad x_2 > 0, \quad -\infty < x_3 < \infty; \\ u\Big|_{x_1=0} = 0, \quad \frac{\partial u}{\partial x_2}\Big|_{x_2=0} = 0; \quad u(x) \to 0 \quad \text{as} \quad |x| \to \infty. \end{cases} \tag{4.4.13}$$

Hint. One should apply the method of even reflection in x_1 and the method of odd reflection in x_2.

Problem 4.4.2. Find the Green functino and write the formula for the solution to the Dirichlet problem in the following domains:

1. In the angular domain, with the angle of magnitude $\alpha = \frac{\pi}{n}$, $n = 3, 4, 5, \ldots$ (above, this was done for $n = 1, 2$);
2. In the octant of the three-dimensional space: $x_1 > 0, \quad x_2 > 0, \quad x_3 > 0$;
3. In the layer: $0 < x_3 < a, \quad -\infty < x_1, \quad x_2 < +\infty$ (investigate the convergence of the obtained series);
4. In a “half” of the layer: $0 < x_3 < a, \quad -\infty < x_1 < \infty, \quad x_2 > 0$;
5. In a “quarter” of the layer: $0 < x_3 < a, \quad x_1 > 0, \quad x_2 > 0$.

Remark 4.4.1. In the previous problem, instead of the Dirichlet condition $u\big|_\Gamma = 0$ one can consider the Neumann condition $\frac{\partial u}{\partial n}\Big|_\Gamma = 0$ at certain parts of the boundary, as in problem (4.4.13). For solutions of such problems one has to use the method of even reflection at these parts of the boundary.

Problem 4.4.3. Find the Green function of the Dirichlet problem in the following domains:

1. In the ball $|x| < \mathbb{R}$. *Hint.* Look for the Green function in the form of the sum of the fundamental solution $-\frac{1}{4\pi}\frac{1}{|x-y|}$ and the potential of

the "reflected" charge $\frac{q}{4\pi}\frac{1}{|x-y^*|}$ of certain magnitude $q>0$, located at the point y^*, "symmetric" to the point y with respect to a sphere:

$$y^* = \frac{R^2}{|y|^2}y \qquad (4.4.14)$$

2. In a semiball $|x|<R, \quad x_3>0$. Hint: Use the method of odd reflection with respect to the plane x_3 to reduce the problem to the ball.
3. In a quarter of the ball $|x|<R, \quad x_2>0, \quad x_3>0$.

4.5 Solution of boundary value problems in two dimensions by the method of the Green function. Application of conformal mappings

The Dirichlet problem in half-plane $\mathbb{R}^2_+ = \{x \in \mathbb{R}^2 : x_2 > 0\}$

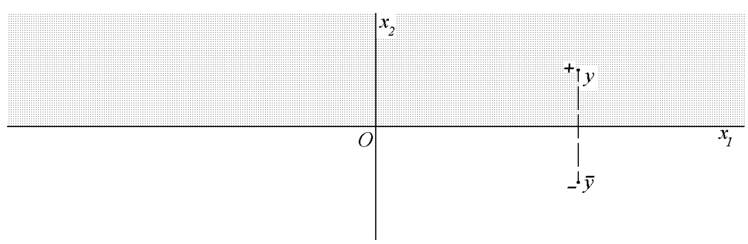


Figure 4.14:

$$\begin{cases} \triangle u(x_1,x_2) = f(x_1,x_2), \quad x_2>0, \quad -\infty<x_1<\infty; \\ u\Big|_{x_2=0} = 0, \quad u(x)\to 0 \quad \text{as} \quad |x|\to\infty, \end{cases} \qquad (4.5.1)$$

$$\text{where} \quad f(x)\in C(\bar{\mathbb{R}}^2_+), \quad f(x)=0 \quad \text{for} \quad |x|>\text{const}. \qquad (4.5.2)$$

The Green function $G(x,y)$ of this problem satisfies the equation

$$\begin{cases} \triangle_x G(x,y) = \delta(x-y), \quad x\in\mathbb{R}^2_+; \\ G\Big|_{x_2=0} = 0, \quad G(x,y)\to 0 \quad \text{as} \quad |x|\to\infty, \end{cases} \qquad (4.5.3)$$

where $y\in\mathbb{R}^2_+$. Similarly to (4.4.3), this function is found by the method of odd reflection applied to the fundamental solution (4.1.20) of the Laplace operator in the plane:

$$G(x,y) = \frac{1}{2\pi}\big(\ln|x-y| - \ln|x-\bar{y}|\big), \quad \text{where} \quad \bar{y} = (y_1,-y_2). \qquad (4.5.4)$$

Thus, the solution to problem (4.5.1) has the form

$$u(x) = \int_{\mathbb{R}^2_+} G(x,y) f(y)\, dy = \frac{1}{2\pi} \int \ln \frac{|x-y|}{|x-\bar{y}|} f(y)\, dy. \tag{4.5.5}$$

The Green function for two-dimensional domains

Unlike in the case of three-dimensional boundary value problems, the Green function for many simply connected two-dimensional domains could be found with the aid of conformal mappings. This is because the Green function $G(x, y)$ is harmonic in x for $x \neq y$, while the conformal mappings map harmonic functions again into harmonic ones.

Let us illustrate the relation of the Green functions to the conformal mappings on a particular example of the boundary value problem (4.5.1). For this, we rewrite formula (4.5.4) in the following form:

$$G(x,y) = \frac{1}{2\pi} \ln\Big(\frac{|x-y|}{|x-\bar{y}|}\Big) = \frac{1}{2\pi} \ln \left| \frac{x-y}{x-\bar{y}} \right|. \tag{4.5.6}$$

Remark 4.5.1. The last equality in (4.5.6) holds under the condition that $\frac{x-y}{x-\bar{y}}$ is understood in the sense of the division of the complex numbers:

$$\frac{x-y}{x-\bar{y}} = \frac{x_1 + ix_2 - y_1 - iy_2}{x_1 + ix_2 - y_1 + iy_2}. \tag{4.5.7}$$

Here $\bar{y} = y_1 - iy_2$ turns out to be the complex conjugate to y.

Let us point out that

1) For each fixed $y \in \mathbb{R}^2_+$ the map

$$x \mapsto z = \Phi_y(x) \equiv \frac{x-y}{x-\bar{y}} \tag{4.5.8}$$

maps the upper half-plane, $x_2 > 0$, conformally into the unit disc $|z| < 1$;

2) under the mapping (4.5.8), the point y is sent to 0:

$$y \mapsto \Phi_y(y) = 0. \tag{4.5.9}$$

Now let us consider a more general case of the Dirichlet problem in the flat simply connected region $\Omega \subset \mathbb{R}^2$ with a piecewise-smooth boundary $\partial\Omega$, that contains at least two points:

$$\begin{cases} \triangle u(x) = f(x), \quad x \in \Omega; \\ u\Big|_{x\in\partial\Omega} = 0 u(x) \to 0 \quad |u(x)| \to \infty, \quad x \in \Omega. \end{cases} \tag{4.5.10}$$

where $f(x) \in C(\bar{\Omega}), \quad f(x) = 0 \quad |x| > \mathrm{const}(y)$.

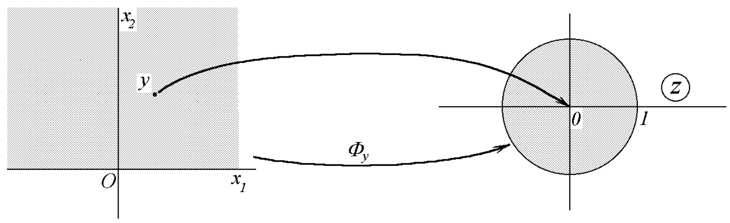


Figure 4.15:

The Green function of this problem by definition satisfies the conditions

$$\begin{cases} \triangle_x G(x,y) = \delta(x-y), \quad x \in \Omega; \\ u\Big|_{x\in\partial\Omega} = 0 G(x,y) \to 0 \quad |u(x)| \to \infty, \quad x \in \Omega. \end{cases} \tag{4.5.11}$$

Here y is a parameter, $y \in \Omega$. It is known from the theory of functions of a complex variable (the Riemann theorem, see [Sha85]) that for any simply connected region $\Omega \subset \mathbb{R}^2$ with the boundary $\partial\Omega$ that contains at least two points there exists a conformal mapping of the region Ω onto the unit disc. Moreover, any a priori fixed point y is mapped to zero. Let $\Phi_y(x)$ be such a map (see Fig. 4.16).

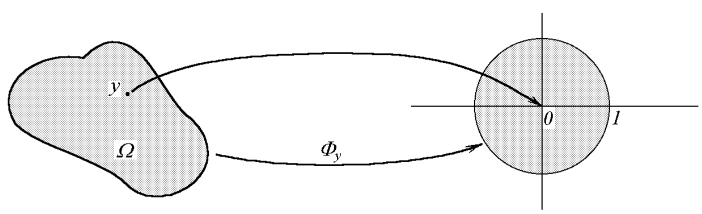


Figure 4.16:

It turns out [CH53] that the Green function (4.5.11) has the form (4.5.6):

$$G(x,y) = \frac{1}{2\pi} \ln |\Phi_y(x)|. \tag{4.5.12}$$

Then we get the solution to the Dirichlet problem (4.5.10):

$$u(x) = \frac{1}{2\pi} \int\limits_{\Omega} \ln |\Phi_y(x)| \, f(y) \, dy. \tag{4.5.13}$$

Problem 4.5.1. Check that function (4.5.12) is the solution to problem (4.5.11).

Hints.

A. $\ln|\Phi_y(x)| = \Re \ln \Phi_y(x)$ is a harmonic function if $\Phi_y(x) \neq 0$, that is, for $x \neq y$;

B. $\ln|\Phi_y(x)|$ at $x = y$ allows the decomposition

$$\ln|\Phi_y(x)| = \ln|x-y| + O(1), \quad x \to y; \tag{4.5.14}$$

C. Use the theorem about a removable singularity to prove that $O(1)$ in (4.5.14) is a harmonic function at $x = y$;

D. The boundary condition $G\Big|_{x\in\partial\Omega} = 0$ is obviously satisfied, since

$$|\Phi_y(x)\Big|_{x\in\partial\Omega}| = 1. \tag{4.5.15}$$

Problem 4.5.2. Let us find the Green function and the formula for the solution of the Dirichlet problem in the strip Ω (for $f(x) \in C(\bar{\Omega})$, $f(x) = 0$ for $|x| > \text{const}$):

$$\begin{cases} \triangle u(x) = f(x), \quad 0 < x_2 < a, \quad -\infty < x_1 < \infty; \\ u\Big|_{x_2=0,a} = 0, \quad u(x) \to 0 \quad \text{when} \quad |x| \to \infty. \end{cases} \tag{4.5.16}$$

Solution. Let us map conformally the strip into a disc (see Fig. 4.17).

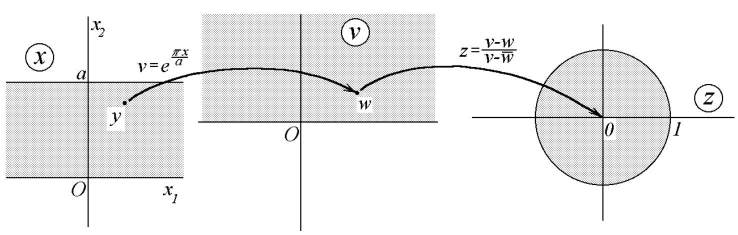


Figure 4.17:

The point y is mapped into the origin: $w = e^{\frac{\pi y}{a}}$,

$$z = \frac{e^{\frac{\pi x}{a}} - e^{\frac{\pi y}{a}}}{e^{\frac{\pi x}{a}} - e^{\frac{\pi \bar{y}}{a}}} \tag{4.5.17}$$

Now, according to (4.5.12),

$$G(x,y) = \frac{1}{2\pi} \ln \left| \frac{e^{\frac{\pi x}{a}} - e^{\frac{\pi y}{a}}}{e^{\frac{\pi x}{a}} - e^{\frac{\pi \bar{y}}{a}}} \right| \tag{4.5.18}$$

and using (4.5.13), the solution of problem (4.5.16) is given by

$$u(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_0^a \ln \left| \frac{e^{\frac{\pi x}{a}} - e^{\frac{\pi y}{a}}}{e^{\frac{\pi x}{a}} - e^{\frac{\pi \bar{y}}{a}}} \right| f(y)\, dy_2\, dy_1. \tag{4.5.19}$$

Let us mention that

$$e^{\frac{\pi x}{a}} = e^{\frac{\pi}{a}(x_1 + i x_2)} = e^{\frac{\pi x_1}{a}} (\cos \frac{\pi}{a} x_2 + i \sin \frac{\pi}{a} x_2). \tag{4.5.20}$$

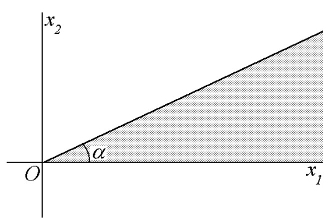


Figure 4.18:

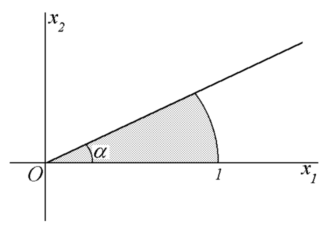


Figure 4.19:

Problem 4.5.3. Find the Green function and write the formula for the solution of the Dirichlet problem in the following regions:

1. Angle of magnitude α (see Fig. 4.18);
2. Disc: $|x| < 1$ (one gets the classical Poisson formula);
3. Half of a disc: $|x| < 1, \quad x_2 > 0$ (use the method of the odd reflection with resect to x_2 to reduce to Problem 2);
4. Sector of a disc: $|x| < 1, \quad 0 < \arg x < \alpha$ (see Fig. 4.19).

Remark 4.5.2. It turns out that, knowing the Green function of the Dirichlet problem in the region Ω, one can solve the homogeneous equations $\triangle u(x) = 0$ in Ω with non-homogeneous boundary conditions $u\Big|_{\partial\Omega} = f(x)$: if $\partial\Omega$ is of class C^2 and $f \in C^2(\partial\Omega)$, then (see the details in [Vla81, p. 429]; [Ole76, p. 48]):

$$u(x) = \int_{\partial\Omega} \frac{\partial G(x,y)}{\partial n_y} f(y)\, dy, \tag{4.5.21}$$

where $\frac{\partial}{\partial n_y}$ is the differentiation in the direction of the external normal to the boundary at the point $y \in \partial\Omega$. This holds for the region Ω of any dimension: $\Omega \subset \mathbb{R}^2, \quad \Omega \subset \mathbb{R}^3$, et cetera.

Problem 4.5.4. Let us find the formula for the solution of the Dirichlet problem in the half-plane

$$\triangle u(x_1,x_2)=0,\quad x_2>0;\quad u(x_1,0)=f(x_1);\quad u(x)\underset{|x|\to\infty}{\longrightarrow}0, \tag{4.5.22}$$

where $f(x_1)\in C(\mathbb{R})$, $f(x_1)=0$ for $|x_1|>\text{const.}$

Solution. The Green function for the half-plane is given by (4.5.4):

$$\begin{aligned} G(x,y) &= \frac{1}{2\pi}\ln\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}-\frac{1}{2\pi}\ln\sqrt{(x_1-y_1)^2+(x_2+y_2)^2}= \\ &= \frac{1}{4\pi}\ln\big((x_1-y_1)^2+(x_2-y_2)^2\big)-\frac{1}{4\pi}\big((x_1-y_1)^2+(x_2+y_2)^2\big). \end{aligned}$$

Taking into account that the external normal n_y to the half-plane $x_2>0$ is represented by the vector $(0,-1)$, we get:

$$\Big(\frac{\partial G(x,y)}{\partial n_y}\Big)\Big|_{y_2=0}=-\Big(\frac{\partial}{\partial y_2}G(x,y)\Big)\Big|_{y_2=0}=$$

$$=\frac{1}{4\pi}\frac{2x_2}{(x_1-y_1)^2+x_2^2}+\frac{1}{4\pi}\frac{2x_2}{(x_1-y_1)^2+x_2^2}=\frac{1}{\pi}\frac{x_2}{(x_1-y_1)^2+x_2^2}.$$

Answer.

$$u(x_1,x_2)=\frac{x_2}{\pi}\int\limits_{-\infty}^{+\infty}\frac{f(y_1)\,dy_1}{(x_1-y_1)^2+x_2^2}. \tag{4.5.23}$$

4.6 Solutions to the hyperbolic equations in the sense of distributions

Let us remind (see Section 1.2), that the solution to the homogeneous d'Alembert equation

$$\frac{\partial^2 u(x,t)}{\partial t^2}=a^2\frac{\partial^2 u(x,t)}{\partial x^2} \tag{4.6.1}$$

can be written as in (3.2.3):

$$u(x,t)=f(x-at)+g(x+at). \tag{4.6.2}$$

If the functions $f(x)$ and $g(x)$ are C^2 (have two continuous derivatives), then $u(x,t)$ from (4.6.2) also possesses the same property. If, instead, $f(x)$ or $g(x)$ are discontinuous, then $u(x,t)$ is also going to be discontinuous. Let us show that in this case the function $u(x,t)$ in (4.6.2) is still going to be a solution of equation (4.6.1) if one considers the derivatives in both sides of (4.6.1) in the sense of distributions (see Remark 1.2.1 of Chapter I). This means that

$$\left\langle\frac{\partial^2 u}{\partial t^2},\varphi(x,t)\right\rangle=a^2\left\langle\frac{\partial^2 u}{\partial x^2},\varphi(x,t)\right\rangle,\quad \forall\varphi\in C_0^\infty(\mathbb{R}^2). \tag{4.6.3}$$

To prove identity (4.6.3), let us remind that, according to the definition of the derivatives of the distributions,

$$\begin{cases} \langle \frac{\partial^2 u}{\partial t^2}, \varphi\rangle = -\langle \frac{\partial u}{\partial t}, \frac{\partial \varphi}{\partial t}\rangle = \langle u, \frac{\partial^2 \varphi}{\partial t^2}\rangle, \\ \langle \frac{\partial^2 u}{\partial x^2}, \varphi\rangle = -\langle \frac{\partial u}{\partial x}, \frac{\partial \varphi}{\partial x}\rangle = \langle u, \frac{\partial^2 \varphi}{\partial x^2}\rangle, \end{cases} \tag{4.6.4}$$

Therefore identity (4.6.3) is equivalent to

$$\left\langle u, \frac{\partial^2 \varphi}{\partial t^2}\right\rangle = a^2 \left\langle u, \frac{\partial^2 \varphi}{\partial x^2}\right\rangle, \tag{4.6.5}$$

or

$$\left\langle u, \frac{\partial^2 \varphi}{\partial t^2} - a^2 \frac{\partial^2 \varphi}{\partial x^2}\right\rangle = 0. \tag{4.6.6}$$

Identity (4.6.6) in the coordinates $\xi = x - at, \quad \eta = x + at$ takes the form (see (1.2.9))

$$\int_{-\infty}^{+\infty}\int u(\xi,\eta)\frac{\partial^2 \varphi}{\partial\xi\partial\eta}\, d\xi\, d\eta = 0. \tag{4.6.7}$$

Decomposition (4.6.2) means that

$$u(\xi,\eta) = f(\xi) + g(\eta). \tag{4.6.8}$$

Substituting (4.6.8) into (4.6.7), we get

$$\int_{-\infty}^{+\infty} f(\xi)\Big(\int_{-\infty}^{+\infty} \frac{\partial^2 \varphi}{\partial\xi\partial\eta}\, d\eta\Big)\, d\xi + \int_{-\infty}^{+\infty} g(\eta)\Big(\int_{-\infty}^{+\infty} \frac{\partial^2 \varphi}{\partial\xi\partial\eta}\, d\xi\Big)\, d\eta = 0. \tag{4.6.9}$$

But this identity is obvious, since

$$\begin{aligned} \int_{-\infty}^{+\infty} \frac{\partial^2 \varphi}{\partial\xi\partial\eta}\, d\eta &= \frac{\partial\varphi}{\partial\xi}(\xi,\eta)\Big|_{\eta=-\infty}^{\eta=+\infty} = 0, \\ \int_{-\infty}^{+\infty} \frac{\partial^2 \varphi}{\partial\xi\partial\eta}\, d\xi &= \frac{\partial\varphi}{\partial\eta}(\xi,\eta)\Big|_{\xi=-\infty}^{\xi=+\infty} = 0 \end{aligned} \tag{4.6.10}$$

due to the compact support of $\varphi(\xi,\eta)$. Thus, equality (4.6.3) is proved.

0.1 Appendix: Classification of the second order linear partial differential equations with constant coefficients

Differential equations with constant coefficients

Let us consider the following equation in $\mathbb{R}^n$:

$$\sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i}^{n} a_i \frac{\partial u}{\partial x_i} + a_0 u(x) = 0, \quad x \in \mathbb{R}^n; \quad a_{ij} = a_{ji}. \tag{0.1.1}$$

Let us bring it to the canonical form, that is, to the form so that $a_{ij} = 0$ for $i \neq j$. To accomplish this, consider the linear change of variables:

$$\begin{cases} y_1 = c_{11}x_1 + \ldots + c_{1n}x_n, \\ \ldots \\ y_n = c_{n1}x_1 + \ldots + c_{nn}x_n, \end{cases} \tag{0.1.2}$$

or, in the vector form,

$$y = Cx. \tag{0.1.3}$$

In the coordinates y_k we have $\frac{\partial u}{\partial x_i} = \sum_{k=1}^n \frac{\partial u}{\partial y_k}\frac{\partial y_k}{\partial x_i} = \sum_1^n C_{ki}\frac{\partial u}{\partial y_k}$, and also $\frac{\partial^2 u}{\partial x_i \partial x_j} = \sum_{k,l=1}^n C_{ki}C_{lj}\frac{\partial^2 u}{\partial y_k \partial y_l}$. Substituting into (0.1.1), we get

$$\sum_{i,j,k,l} a_{ij}C_{ki}C_{lj}\frac{\partial^2 u}{\partial y_k \partial y_l} + \ldots = 0, \tag{0.1.4}$$

where dots denote terms that contain lower order derivatives of the function u. We can write (0.1.4) as follows:

$$\sum_{k,l=1}^{n} b_{kl}\frac{\partial^2 u}{\partial y_k \partial y_l} + \ldots = 0, \tag{0.1.5}$$

where

$$b_{kl} = \sum_{i,j} a_{i,j}C_{kl}C_{lj}. \tag{0.1.6}$$

In the matrix form, $b = CaC^*$, where a is the matrix (a_{ij}), while C^* is the transpose of C. This formula resembles the transformation law for the matrix of the quadratic form

$$(a\xi, \xi) = \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j. \tag{0.1.7}$$

Namely, if one makes the change of variables

$$\xi = d\eta, \quad d = (d_{ij})_{i,j=1,\ldots,n}, \tag{0.1.8}$$

then, taking $d^* = C$, one gets

$$(a\xi, \xi) = (a\, d\eta, d\eta) = (d^* a\, d\eta, \eta) = (CaC^* \eta, \eta) = (b\eta, \eta). \tag{0.1.9}$$

Therefore, if the change of variables (0.1.8) brings the quadratic form to the diagonal form $(a\xi, \xi) = \sum_1^n b_k \eta_k^2$ (we know from the Linear Algebra that such a change of variables exists), then the change of variables (0.1.3) with the matrix $C = d^*$ brings the differential equation (0.1.1) to form (0.1.5) with the same diagonal matrix b:

$$\sum_1^n b_k \frac{\partial^2 u}{\partial y_k^2} + \ldots = 0. \tag{0.1.10}$$

After this is accomplished, one the following is possible:
I. det $a \neq 0$. Then equation (0.1.1) is called nondegenerage, and all b_k in (0.1.10) could be made equal to ± 1. Then there are three possibilities:

A. All the coefficients b_k are of the same sign (all are equal to $+1$ or instead all are equal to -1), then equation (0.1.10) has the form $\frac{\partial^2 u}{\partial y_1} + \ldots + \frac{\partial^2 u}{\partial y_n^2} + \ldots = 0$ and is called elliptic. An example is the Laplace equation (1.8.24);

B. All the coefficients b_k but one are of the same sign, so that (0.1.10) takes the form

$$\frac{\partial^2 u}{\partial y_k^2} + \ldots + \frac{\partial^2 u}{\partial y_{n-1}^2} - \frac{\partial^2 u}{\partial y_n^2} + \ldots = 0 \tag{0.1.11}$$

and is called hyperbolic. An example is the wave equation (1.7.1);

C. Some of the coefficients b_k (more than one) are positive, while others (also more than one) are negative; then (0.1.10) has the form

$$\frac{\partial^2 u}{\partial y_1^2} + \frac{\partial^2 u}{\partial y_2^2} \ldots - \frac{\partial^2 u}{\partial y_{n-1}^2} - \frac{\partial^2 u}{\partial y_n^2} + \ldots = 0 \tag{0.1.12}$$

and is called ultrahyperbolic. This is only possible if $n \geq 4$.

II. det $a = 0$. Then equation (0.1.1) is called degenerate, or parabolic in the general sense. An example is the heat equation (1.8.18).

Problem 0.1.1. Find the canonical form and the change of variables (0.1.2) for the equation

$$\frac{\partial^2 u}{\partial x_1^2} + 4\frac{\partial^u}{\partial x_1 \partial x_2} - 3\frac{\partial^2 u}{\partial x_3^2} = 0. \tag{0.1.13}$$

Solution. Compose the quadratic form (0.1.7) and bring it to the diagonal form:

$$\xi^2 + 4\xi_1\xi_2 - 3\xi_3^2 = (\xi_1 + 2\xi_2)^2 - 4\xi_2^2 - 3\xi_3^2 = \eta_1^2 - \eta_2^2 - 3\eta_3^2. \tag{0.1.14}$$

Therefore, equation (0.1.13) is of hyperbolic type, as in (0.1.11). The change of variables (0.1.8), or, rather, the inverse to it, has the form

$$\begin{cases} \eta_1 = \xi_1 + 2\xi_2, \\ \eta_2 = 2\xi_2, \\ \eta_3 = \xi_3. \end{cases} \tag{0.1.15}$$

To bring these relations to the form as in (0.1.8), one needs to solve equations (0.1.15):

$$\begin{cases} \xi_2 = \frac{\eta_2}{2}, \\ \xi_3 = \eta_3, \\ \xi_1 = \eta_1 - 2\xi_2 = \eta_1 - \eta_2. \end{cases}$$

From here we get the matrix d:

$$d = \begin{pmatrix} 1 & -1 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and, consequently,

$$C = d^* = \begin{pmatrix} 1 & 0 & 0 \\ -1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore, substitution (0.1.2) has the form

$$\begin{cases} y_1 = x_1, \\ y_2 = -x_1 + \frac{1}{2}x_2, \\ y_3 = x_3. \end{cases} \tag{0.1.16}$$

According to (0.1.14), the canonical form of equation (0.1.13) is as follows:

$$\frac{\partial^2 u}{\partial y_1^2} - \frac{\partial^2 u}{\partial y_2^2} - 3\frac{\partial^2 u}{\partial y_3^2} = 0. \tag{0.1.17}$$

Problem 0.1.2. Find the canonical form and the change of variables (0.1.2) for equations

$$\frac{\partial^2 u}{\partial x_1 \partial x_2} + \frac{\partial^2 u}{\partial x_2 \partial x_3} + \frac{\partial^2 u}{\partial x_3 \partial x_1} = 0 \tag{0.1.18}$$

$$\frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_1 \partial x_2} + 6\frac{\partial^2 u}{\partial x_1 \partial x_3} = 0. \tag{0.1.19}$$

Equations with variable coefficients

Now we assume that the coefficients in (0.1.1) are varying:

$$\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \ldots = 0. \tag{0.1.20}$$

Then for each fixed $x_0 \in \mathbb{R}^n$ one can consider the equation with the constant coefficients, obtained from the variable coefficients "frozen" at the point x_0:

$$\sum_{i,j=1}^{n} a_{ij}(x_0)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \ldots = 0. \tag{0.1.21}$$

The type of this equation is called the type of equation (0.1.20) at the point x_0. The example is the Tricomi equation

$$y\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{0.1.22}$$

which is elliptic in the half-plane $y > 0$, hyperbolic in the half-plane $y < 0$, and degenerate on the line $y = 0$.

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