Computer Algebra for Lattice Path Combinatorics

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Inria

Computer Algebra in Combinatorics

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Overview

Part 1: General presentation

Part 2: Guess'n'Prove

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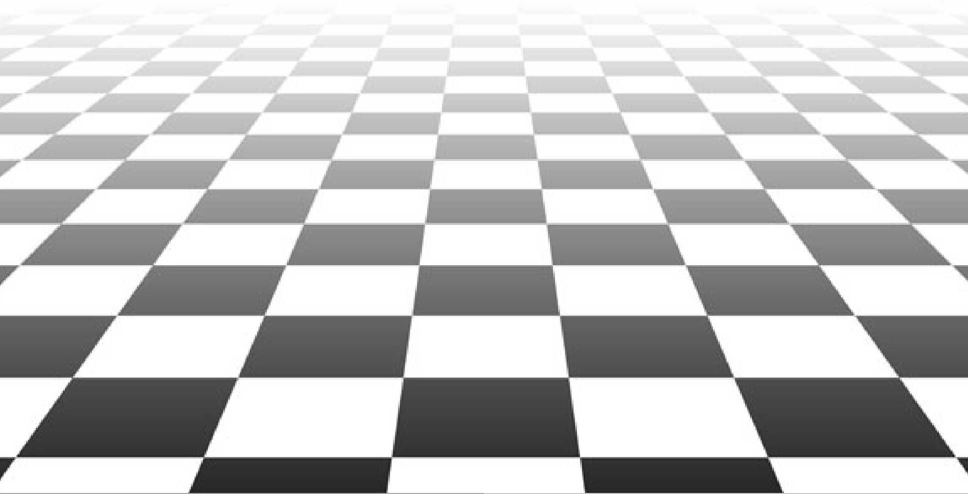

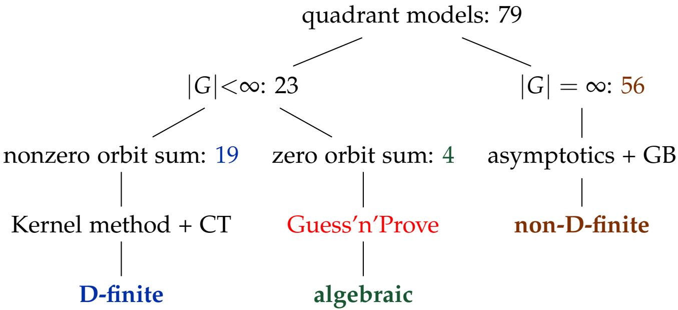
quadrant models: 79
$|G| < \infty$: 23
$|G| = \infty$: 56
nonzero orbit sum: 19
zero orbit sum: 4
asymptotics + GB
Kernel method + CT
Guess'n'Prove
non-D-finite
D-finite
algebraic

Summary of Part 1: Classification of 2D non-singular walks

The Main Theorem Let $\mathfrak{S}$ be a 2D non-singular model with small steps. The following assertions are equivalent:

(1) The full generating function $F_{\mathfrak{S}}(t;x,y)$ is D-finite

(2) the excursions generating function $F_{\mathfrak{S}}(t;0,0)$ is D-finite

(3) the excursions sequence $[t^n]\, F_{\mathfrak{S}}(t;0,0)$ is $\sim K \cdot \rho^n \cdot n^\alpha$, with $\alpha \in \mathbb{Q}$

(4) the group $\mathcal{G}_{\mathfrak{S}}$ is finite (and $|\mathcal{G}_{\mathfrak{S}}| = 2 \cdot \min\{\ell \in \mathbb{N}^\star \mid \frac{\ell}{\alpha+1} \in \mathbb{Z}\}$)

(5) the step set $\mathfrak{S}$ has either an axial symmetry, or zero drift and cardinality different from 5.

Proof

(1) $\Rightarrow$ (2) Easy

(2) $\Rightarrow$ (3) [Denisov, Wachtel, 2013] + [Katz '70, Chudnovsky '85, André '89]

(3) $\Rightarrow$ (4) [B., Raschel, Salvy, 2013]

(4) $\Rightarrow$ (1) [Bousquet-Mélou, Mishna, 2010] + [B., Kauers, 2010]

(5) $\Leftrightarrow$ (4) A posteriori observation

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(5) the step set $\mathfrak{S}$ has either an axial symmetry, or zero drift and cardinality different from 5.

Moreover, under (1)–(5), $F_{\mathfrak{S}}(t;x,y)$ is algebraic if and only if the model $\mathfrak{S}$ has positive covariance $\sum_{(i,j)\in\mathfrak{S}} ij - \sum_{(i,j)\in\mathfrak{S}} i \cdot \sum_{(i,j)\in\mathfrak{S}} j > 0$, and iff it has $\mathrm{OS}=0$.

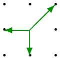
In this case, $F_{\mathfrak{S}}(t;x,y)$ is expressible using nested radicals.
If not, $F_{\mathfrak{S}}(t;x,y)$ is expressible using iterated integrals of ${}_2F_1$ expressions.

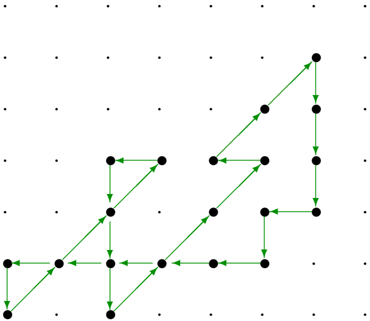
▷ Proof of the last statements: [B., Chyzak, van Hoeij, Kauers, Pech, 2017]

Two important models: Kreweras and Gessel walks

$\mathfrak{S} = \{\downarrow, \leftarrow, \nearrow\}$ $\qquad F_{\mathfrak{S}}(t;x,y) \equiv K(t;x,y)$

$\mathfrak{S} = \{\nearrow, \swarrow, \leftarrow, \rightarrow\}$ $\qquad F_{\mathfrak{S}}(t;x,y) \equiv G(t;x,y)$





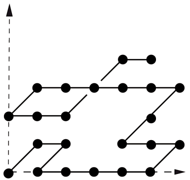
Example: A Kreweras excursion.

Gessel's conjecture

- Gessel walks: walks in $\mathbb{N}^2$ using only steps in $\mathfrak{S} = \{\nearrow, \swarrow, \leftarrow, \rightarrow\}$
- $g(n; i, j)$ = number of walks from $(0, 0)$ to (i, j) with n steps in $\mathfrak{S}$

Question: Find the nature of the generating function

$$G(t; x, y) = \sum_{i,j,n=0}^{\infty} g(n; i, j)\, x^i y^j t^n \in \mathbb{Q}[[x, y, t]]$$

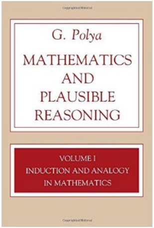


Theorem (B.-Kauers, 2010) $G(t; x, y)$ is an algebraic function†.

$\rightarrow$ Effective, computer-driven discovery and proof

† Minimal polynomial $P(x, y, t, G(t; x, y)) = 0$ has $> 10^{11}$ terms; ≈ 30 Gb (!)

First guess, then prove [Pólya, 1954]



Guessing and Proving

George Pólya

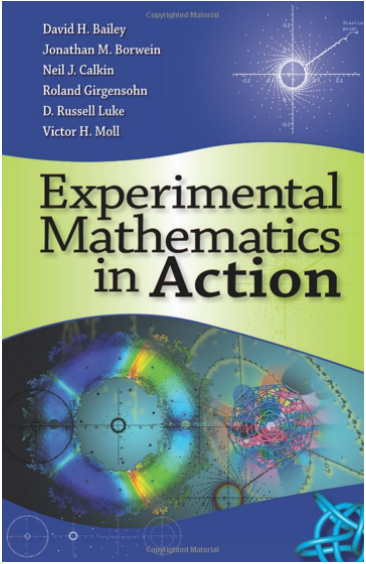
What is "scientific method"? Philosophers and non-philosophers have discussed this question and have not yet finished discussing it. Yet as a first introduction it can be described in three syllables:

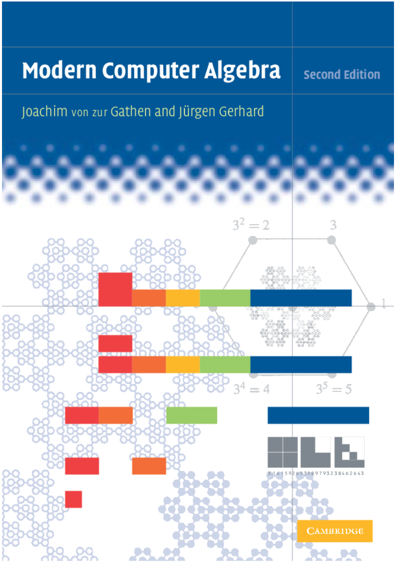
Guess and test.

Mathematicians too follow this advice in their research although they sometimes refuse to confess it. They have, however, something which the other scientists cannot really have. For mathematicians the advice is

First guess, then prove.

Personal bias: Experimental Mathematics using Computer Algebra





Guess'n'Prove for

–PROVING ALGEBRAICITY–

Methodology for proving algebraicity

Experimental mathematics –Guess'n'Prove– approach:

(S1) Generate data

(S2) Conjecture

(S3) Prove

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compute a high order expansion of the series $F_{\mathfrak{S}}(t;x,y)$;

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guess a candidate for the minimal polynomial of $F_{\mathfrak{S}}(t;x,y)$, using Hermite-Padé approximation;

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rigorously certify the minimal polynomials, using (exact) polynomial computations.

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+ Efficient Computer Algebra

Step (S1): high order series expansions

$f_{\mathfrak{S}}(n;i,j)$ satisfies the recurrence with constant coefficients

$$f_{\mathfrak{S}}(n+1;i,j) = \sum_{(u,v)\in\mathfrak{S}} f_{\mathfrak{S}}(n;i-u,j-v) \quad \text{for} \quad n,i,j \geq 0$$

+ initial conditions $f_{\mathfrak{S}}(0;i,j) = \delta_{0,i,j}$ and $f_{\mathfrak{S}}(n;-1,j) = f_{\mathfrak{S}}(n;i,-1) = 0$.

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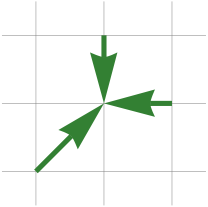
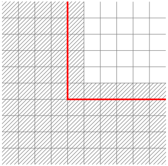
Example: for the Kreweras walks,

$$\begin{aligned} k(n+1;i,j) &= k(n;i+1,j) \\ &\quad + k(n;i,j+1) \\ &\quad + k(n;i-1,j-1) \end{aligned}$$





Step (S1): high order series expansions

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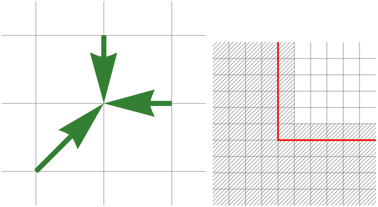

▷ Recurrence is used to compute $F_{\mathfrak{S}}(t;x,y) \bmod t^N$ for large N.

$$\begin{aligned} K(t;x,y) &= 1 + xyt + (x^2y^2 + y + x)t^2 + (x^3y^3 + 2xy^2 + 2x^2y + 2)t^3 \\ &+ (x^4y^4 + 3x^2y^3 + 3x^3y^2 + 2y^2 + 6xy + 2x^2)t^4 \\ &+ (x^5y^5 + 4x^3y^4 + 4x^4y^3 + 5xy^3 + 12x^2y^2 + 5x^3y + 8y + 8x)t^5 + \cdots \end{aligned}$$

Step (S2): guessing equations for $F_{\mathfrak{S}}(t;x,y)$, a first idea

In terms of generating functions, the recurrence on $k(n;i,j)$ reads

$$\boxed{\begin{aligned}(xy-(x+y+x^2y^2)t)&K(t;x,y)\\ &=xy-xt\,K(t;x,0)-yt\,K(t;0,y)\end{aligned}} \qquad \text{(KerEq)}$$

▷ A similar kernel equation holds for $F_{\mathfrak{S}}(t;x,y)$, for any $\mathfrak{S}$-walk.

Corollary. $F_{\mathfrak{S}}(t;x,y)$ is algebraic (resp. D-finite) if and only if $F_{\mathfrak{S}}(t;x,0)$ and $F_{\mathfrak{S}}(t;0,y)$ are both algebraic (resp. D-finite).

▷ Crucial simplification: equations for $G(t;x,y)$ are huge ($\approx$ 30 Gb)

Step (S2): guessing equations for $F_{\mathfrak{S}}(t;x,0)$ and $F_{\mathfrak{S}}(t;0,y)$

Task 1: Given the first N terms of $S = F_{\mathfrak{S}}(t;x,0) \in \mathbb{Q}[x][[t]]$, search for a differential equation satisfied by S at precision N:

$$c_r(x,t) \cdot \frac{\partial^r S}{\partial t^r} + \cdots + c_1(x,t) \cdot \frac{\partial S}{\partial t} + c_0(x,t) \cdot S = 0 \bmod t^N.$$

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- Both tasks amount to linear algebra in size N over $\mathbb{Q}(x)$.
- In practice, we use modular Hermite-Padé approximation (Beckermann-Labahn algorithm) combined with (rational) evaluation-interpolation and rational number reconstruction.
- Fast (FFT-based) arithmetic in $\mathbb{F}_p[t]$ and $\mathbb{F}_p[t]\langle \frac{t}{\partial t} \rangle$.

Using $N = 80$ terms of $K(t;x,0)$, one can guess

▷ a linear differential equation of order 4, degrees $(14,11)$ in (t,x), such that

$$t^3 \cdot (3t-1) \cdot (9t^2+3t+1) \cdot (3t^2+24t^2x^3-3xt-2x^2) \cdot$$
$$\cdot (16t^2x^5+4x^4-72t^4x^3-18x^3t+5t^2x^2+18xt^3-9t^4) \cdot$$
$$\cdot (4t^2x^3-t^2+2xt-x^2) \cdot \frac{\partial^4 K(t;x,0)}{\partial t^4} + \cdots$$
$$= 0 \bmod t^{80}$$

▷ a polynomial of tridegree $(6,10,6)$ in (T,t,x)

$$\mathcal{P}_{x,0} = x^6t^{10}T^6 - 3x^4t^8(x-2t)T^5 +$$
$$+ x^2t^6\left(12t^2+3t^2x^3-12xt+\frac{7}{2}x^2\right)T^4 + \cdots$$

such that $\mathcal{P}_{x,0}(K(t;x,0),t,x) = 0 \bmod t^{80}$.

Step (S2): guessing equations for $G(t;x,0)$ and $G(t;0,y)$

Using $N = 1200$ terms of $G(t;x,y)$, our guesser found candidates

- $\mathcal{P}_{x,0}$ in $\mathbb{Z}[T,t,x]$ of degree $(24,43,32)$, coefficients of 21 digits
- $\mathcal{P}_{0,y}$ in $\mathbb{Z}[T,t,y]$ of degree $(24,44,40)$, coefficients of 23 digits

such that

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▹ Guessing $\mathcal{P}_{x,0}$ by undetermined coefficients would have required to solve a dense linear system of size $\approx 100\,000$, and ≈ 1000 digits entries!

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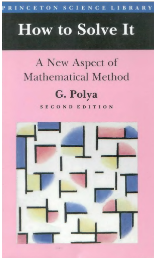
▷ Guessing $\mathcal{P}_{x,0}$ by undetermined coefficients would have required to solve a dense linear system of size ≈ 100 000, and ≈ 1000 digits entries!

▷ [B., Kauers '09] actually first guessed differential equations†, then computed their p-curvatures to empirically certify them. This led them suspect the algebraicity of $G(t;x,0)$ and $G(t;0,y)$, using a conjecture of Grothendieck's (on differential equations modulo p) as an oracle.

† of order 11, and bidegree $(96,78)$ for $G(t;x,0)$, and $(68,28)$ for $G(t;0,y)$

Guessing is good, proving is better [Pólya, 1957]



Guessing and Proving

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Step (S3): warm-up – Gessel excursions are algebraic

Theorem. $g(t) := G(\sqrt{t}; 0, 0) = \sum_{n=0}^{\infty} \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} (16t)^n$ is algebraic.

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3. $r(t) = \sum_{n=0}^{\infty} r_n t^n$ being algebraic, it is D-finite, and so is (r_n):

$$(n+2)(3n+5)r_{n+1} - 4(6n+5)(2n+1)r_n = 0, \qquad r_0 = 1$$

$\Rightarrow$ solution $r_n = \frac{(5/6)_n(1/2)_n}{(5/3)_n(2)_n}16^n = g_n$, thus $g(t) = r(t)$ is algebraic.

Step (S3): rigorous proof for Kreweras walks 

1. Setting $y_0 = \frac{x-t-\sqrt{x^2-2tx+t^2(1-4x^3)}}{2tx^2} = t + \frac{1}{x}t^2 + \frac{x^3+1}{x^2}t^3 + \cdots$ in the kernel equation

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shows that $U = K(t;x,0)$ satisfies the reduced kernel equation

$$\boxed{0 = x \cdot y_0 - x \cdot t \cdot U(t,x) - y_0 \cdot t \cdot U(t,y_0)} \qquad \text{(RKerEq)}$$

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2. (RKerEq) admits a unique solution in $\mathbb{Q}[[x,t]]$, namely $U = K(t;x,0)$.

3. The guessed candidate $\mathcal{P}_{x,0}(T,t,x)$ has a root $H(t,x)$ in $\mathbb{Q}[[x,t]]$.

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shows that $U = K(t;x,0)$ satisfies the reduced kernel equation

$$\boxed{0 = x \cdot y_0 - x \cdot t \cdot U(t,x) - y_0 \cdot t \cdot U(t,y_0)} \qquad \text{(RKerEq)}$$

2. (RKerEq) admits a unique solution in $\mathbb{Q}[[x,t]]$, namely $U = K(t;x,0)$.

3. The guessed candidate $\mathcal{P}_{x,0}(T,t,x)$ has a root $H(t,x)$ in $\mathbb{Q}[[x,t]]$.

4. $U = H(t,x)$ also satisfies (RKerEq) Resultant computations!

Step (S3): rigorous proof for Kreweras walks 

1. Setting $y_0 = \frac{x-t-\sqrt{x^2-2tx+t^2(1-4x^3)}}{2tx^2} = t + \frac{1}{x}t^2 + \frac{x^3+1}{x^2}t^3 + \cdots$ in the kernel equation

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4. $U = H(t,x)$ also satisfies (RKerEq) Resultant computations!

5. Uniqueness $\implies H(t,x) = K(t;x,0) \implies K(t;x,0)$ is algebraic!

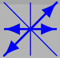
Algebraicity of Kreweras walks: a computer proof in a nutshell

```
# HIGH ORDER EXPANSION (S1)
> st,bu:=time(),kernelopts(bytesused):
> f:=proc(n,i,j) option remember;
    if i<0 or j<0 or n<0 then 0
    elif n=0 then if i=0 and j=0 then 1 else 0 fi
    else f(n-1,i-1,j-1)+f(n-1,i,j+1)+f(n-1,i+1,j) fi
  end:
> S:=series(add(add(f(k,i,0)*x^i,i=0..k)*t^k,k=0..80),t,80):

# GUESSING (S2)
> libname:=".",libname:gfun:-version();
                                  3.76
> P:=subs(Fx0(t)=T,gfun:-seriestoalgeq(S,Fx0(t))[1]):

# RIGOROUS PROOF (S3)
> ker := (T,t,x) -> (x+T+x^2*T^2)*t-x*T:
> pol := unapply(P,T,t,x):
> p1 := resultant(pol(z-T,t,x),ker(t*z,t,x),z):
> p2 := subs(T=x*T,resultant(numer(pol(T/z,t,z)),ker(z,t,x),z)):
> normal(primpart(p1,T)/primpart(p2,T));
                                    1

# time (in sec) and memory consumption (in Mb)
> trunc(time()-st),trunc((kernelopts(bytesused)-bu)/1000^2);
                                  8, 785
```

Step (S3): rigorous proof for Gessel walks 

Same strategy, but several complications:

- stepset diagonal symmetry is lost: $G(t;x,y) \neq G(t;y,x)$;
- $G(t;0,0)$ occurs in (KerEq) (because of the step $\swarrow$);
- equations are $\approx 5\,000$ times bigger.

$\longrightarrow$ replace equation (RKerEq) by a system of 2 reduced kernel equations.

$\longrightarrow$ fast algorithms needed (e.g., [B., Flajolet, Salvy, Schost, 2006] for computations with algebraic series).

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Fast computation of special resultants

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INSIDE THE BOX

–Hermite-Padé approximants–

Definition: Given a column vector $\mathbf{F} = (f_1, \ldots, f_n)^T \in \mathbb{K}[[x]]^n$ and an n-tuple $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{N}^n$, a Hermite-Padé approximant of type $\mathbf{d}$ for $\mathbf{F}$ is a row vector $\mathbf{P} = (P_1, \ldots, P_n) \in \mathbb{K}[x]^n$, $(\mathbf{P} \neq 0)$, such that:

(1) $\mathbf{P} \cdot \mathbf{F} = P_1 f_1 + \cdots + P_n f_n = O(x^\sigma)$ with $\sigma = \sum_i (d_i + 1) - 1$,

(2) $\deg(P_i) \leq d_i$ for all i.

σ is called the order of the approximant $\mathbf{P}$.

▷ Very useful concept in number theory (irrationality/transcendence):

- [Hermite, 1873]: e is transcendent.
- [Lindemann, 1882]: π is transcendent; so does e^α for any $\alpha \in \overline{\mathbb{Q}} \setminus \{0\}$.
- [Apéry, 1978; Beukers, 1981]: $\zeta(3) = \sum_{n \geq 1} \frac{1}{n^3}$ is irrational.
- [Rivoal, 2000]: there exist infinite values of k such that $\zeta(2k+1) \notin \mathbb{Q}$.

Worked example

Let us compute a Hermite-Padé approximant of type $(1,1,1)$ for $(1, C, C^2)$, where $C(x) = 1 + x + 2x^2 + 5x^3 + 14x^4 + 42x^5 + O(x^6)$.
This boils down to finding $\alpha_0, \alpha_1, \beta_0, \beta_1, \gamma_0, \gamma_1$ (not all zero) such that

$$\alpha_0 + \alpha_1 x + (\beta_0 + \beta_1 x)(1 + x + 2x^2 + 5x^3 + 14x^4) + (\gamma_0 + \gamma_1 x)(1 + 2x + 5x^2 + 14x^3 + 42x^4) = O(x^5)$$

Identifying coefficients, this is equivalent to a homogeneous linear system:

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 2 & 1 \\ 0 & 0 & 2 & 1 & 5 & 2 \\ 0 & 0 & 5 & 2 & 14 & 5 \\ 0 & 0 & 14 & 5 & 42 & 14 \end{bmatrix} \times \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \beta_0 \\ \beta_1 \\ \gamma_0 \\ \gamma_1 \end{bmatrix} = 0 \iff \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 2 & 1 & 5 \\ 0 & 0 & 5 & 2 & 14 \\ 0 & 0 & 14 & 5 & 42 \end{bmatrix} \times \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \beta_0 \\ \beta_1 \\ \gamma_0 \end{bmatrix} = -\gamma_1 \begin{bmatrix} 0 \\ 1 \\ 2 \\ 5 \\ 14 \end{bmatrix}.$$

By homogeneity, one can choose $\gamma_1 = 1$.
Then, the violet minor shows that one can take $(\beta_0, \beta_1, \gamma_0) = (-1, 0, 0)$.
The other values are $\alpha_0 = 1$, $\alpha_1 = 0$.

Thus the approximant is $(1, -1, x)$, which corresponds to $P = 1 - y + xy^2$ such that $P(x, C(x)) = 0 \bmod x^5$.

- Hermite-Padé approximants of $n = 2$ power series are related to Padé approximants, i.e. to approximation of series by rational functions
- algebraic approximants = Hermite-Padé approximants for $f_\ell = A^{\ell-1}$, where $A \in \mathbb{K}[[x]]$ seriestoalgeq, listtoalgeq
- differential approximants = Hermite-Padé approximants for $f_\ell = A^{(\ell-1)}$, where $A \in \mathbb{K}[[x]]$ seriestodiffeq, listtodiffeq

```
> listtoalgeq([1,1,2,5,14,42,132,429],y(x));
                                              2
                         [1 - y(x) + x y(x) , ogf]

> listtodiffeq([1,1,2,5,14,42,132,429],y(x))[1];
                                     / 2        \
                      /d        \    |d         |
[{-2 y(x) + (2 - 4 x) |-- y(x)| + x |--- y(x)|,
                      \dx       /    |  2       |
                                     \dx        /

                                         y(0) = 1, D(y)(0) = 1}, egf]
```

Existence and naive computation

Theorem For any vector $\mathbf{F} = (f_1, \ldots, f_n)^T \in \mathbb{K}[[x]]^n$ and for any n-tuple $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{N}^n$, there exists a Hermite-Padé approx. of type $\mathbf{d}$ for $\mathbf{F}$.

Proof: The undetermined coefficients of $P_i = \sum_{j=0}^{d_i} p_{i,j} x^j$ satisfy a linear homogeneous system with $\sigma = \sum_i (d_i + 1) - 1$ eqs and $\sigma + 1$ unknowns.

Corollary Computation in $O(\sigma^\omega)$, for $2 \leq \omega \leq 3$ (linear algebra exponent)

▷ There are better algorithms (the linear system is structured, Sylvester-like):

- Derksen's algorithm (Euclidean-like elimination) $O(\sigma^2)$
- Beckermann-Labahn algorithm (DAC) $\tilde{O}(\sigma) = O(\sigma \log^2 \sigma)$
- structured linear algebra algorithms for Toeplitz-like matrices $\tilde{O}(\sigma)$

Quasi-optimal computation

Theorem [Beckermann, Labahn, 1994] One can compute a Hermite-Padé approximant of type $(d, \ldots, d)$ for $\mathbf{F} = (f_1, \ldots, f_n)$ in $\tilde{O}(n^\omega d)$ ops. in $\mathbb{K}$.

Ideas:

- Compute a whole matrix of approximants
- Exploit divide-and-conquer

Algorithm:

1. If $\sigma = n(d+1) - 1 \leq$ threshold, call the naive algorithm
2. Else:
   1. recursively compute $\mathbf{P}_1 \in \mathbb{K}[x]^{n \times n}$ s.t. $\mathbf{P}_1 \cdot \mathbf{F} = O(x^{\sigma/2})$, $\deg(\mathbf{P}_1) \approx \frac{d}{2}$
   2. compute "residue" $\mathbf{R}$ such that $\mathbf{P}_1 \cdot \mathbf{F} = x^{\sigma/2} \cdot \left(\mathbf{R} + O(x^{\sigma/2})\right)$
   3. recursively compute $\mathbf{P}_2 \in \mathbb{K}[x]^{n \times n}$ s.t. $\mathbf{P}_2 \cdot \mathbf{R} = O(x^{\sigma/2})$, $\deg(\mathbf{P}_2) \approx \frac{d}{2}$
   4. return $\mathbf{P} := \mathbf{P}_2 \cdot \mathbf{P}_1$

▷ The precise choices of degrees is a delicate issue
▷ Corollary: Gcd, extended gcd, Padé approximants in $\tilde{O}(d)$ ops. in $\mathbb{K}$.
▷ Extensions to order bases, over Ore domains: George Labahn's talk.

Guess'n'Prove for

–TRANSCENDENCE–

Stanley's problem

Design an algorithm suitable for computer implementations which decides if a D-finite power series —represented by a linear differential equation with polynomial coefficients and suitable initial conditions— is transcendental, or not.

[Stanley, 1980]

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[Stanley, 1980]

E.g.,

$$f = \ln(1-t) = -t - \frac{t^2}{2} - \frac{t^3}{3} - \frac{t^4}{4} - \frac{t^5}{5} - \frac{t^6}{6} - \cdots$$

is D-finite and can be represented by the second-order equation

$$\left((t-1)\partial_t^2 + \partial_t\right)(f) = 0, \quad f(0) = 0, f'(0) = -1.$$

The algorithm should recognize that f is transcendental.

Design an algorithm suitable for computer implementations which decides if a D-finite power series —represented by a linear differential equation with polynomial coefficients and suitable initial conditions— is transcendental, or not.

[Stanley, 1980]

▷ Notation: For a D-finite series f, we write $L_f^{\min}$ for its *differential resolvent*, i.e. the least order monic differential operator in $\mathbb{Q}(t)\langle\partial_t\rangle$ that cancels f.

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▷ Warning: $L_f^{\min}$ is not known a priori; only some multiple L of it is given.

▷ Difficulty: $L_f^{\min}$ might not be irreducible. E.g., $L_{\ln(1-t)}^{\min} = \left(\partial_t + \frac{1}{t-1}\right)\partial_t$.

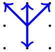
Three examples

(A) Apéry's power series [Apéry, 1978] (used in his proof of $\zeta(3) \notin \mathbb{Q}$)

$$\sum_n \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 t^n = 1 + 5\,t + 73\,t^2 + 1445\,t^3 + 33001\,t^4 + \cdots$$

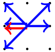
(B) GF of trident walks in the quarter plane

$$\sum_n a_n t^n = 1 + 2\,t + 7\,t^2 + 23\,t^3 + 84\,t^4 + 301\,t^5 + 1127\,t^6 + \cdots,$$

where $a_n = \#\Big\{$  – walks of length n in $\mathbb{N}^2$ starting at $(0,0)\Big\}$

(C) GF of a quadrant model with repeated steps

$$\sum_n a_n t^n = 1 + t + 4\,t^2 + 8\,t^3 + 39\,t^4 + 98\,t^5 + 520\,t^6 + \cdots,$$

where $a_n = \#\Big\{$  – walks of length n in $\mathbb{N}^2$ from $(0,0)$ to $(\star,0)\Big\}$

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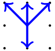
where $a_n = \#\Big\{$  – walks of length n in $\mathbb{N}^2$ starting at $(0,0)\Big\}$

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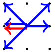
where $a_n = \#\Big\{$  – walks of length n in $\mathbb{N}^2$ from $(0,0)$ to $(\star,0)\Big\}$

Question: *How to prove that these three power series are transcendental?*

Singer's algorithm

Problem: Decide if a D-finite power series $f \in \mathbb{Q}[[t]]$, given by a differential equation $L(f) = 0$ and sufficiently many initial terms, is transcendental.

1. Compute L^{alg}, the (right) factor of L whose solution space is spanned by all algebraic solutions of L [Singer, 2014]
2. Decide if L^{alg} annihilates f

▷ Benefit: Solves (in principle) Stanley's problem.

▷ Drawbacks: Step 1 involves *impractical bounds* and *requires ODE factorization*

▷ ODE factorization is effective
[Schlesinger, 1897], [Singer, 1981], [Grigoriev, 1990], [van Hoeij, 1997]

▷ . . . but possibly extremely costly [Grigoriev, 1990] $\exp\left((\text{bitsize}(L)2^n)^{2^n}\right)$

A different method: the basic idea

Problem: Decide if a D-finite power series $f \in \mathbb{Q}[[t]]$, given by a differential equation $L(f) = 0$ and sufficiently many initial terms, is transcendental.

Basic remark: If $L_f^{\min}$ has a logarithmic singularity, then f is transcendental.
(f algebraic implies basis of algebraic solutions for $L_f^{\min}$ [Tannery, 1875].)

▷ Pros and cons: Avoids factorization of L, but requires to compute $L_f^{\min}$.

Ex. (A): Apéry's power series

$$f(t) = \sum_n a_n t^n, \quad \text{where } a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

▷ Creative telescoping:

$$(n+1)^3 a_n - (2n+3)(17n^2+51n+39)a_{n+1} + (n+2)^3 a_{n+2} = 0, \quad a_0 = 1,\ a_1 = 5$$

▷ Conversion from recurrence to differential equation $L(f) = 0$, where

$$L = (t^4 - 34t^3 + t^2)\partial_t^3 + (6t^3 - 153t^2 + 3t)\partial_t^2 + (7t^2 - 112t + 1)\partial_t + t - 5$$

▷ $L_f^{\min} = \frac{1}{t^4 - 34t^3 + t^2} L$ using L irreducible, or cf. new algorithm

▷ Basis of formal solutions of $L_f^{\min}$ at $t = 0$:

$$\left\{ 1 + 5t + O(t^2),\ \ln(t) + (5\ln(t) + 12)t + O(t^2),\ \ln(t)^2 + (5\ln(t)^2 + 24\ln(t))t + O(t^2) \right\}$$

▷ Conclusion: f is transcendental

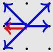
Ex. (B): Nature of $F(t;1,1)$ for SSW [B., Chyzak, van Hoeij, Kauers, Pech, 2016]

	OEIS	$\mathfrak{S}$	algebraic?	asymptotics		OEIS	$\mathfrak{S}$	algebraic?	asymptotics
1	A005566		N	$\frac{4}{\pi}\frac{4^n}{n}$	13	A151275		N	$\frac{12\sqrt{30}}{\pi}\frac{(2\sqrt{6})^n}{n^2}$
2	A018224		N	$\frac{2}{\pi}\frac{4^n}{n}$	14	A151314		N	$\frac{\sqrt{6}\lambda\mu C^{5/2}}{5\pi}\frac{(2C)^n}{n^2}$
3	A151312		N	$\frac{\sqrt{6}}{\pi}\frac{6^n}{n}$	15	A151255		N	$\frac{24\sqrt{2}}{\pi}\frac{(2\sqrt{2})^n}{n^2}$
4	A151331		N	$\frac{8}{3\pi}\frac{8^n}{n}$	16	A151287		N	$\frac{2\sqrt{2}A^{7/2}}{\pi}\frac{(2A)^n}{n^2}$
5	A151266		N	$\frac{1}{2}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{1/2}}$	17	A001006		Y	$\frac{3}{2}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{3/2}}$
6	A151307		N	$\frac{1}{2}\sqrt{\frac{5}{2\pi}}\frac{5^n}{n^{1/2}}$	18	A129400		Y	$\frac{3}{2}\sqrt{\frac{3}{\pi}}\frac{6^n}{n^{3/2}}$
7	A151291		N	$\frac{4}{3\sqrt{\pi}}\frac{4^n}{n^{1/2}}$	19	A005558		N	$\frac{8}{\pi}\frac{4^n}{n^2}$
8	A151326		N	$\frac{2}{\sqrt{3\pi}}\frac{6^n}{n^{1/2}}$					
9	A151302		N	$\frac{1}{3}\sqrt{\frac{5}{2\pi}}\frac{5^n}{n^{1/2}}$	20	A151265		Y	$\frac{2\sqrt{2}}{\Gamma(1/4)}\frac{3^n}{n^{3/4}}$
10	A151329		N	$\frac{1}{3}\sqrt{\frac{7}{3\pi}}\frac{7^n}{n^{1/2}}$	21	A151278		Y	$\frac{3\sqrt{3}}{\sqrt{2}\Gamma(1/4)}\frac{3^n}{n^{3/4}}$
11	A151261		N	$\frac{12\sqrt{3}}{\pi}\frac{(2\sqrt{3})^n}{n^2}$	22	A151323		Y	$\frac{\sqrt{2}3^{3/4}}{\Gamma(1/4)}\frac{6^n}{n^{3/4}}$
12	A151297		N	$\frac{\sqrt{3}B^{7/2}}{2\pi}\frac{(2B)^n}{n^2}$	23	A060900		Y	$\frac{4\sqrt{3}}{3\Gamma(1/3)}\frac{4^n}{n^{2/3}}$

- ▷ Computer-driven discovery and proof; no human proof yet in some cases
- ▷ Proof uses creative telescoping, ODE factorization, Singer's algorithm
- ▷ For models 5–10, asymptotics do not conclude.

Ex. (C): a difficult quadrant model with repeated steps

Theorem [B., Bousquet-Mélou, Kauers, Melczer, 2016]

Let $a_n = \#\Big\{$  $-$ walks of length n in $\mathbb{N}^2$ from $(0,0)$ to $(\star,0)\Big\}$. Then
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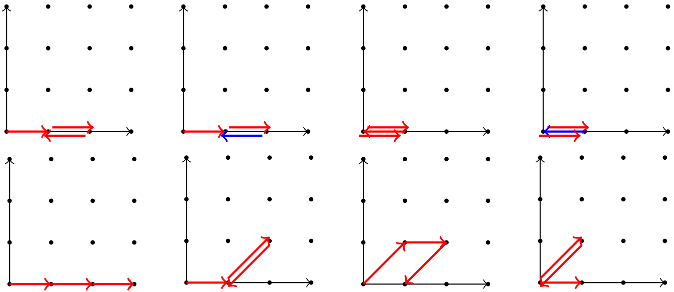

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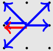
Let $a_n = \#\Big\{$  $-$ walks of length n in $\mathbb{N}^2$ from $(0,0)$ to $(\star,0)\Big\}$. Then $f(t) = \sum_n a_n t^n = 1 + t + 4\,t^2 + 8\,t^3 + 39\,t^4 + 98\,t^5 + \cdots$ is transcendental.

Proof:

1. Discover and certify a differential equation L for $f(t)$ of order 11 and degree 73 — (high-tech) Guess'n'Prove
2. If $\operatorname{ord}(L_f^{\min}) \leq 10$, then $\deg_t(L_f^{\min}) \leq 580$ — apparent singularities
3. Rule out this possibility — differential Hermite-Padé approximants
4. Thus, $L_f^{\min} = L$
5. L has a log singularity at $t = 0$, thus f is transcendental □

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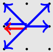
Let $a_n = \# \Big\{$  $-$ walks of length n in $\mathbb{N}^2$ from $(0,0)$ to $(\star,0)\Big\}$. Then $f(t) = \sum_n a_n t^n = 1 + t + 4\,t^2 + 8\,t^3 + 39\,t^4 + 98\,t^5 + \cdots$ is transcendental.

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3. Rule out this possibility — [Beckermann, Labahn, 1994]
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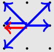
5. L has a log singularity at $t = 0$, thus f is transcendental $\square$

Ex. (C): a difficult quadrant model with repeated steps

Theorem [B., Bousquet-Mélou, Kauers, Melczer, 2016]

Let $a_n = \# \Big\{$  – walks of length n in $\mathbb{N}^2$ from $(0,0)$ to $(\star,0)\Big\}$. Then $f(t) = \sum_n a_n t^n = 1 + t + 4\,t^2 + 8\,t^3 + 39\,t^4 + 98\,t^5 + \cdots$ is transcendental.

Proof:

1. Discover and certify a differential equation L for $f(t)$ of order 11 and degree 73 — (high-tech) Guess'n'Prove
2. If $\mathrm{ord}(L_f^{\min}) \leq 10$, then $\deg_t(L_f^{\min}) \leq 580$ — apparent singularities
3. Rule out this possibility — differential Hermite-Padé approximants
4. Thus, $L_f^{\min} = L$
5. L has a log singularity at $t = 0$, thus f is transcendental □

▷ Computer-driven discovery and proof; no human proof yet.
▷ All other transcendence criteria / algorithms fail or do not terminate.

Central sub-task: computation of $L_f^{\min}$

Problem: Given a D-finite power series $f \in \mathbb{Q}[[t]]$ by a differential equation $L(f) = 0$ and sufficiently many initial terms, compute its resolvent $L_f^{\min}$.

▷ Why isn't this easy? After all, it is just a differential analogue of:

Given an algebraic power series $f \in \mathbb{Q}[[t]]$
by an algebraic equation $P(t, f) = 0$ and sufficiently many initial terms,
compute its minimal polynomial P_f^{min}.

▷ $L_f^{\min}$ is a (right) factor of L, but contrary to the commutative case:

- factorization of diff. operators is not unique $\quad \partial_t^2 = (\partial_t + \frac{1}{t-c})(\partial_t - \frac{1}{t-c})$
- ... and it is difficult to compute
- $\deg_t L_f^{\min} > \deg_t L$, due to apparent singularities $\quad (t\partial_t - N) \mid \partial_t^{N+1}$

▷ $\deg_t L_f^{\min}$ can be bounded w.r.t. n and local data of L via Fuchs' relation

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▷ More on apparent singularities and desingularization in Moulay Barkatou's and Maximilian Jaroschek's talks

▷ $L_f^{\min}$ useful in other contexts, e.g. in number theory: Tanguy Rivoal's talk

Summary

- Guess'n'Prove is a powerful method, especially when combined with efficient computer algebra

- It is robust: it can be used to uniformly prove
  - ☺ D-finiteness in all the cases with finite group
  - ☺ algebraicity in all the cases with finite group and zero orbit sum
  - ☺ transcendence in all the cases with finite group and nonzero orbit sum

- Brute-force and/or use of naive algorithms = hopeless.
  E.g. size of algebraic equations for $G(t;x,y) \approx 30\,\text{Gb}$.

BACK TO THE EXERCISE

–A hint–

The exercise

Let $\mathfrak{S} = \{\uparrow, \leftarrow, \searrow\}$. A $\mathfrak{S}$-walk is a path in $\mathbb{Z}^2$ using only steps from $\mathfrak{S}$. Show that, for any integer n, the following quantities are equal:

(i) the number a_n of $\mathfrak{S}$-walks of length n confined to the upper half plane $\mathbb{Z} \times \mathbb{N}$ that start and end at the origin $(0,0)$;

(ii) the number b_n of $\mathfrak{S}$-walks of length n confined to the quarter plane $\mathbb{N}^2$ that start at the origin $(0,0)$ and finish on the diagonal $x = y$.

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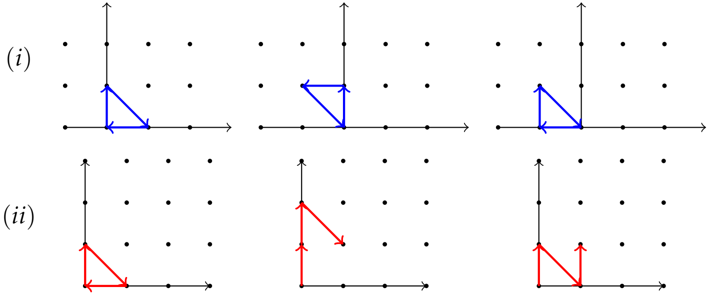
For instance, for $n = 3$, this common value is $a_3 = b_3 = 3$:



A recurrence relation for $\{\uparrow, \leftarrow, \searrow\}$-walks in $\mathbb{Z} \times \mathbb{N}$

$h(n;i,j)$ = nb. of $\{\uparrow, \leftarrow, \searrow\}$-walks in $\mathbb{Z} \times \mathbb{N}$ of length n from $(0,0)$ to (i,j)
The numbers $h(n;i,j)$ satisfy

$$h(n;i,j) = \begin{cases} 0 & \text{if } j < 0 \text{ or } n < 0, \\ \mathbb{1}_{i=j=0} & \text{if } n = 0, \\ \sum_{(i',j') \in \mathfrak{S}} h(n-1; i-i', j-j') & \text{otherwise.} \end{cases}$$

```
> h:=proc(n,i,j)
  option remember;
    if j<0 or n<0 then 0
    elif n=0 then if i=0 and j=0 then 1 else 0 fi
    else h(n-1,i,j-1) + h(n-1,i+1,j) + h(n-1,i-1,j+1) fi
  end:

> A:=series(add(h(n,0,0)*t^n, n=0..12), t,12);
```

$$A = 1 + 3t^3 + 30t^6 + 420t^9 + O(t^{12})$$

A recurrence relation for $\{\uparrow, \leftarrow, \searrow\}$-walks in $\mathbb{N}^2$

$q(n;i,j)$ = nb. of $\{\uparrow, \leftarrow, \searrow\}$-walks in $\mathbb{N}^2$ of length n from $(0,0)$ to (i,j)
The numbers $q(n;i,j)$ satisfy

$$q(n;i,j) = \begin{cases} 0 & \text{if } i<0 \text{ or } j<0 \text{ or } n<0, \\ \mathbb{1}_{i=j=0} & \text{if } n=0, \\ \sum_{(i',j')\in\mathfrak{S}} q(n-1;i-i',j-j') & \text{otherwise.} \end{cases}$$

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  end:

> B:=series(add(add(q(n,k,k), k=0..n)*t^n, n=0..12), t,12);
```

$$B = 1 + 3t^3 + 30t^6 + 420t^9 + O(t^{12})$$

Guessing the answer

```
> seriestorec(A, u(n))[1];
        2                       2
{(-27 n  - 81 n - 54) u(n) + (n  + 9 n + 18) u(n + 3),

                                      u(0) = 1, u(1) = 0, u(2) = 0}

> rsolve(%, u(n)):

> A:=sum(subs(n=3*n, op(2,%))*t^(3*n), n=0..infinity);
                                                        3
              A := hypergeom([1/3, 2/3], [2], 27 t )
```

▷ Thus, differential guessing predicts

$$A(t) = B(t) = {}_2F_1\left(\begin{matrix}1/3 \quad 2/3\\ 2\end{matrix}\,\middle|\, 27\,t^3\right) = \sum_{n=0}^{\infty} \frac{(3n)!}{n!^3}\,\frac{t^{3n}}{n+1}.$$

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```

▷ This can be algorithmically proved using creative telescoping

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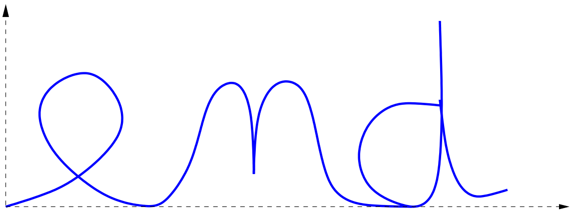
This is the



Thanks for your attention!