Arctic curves in path models from the Tangent Method

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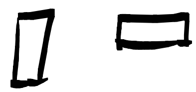
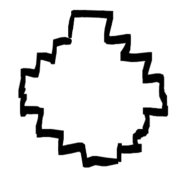
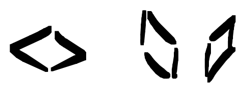
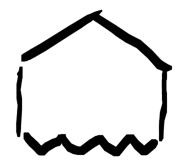
ESI '17

0. INTRODUCTION

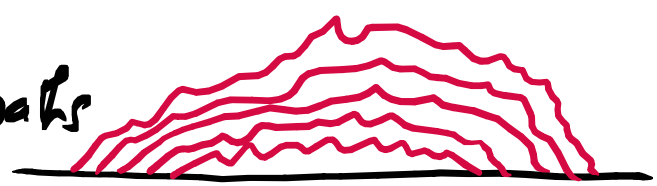
Combinatorial problems :

(1) Tiling of finite domain with finite set of tiles

Ex : a. dominos  2x1 1x2 , Aztec diamond 

b. rhombi  , half-hexagon 

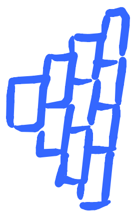
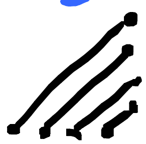
(2) Non-intersecting lattice paths $\{a_i\} \to \{e_i\}$.

Ex Dyck paths 

Asymptotics

- appearance of different phases $\to$ order "crystal"
  $\to$ disorder "liquid"

Ex

-  domino crystal
-  path crystal

- At large size ($\mathcal{D} \to N\mathcal{D}$, $N \to \infty$) there is a sharp separation between these phases

= ARCTIC CURVE

Enumeration

- reformulate all problems in terms of NILP
- Use Gessel–Viennot–Lindström determinant

$$\boxed{Z_{\{a_i\}\to\{e_i\}} = \det_{1\le i,j\le n}\left(Z_{a_i\to e_j}\right)}$$

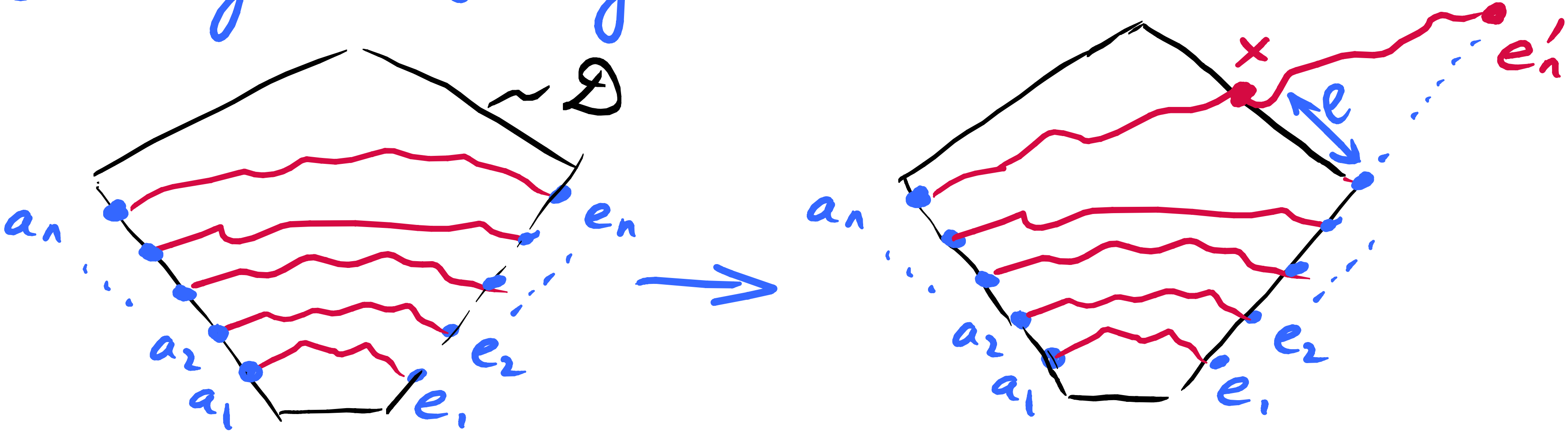
= partition function of paths from $a_1 \ldots a_n$ to $e_1 \ldots e_n$

$$Z_{a\to b} = \sum_{\substack{\text{paths} \\ a\to b}} w(\text{path}) \leftarrow \text{Weight} = \prod_{\text{edges}} w(\text{edge})$$

- Calculate det by LU decomposition of $A = \left(Z_{a_i\to e_j}\right)$
- use generating functions as a tool

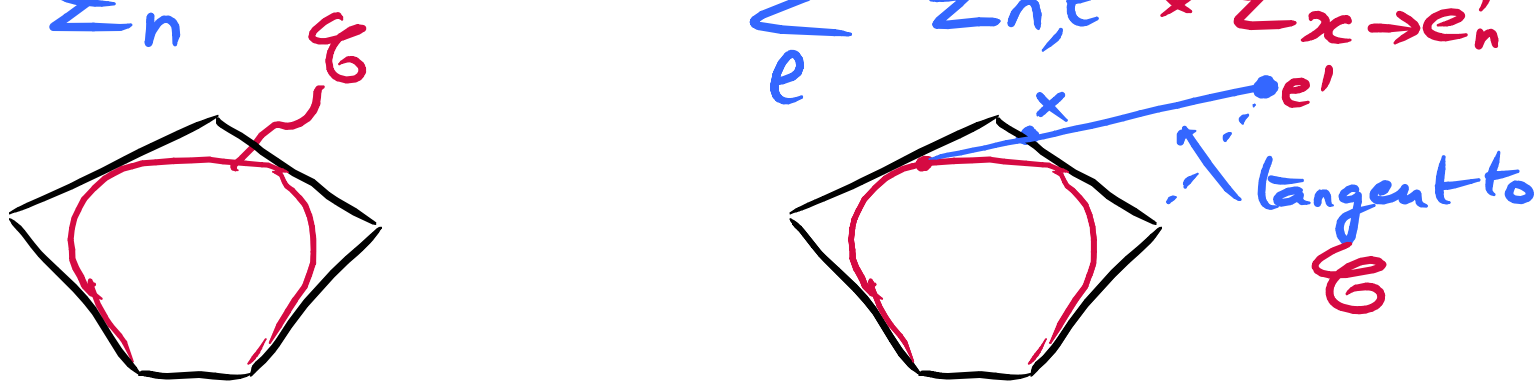
1. TANGENT METHOD [Colomo–Sportiello 16]

- Change the setting



$$Z_n \qquad \sum_e Z_{n,e} \times Z_{x \to e'_n}$$

- large size



Relies on 2 properties:

1. "left to its own devices, a directed random path with fixed endpoints is most likely to follow a straight line"

2. "The line followed by the external path away from the others is tangent to the arctic curve $\mathcal{C}$"

1. can be proved rigorously.
2. still an assumption

Proof of 1.

directed paths on $\mathbb{Z}^2$

- allow steps (s_i, t_i) with $t_i > 0$ say, weight w_i
- $P(z,w) := \sum_i w_i \, z^{s_i} w^{t_i}$ Newton polynomial

$$Z_{\substack{(00)\to(ab)\\ N\text{ steps}}} = \frac{1}{1-t\,P(z,w)}\Bigg|_{z^a w^b t^N}$$

cut with a line L

intersection w path (x,y)

$$Z_{(00)\to(ab)} = \sum_{(x,y)\in L} Z_{(00)\to(xy)} Z_{(xy)\to(ab)}$$

$$= \sum Z_{(00)\to(xy)} \; Z_{(00)\to(a-x,b-y)}$$

$$Z_{(00)\to(xy)} = \oint \frac{dz}{z^{x+1}} \frac{dw}{w^{y+1}} \frac{1}{1-P(z,w)} = \oint \frac{dz}{z}\frac{dw}{w} e^{S_{xy}}$$

$$S_{xy}^{zw} = -\log(1-P(z,w)) - x\log z - y\log w$$

<u>large scale</u> $(x,y) = n(\xi,\eta)$ $\quad (a,b) = n(\alpha,\beta)$ $\quad n\to\infty$

$$S_{xy}^{zw} = -n\left(\xi\log z + \eta\log w + \frac{1}{n}\log(1-P(z,w))\right)$$

$$Z_{(00)\to n(\alpha,\beta)} \underset{n\to\infty}{\sim} \int d\xi\, d\eta \oint \frac{dz}{z}\frac{dw}{w}\oint\frac{dz'}{z'}\frac{dw'}{w'}\, e^{\underbrace{S_{xy}^{zw} + S_{a-x,b-y}^{z',w'}}_{S}}$$

dominated by saddle-point $\partial_\xi S = \partial_\eta S = 0 \Rightarrow \begin{cases} z_0 = z_0' \\ w_0 = w_0' \end{cases}$

$$\partial_z S = \partial_w S = \partial_{z'} S = \partial_{w'} S = 0 \Rightarrow$$

$$\frac{\xi_0}{z_0} - \frac{1}{n}\frac{\partial_z P_0}{1-P_0} = \frac{\alpha-\xi_0}{z_0'} - \frac{1}{n}\frac{\partial_{z'} P_0}{1-P_0} = 0$$

$$\frac{\eta_0}{w_0} - \frac{1}{n}\frac{\partial_w P_0}{1-P_0} = \frac{\beta-\eta_0}{w_0'} - \frac{1}{n}\frac{\partial_{w'} P_0}{1-P_0} = 0$$

$$\Rightarrow \quad \xi_0/\eta_0 = \alpha-\xi_0/\beta-\eta_0 \Leftrightarrow \beta\xi_0 - \alpha\eta_0 = 0$$

(00) (xy) (ab) aligned!

qed.

APPLYING THE TANGENT METHOD:

Summary: 1. Compute the "escaping path" partition function $Z_{n,\ell}$ and "1pt function"

$$\boxed{H_{n,\ell} := \frac{Z_{n,\ell}}{Z_{n,0}}}$$

$\leftarrow$ escaping point at ℓ

$\leftarrow$ original partition function

2. Compute the free path partition function

$Y_{\ell,e'}$ = single path from $\ell \to e'$

3. Scaling estimate $\sum_{\ell} H_{n,\ell}\, Y_{\ell,e'}$

$n\to\infty \quad \ell = n\xi \quad e' = na$

then $\sum_{\ell} H_{n,\ell}\, Y_{\ell,\ell'} \sim \int d\zeta\, e^{n\underbrace{(S_0(\zeta)+S_1(\zeta,a))}_{S}}$

by saddle point $\to$ most likely $\zeta=\zeta_0 = \text{fctn}(a)$

$\Rightarrow$ tangent line thru $n\zeta$ and na

$\Rightarrow$ 6 as envelope, for varying a.

- We must estimate $H_{n,\ell=n\zeta}$ at large n

<u>Do exact enumeration first.</u> $Z_{n,0} = \text{LGV}\det = \det\limits_{0\le i,j\le n}(A_{ij})$

$Z_{n,\ell} = \det\limits_{0\le i,j\le n}(\tilde{A}_{ij})$ where $\begin{cases}\tilde{A}_{ij} = A_{ij} & j<n\\ \tilde{A}_{i,n} = Z_{i\to e'}\end{cases}$

only the last column differs

LU decomposition: $A = L \cdot U$ ← upper triangular; L ← uni-lower triangular; $\det A = \det U$

Then $L^{-1} \tilde{A} = \tilde{U}$ with $\tilde{U}$ upper triangular

same L!

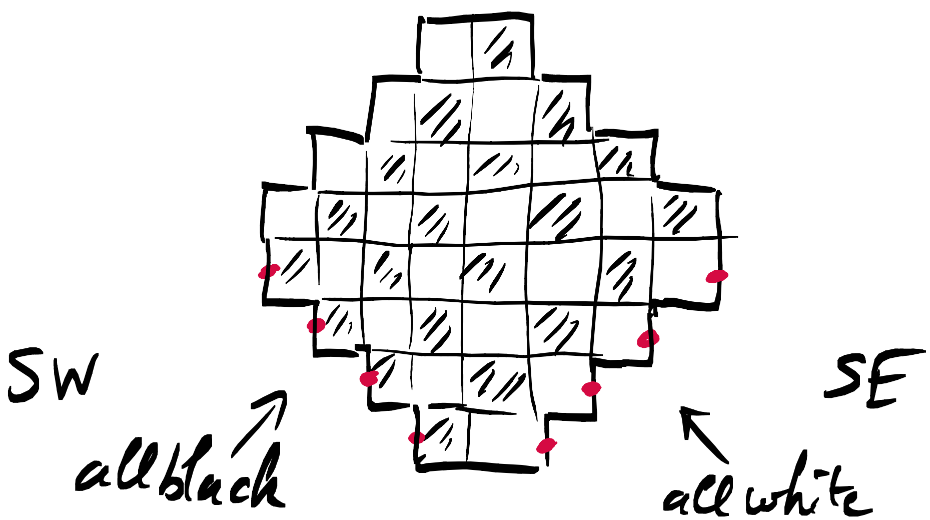
$\tilde{U}_{ij} = U_{ij} \quad j<n$

and $$\boxed{H_{n,\ell} = \frac{\det(\tilde{A})}{\det(A)} = \frac{\tilde{U}_{n,n}}{U_{n,n}}}$$

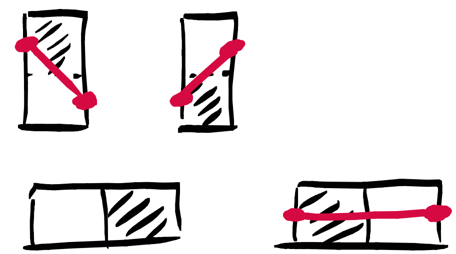
It all boils down to LU decomposition

<u>NB</u>: $\tilde{U}_{n,n} = \sum_i (L^{-1})_{n,i} \cdot \tilde{A}_{i,n}$ = alternating sum, not good for large n estimates → TURN IT INTO a >0 sum!!

2. Domino Tilings of the Aztec Diamond

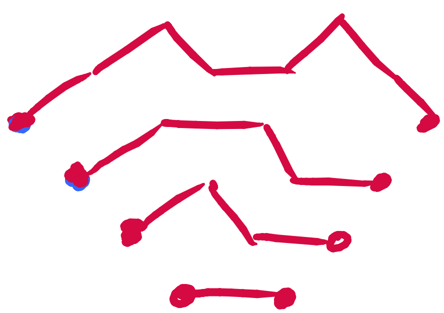


PATHS



$\Updownarrow$

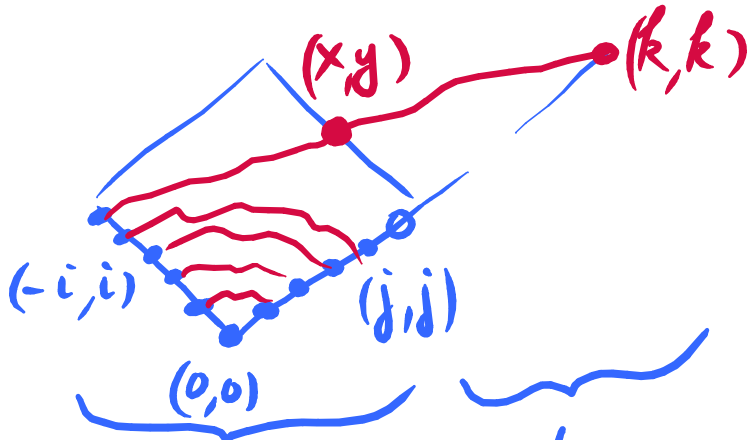
large Schröder NILP



steps (1,1) (1,-1) (2,0)

$\Rightarrow$ partition function $\boxed{Z_n = 2^{n(n+1)/2}}$

Tangent method:



$(x,y) = (\ell, 2n-\ell)$

exit point of the escaping path

$$\frac{1}{Z_n}\sum_{\ell} Z_{n,\ell}\, Y_{\ell,k} = \sum_{\ell} H_{n,\ell}\, Y_{\ell,k}$$

$\left.\begin{array}{ll} \bullet\ Z_n = Z_{n,0} & \text{partition function} \\ \bullet\ H_{n,\ell} = \dfrac{Z_{n,\ell}}{Z_n} & \text{1pt function} \end{array}\right\}$ use LGV.

$Z_n)$

LGV matrix:

$$A_{ij} = \frac{1}{1-z-w-zw}\Big|_{z^i w^j} = \sum_{p=0}^{\mathrm{Min}(i,j)} \frac{(i+j-p)!}{p!\,(i-p)!\,(j-p)!}$$

LU decomposition

$$L_{ij} = \frac{1}{1-z(1+w)}\Big|_{z^i w^j} = \binom{i}{j} \qquad (L^{-1})_{ij} = (-1)^{i+j}\binom{i}{j}$$

$$U_{ij} = \frac{1}{1-w(1+2z)}\Big|_{z^i w^j} = 2^i \binom{j}{i}$$

Partition function:

$$\boxed{Z_n = \det A = \prod_0^n U_{ii} = 2^{n(n+1)/2}}$$

$Z_{n,\ell}$

LGV matrix:

$$\tilde{A}_{ij} = \begin{cases} A_{ij} & j<n \\ A_{i+\ell-n,n} & j=n \end{cases}$$

LU decomposition:

$$L^{-1}\tilde{A} = \tilde{U}$$

$$\tilde{U}_{ij} = \begin{cases} U_{ij} & j<n \\ \sum_{k}^{n} L^{-1}_{ik}\tilde{A}_{kn} & j=n \end{cases}$$

1-pt function:

$$\boxed{H_{n,\ell} = \frac{\det\tilde{A}}{\det A} = \frac{\tilde{U}_{n,n}}{U_{n,n}} = \frac{1}{2^n}\sum_{j=0}^{\ell}\binom{n}{j}}$$

Proof:

$$\tilde{U}_{n,\ell} = \sum_i \underbrace{L^{-1}_{n,i}}_{(-1)^{n+i}\binom{n}{i}} \tilde{A}_{i,n} = \sum_i L^{-1}_{n,i} \underbrace{A_{i+\ell-n,n}}_{\left.\frac{1}{1-z-w-zw}\right|_{z^{i+\ell-n}w^{n}}}$$

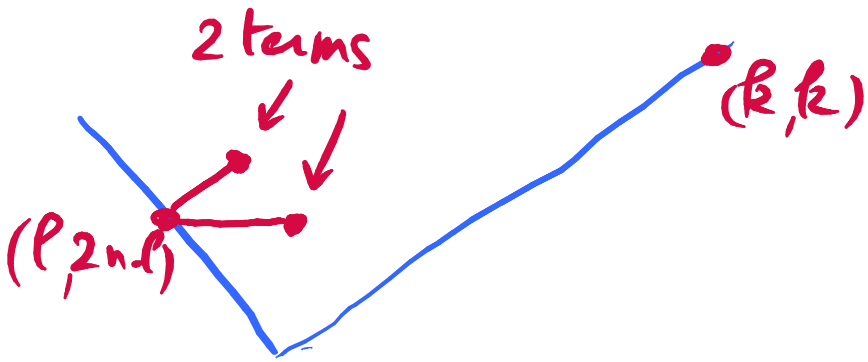
$$= \sum_i (-z)^{n-i}\binom{n}{i} \left.\frac{1}{1-z-w-zw}\right|_{z^{\ell}w^{n}}$$

$$= \left.\frac{(1-z)^n}{1-z-w-zw}\right|_{z^{\ell}w^{n}} = \left.(1-z)^n \frac{(1+z)^n}{(1-z)^{n+1}}\right|_{z^{\ell}} = \left.\frac{(1+z)^n}{1-z}\right|_{z^{\ell}}$$

$$= \sum_{0}^{\ell} \binom{n}{j} \qquad \text{qed.}$$

$Y_{\ell,k}$

Single path from $(\ell, 2n-\ell) \longrightarrow (k,k)$ exiting diamond



$$\boxed{Y_{\ell,k} = A_{n-\ell,\,k-n-1} + A_{n-\ell-1,\,k-n-1}}$$

Tangent method: asymptotics of $\sum_\ell H_{n,\ell} Y_{\ell,k}$

Scaling: $\boxed{n \text{ large} \quad \ell = n\xi \quad k = nz \quad \xi \in (0,1),\ z > 1.}$

$Y_{n,k}$ $\quad Y_{n\xi, nx} \sim 2A_{n(1-\xi), n(z-1)} \sim \int_0^{\mathrm{Min}(1-\xi, x-1)} d\theta \, e^{S_0(\theta,\xi,z)}$

$$\boxed{S_0(\theta,\xi,z) \underset{\text{Stirling}}{=} (z-\xi-\theta)\log(z-\xi-\theta) - \theta\log\theta - (1-\xi-\theta)\log(1-\xi-\theta) - (z-1-\theta)\log(z-1-\theta)}$$

$H_{n\ell}$ $\quad H_{n,n\xi} \sim \frac{1}{2^n}\sum_{j=0}^{\ell}\binom{n}{j} \sim \int_0^{\xi} d\varphi \, e^{nS_1(\varphi,z)}$

$$\boxed{S_1(\varphi,z) = -\varphi\log\varphi - (1-\varphi)\log(1-\varphi) - \log 2}$$

Saddle point:

total action: $S = S_0 + S_1 \quad (\varphi, \theta, \zeta, z)$

$$\frac{\partial S}{\partial \varphi} = 0 \implies \varphi_0 = \frac{1}{2}$$

$$\begin{cases} (1)\ \zeta > \frac{1}{2} & \text{then} \quad S_1(\varphi_0, z) = 0 \quad \text{and } H_{n,n\zeta} \sim 1 \\ (2)\ \zeta < \frac{1}{2} & \text{then} \quad S_1(\zeta, z) \text{ dominates } = -\zeta \log \zeta - (1-\zeta) \log(1-\zeta) - \log 2 \end{cases}$$

$$\begin{cases} (1)\ \zeta > \frac{1}{2} & S = S_0(\theta, \zeta, z) \\ (2)\ \zeta < \frac{1}{2} & S = S_0(\theta, \zeta, z) + S_1(\zeta, z) \end{cases}$$

Now extremize S over θ, ζ : $\dfrac{\partial S}{\partial \theta} = \dfrac{\partial S}{\partial \zeta} = 0$

(1) no solution

(2) $(1-\zeta-\theta)(z-1-\theta) = \theta(z-\zeta-\theta)$ and $(1-\zeta-\theta)(1-\zeta) = (z-\zeta-\theta)\zeta$

$$\Rightarrow \boxed{\zeta_0(z) = \frac{1}{2z}}$$

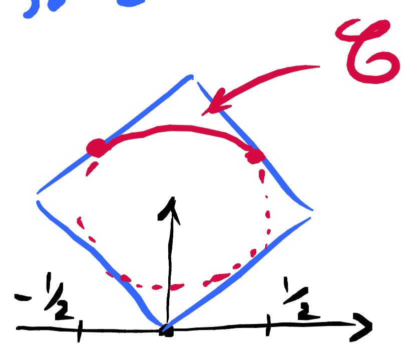
most likely exit point $= (\zeta_0 , 2-\zeta_0)$

Tangent Family:

$$L(x,y) = y - \frac{2-\zeta_0-z}{\zeta_0-z}\, x + 2z\, \frac{1-\zeta_0}{\zeta_0-z} = 0$$

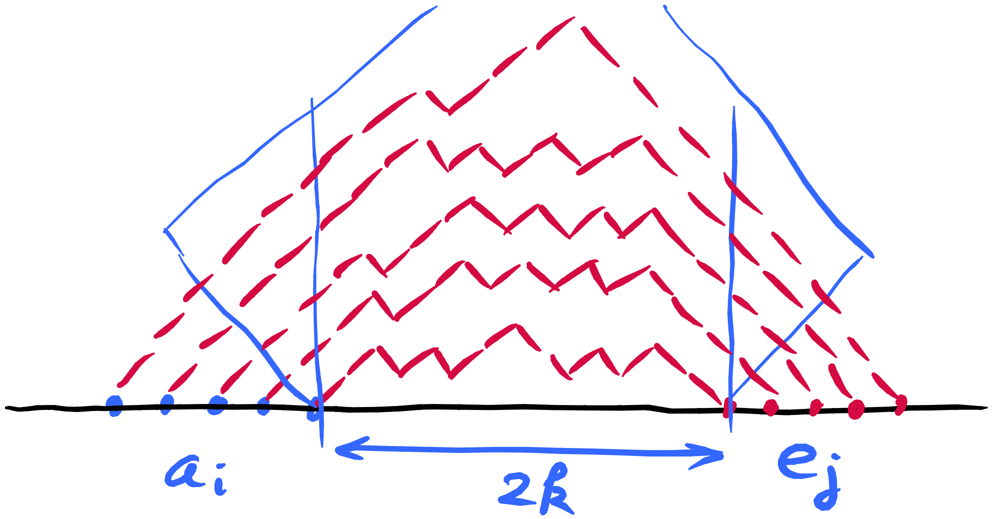
Envelope $\qquad \dfrac{\partial L}{\partial z} = L = 0$

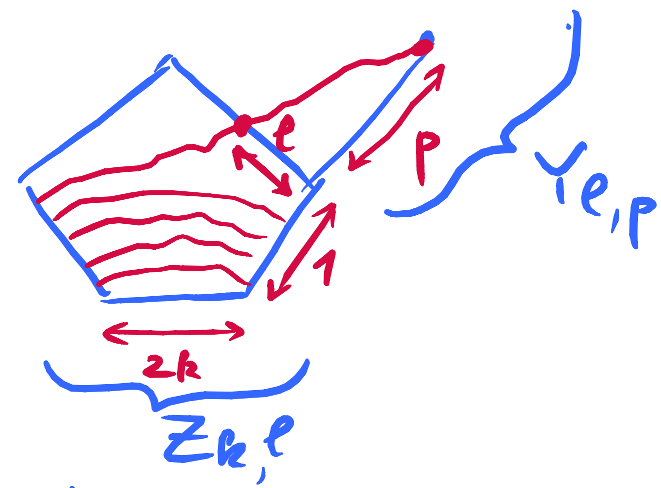
$\mathcal{C}$: $\boxed{x^2 + (y-1)^2 = \frac{1}{2}}$ $\qquad x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$



3. DYCK PATHS

Tangent Method setting:





Z_n

LGV matrix $\quad A_{i,j} = C_{i+j+k} \quad 0 \le i,j \le n$

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$ Catalan number.

[Krattenthaler] $\rightarrow$ $\det(C_{i+\alpha_i})$ formula

LU decomposition:

$$\boxed{L_{ij} = \frac{(2k+2i)!\,(k+j)!\,(k+2j+1)!}{(2k+2j)!\,(k+i)!\,(k+i+j+1)!}\binom{i}{j}}$$

$$\boxed{(L^{-1})_{ij} = (-1)^{i+j}\,\frac{(2k+2i)!\,(k+j)!\,(k+i+j)!}{(2k+2j)!\,(k+i)!\,(k+2i)!}\binom{i}{j}}$$

$$\Rightarrow \boxed{U_{n,n} = \frac{(2n+1)!\,(2n+2k)!}{(k+2n+1)!\,(k+2n)!}}$$

From combinatorial identity:

$$\sum_{m=0}^{i}(-1)^{m+i}\binom{k+m+i}{i-j}\binom{k+i+j+1}{i-m}\binom{k+m}{m}\binom{2k+2m+2j}{2j} = \begin{cases}(2i+1)\binom{k+i}{k} & (i=j)\\ 0 & (i>j)\end{cases}$$

$Z_{n,\ell}$

LGV matrix:

$$\tilde{A}_{i,j} = \begin{cases} A_{ij} & (j<n) \\ \dfrac{n+\ell+1}{n+i+k+1}\dbinom{2k+2i+n-\ell}{k+i-\ell} & (j=n) \end{cases}$$

1pt function:

$$H_{n,k,\ell} = \frac{\det(\tilde{A})}{\det(A)} = \frac{\tilde{U}_{n,n}}{U_{n,n}}$$

$\boxed{\text{THM} \quad H_{n,k,\ell} = \dfrac{1}{\dbinom{2n+2k}{n+\ell}} \displaystyle\sum_{s=0}^{n} \dbinom{n+\ell+1}{2n+1-2s}\dbinom{2n+k-s}{n+\ell}}$

follows from a very non-trivial combinatorial identity:

$$(1)\quad \frac{\tilde{U}_{nn}}{U_{n,n}} = \frac{(n+\ell+1)\; n!\,(2n+k+1)!}{(2k+1)!\,(n+k)!} \sum_{r=0}^{n} (-1)^{n+r} \frac{(k+r)!\,(n+k+r)!\,(2k+2r+n-\ell)!}{r!\,(2k+2r)!\,(n-r)!\,(n+k+r+1)!\,(k+r-\ell)!}$$

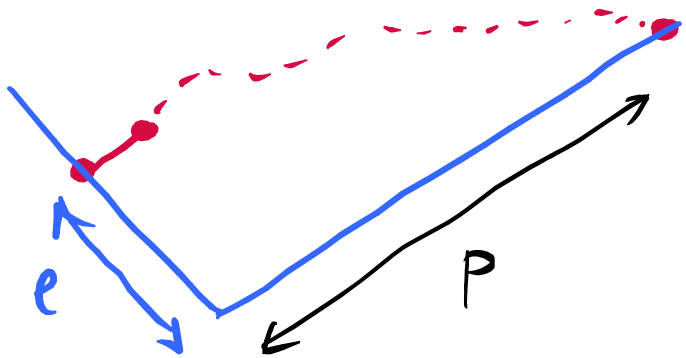
alternating sum

$$(2)\quad = \frac{(n+\ell+1)!\,(2k+n-\ell)!}{(2n+2k)!} \sum_{s=0}^{n} \frac{(2n+k-s)!}{(\ell+2s-n)!\,(n+k-\ell-s)!\,(2n+\ell-2s)!}$$

positive sum

Proof: by interpolation.

$Y_{e,p}$



$$\boxed{Y_{e,p} = \binom{p+e-1}{e}}$$

Tangent Method: asymptotics of $\sum_\ell H_{n,k,\ell}\, Y_{\ell,p}$

Scaling: n large, $\ell = \xi n$, $k = x n$, $p = y n$

$Y_{n,\ell}$ $$Y_{\xi n, yn} \sim e^{n S_0(\xi, y)}$$

$$\boxed{S_0(\xi,y) = (y+\xi)\log(y+\xi) - \xi\log\xi - y\log y}$$

$H_{n,k,\ell}$ $$H_{n,xn,\xi n} \sim \int_0^x d\sigma\, e^{S_1(\sigma,\xi,x)}$$

$$\boxed{\begin{aligned} S_1(\sigma,\xi,x) = {} & (1+\xi)\log(1+\xi) + (2x+1-\xi)\log(2x+1-\xi) - (2x+2)\log(2x+2) \\ & + (x+2-\sigma)\log(x+2-\sigma) - (\xi+2\sigma-1)\log(\xi+2\sigma-1) \\ & - (1+x-\sigma-\xi)\log(1+x-\sigma-\xi) - (2-2\sigma)\log(2-2\sigma) \end{aligned}}$$

(and constraints $0\le\sigma\le1$ $\quad 0\le\zeta\le1+x$ $\quad \zeta+2\sigma\ge1$)

Extremization : $S=S_0+S_1 \qquad \partial_\sigma S=\partial_\zeta S=0$

$$\Rightarrow \sigma = \frac{2+3x-(2+x)\zeta}{3+4x-\zeta} = 1 - \frac{x+(2+x)\zeta}{3+4x-\zeta} \le 1$$

- if $\sigma\ge0 \iff \zeta<\frac{2+3x}{2+x}$ then $\zeta=2x+1$ no solution.
- hence $\sigma=0$ and $\partial_\zeta S=0 \Rightarrow \boxed{(1+\zeta)(y+\zeta)(1+x-\zeta)=\zeta(1+2x-\zeta)(\zeta-1)}$ (*)

  $\zeta>\frac{2+3x}{2+x}$

Tangent family $L=v-au-b=0 \quad a=\frac{y-\zeta}{y+\zeta} \quad b=2\frac{(1+x+y)\zeta-y}{y+\zeta}$

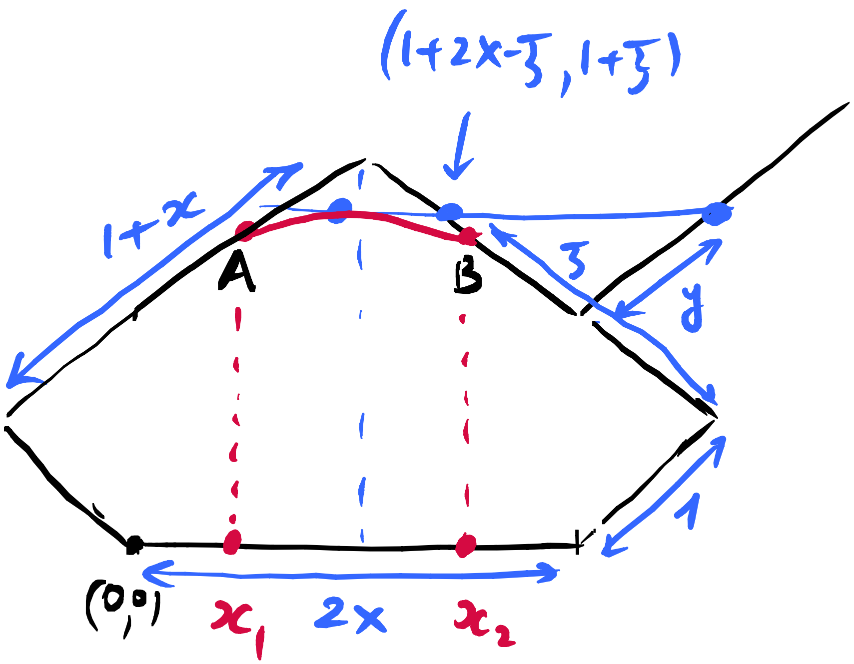
$\zeta=\zeta(x,y)$ solution of (*)

(*) $\Rightarrow y = \dfrac{\zeta\left((2+x)\zeta-(2+3x)\right)}{(1+\zeta)(1+x-\zeta)}$ envelope : $\dfrac{\partial L}{\partial\zeta}=L=0$

Arctic Curve (ellipse)

$(\mathscr{C})$

$$\boxed{x^2 v^2 + 4(1+x)\,u(u-2x) = 0}$$



Tangency points A, B

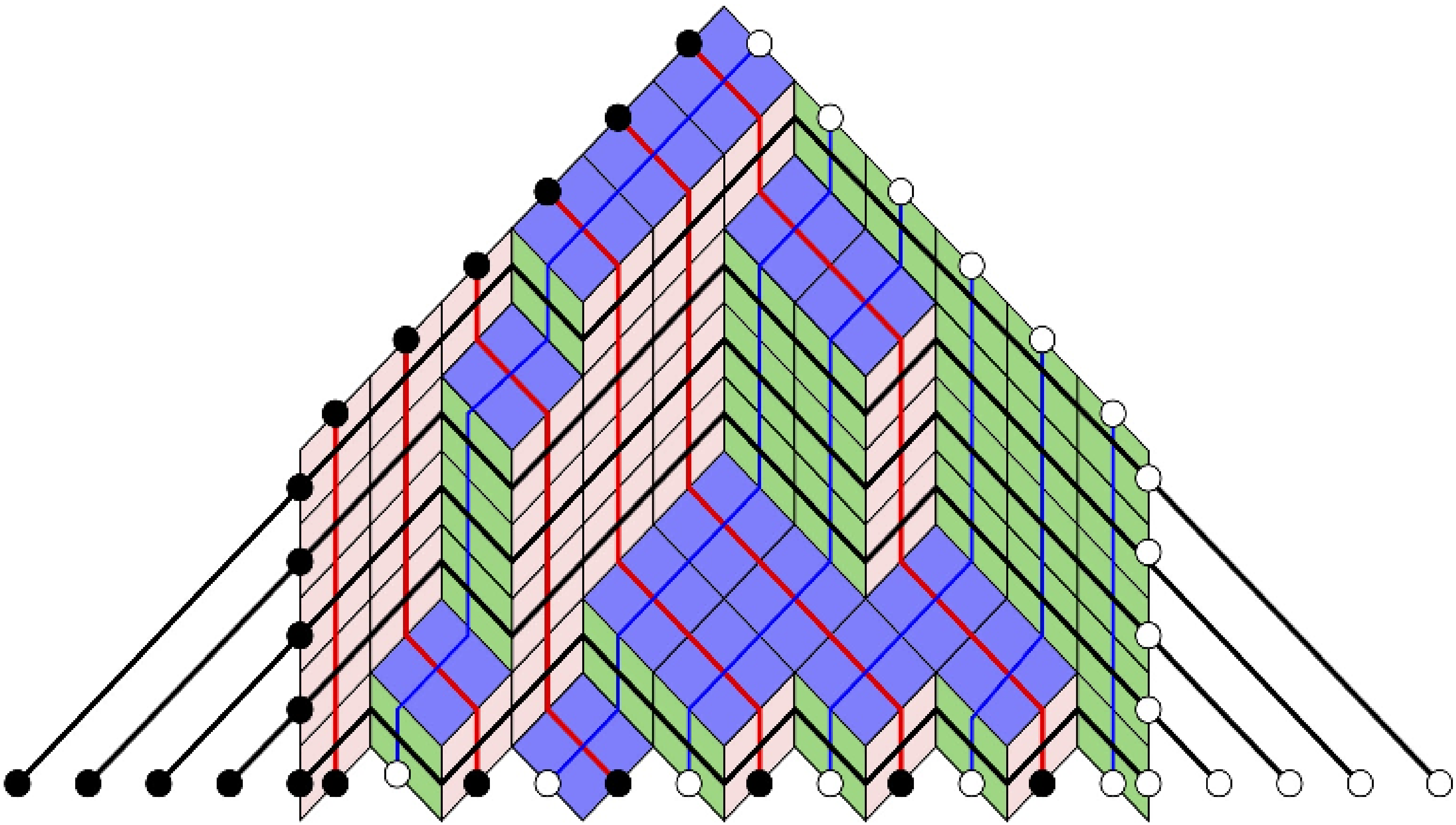
- A: $y \to \infty$ $\quad \xi = 1+x$

$$x_1 = \frac{2x}{2+x}$$

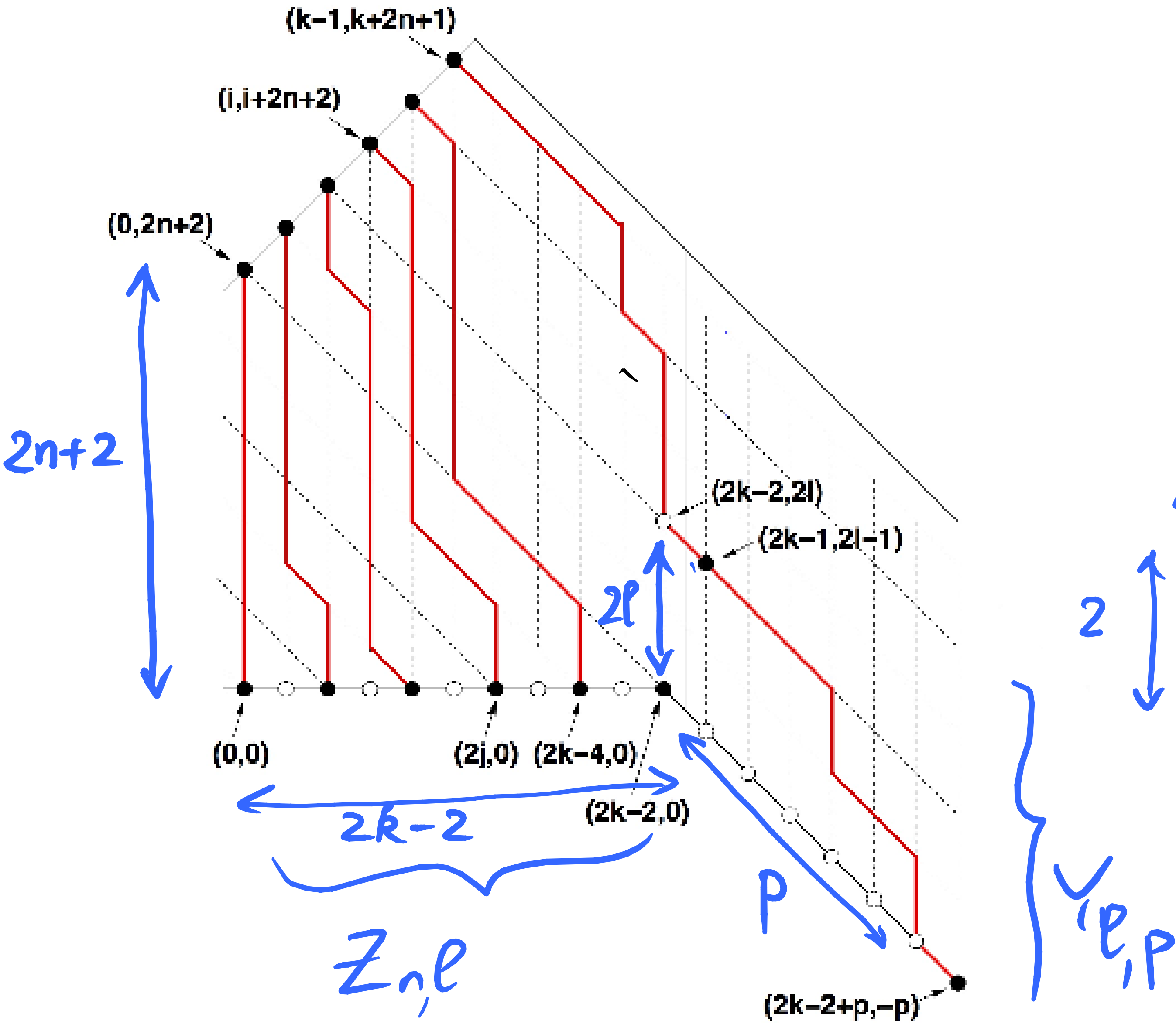
- B: $y \to 0$ $\quad \xi = \frac{2+3x}{2+x}$

$$x_2 = \frac{2x(1+x)}{2+x}$$

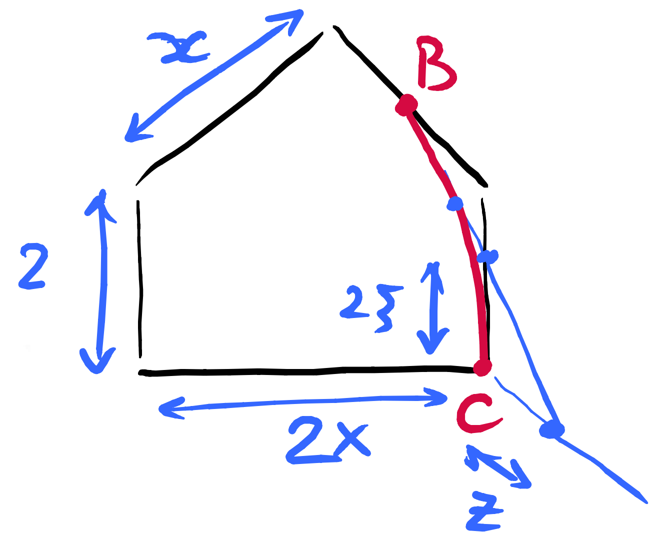
What about the rest of the curve ? TILING problem !



TANGENT METHOD AGAIN



⇒ rest of the arctic ellipse



2n)

Gessel-Viennot matrix

$$A_{ij} = \binom{j+n+1}{2j-i} \quad (0 \le i,j \le k-1)$$

LU decomposition

$$L_{ij} = \frac{i!\,(j+2n+2)!}{(i-j)!\,(2j-i)!\,(i+2n+2)!}$$

$$(L^{-1})_{ij} = (-1)^{i+j} \frac{(j+2n+2)!\,(2i-j-1)!}{(i+2n+2)!\,(i-j)!\,(j-1)!} \quad (0 \le i,j \le k-1)$$

$$U_{k-1,k-1} = \frac{(2n+2k)!\,(k-1)!}{(2n+k+1)!\,(2k-2)!}$$

$Z_{n,\ell}$)

Gessel-Viennot matrix

$$\tilde{A}_{ij} := \begin{cases} A_{ij} & (0 \le j < k-1) \\ \binom{n+k-\ell}{2k-2-i} & (j = k-1) \end{cases}$$

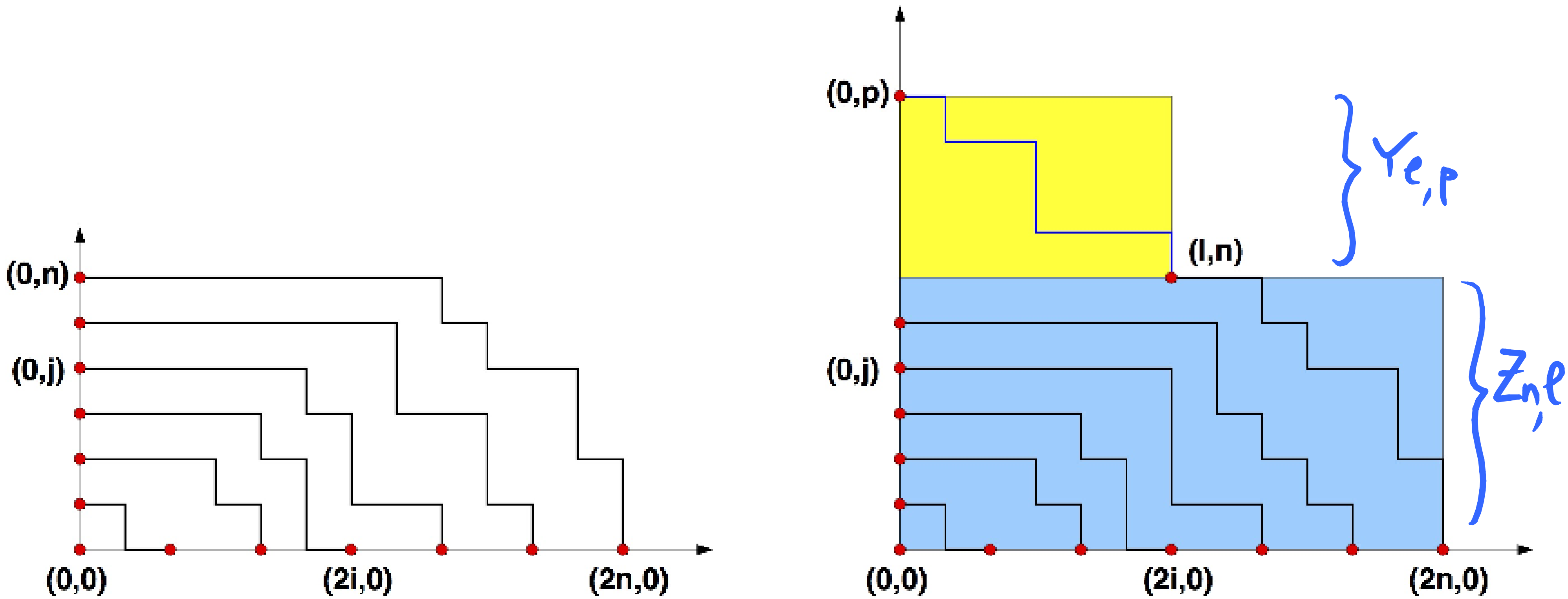
$$\tilde{U}_{k-1,k-1} = \sum_i (L^{-1})_{k-1,i} \, \tilde{A}_{i,k-1} = \text{signed sum}$$

THM

$$\boxed{H_{n,\ell} = \frac{\tilde{U}_{k-1,k-1}}{U_{k-1,k-1}} = \frac{2}{\binom{2n+2k}{2n+3}} \sum_{s=\ell}^{n+1} \binom{k+n-s}{k-2} \binom{k+n+s-1}{k-2}}$$

(6) = remainder of ellipse.

4. ANOTHER PATH MODEL



(a) (b)

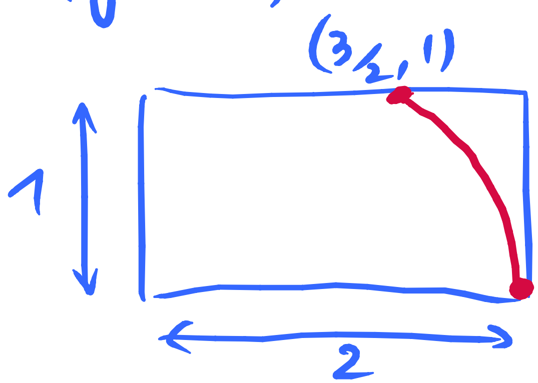
$$Z_n = 2^{n(n+1)/2} \quad \text{(same as domino tilings!)}$$

$$H_{n,\ell} = \frac{1}{2^n} \sum_{i=0}^{\min(n,2n-\ell)} \binom{n}{i}$$

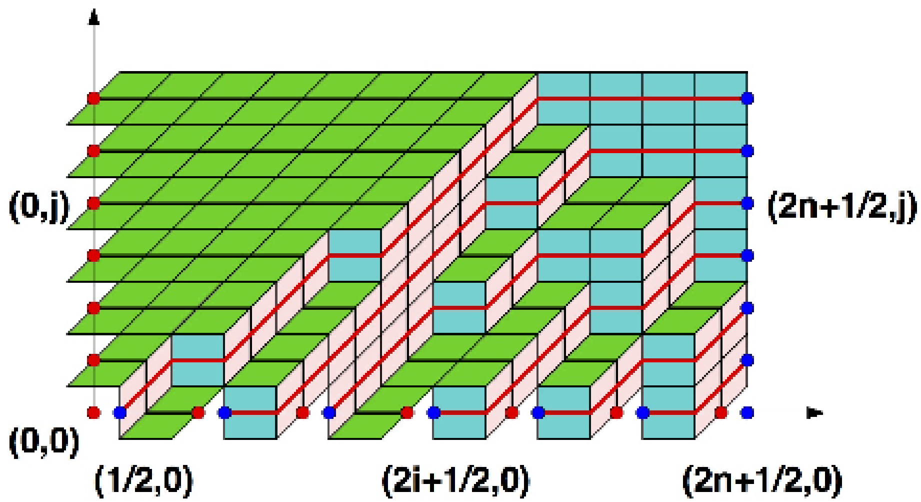
Arctic Curve

(6) $4x^2 + y^2 - 4xy - 8x + 8y = 0$

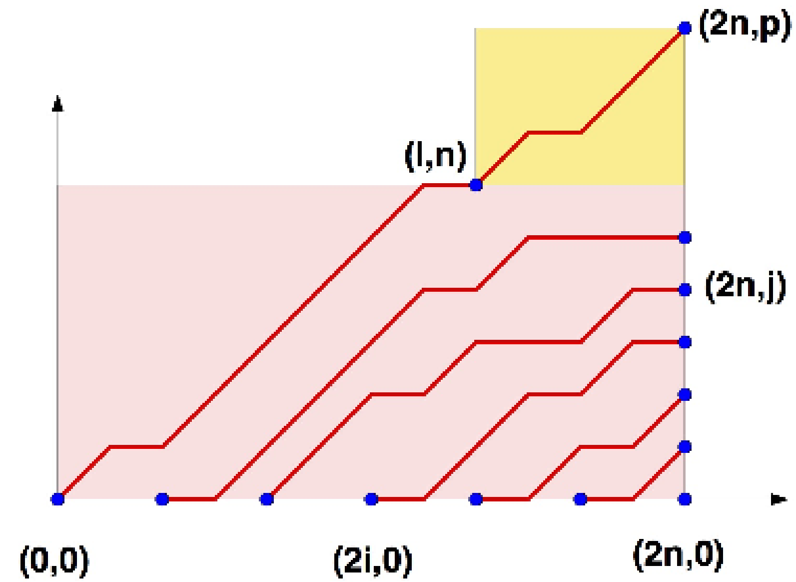
$$(y-2x)^2 = 8(x-y) \quad \text{parabola!}$$



Equivalent Tiling problem:

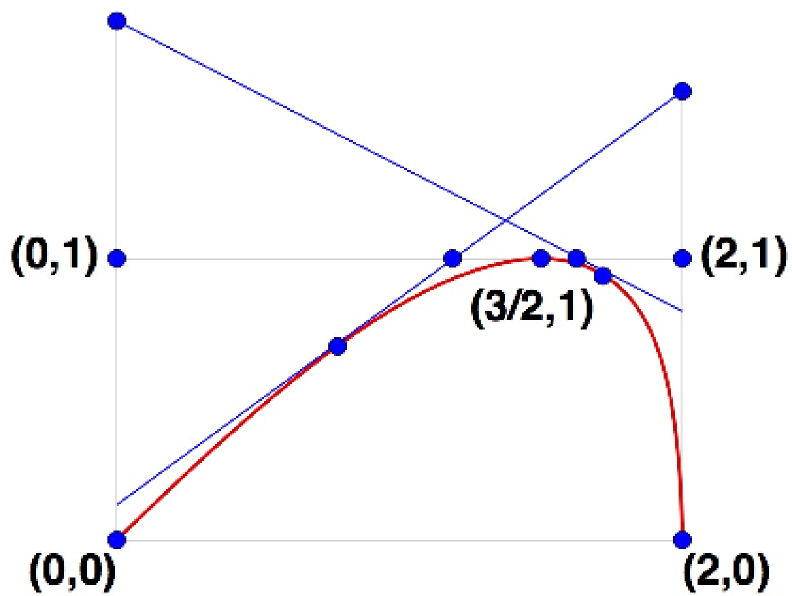


(a)



(b)

Arctic Curve

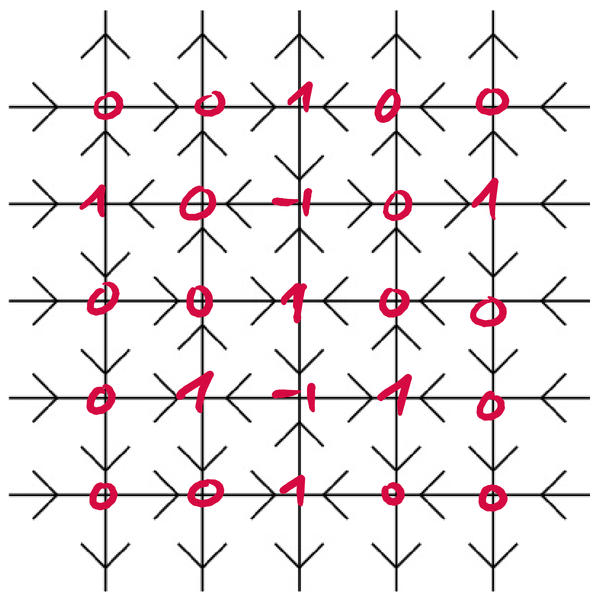


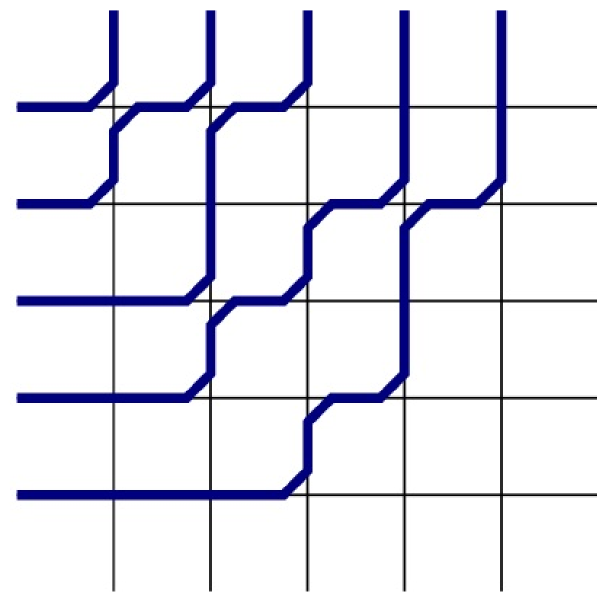
5. Vertically Symmetric Alternating sign matrices

ASM

|||

6 Vertex



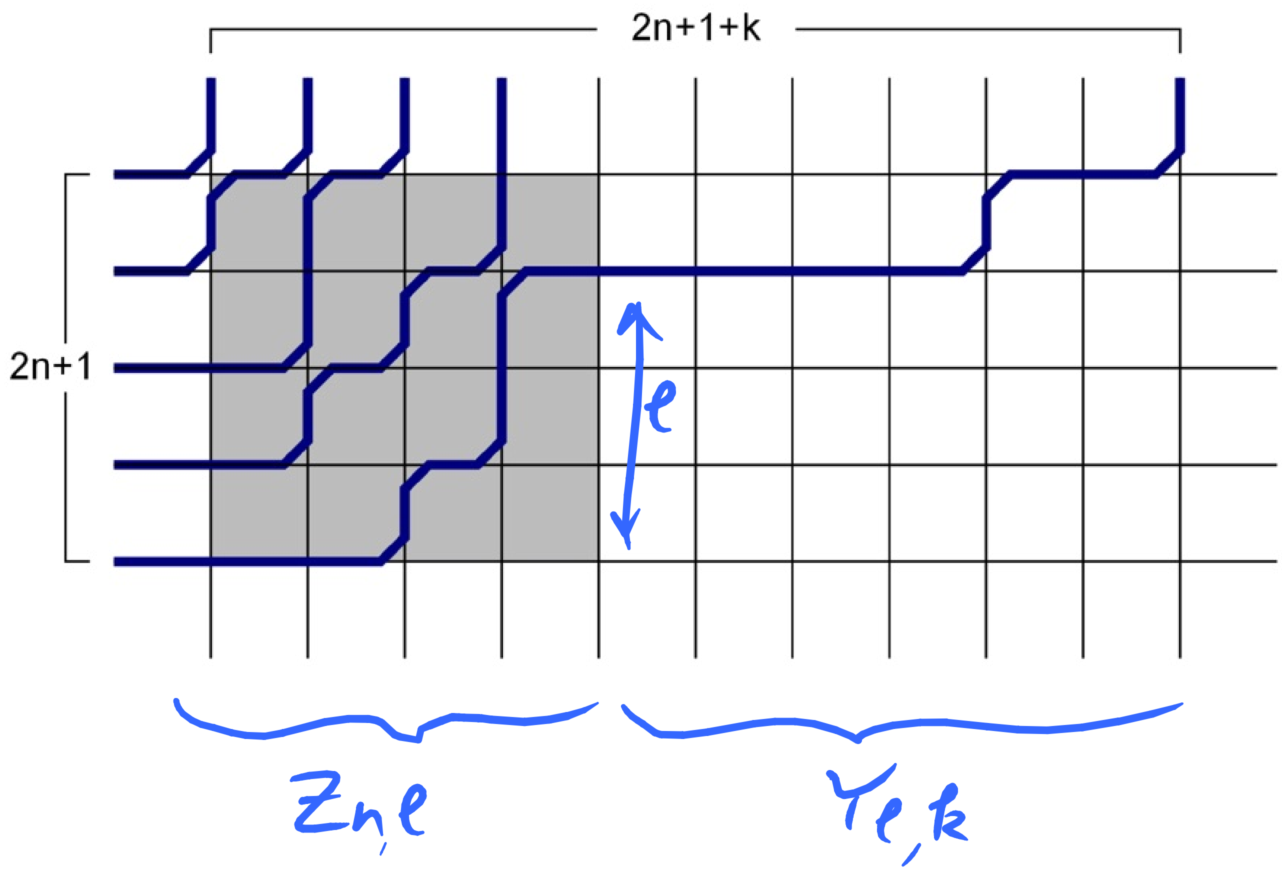


VSASM

symmetric wrt vertical line.

OSCULATING PATHS

TANGENT METHOD



Crucial relation [Razumov-Stroganov 04]

$$\frac{1}{N_{VSASM}(2n+1)} \sum_{\ell=1}^{2n} N_{VSASM}(2n+1,\ell)\, t^{\ell-1}$$

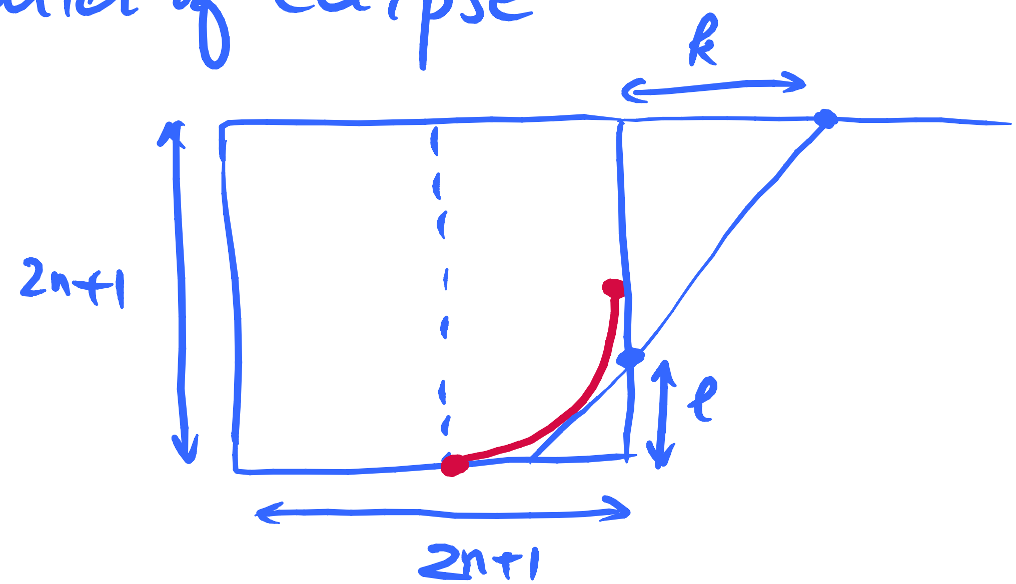
← position of 1 in last column

$$= \frac{1}{N_{ASM}(2n-1)} \frac{t}{1+t} \sum_{i=1}^{2n} N_{ASM}(2n,i)\, t^{i-1}$$

$$H_{n,\ell} = \frac{N_{VSASM}(2n+1,\ell)}{N_{VSASM}(2n+1)} \qquad Y_{\ell,k} = \sum_{i=0}^{\min(k-1,2n+1-\ell)} \binom{k-1}{i}\binom{2n+1-\ell}{i}$$

(8) $\boxed{4\left((x-1)^2 + y^2 - xy\right) + 4(x-1) + 8y + 1 = 0}$

$\Rightarrow$ quarter of ellipse



complete by symmetry = <u>same result as ASMs</u>

CONCLUSION

- It works, but why?

→ must show tangency to $\mathscr{C}$

- Beyond NILP = it still works. Why?

  and what kind of interaction can we allow

- many other examples
  - Osculating Schröder
  - inhomogeneous weights
  - fused 6V ...

DANKE!

Reference: arXiv:1711.03182 [math-ph]